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Distributive Lattices, Bipartite Graphs and Alexander Duality

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Abstract. A certain squarefree monomial ideal H_P arising from a finite partially ordered set P will be studied from viewpoints of both commutative algbera and combinatorics. First, it is proved that the defining ideal of the Rees algebra of H_P possesses a quadratic Gröbner basis. Thus in particular all powers of H_P have linear resolutions. Second, the minimal free graded resolution of H_P will be constructed explicitly and a combinatorial formula to compute the Betti numbers of H_P will be presented. Third, by using the fact that the Alexander dual of the simplicial complex Δ whose Stanley–Reisner ideal coincides with H_P is Cohen–Macaulay, all the Cohen–Macaulay bipartite graphs will be classified.

Introduction

Let P be a finite partially ordered set (*poset* for short) and write $\mathcal{J}(P)$ for the finite poset which consists of all poset ideals of P, ordered by inclusion. Here a *poset ideal* of P is a subset I of P such that if $x \in I$, $y \in P$ and $y \leq x$, then $y \in I$. In particular the empty set as well as P itself is a poset ideal of P. It follows easily that $\mathcal{J}(P)$ is a finite distributive lattice [12, p. 106]. Conversely, Birkhoff's fundamental structure theorem [12, Theorem 3.4.1] guarantees that, for any finite distributive lattice $\mathcal{L}$, there exists a unique poset P such that $\mathcal{L} = \mathcal{J}(P)$.

Let P be a finite poset with $|P| = n$, where $|P|$ is the cardinality of P, and let $S = K[\{x_p, y_p\}_{p \in P}]$ denote the polynomial ring in $2n$ variables over a field K with each $\deg x_p = \deg y_p = 1$.

We associate each poset ideal I of P with the squarefree monomial

$$u_I = \left(\prod_{p \in I} x_p\right)\left(\prod_{p \in P \setminus I} y_p\right)$$

of S of degree n. In particular $u_P = \prod_{p \in P} x_p$ and $u_\emptyset = \prod_{p \in P} y_p$.

The normal affine semigroup ring $K[\{u_I\}_{I\in\mathcal{J}(P)}]$ is studied in [9] from viewpoints of both commutative algebra and combinatorics.

In the present paper, however, we are interested in the squarefree monomial ideal

$$H_P = \big(\{u_I\}_{I\in\mathcal{J}(P)}\big)$$

of S generated by all u_I with $I \in \mathcal{J}(P)$.

The outline of the present paper is as follows. First, in Section 1 we study the Rees algebra $\mathcal{R}(H_P)$ of H_P and establish our fundamental Theorem 1.1 which says that the defining ideal of $\mathcal{R}(H_P)$ possesses a reduced Gröbner basis consisting of quadratic binomials whose initial monomials are squarefree. Thus $\mathcal{R}(H_P)$ turns out to be normal and Koszul (Corollary 1.2), and all powers of H_P have linear resolutions (Corollary 1.3).

Second, in Section 2 the minimal graded free S-resolution of H_P is constructed explicitly. See Theorem 2.1. The resolution tells us how to compute the Betti numbers $\beta_i(H_P)$ of H_P in terms of the combinatorics of the distributive lattice $\mathcal{L} = \mathcal{J}(P)$. In fact, if $b_i(\mathcal{L})$ is the number of intervals $[I, J]$ of $\mathcal{L} = \mathcal{J}(P)$ which are Boolean lattices of rank i, then the ith Betti number $\beta_i(H_P)$ of H_P coincides with $b_i(\mathcal{L})$. See Corollary 2.2. (A Boolean lattice of rank i is the distributive lattice B_i which consists of all subsets of $\{1, \dots, i\}$, ordered by inclusion.) Thus in particular for a finite distributive lattice $\mathcal{L} = \mathcal{J}(P)$, one has $\sum_{i\geq 0}(-1)^i b_i(\mathcal{L}) = 1$. See Corollary 2.3. In addition, it is shown that the ideal H_P is of height 2 and a formula to compute the multiplicity of S/H_P will be given. See Proposition 2.4 (and Corollary 2.5).

Let Δ_P denote the simplicial complex on the vertex set $\{x_p, y_p\}_{p\in P}$ such that the squarefree monomial ideal H_P coincides with the Stanley–Reisner ideal I_{Δ_P}. In Section 3 the Alexander dual $\Delta_P^\vee$ of Δ_P will be studied. Since the Stanley–Reisner ideal $H_P = I_{\Delta_P}$ has a linear resolution, it follows from [4, Theorem 3] that $\Delta_P^\vee$ is Cohen–Macaulay. It will turn out that the Stanley–Reisner ideal $I_{\Delta_P^\vee}$ of $\Delta_P^\vee$ is an edge ideal of a finite bipartite graph. Somewhat surprisingly, this simple observation enables us to classify all Cohen–Macaulay bipartite graphs. In fact, Theorem 3.4 says that a finite bipartite graph G is Cohen–Macaulay if and only if G comes from the comparability graph of a finite poset.

1. Monomial ideals arising from distributive lattices

Work with the same notation as in Introduction. Let P be a finite poset with $|P| = n$ and $S = K[\{x_p, y_p\}_{p\in P}]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_p = \deg y_p = 1$. Recall that we associate each poset ideal I of P with the squarefree monomial $u_I = (\prod_{p\in I} x_p)(\prod_{p\in P\setminus I} y_p)$ of S of degree n, and introduce the ideal $H_P = (\{u_I\}_{I\in\mathcal{J}(P)})$ of S.

Let $\mathcal{R}(H_P)$ denote the Rees algebra of H_P and $\mathcal{W}_P$ the defining ideal of $\mathcal{R}(H_P)$. In other words, $\mathcal{R}(H_P)$ is the affine semigroup ring

$$\mathcal{R}(H_P) = K\big[\{x_p, y_p\}_{p\in P}, \{u_I t\}_{I\in\mathcal{J}(P)}\big] \quad (\subset K[\{x_p, y_p\}_{p\in P}, t])$$

and $\mathcal{W}_P$ is the kernel of the surjective ring homomorphism $\varphi : K[\mathbf{x}, \mathbf{y}, \mathbf{z}] \to \mathcal{R}(H_P)$, where

$$K[\mathbf{x}, \mathbf{y}, \mathbf{z}] = K\left[\{x_p, y_p\}_{p \in P}, \{z_I\}_{I \in \mathcal{J}(P)}\right]$$

is the polynomial ring over K and where φ is defined by setting $\varphi(x_p) = x_p$, $\varphi(y_p) = y_p$ and $\varphi(z_I) = u_I t$.

For the convenience of our discussion, in the remainder of the present section, we will use the notation $P = \{p_1, \ldots, p_n\}$ and write x_i, y_i instead of x_{p_i}, y_{p_i}. Let $<_{lex}$ denote the lexicographic order [5, p. 329] on S induced by the ordering $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ and $<^\sharp$ the reverse lexicographic order [5, p. 330] on $K[\{z_I\}_{I \in \mathcal{J}(P)}]$ induced by an ordering of the variables z_I's such that $z_I > z_J$ if $J \subset I$ in $\mathcal{J}(P)$. We then introduce the new monomial order $<^\sharp_{\rm lex}$ on T by setting

$$\left(\prod_{i=1}^n x_i^{a_i} y_i^{b_i}\right)\left(z_{I_1} \cdots z_{I_q}\right) <^\sharp_{lex} \left(\prod_{i=1}^n x_i^{a'_i} y_i^{b'_i}\right)\left(z_{I'_1} \cdots z_{I'_{q'}}\right)$$

if either

(i) $\prod_{i=1}^n x_i^{a_i} y_i^{b_i} <_{lex} \prod_{i=1}^n x_i^{a'_i} y_i^{b'_i}$

or

(ii) $\prod_{i=1}^n x_i^{a_i} y_i^{b_i} = \prod_{i=1}^n x_i^{a'_i} y_i^{b'_i}$ and $z_{I_1} \cdots z_{I_q} <^\sharp z_{I'_1} \cdots z_{I'_{q'}}$.

Theorem 1.1 *The reduced Gröbner basis $\mathcal{G}_{<^\sharp_{lex}}(\mathcal{W}_P)$ of the defining ideal $\mathcal{W}_P \subset K[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ with respect to the monomial order $<^\sharp_{\rm lex}$ consists of quadratic binomials whose initial monomials are squarefree.*

Proof: The reduced Gröbner basis of $\mathcal{W}_P \cap K[\{z_I\}_{I \in \mathcal{J}(P)}]$ with respect to the reverse lexicographic order $<^\sharp$ coincides with $\mathcal{G}_{<^\sharp_{lex}}(\mathcal{W}_P) \cap K[\{z_I\}_{I \in \mathcal{J}(P)}]$. It follows from [9] that $\mathcal{G}_{<^\sharp_{lex}}(\mathcal{W}_P) \cap K[\{z_I\}_{I \in \mathcal{J}(P)}]$ consists of those binomials

$$z_I z_J - z_{I \wedge J} z_{I \vee J}$$

such that I and J are incomparable in the distributive lattice $\mathcal{J}(P)$.

It is known [14, Corollary 4.4] that the reduced Gröbner basis of $\mathcal{W}_P$ consists of irreducible binomials of $K[\mathbf{x}, \mathbf{y}, \mathbf{z}]$. Let

$$f = \left(\prod_{i=1}^n x_i^{a_i} y_i^{b_i}\right)\left(z_{I_1} \cdots z_{I_q}\right) - \left(\prod_{i=1}^n x_i^{a'_i} y_i^{b'_i}\right)\left(z_{I'_1} \cdots z_{I'_q}\right)$$

be an irreducible binomial of $K[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ belonging to $\mathcal{G}_{<^\sharp_{lex}}(\mathcal{W}_P)$ with

$$\left(\prod_{i=1}^{n} x_i^{a_i} y_i^{b_i}\right)(z_{I_1} \cdots z_{I_q})$$

its initial monomial, where $z_{I_1} \leq \cdots \leq z_{I_q}$ and $z_{I'_1} \leq \cdots \leq z_{I'_q}$.

Let $f \notin K[\{z_I\}_{I \in \mathcal{J}(P)}]$. Let j denote an integer for which $I'_j \not\subset I_j$. Such an integer exists. In fact, if $I'_j \subset I_j$ for all j, then each $a_i = 0$ and each $b'_i = 0$. This is impossible since $(\prod_{i=1}^{n} x_i^{a_i} y_i^{b_i})(z_{I_1} \cdots z_{I_q})$ is the initial monomial of f.

Let $p_i \in I'_j \setminus I_j$. Then p_i belongs to each of $I'_j, I'_{j+1}, \ldots, I'_q$, and does not belong to each of $I_1, I_2, \ldots, I_j$. Hence $a_i > 0$.

Let $p_{i_0} \in P$ with $p_{i_0} \in I'_j \setminus I_j$ such that $I_j \cup \{p_{i_0}\} \in \mathcal{J}(P)$. Thus $a_{i_0} > 0$. Let $J = I_j \cup \{p_{i_0}\}$. Then the binomial $g = x_{i_0} z_{I_j} - y_{i_0} z_J$ belongs to $\mathcal{W}_P$ with $x_{i_0} z_{I_j}$ its initial monomial. Since $x_{i_0} z_{I_j}$ divides the initial monomial of f, it follows that the initial monomial of f must coincides with $x_{i_0} z_I$, as desired. □

It is well known that a homogeneous affine semigroup ring whose defining ideal has an initial ideal which is generated by squarefree (resp. quadratic) monomials is normal (resp. Koszul). See, e.g., [14, Proposition 13.15] and [6].

Corollary 1.2 *Let P be an arbitrary finite poset. Then the Rees algebra $\mathcal{R}(H_P)$ is normal and Koszul.*

On the other hand, Stefan Blum [2] proved that if the Rees algebra of an ideal is Koszul, then all powers of the ideal have linear resolutions.

Corollary 1.3 *Let P be an arbitrary finite poset. Then all powers of H_P have linear resolutions.*

2. The free resolution and Betti numbers of H_P

Corollary 1.3 says that the monomial ideal H_P arising from a finite poset P has a linear resolution. The main purpose of the present section is to construct a minimal graded free S-resolution $\mathbb{F} = \mathbb{F}_P$ of H_P explicitly.

Let P be a finite poset with $|P| = n$ and $S = K[\{x_p, y_p\}_{p \in P}]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_p = \deg y_p = 1$. Recall that, for each poset ideal I of P, we associate the squarefree monomial $u_I = (\prod_{p \in I} x_p)(\prod_{p \in P \setminus I} y_p)$ of S of degree n. Let H_P denote the ideal of S generated by all u_I with $I \in \mathcal{J}(P)$.

The maximal elements of a poset ideal I of P are called the *generators* of I. Let $M(I)$ denote the set of generators of I.

The construction of a minimal graded free S-resolution $\mathbb{F} = \mathbb{F}_P$ of H_P is achieved as follows: For all $i \geq 0$ let $\mathbb{F}_i$ denote the free S-module with basis

$$e(I, T),$$

where

$$I \in \mathcal{J}(P),\ T \subset P,\ I \cap T \subset M(I),\ |I \cap T| = i \quad \text{and} \quad |I \cup T| = n + i.$$

Extending the partial order on P to a total order, we define for $i > 0$ the differential

$$\partial :\ \mathbb{F}_i \rightarrow \mathbb{F}_{i-1}$$

by

$$\partial(e(I, T)) = \sum_{p \in I \cap T} (-1)^{\sigma(I \cap T, p)}(x_p e(I \backslash \{p\}, T) - y_p e(I, T \backslash \{p\})),$$

where for a subset $Q \subset P$ and $p \in Q$ we set $\sigma(Q, p) = |\{q \in Q : q < p\}|$.
With the notation introduced we have

Theorem 2.1 *The complex $\mathbb{F}$ is a graded minimal free S-resolution of H_P.*

Proof: We define an augmentation $\varepsilon : \mathbb{F}_0 \rightarrow H_P$ by setting

$$\varepsilon(e(I, T)) = u_I$$

for all $e(I, T) \in \mathbb{F}_0$. Note that if $e(I, T)$ is a basis element of $\mathbb{F}_0$, then $T = [n] \backslash I$, so that ε is well defined.
We first show that

$$\cdots \xrightarrow{\ \partial\ } \mathbb{F}_1 \xrightarrow{\ \partial\ } \mathbb{F}_0 \xrightarrow{\ \varepsilon\ } H_P \longrightarrow 0$$

is a complex.
Let $e(I, T) \in \mathbb{F}_1$ with $I \cap T = \{p\}$. Then

$$\begin{aligned}(\varepsilon \circ \partial)(e(I, T)) &= x_p \varepsilon(e(I \backslash \{p\}, T)) - y_p \varepsilon(e(I, T \backslash \{p\})) \\ &= x_p u_{I \backslash \{p\}} - y_p u_I = 0.\end{aligned}$$

Thus $\partial \circ \varepsilon = 0$, as desired.

Next we show that $\partial \circ \partial = 0$. Let $e(I, T) \in \mathbb{F}_{i+1}$ and set $L = I \cap T$. Then

$$
\begin{aligned}
&\partial \circ \partial(e(I, T)) \\
&= \sum_{p \in L} (-1)^{\sigma(L,p)} (x_p \partial(e(I \setminus \{p\}, T)) - y_p \partial(e(I, T \setminus \{p\})) \\
&= \sum_{p \in L} (-1)^{\sigma(L,p)} \Bigg[x_p \Bigg(\sum_{q \in L, q \neq p} (-1)^{\sigma(L \setminus \{p\}, q)} \\
&\quad \times (x_q e(I \setminus \{p, q\}, T) - y_q e(I \setminus \{p\}, T \setminus \{q\})) \Bigg) \\
&\quad - y_p \Bigg(\sum_{q \in L, q \neq p} (-1)^{\sigma(L \setminus \{p\}, q)} (x_q e(I \setminus \{q\}, T \setminus \{p\}) - y_q e(I, T \setminus \{p, q\})) \Bigg) \Bigg] \\
&= \sum_{p,q \in L, p \neq q} (-1)^{\sigma(L,p) + \sigma(L \setminus \{p\}, q)} x_p x_q e(I \setminus \{p, q\}, T) \\
&\quad - \sum_{p,q \in L, p \neq q} (-1)^{\sigma(L,p) + \sigma(L \setminus \{p\}, q)} x_p y_q e(I \setminus \{p\}, T \setminus \{q\}) \\
&\quad - \sum_{p,q \in L, p \neq q} (-1)^{\sigma(L,p) \sigma(L \setminus \{p\}, q)} x_q y_p e(I \setminus \{q\}, T \setminus \{p\}) \\
&\quad + \sum_{p \in L, p \neq q} (-1)^{\sigma(L,p) + \sigma(L \setminus \{p\}, q)} y_p y_q e(I, T \setminus \{p, q\}) \\
&= 0.
\end{aligned}
$$

The last equality holds since $(-1)^{\sigma(L,p)+\sigma(L\setminus\{p\},q)} = -(-1)^{\sigma(L,q)+\sigma(L\setminus\{q\},p)}$.

In order to prove that the augmented complex

$$
\cdots \longrightarrow \mathbb{F}_1 \xrightarrow{\partial} \mathbb{F}_0 \xrightarrow{\varepsilon} H_P \longrightarrow 0
$$

is exact we show:

(1) $H_0(\mathbb{F}) = H_P$,
(2) $\mathbb{F}$ is acyclic.

For the proof of (1) we note that the Taylor relations

$$
r_{I,J} = x_{J \setminus I} y_{I \setminus J} e(I) - x_{I \setminus J} y_{J \setminus I} e(J), \qquad I, J \in \mathcal{J}(P)
$$

generate the first syzygy module of H_P. Here we set for simplicity $e(I)$ for the basis element $e(I, P\setminus I)$ in $\mathbb{F}_0$, and denote by $x_A y_B$ the monomial $\prod_{p \in A} x_p \prod_{q \in B} y_q$.

Observe that

$$
r_{I,J} = x_{J \setminus I} r_{I,I \cap J} - x_{I \setminus J} r_{J,I \cap J}.
$$

Hence it suffices to show that $r_{I,J} \in \partial(\mathbb{F}_1)$ for all $I, J \in L$ with $J \subset I$. To this end we choose a sequence $J = I_0 \subset I_1 \subset \ldots I_{m-1} \subset I_m = I$ of poset ideals such that $I_j = I_{j-1} \cup \{p_j\}$ for $j = 1, \ldots, m$. Then

$$r_{I,J} = \sum_{j=1}^{m} \left(\prod_{k=j+1}^{m} x_{p_k} \prod_{k=1}^{j-1} y_{p_k} \right) r_{I_j, I_{j-1}}.$$

The assertion follows since $r_{I_j, I_{j-1}} = -\partial(e(I_j, P \setminus I_{j-1}))$ for all j.

We prove (2), that is, the acyclicity of $\mathbb{F}$ by induction on $|P|$. If $P = \{p\}$, then $H_P = (x_p, y_p)$, and $\mathbb{F}$ can be identified with the Koszul complex associated with $\{x_p, y_p\}$, and hence is acyclic.

Suppose now that $|P| > 1$. Let $q \in P$ be a maximal element and let Q be the subposet $P \setminus \{q\}$. We define a map

$$\phi \colon \mathbb{F}_Q \to \mathbb{F}_P, \quad e_i(I, T) \mapsto e_i(I, T \cup \{q\})$$

It is clear that ϕ is an injective map of complexes whose induced map $H_Q = H_0(\mathbb{F}_Q) \to H_0(\mathbb{F}_P) = H_P$ is multiplication by y_q. Let $\mathbb{G}$ be the quotient complex $\mathbb{F}_P/\mathbb{F}_Q$. Since the multiplication map is injective, the short exact sequence of complexes

$$0 \longrightarrow \mathbb{F}_Q \longrightarrow \mathbb{F}_P \longrightarrow \mathbb{G} \longrightarrow 0$$

induces the long exact homology sequence

$$\cdots \longrightarrow H_2(\mathbb{G}) \longrightarrow H_1(\mathbb{F}_Q) \longrightarrow H_1(\mathbb{F}_P) \longrightarrow H_1(\mathbb{G}) \longrightarrow 0$$

By induction hypothesis, $H_i(\mathbb{F}_Q) = 0$ for $i > 0$. Hence it suffices to show that $H_i(\mathbb{G}) = 0$ for $i > 0$.

The principal order ideal (q) consists of all $p \in P$ with $p \leq p$. Let R be the subposet $P \setminus (q)$, and let $\mathbb{C}$ be the mapping cone of the complex homomorphism

$$\mathbb{F}_R \xrightarrow{-y_q} \mathbb{F}_R.$$

Then we get an exact sequence

$$0 \longrightarrow \mathbb{F}_R \longrightarrow \mathbb{C} \longrightarrow \mathbb{F}_R[-1] \longrightarrow 0$$

Here $\mathbb{F}_R[-1]$ is the complex $\mathbb{F}_R$ shifted to the 'left', that is, $(\mathbb{F}_R[-1])_i = (\mathbb{F}_R)_{i-1}$ for all i.

By our induction hypothesis $\mathbb{F}_R$ is acyclic. Thus from the long exact sequence

$$\begin{array}{ccccccccc} H_1(\mathbb{C}) & \longrightarrow & H_0(\mathbb{F}_R) & \xrightarrow{-y_q} & H_0(\mathbb{F}_R) & \longrightarrow & H_0(\mathbb{C}) & \longrightarrow & 0 \\ \cdots & \longrightarrow & H_2(\mathbb{C}) & \longrightarrow & H_1(\mathbb{F}_R) & \xrightarrow{-y_q} & H_1(\mathbb{F}_R) & \longrightarrow & \end{array}$$

we deduce that $H_i(\mathbb{C}) = 0$ for $i > 1$. We also get $H_1(\mathbb{C}) = 0$, since $H_0(\mathbb{F}_R) = H_R$, and since multiplication by y_q is injective on H_R. Thus we see that $\mathbb{C}$ is acyclic.

We now claim that $\mathbb{C} \cong \mathbb{G}$, thereby proving that $\mathbb{G}$ is acyclic, as desired.

In order to prove this claim we first notice that $\mathbb{C}_i = (\mathbb{F}_R)_{i-1} \oplus (\mathbb{F}_R)_i$ for $i \geq 0$ (where $(\mathbb{F}_R)_{-1} = 0$). Thus if $r = |R|$, then $\mathbb{C}_i$ has the basis $\mathcal{C}_i = \mathcal{B}_{i-1} \cup \mathcal{B}_i$, where

$$\mathcal{B}_i = \{e(I, T)\colon I \in L(R),\ T \subset R,\ I \cap T \subset M(I), |I \cap T| = i, |I \cup T| = r + i\}.$$

On the other hand $\mathbb{G}_i$ has the basis

$$\mathcal{G}_i = \{e(I, T)\colon I \in L(P),\ (q) \subset I,\ T \subset P,\ I \cap T \subset M(I), |I \cap T| = i, \\ |I \cup T| = n + i\}.$$

Let $\psi_i : \mathbb{C}_i \to \mathbb{G}_i$ be the S-linear homomorphism with

$$\psi_i(e(I, T)) = \begin{cases} e(I \cup (q), T \cup \{q\}) & \text{if } e(I, T) \in \mathcal{B}_{i-1}; \\ e(I \cup (q), T) & \text{if } e(I, T) \in \mathcal{B}_i. \end{cases}$$

It is easy to see that all ψ_i are bijections and induce an isomorphism of complexes. □

Suppose P is of cardinality n and P is an antichain, i.e., any two elements of P are incomparable. Then $B_n = \mathcal{J}(P)$ is called the *Boolean lattice of rank* n.

Let now $\mathcal{L}$ be an arbitrary finite distributive lattice, and let $I, J \in \mathcal{L}$ with $I \leq J$. Then the set

$$[I, J] = \{M \in \mathcal{L}\colon I \leq M \leq J\}$$

is called an *interval* in $\mathcal{L}$. The interval $[I, J]$ with the induced partial order is again a distributive lattice. Let $b_i(\mathcal{L})$ denote the number of intervals of $\mathcal{L}$ which are isomorphic to Boolean lattices of rank i. In particular, $b_0(\mathcal{L}) = |\mathcal{L}|$. These numbers have an algebraic interpretation.

Recall that for a graded S-module M,

$$\beta_i(M) = \dim_K \operatorname{Tor}_i^S(M, K)$$

is called the ith *Betti-number of* M. If $\mathbb{F}$ is a graded minimal free resolution of M, then $\beta_i(M)$ is nothing but the rank of $\mathbb{F}_i$.

Corollary 2.2 *Let P be a finite poset, $\mathcal{L} = \mathcal{J}(P)$ the distributive lattice and H_P the squarefree monomial ideal arising from P. Then*

(a) $b_i(\mathcal{L}) = \beta_i(H_P)$ *for all* i;
(b) *the following three numbers are equal*:

(i) *the projective dimension of* H_P;
(ii) *the maximum of the ranks of Boolean lattices which are isomorphic to an interval of* $\mathcal{L}$;
(iii) *the Sperner number of* P, *i.e.*, *the maximum of the cardinalities of antichains of* P.

Proof: (a) For each $i \geq 0$, let $\mathcal{J}_i$ be the set of pairs (I, S), where $I \in \mathcal{L}$, $S \subset M(I)$ and $|S| = i$, and let $\mathcal{B}_i$ be the set of basis elements $e(I, T)$ of $\mathbb{F}_i$. Then

$$\mathcal{B}_i \longrightarrow \mathcal{J}_i, \quad e(I, T) \mapsto (I, I \cap T)$$

establishes a bijection between these two sets.

Since for each $(I, S) \in \mathcal{J}_i$, the elements in S are pairwise incomparable it is clear that $[I \setminus S, I]$ is isomorphic to a Boolean lattice of rank i.

Conversely, suppose $[J, I]$ is isomorphic to a Boolean lattice of rank i. Then $S = I \setminus J$ is of a set of cardinality i, and $J \cup T \in \mathcal{L}$ for all subsets $T \subset S$.

Suppose that $S \not\subset M(I)$. Then there exists, $q \in S$ and $p \in I$ such that $p > q$. If $p \in J$, then $q \in J$, a contradiction. Thus $p \in S$, and hence $(J, p) \in \mathcal{L}$. This is again a contradiction, because it would imply that $q \in (J, p)$. Hence we have shown that $(I, S) \in \mathcal{J}_i$.

It follows that the assignment $e(I, T) \mapsto [I \setminus (I \cap T), I]$ establishes a bijection between the basis of $\mathbb{F}_i$ and the intervals of $[J, I]$ in $\mathcal{L}$ which are isomorphic to Boolean lattices.

(b) is an immediate consequence of (a) and its proof. □

Corollary 2.3 *Let* $\mathcal{L}$ *be a finite distributive lattice. Then*

$$\sum_{i \geq 0} (-1)^{i+1} b_i(\mathcal{L}) = 1.$$

Corollary 2.3 is a special case of [12, Exercise 3.19 (b)] and the resolution constructed in Theorem 2.1 is the cellular resolution [1] of the cubical complex appearing in Topological Remark [12, pp. 178–179]. In the forthcoming paper [8], we construct such the resolutions in more general contexts and show that these resolutions are cellular in some cases.

Let Δ_P be the simplicial complex attached to the squarefree monomial ideal H_P. In the next section we will see (Lemma 3.1) that the Stanley–Reisner ideal attached to the Alexander dual $\Delta_P^\vee$ is generated by the monomials $x_p y_q$ such that $p \leq q$. Hence for the Stanley–Reisner ideal of Δ_P we have

$$I_{\Delta_P} = \bigcap_{p,q \in P,\, p \leq q} (x_p, y_q).$$

In particular we get

Proposition 2.4 *Let* P *be a finite poset. Then the squarefree monomial ideal* H_P *is of height* 2, *and the multiplicity of* S/H_P *is given by*

$$e(S/H_P) = |\{(p, q) \colon\ p, q \in P,\ \ p \leq q\}|.$$

Let $I \subset S$ be an arbitrary graded ideal with graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j=1}^{\beta_s} S(-a_{sj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_1} S(-a_{1j}) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Suppose the height of I equals h. Then by a formula of Peskine and Szpiro [11] one has

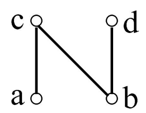
$$e(S/I) = \frac{(-1)^h}{h!}\sum_{i=1}^{s}(-1)^i \sum_{j=1}^{\beta_i} a_{ij}^h.$$

Applying this formula in our situation and using Corollary 2.2 and Proposition 2.4 we get

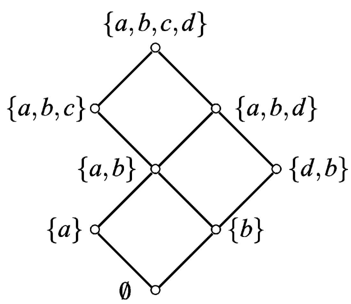
Corollary 2.5 *Let P be a finite poset with $|P| = n$, and let $\mathcal{L} = \mathcal{J}(P)$ be the distributive lattice. Then*

$$|\{(p, q)\colon\ p, q \in P,\ \ p \leq q\}| = \frac{1}{2}\sum_{i\geq 0}(-1)^{i+1} b_i(\mathcal{L})(n+i)^2.$$

We close this section with an example. Let P be the poset with Hasse diagram



The distributive lattice $\mathcal{L} = \mathcal{J}(P)$ has the Hasse diagram



Thus $H_P = (uvwx, avwx, buwx, abwx, bduw, abcx, abdw, abcd)$. Here we use for convenience the indeterminates a, b, c, d, u, v, w, x instead of x_p and y_p. The free resolution of H_P is given by

$$0 \longrightarrow S^3(-6) \longrightarrow S^{10}(-5) \longrightarrow S^8(-4) \longrightarrow H_P \longrightarrow 0.$$

We see from the Hasse diagram that the ith Betti number of H_P coincides with number of intervals of $\mathcal{L}$ which are isomorphic to Boolean lattices of rank i. The number of pairs (p, q) in the poset P with $p \leq q$ is equal to 7, and this is also the number we get from Corollary 2.5, namely $(1/2)(-8 \cdot 16 + 10 \cdot 25 - 3 \cdot 36) = 7$.

3. Alexander duality and Cohen–Macaulay bipartite graphs

We refer the reader to, e.g., [3, 10, 13] for fundamental information about Stanley–Reisner rings.

Let $P = \{p_1, \ldots, p_n\}$ be a finite poset and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_i = 1$. We will use the notation x_i, y_i instead of x_{p_i}, y_{p_i}, and set $V_n = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$.

Recall that H_P is the ideal of S which is generated by those squarefree monomials $u_I = (\prod_{p_i \in I} x_i)(\prod_{p_i \in P \setminus I} y_i)$ with $I \in \mathcal{J}(P)$. It then follows that there is a unique simplicial complex Δ_P on V_n such that the *Stanley–Reisner ideal* I_{Δ_P} coincides with H_P. We study the *Alexander dual* $\Delta_P^\vee$ of Δ_P, which is the simplicial complex

$$\Delta_P^\vee = \{V_n \setminus F \ : \ F \notin \Delta_P\}$$

on V_n.

Lemma 3.1 *The Stanley–Reisner ideal of $\Delta_P^\vee$ is generated by those squarefree quadratic monomials $x_i y_j$ such that $p_i \leq p_j$ in P.*

Proof: Let $w = x_1 \ldots x_n y_1 \ldots y_n$. If u is a squarefree monomial of S, then we write $\mathrm{supp}(u)$ for the support of u, i.e., $\mathrm{supp}(u) = \{x_i : x_i \text{ divides } u\} \cup \{y_j : y_j \text{ divides } u\}$. Now since $\{\mathrm{supp}(u_I) : I \in \mathcal{J}(P)\}$ is the set of minimal nonfaces of Δ_P, it follows that $\{\mathrm{supp}(w/u_I) : I \in \mathcal{J}(P)\}$ is the set of facets (maximal faces) of $\Delta_P^\vee$. Our work is to find the minimal nonfaces of $\Delta_P^\vee$. Since $\mathrm{supp}(w/u_\emptyset) = x_1 \cdots x_n$ and $\mathrm{supp}(w/u_P) = y_1 \cdots y_n$, both $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are faces of $\Delta_P^\vee$. Let $F \subset V_n$ be a nonfaces of $\Delta_P^\vee$. Let $F_x = F \cap \{x_1, \ldots, x_n\}$ and $F_y = \{x_j : y_j \in F\}$. Then $F_x \neq \emptyset$ and $F_y \neq \emptyset$. Since $\{x_i, y_i\}$ is a minimal nonface of $\Delta_P^\vee$, we will assume that $F_x \cap F_y = \emptyset$. Since F is a nonface, there exists *no* poset ideal I of P with $F_x \cap \{x_i : p_i \in I\} = \emptyset$ and $F_y \subset \{x_i : p_i \in I\}$. Hence there are $x_i \in F_x$ and $x_j \in F_y$ such that $p_i < p_j$. Thus $\{x_i, y_j\}$ is a nonface of $\Delta_P^\vee$. Hence the set of minimal nonfaces of $\Delta_P^\vee$ consists of those 2-element subsets $\{x_i, y_j\}$ of V_n such that $p_i \leq p_j$ in P, as required. □

Let G be a finite graph on the vertex set $[N] = \{1, \ldots, N\}$ with no loops and no multiple edges. We will assume that G possesses no isolated vertex, i.e., for each vertex i there is an edge e of G with $i \in e$. A *vertex cover* of G is a subset $C \subset [N]$ such that, for each edge

$\{i, j\}$ of G, one has either $i \in C$ or $j \in C$. Such a vertex cover C is called *minimal* if no subset $C' \subset C$ with $C' \neq C$ is a vertex cover of G. We say that a finite graph G is *unmixed* if all minimal vertex covers of G have the same cardinality.

Let $K[\mathbf{z}] = K[z_1, \dots, z_N]$ denote the polynomial ring in N variables over a field K. The *edge ideal* of G is the ideal $I(G)$ of $K[\mathbf{z}]$ generated by those squarefree quadratic monomials $z_i z_j$ such that $\{i, j\}$ is an edge of G. A finite graph G on $[N]$ is called *Cohen–Macaulay* over K if the quotient ring $K[\mathbf{z}]/I(G)$ is Cohen–Macaulay. Every Cohen–Macaulay graph is unmixed ([15, Proposition 6.1.21]).

A finite graph G on $[N]$ is *bipartite* if there is a partition $[N] = W \cup W'$ such that each edge of G is of the form $\{i, j\}$ with $i \in W$ and $j \in W'$. A basic fact on the graph theory says that a finite graph G is bipartite if and only if G possesses no cycle of odd length. A *tree* is a connected graph with no cycle. A tree is Cohen–Macaulay if and only if it is unmixed ([15, Corollary 6.3.5]).

Given a finite poset $P = \{p_1, \dots, p_n\}$, we write $G(P)$ for the bipartite graph on the vertex set $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ whose edges are those $\{x_i, y_j\}$ such that $p_i \leq p_j$ in P. Lemma 3.1 says that the Stanley–Reisner ideal of $\Delta_P^\vee$ is equal to the edge ideal of $G(P)$. Since the Stanley–Reisner ideal $H_P = I_{\Delta_P}$ has a linear resolution, it follows from [4, Theorem 3] that $\Delta_P^\vee$ is Cohen–Macaulay. Then [15, Theorem 6.4.7] says that $\Delta_P^\vee$ is shellable. Hence I_{Δ_P} has linear quotients (e.g., [7]).

Corollary 3.2 *The Alexander dual $\Delta_P^\vee$ is shellable and the ideal H_P has linear quotients.*

We now turn to the problem of classifying the Cohen–Macaulay bipartite graphs by using the Alexander dual $\Delta_P^\vee$.

Let G be a finite bipartite graph on the vertex set $W \cup W'$ with $W = \{i_1, \dots, i_s\}$ and $W' = \{j_1, \dots, j_t\}$, where $s \leq t$. For each subset U of W, we write $N(U)$ for the set of those vertices $j \in W'$ for which there is a vertex $i \in U$ such that $\{i, j\}$ is an edge of G. The well-known "marriage theorem" in graph theory says that if $|U| \leq |N(U)|$ for all subsets U of W, then there is a subset $W'' = \{j_{\ell_1}, \dots, j_{\ell_s}\} \subset W'$ with $|W''| = s$ such that $\{i_k, j_{\ell_k}\}$ is an edge of G for $k = 1, 2, \dots, s$.

Let G be a finite bipartite graph on the vertex set $W \cup W'$ and suppose that G is unmixed. Since each of W and W' is a minimal vertex cover, one has $|W| = |W'|$. Let $W = \{x_1, \dots, x_n\}$ and $W' = \{y_1, \dots, y_n\}$. Since $(W \setminus U) \cup N(U)$ is a vertex cover of G for all subsets U of W and since G is unmixed, it follows that $|U| \leq |N(U)|$ for all subsets U of W. Thus the marriage theorem enables us to assume that G satisfies the condition as follows: ($\sharp$) $\{x_i, y_i\}$ is an edge of G for all $1 \leq i \leq n$.

Lemma 3.3 *Work with the same notation as above and, furthermore, suppose that G is a Cohen–Macaulay graph. Then, after a suitable change of the labeling of variables $y_1, \dots, y_n$, the edge set of G satisfies the condition ($\sharp$) together with the condition as follows: ($\sharp\sharp$) if $\{x_i, y_j\}$ is an edge of G, then $i \leq j$.*

Proof: Let Δ be the Cohen–Macaulay complex on the vertex set $W \cup W'$ whose Stanley–Reisner ideal I_Δ coincides with $I(G)$. Recall that every Cohen–Macaulay complex is strongly connected and that all links of a Cohen–Macaulay complex are again Cohen–Macaulay. Since both W and W' are facets of Δ, it follows (say, by induction on n) that, after a suitable change of the labeling of variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$, the subset $F_i = \{y_1, \dots, y_i, x_{i+1}, \dots, x_n\}$ is a facet of Δ for each $0 \leq i \leq n$, where $F_0 = W$ and $F_n = W'$. In particular $\{x_i, y_j\}$ cannot be an edge of G if $j < i$. In other words, the edge set of G satisfies the conditions ($\sharp$) and ($\sharp\sharp$), as required. □

Theorem 3.4 *Let G be a finite bipartite graph on the vertex set $W \cup W'$, where $W = \{x_1, \dots, x_n\}$ and $W' = \{y_1, \dots, y_n\}$, and suppose that the edge set of G satisfies the conditions ($\sharp$) and ($\sharp\sharp$). Then G is a Cohen–Macaulay graph if and only if the following condition ($\sharp\sharp\sharp$) is satisfied*:

($\sharp\sharp\sharp$) *If $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges of G with $i < j < k$, then $\{x_i, y_k\}$ is an edge of G.*

Proof: ("Only if") Let G be a Cohen–Macaulay graph satisfying ($\sharp$) and ($\sharp\sharp$) and Δ the Cohen–Macaulay complex on the vertex set $W \cup W'$ whose Stanley–Reisner ideal coincides with $I(G)$. Let $\{x_i, y_j\}$ and $\{x_j, y_k\}$ be edges of G with $i < j < k$ and suppose that $\{x_i, y_k\}$ is not an edge of G. Since every Cohen–Macaulay complex is pure and since $\{x_i, y_k\}$ is a face of Δ, it follows that there is an n-element subset $F \subset W \cup W'$ of G with $\{x_i, y_k\} \subset F$ such that F is independent in G, i.e., no 2-element subset of F is an edge of G. One has $y_j \notin F$ and $x_j \notin F$ since $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges of G. Since $\{x_\ell, y_\ell\}$ is an edge of G for each $1 \leq \ell \leq n$, the independent subset F can contain both x_i and y_i for no $1 \leq i \leq n$. Thus to find such an n-element independent set F is impossible.

("If") Now, suppose that a finite bipartite graph G on the vertex set $W \cup W'$ satisfies the conditions ($\sharp$), ($\sharp\sharp$) together with ($\sharp\sharp\sharp$). Let $\leq$ denote the binary relation on $P = \{p_1, \dots, p_n\}$ defined by setting $p_i \leq p_j$ if $\{x_i, y_j\}$ is an edge of G. By ($\sharp$) one has $p_i \leq p_i$ for each $1 \leq i \leq n$. By ($\sharp\sharp$) if $p_i \leq p_j$ and $p_j \leq p_i$, then $p_i = p_j$. By ($\sharp\sharp\sharp$) if $p_i \leq p_j$ and $p_j \leq p_k$, then $p_i \leq p_k$. Thus $\leq$ is a partial order on P. Lemma 3.1 then guarantees that $G = G(P)$. Hence G is Cohen–Macaulay, as desired. □

Corollary 3.5 *Let G be a finite bipartite graph and Δ the simplicial complex whose Stanley–Reisner ring coincides with $I(G)$. Then G is Cohen–Macaulay if and only if Δ is pure and strongly connected.*

Work with the same situation as in the "if" part of the proof of Theorem 3.4. Let com(P) denote the *comparability graph* of P, i.e., com(P) is the finite graph on $\{p_1, \dots, p_n\}$ whose edges are those $\{p_i, p_j\}$ with $i \neq j$ such that p_i and p_j are comparable in P. It then follows from [15, pp. 184–185] that the Cohen–Macaulay type of the Cohen–Macaulay ring $S/I(G)$, where $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$, is the number of maximal independent subsets of com(P), i.e., the number of maximal antichains of P. Hence G is Gorenstein, i.e., $S/I(G)$ is a Gorenstein ring, if and only if P is an antichain.

Corollary 3.6 *A Cohen-Macaulay bipartite graph G is Gorenstein if and only if G is the disjoint union of edges.*

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