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DISTURBANCE DECOUPLING BY MEASUREMENT FEEDBACK WITH STABILITY OR POLE PLACEMENT*

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Abstract. In this paper we solve the disturbance decoupling problem by measurement feedback and requiring stability or pole placement on the closed loop system. The problem is attacked using the geometric approach through the concepts of $A(\text{mod}\,\mathcal{B})$-invariant and controllability subspaces and their duals, $A|\mathcal{K}$-invariant and complementary observability subspaces. The solution of this problem has an interesting structure consisting of a feedback processor which decomposes into (i) a disturbance decoupling loop; (ii) a disturbance input stabilization or pole placement loop, and (iii) a controlled output stabilization or pole placement loop.

1. Introduction. Consider the dynamical system with signal flow graph depicted in Fig. 1.

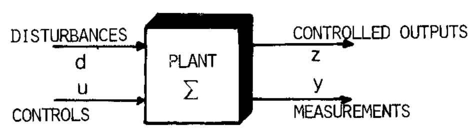

FIG. 1

If this system is controlled by means of the feedback processor shown in Fig. 2,

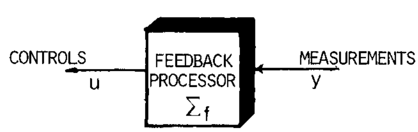

FIG. 2

then one obtains the closed loop system shown in Fig. 3.

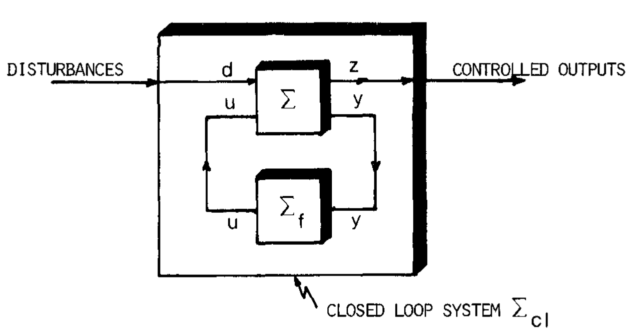

FIG. 3

One of the most easily motivated control synthesis questions is the problem of designing a feedback processor such that in the closed loop system,

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(i) The disturbances are completely decoupled from the controlled outputs; and
(ii) the closed loop system is internally stable or
(ii)′ (in the linear case) the closed loop poles may be arbitrarily assigned.

We will call these problems, following the acronym perversion propagated in [1]: DDPM (for (i)), the disturbance decoupling problem with measurement feedback; DDPMS (for (i) and (ii)), the disturbance decoupling problem with measurement feedback and stability; and DDPMPP (for (i) and (ii))′, the disturbance decoupling problem with measurement feedback and pole placement.

The disturbance decoupling problem in all its variations has been studied extensively before, and has motivated much of the development of the geometric approach in linear (and recently also in nonlinear) system theory. However, the early papers in this area have primarily been about disturbance decoupling using state feedback with or without stability or pole placement requirements [1, §§ 4.3, 5.6]. These results are based on the concepts of $A(\text{mod}\,\mathcal{B})$-invariant and controllability subspaces. There have also been a number of papers on DDEP, the disturbance decoupled estimation problem, or, what amounts to the same thing, the unknown input observer design problem (see [2], [3], and for earlier references, [4], [5], [6]). This problem will be treated in § 4. The crucial concepts in this context are those of $A|\mathcal{K}$-invariant and complementary observability subspaces. These are the duals of $A(\text{mod}\,\mathcal{B})$-invariant and controllability subspaces. They may be introduced by formal dualization (see [1, Ex. 5.17], or [2], where DDEP is solved this way) but they can also be defined directly, in a more intrinsic way in connection with observer synthesis questions [3], [5], [6].

In most industrial applications it will not be possible to assume that all the state variables are measured. Consequently, there is a direct practical motivation for studying the disturbance decoupling problem in the context of measurement feedback. Recently, in fact, DDPM has been solved in [7] and in [3]. Actually DDPM had already been formulated by Basile and Marro who, for this purpose, introduced the notions of *controlled* and *conditioned* invariant subspaces (we will call these $A(\text{mod}\,\mathcal{B})$- and $A|\mathcal{K}$-invariant subspaces) and they actually obtained as necessary conditions the conditions which, as shown in [3], [7], are in fact sufficient and hence lead to a synthesis for DDPM.

In all of the above references, the stability or pole placement question was not considered. It goes without saying that in applications one will need to consider also the stability aspects. In the present paper we will solve this problem (see (ii) and (iii) of our theorem).

It is quite surprising that DDPMS and DDPMPP have not been solved before even though their solution has been very much in reach, through the combined results in the work of Wonham [1], Basile and Marro [6] and the compensator design by output feedback of Brasch and Pearson [8] (see also [1, § 2.8]). The solution which we have obtained is in a sense what could have been conjectured from [3] or [7]. However, the resulting synthesis is a rather intricate and complex one.

We have attempted to make the paper reasonably self contained. Given the potential practical interest in this problem, one could hope that this true culmination of the disturbance decoupling circle of ideas ought to serve as the theoretical basis for some convincing specific applications.

We would like to emphasize that the disturbances could be also state or parameter dependent. The theorem which will be obtained also gives disturbance decoupling when the disturbance is of the form $d((\mathcal{F}x(\cdot))(t), \alpha, t)$, with $\mathcal{F}$ an (unknown) dynamic function of the state and α an unknown parameter.

2. Mathematical problem formulation. Consider the plant equations given by

$$\Sigma: \qquad \dot{x} = Ax + Bu + Gd, \qquad y = Cx, \qquad z = Hx,$$

with $x \in \mathbb{R}^n =: \mathscr{X}$, the state, $u \in \mathbb{R}^m =: \mathscr{U}$, the control, $d \in \mathbb{R}^q =: \mathscr{D}$, the disturbance, $y \in \mathbb{R}^p =: \mathscr{Y}$, the measurement and $z \in \mathbb{R}^l =: \mathscr{Z}$, the controlled output.

The DDPM problem is to find (real) feedback matrices $\{F, E, M, N\}$ defining the feedback processor

$$\Sigma_f: \qquad \dot{w} = Fw + Ey, \qquad u = Mw + Ny,$$

with $w \in \mathbb{R}^k =: \mathscr{W}$, the state of the feedback processor, such that the closed loop system $\Sigma_{cl} := \Sigma \times \Sigma_f|\text{feedback}$:

$$\Sigma_{cl}: \qquad \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A + BNC & BM \\ EC & F \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} d, \qquad z = [H \mid 0] \begin{bmatrix} x \\ w \end{bmatrix},$$

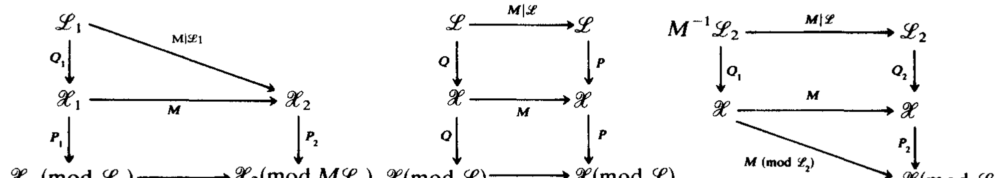
which may be written compactly as $\dot{x}^e = A^e x^e + G^e d$, $z = H^e x^e$, has zero transfer function, $H^e(Is - A^e)^{-1} G^e = 0$; i.e., the controlled output z is influenced only by the initial conditions and not by the disturbances d.

DDPMS requires in addition some conditions on the spectrum of A^e, $\sigma(A^e)$. This stability requirement is modelled, as usual, by requiring $\sigma(A^e) \subset \mathbb{C}_g$ with $\mathbb{C}_g$ a given subset of the complex plane $\mathbb{C}$ which is symmetric ($\{\lambda \in \mathbb{C}_g\} \Leftrightarrow \{\bar{\lambda} \in \mathbb{C}_g\}$; $^-$ denotes the complex conjugate), and which contains at least one point of the real axis. Simple asymptotic stability is thus obtained by taking $\mathbb{C}_g = \{\lambda \in \mathbb{C} \mid \text{Re}\, \lambda < 0\}$.

DDPMPP requires pole placement in the sense that for any $\mathbb{C}_g$ which is symmetric and contains at least one point of the real axis it should be possible to achieve $\sigma(A^e) \subset \mathbb{C}_g$. (The results essentially imply that the closed loop characteristic polynomial can be chosen arbitrarily, provided that this characteristic polynomial have a sufficiently high degree and can be factored into two real factors of the right degree. These details we leave to the reader to fill in.

Some notation.

1. We will throughout use lower case letters for vectors, capitals for matrices and linear operators, and script for linear subspaces and vector spaces. If $M: \mathscr{X}_1 \to \mathscr{X}_2$ and $\mathscr{L}_1 \subset \mathscr{X}_1$, then $M|\mathscr{L}_1: \mathscr{L}_1 \to \mathscr{X}_2$, denotes $l_1 \mapsto Ml_1$, while $M(\text{mod}\, \mathscr{L}_1): \mathscr{X}_1(\text{mod}\, \mathscr{L}_1) \to \mathscr{X}_2(\text{mod}\, M\mathscr{L}_1)$ denotes $x_1(\text{mod}\, \mathscr{L}_1) \mapsto (Mx_1)(\text{mod}\, M\mathscr{L}_1)$. If $\mathscr{L}_2 \subset \mathscr{X}_2$, then $M|\mathscr{L}_2: M^{-1}\mathscr{L}_2 \to \mathscr{X}_2$ denotes $l_2 \mapsto Ml_2$, while $M(\text{mod}\, \mathscr{L}_2): \mathscr{X}_1 \to \mathscr{X}_2(\text{mod}\, \mathscr{L}_2)$ denotes $x_1 \mapsto (Mx_1)(\text{mod}\, \mathscr{L}_2)$. If $M: \mathscr{X} \to \mathscr{X}$ and $\mathscr{L} \subset \mathscr{X}$ is M-invariant then $M|\mathscr{L}: \mathscr{L} \to \mathscr{L}$ denotes $l \mapsto Ml$, while $M(\text{mod}\, \mathscr{L})$ denotes $x(\text{mod}\, \mathscr{L}) \mapsto (Mx)(\text{mod}\, \mathscr{L})$. With Q's representing canonical injections ($Q: x \mapsto x$) and P's canonical projections ($P: x \mapsto x(\text{mod}\, \mathscr{L})$) these definitions may be visualized in the commutative diagrams



If $M:\mathscr{X}\to\mathscr{X}$ and $\mathscr{L}_1,\mathscr{L}_2\subset\mathscr{X}$, then $\mathscr{L}_1$ is said to be $M(\text{mod}\,\mathscr{L}_2)$*-invariant* if, for all $l_1\in\mathscr{L}_1$, $Ml_1\in\mathscr{L}_1(\text{mod}\,\mathscr{L}_2)$. It is said to be $M|\mathscr{L}_2$*-invariant* if, for all $l\in\mathscr{L}_1\cap\mathscr{L}_2$, $Ml\in\mathscr{L}_2$. Thus $\mathscr{L}_1$ is $M(\text{mod}\,\mathscr{L}_2)$-invariant iff $M\mathscr{L}_1\subset\mathscr{L}_1+\mathscr{L}_2$ and $M|\mathscr{L}_2$-invariant iff $M(\mathscr{L}_1\cap\mathscr{L}_2)\subset\mathscr{L}_2$. These concepts, which are very natural in the context of a linear algebra, also turn out to have very natural system theoretical interpretations!

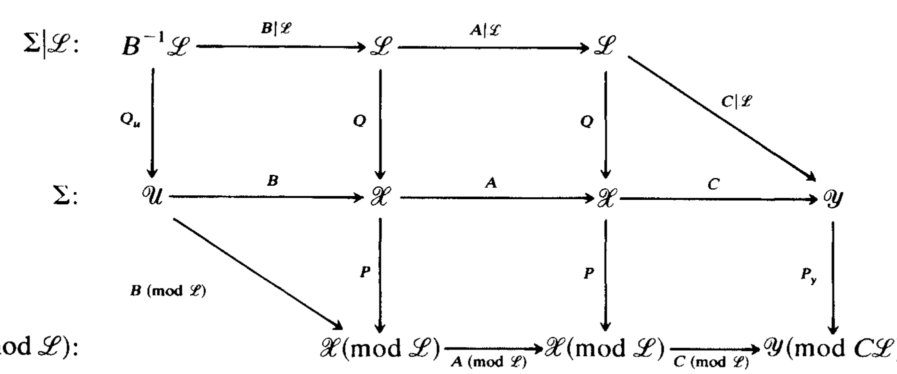
The spectrum of $M:\mathscr{X}\to\mathscr{X}$ is denoted by $\sigma(M)$. It is a set with multiplicity. The characteristic polynomial of M will be denoted by χ_M.

2. Consider the system Σ: $\dot{x}=Ax+Bu$, $y=Cx$, which we will sometimes denote by $\Sigma(A,B,C)$. Let $\mathscr{B}:=\text{Im}\,B$ and $\mathscr{K}:=\text{Ker}\,C$. Then $\mathscr{R}=\langle A|\mathscr{B}\rangle:=\sum_{i=0}^{n-1}A^i\mathscr{B}$ denotes the reachable subspace, while $\mathscr{N}=\langle\mathscr{K}|A\rangle:=\bigcap_{i=0}^{n-1}A^{-i}\mathscr{K}$ denotes the unobservable subspace. Both $\mathscr{R}$ and $\mathscr{N}$ are A-invariant subspaces. In fact, they are respectively the infimal A-invariant subspace containing $\mathscr{B}$ and the supremal A-invariant subspace contained in $\mathscr{K}$. The system is reachable iff $\mathscr{R}=\mathscr{X}$, and observable iff $\mathscr{N}=\{0\}$. If both conditions hold, then we will call the system *minimal*. If $A(\text{mod}\,\mathscr{R})$ and $A|\mathscr{N}$ are stable (relative to some $\mathbb{C}_g$), then we will call Σ *stabilizable and detectable.*

The *reachability index* of Σ, κ_Σ, is defined as the smallest integer l such that $\sum_{i=0}^{l-1}A^i\mathscr{B}=\mathscr{R}$, while the *observability index* of Σ, ν_Σ, is defined as the smallest integer l such that $\bigcap_{i=0}^{l-1}A^{-i}\mathscr{K}=\mathscr{N}$.

It is well known that $\{\Sigma$ stabilizable and detectable (relative to $\mathbb{C}_g$)$\}\Leftrightarrow\{$there exists a feedback compensator such that the closed loop system is stable (relative to $\mathbb{C}_g$)$\}$ and that $\{\Sigma$ minimal$\}\Leftrightarrow\{$for any $\mathbb{C}_g$ there exists a feedback compensator such that the closed loop poles are contained in $\mathbb{C}_g\}$. The required dimension of the feedback compensator achieving these properties is bounded above by $(\min(\kappa_\Sigma,\nu_\Sigma)-1)$.

Let $\mathscr{L}$ be A-invariant. Then one may define the system $\Sigma|\mathscr{L}$, by $\Sigma|\mathscr{L}:=\{A',B',C'\}$ with $A':=A|\mathscr{L}$, $B':=B|\mathscr{L}$ and $C':=C|\mathscr{L}$. Similarly the system $\Sigma(\text{mod}\,\mathscr{L})$ is defined by $\Sigma(\text{mod}\,\mathscr{L}):=\{A'',B'',C''\}$ with $A'':=A(\text{mod}\,\mathscr{L})$, $B'':=B(\text{mod}\,\mathscr{L})$ and $C'':=C(\text{mod}\,\mathscr{L})$. These are illustrated in the commutative diagram.



3. We will use, as standard notation, A_F for $A+BF$ and A^H for $A+HC$.

3. DDP. Consider the linear system Σ: $\dot{x}=Ax+Bu$, $y=Cx$. Let Σ_x denote all state trajectories of this system. Formally, $\Sigma_x:=\{x:\mathbb{R}\to\mathscr{X}|x$ abs. cont. and $\dot{x}(t)-Ax(t)\in\mathscr{B}:=\text{Im}\,B$ a.e.$\}$. A subspace $\mathscr{V}$ is said to be a *controlled invariant* subspace if for all $x_0\in\mathscr{V}$ there exists $x\in\Sigma_x$ such that $x(0)=x_o$ and $x(t)\in\mathscr{V}$ for all t. A subspace $\mathscr{R}$ is said to be a *controllability subspace* if for all $x_0,x_1\in\mathscr{R}$ there exists $T>0$ and $x\in\Sigma_x$ such that $x(0)=x_0$, $x(T)=x_1$, and $x(t)\in\mathscr{R}$ for all t. We will denote the set of all controlled invariant subspaces by $\underline{\mathscr{V}}$ and the set of all controllability subspaces by $\underline{\mathscr{R}}$.

It is well known [1, Chaps. 4, 5] that $\{\mathscr{V}$ is controlled invariant$\}\Leftrightarrow\{\mathscr{V}$ is $A(\text{mod}\,\mathscr{B})$-invariant$\}\Leftrightarrow\{A\mathscr{V}\subset\mathscr{V}+\mathscr{B}\}\Leftrightarrow\{$there exists F such that $\mathscr{V}$ is A_F-invariant$\}$. The family of all such F's will be denoted by $\underline{F}(\mathcal{V})$. Furthermore: {$\mathcal{R}$ is a controllability subspace}$\Leftrightarrow$ {there exists F and $\mathcal{B}_1 \subset \mathcal{B}$ such that $\langle A_F|\mathcal{B}_1\rangle = \mathcal{R}$}$\Leftrightarrow${$\mathcal{R} \in \underline{\mathcal{V}}$, and for any real polynomial p of degree $= \dim \mathcal{R}$, there exists F such that $\chi_{A_F|\mathcal{R}} = p$}$\Leftrightarrow${$\mathcal{R} \in \underline{\mathcal{V}}$ and $\Sigma(A_F, B, -)|\mathcal{R}$ is controllable for all $\underline{F}(\mathcal{R})$}.

Finally, if $\mathcal{V} \in \underline{\mathcal{V}}$ is such that there is an $F \in \underline{F}(\mathcal{V})$ such that $\sigma(A_F|\mathcal{V}) \subset \mathbb{C}_g$, then we call $\mathcal{V}$ a *stabilizable* controlled invariant subspace (relative to $\mathbb{C}_g$). The family of all stabilizable subspaces is denoted by $\underline{\mathcal{V}}_g$.

It is well known and easy to prove that $\underline{\mathcal{V}}$, $\underline{\mathcal{R}}$, and $\underline{\mathcal{V}}_g$ are closed under subspace addition, and thus there exists a supremal element of all elements of $\underline{\mathcal{V}}$, $\underline{\mathcal{R}}$, and $\underline{\mathcal{V}}_g$ contained in any given subspace $\mathcal{L}$ of $\mathcal{X}$. These subspaces will be denoted by $\mathcal{V}^*_{\mathcal{L}}$, $\mathcal{R}^*_{\mathcal{L}}$, and $\mathcal{V}^*_{g,\mathcal{L}}$, respectively. We recall the following algorithms for computing $\mathcal{V}^*_{\mathcal{L}}$ and $\mathcal{R}^*_{\mathcal{L}}$.

Algorithm (ISA) (the invariant subspace algorithm; see [1, p. 91]):

$$\mathcal{V}^{\mu+1}_{\mathcal{L}} := \mathcal{L} \cap A^{-1}(\mathcal{V}^{\mu}_{\mathcal{L}} + \mathcal{B}); \qquad \mathcal{V}^0 = \mathcal{X}.$$

Algorithm (ACSA) (the almost controllability subspace algorithm; see [1, p. 106] and [9], [10]:

$$\mathcal{R}^{\mu+1}_{\mathcal{L}} := \mathcal{L} \cap (A\mathcal{R}^{\mu}_{\mathcal{L}} + \mathcal{B}); \qquad \mathcal{R}^0 = \{0\}.$$

The sequence $\mathcal{V}^{\mu}_{\mathcal{L}}$ reaches, strictly decreasingly, its limit $\mathcal{V}^{\infty}_{\mathcal{L}} = \mathcal{V}^{\dim \mathcal{L}+1}_{\mathcal{L}}$ and $\mathcal{R}^{\mu}_{\mathcal{L}}$ reaches, strictly increasingly, its limit $\mathcal{R}^{\infty}_{\mathcal{L}} = \mathcal{R}^{\dim \mathcal{L}}_{\mathcal{L}}$. Furthermore

$$\mathcal{V}^*_{\mathcal{L}} = \mathcal{V}^{\infty}_{\mathcal{L}} \quad \text{and} \quad \mathcal{R}^*_{\mathcal{L}} = \mathcal{V}^{\infty}_{\mathcal{L}} \cap \mathcal{R}^{\infty}_{\mathcal{L}} = \mathcal{R}^{\infty}_{\mathcal{V}^{\infty}_{\mathcal{L}}}.$$

Computing a corresponding feedback matrix F such that $A_F\mathcal{V}^*_{\mathcal{L}} \subset \mathcal{V}^*_{\mathcal{L}}$ requires solving a set of linear equations. Finding an F such that

$$A_F\mathcal{R}^*_{\mathcal{L}} \subset \mathcal{R}^*_{\mathcal{L}} \quad \text{and} \quad \chi_{A_F|\mathcal{R}^*_{\mathcal{L}}} = p,$$

requires a standard pole placement computation.

One of the main applications of the above concepts is the disturbance decoupling problem. The main results are summarized in the following proposition.

PROPOSITION 1. (See [1, §'s 4.3 and 5.6]). *Consider* $\dot{x} = Ax + Bu + Gd$, $z = Hx$ *and the control law* $u = Fx$. *Then*:

(i) DDP. *There exists F such that* $H(Is - A_F)^{-1}G = 0$ *iff* $\operatorname{Im} G \subset \mathcal{V}^*_{\operatorname{Ker} H}$.

(ii) DDPS. *There exists F such that* $H(Is - A_F)^{-1}G = 0$ *and* $\sigma(A_F) \subset \mathbb{C}_g$ *iff* (A, B) *is stabilizable* (*relative to* $\mathbb{C}_g$) *and* $\operatorname{Im} G \subset \mathcal{V}^*_{g,\operatorname{Ker} H}$.

(iii) DDPPP. *For any $\mathbb{C}_g$ there exists F such that* $H(Is - A_F)^{-1}G = 0$ *and* $\sigma(A_F) \subset \mathbb{C}_g$, *iff* (A, B) *is reachable and* $\operatorname{Im} G \subset R^*_{\operatorname{Ker} H}$.

An important refinement of the above proposition occurs when one allows a feedforward term in the control.

PROPOSITION 2. (See [1, Ex. 4.10, 5.12]). *Consider* $\dot{x} = Ax + Bu + Gd$, $z = Hx$ *and the control law* $u = Fx + Rd$. *Then*:

(i) DDP′. *There exist F, R such that* $H(Is - A_F)^{-1}(G + BR) = 0$ *iff* $\operatorname{Im} G \subset \mathcal{V}^*_{\operatorname{Ker} H} + \mathcal{B}$.

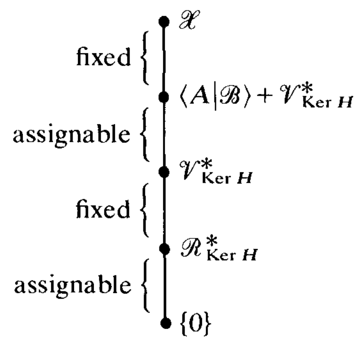
(ii) DDPS′. *There exist F, R such that* $H(Is - A_F)^{-1}(G + BR) = 0$, *and* $\sigma(A_F) \subset \mathbb{C}_g$, *iff* (A, B) *is stabilizable* (*relative to* $\mathbb{C}_g$) *and* $\operatorname{Im} G \subset \mathcal{V}^*_{g,\operatorname{Ker} H} + \mathcal{B}$.

(iii) DDPPP′. *For any $\mathbb{C}_g$ there exist F, R such that* $H(Is - A_F)^{-1}(G + BR) = 0$ *and* $\sigma(A_F) \subset \mathbb{C}_g$, iff (A, B) *is reachable and* $\operatorname{Im} G \subset \mathcal{R}^*_{\operatorname{Ker} H} + \mathcal{B}$.

Proof of Propositions 1 and 2. Because of the references given it suffices to prove (iii) which, however, follows directly from the fact that $\mathcal{V}^*_{g,\operatorname{Ker} H} = \mathcal{R}^*_{\operatorname{Ker} H}$ whenever $\mathbb{C}_g \cap \sigma((A_F|\mathcal{V}^*_{\operatorname{Ker} H}) \bmod \mathcal{R}^*_{\operatorname{Ker} H}) = \varnothing$, for any $F \in \underline{F}(\mathcal{V}^*_{\operatorname{Ker} H})$.

To contrast with what is to come we summarize some of the main features of the above results.

1. The situation with the spectrum may be illustrated (see [11]) as follows:



2. We use the following notion of genericity. Consider all (A, B, C, G, H) belonging to a given algebraic variety Z. Let $Z = \bigcup_{i=1}^{N} Z_i$, be a decomposition of Z into its irreducible components. Let $\bar{\mathscr{S}}$ denote all (A, B, C, G, H) for which a given problem (e.g., DDP) is not solvable. Then we will say that the problem is *generically solvable* iff $Z_i \cap \bar{\mathscr{S}}$ is a proper subvariety of Z_i for all i.

If we consider all elements of (A, B, C, G, H) to be free, then DDP is never generically solvable; DDP′ is generically solvable iff

$$\#\text{ controls} \geqq \#\text{ controlled outputs}.$$

This condition also holds for the generic solvability of DDP, if we consider the subclass of systems with $HG = 0$. For DDPPP′ the condition becomes

$$\#\text{ controls} > \#\text{ controlled outputs},$$

while DDPPP needs again the added a priori assumption $HG = 0$.

4. DDEP. The dual notion of controlled invariance is that of conditioned invariance which has been introduced in [6] and further studied in [3], [12] (see also [13] and [1, Ex. 5.17]). We prefer the following definition.

DEFINITION. Consider the system $\dot{x} = Ax$, $y = Cx$. A subspace $\mathscr{S} \subset X$ is said to be *conditionally invariant* if there exist matrices F, E such that $z := x(\text{mod } \mathscr{S})$ satisfies $\dot{z} = Fz + Ey$.

This definition may seem a bit "ad hoc". In fact, its discrete time analogue may be introduced in a more intrinsic way by defining $\mathscr{S}$ to be *conditionally invariant* for $x(t+1) = Ax(t)$, $y(t) = Cx(t)$, if there exists f such that $x(t+1)(\text{mod } \mathscr{S}) = f(x(t)(\text{mod } \mathscr{S}), y(t))$.

The following conditions are equivalent: $\{\mathscr{S}$ is a conditioned invariant subspace$\}\Leftrightarrow$ $\{\mathscr{S}$ is $A|\text{Ker } C$ invariant$\}:\Leftrightarrow\{A(\mathscr{S} \cap \text{Ker } C) \subset \mathscr{S}\}\Leftrightarrow\{L$ exists such that $A^L\mathscr{S} \subset \mathscr{S}\}$ (L is related to F, E in the above definition by $F = A^L(\text{mod } \mathscr{S})$, and $E = -L(\text{mod } \mathscr{S})$). Indeed, assume that $\mathscr{S}$ is a conditioned invariant subspace. Then if $x \in \text{Ker } C$, it follows that $(\dot{x})(\text{mod } \mathscr{S}) = (d/dt)(x(\text{mod } \mathscr{S})) = Fx(\text{mod } \mathscr{S}) = (Ax)(\text{mod } \mathscr{S})$, which shows that $A(\mathscr{S} \cap \text{Ker } C) \subset \mathscr{S}$. A simple linear algebra calculation shows that this implies the existence of L such that $A^L\mathscr{S} \subset \mathscr{S}$. For such an L there holds (for $\dot{x} = Ax$, $y = Cx$)

$$\begin{aligned}\frac{d}{dt}(x(\text{mod } \mathscr{S})) &= (\dot{x})(\text{mod } \mathscr{S}) = (Ax)(\text{mod } \mathscr{S}) \\ &= (A^L x)(\text{mod } \mathscr{S}) - L(\text{mod } \mathscr{S})y \\ &= A^L(\text{mod } \mathscr{S})x(\text{mod } \mathscr{S}) - L(\text{mod } \mathscr{S})y,\end{aligned}$$

which shows the equivalence of the above statements.

The class of all conditionally invariant subspaces will be denoted by $\underline{\mathcal{S}}$ and for $\mathcal{S} \in \underline{\mathcal{S}}$, $\underline{\mathcal{L}}(\mathcal{S}) := \{L | A^L \mathcal{S} \subset \mathcal{S}\}$.

It follows from the definitions that $A|\mathrm{Ker}\, C$-invariant subspaces are immediately related to the construction of observers. For the stability properties of conditionally invariant subspaces it is not $\sigma(A^L|\mathcal{S})$ but $\sigma(A^L(\mathrm{mod}\, \mathcal{S}))$ which is relevant. Indeed, consider the data processor

$$\dot{z} = A^L(\mathrm{mod}\, \mathcal{S})z - L(\mathrm{mod}\, \mathcal{S})y,$$

as estimator for $x(\mathrm{mod}\, \mathcal{S})$ in $\dot{x} = Ax$, $y = Cx$. Define $e := z - x(\mathrm{mod}\, \mathcal{S})$ and note that in this case we need not have $z(t) = x(t)(\mathrm{mod}\, \mathcal{S})$, since it is not assumed that $z(0) = x(0)(\mathrm{mod}\, \mathcal{S})$. Then e is governed by $\dot{e} = A^L(\mathrm{mod}\, \mathcal{S})e$. Consequently, the error dynamics are governed by $\sigma(A^L(\mathrm{mod}\, \mathcal{S}))$, which leads naturally to the following definition.

DEFINITION. A conditionally invariant subspace $\mathcal{S}$ is said to be a *complementary observability* subspace if for any given real polynomial p of degree $n - \dim \mathcal{S}$, there exists $L \in \underline{\mathcal{L}}(\mathcal{S})$ such that

$$\chi_{A^L(\mathrm{mod}\, \mathcal{S})} = p.$$

It is said to be a *complementary detectability* subspace (relative $\mathbb{C}_g$) if there exists $L \in \underline{\mathcal{L}}(\mathcal{S})$, such that $\sigma(A^L(\mathrm{mod}\, \mathcal{S})) \subset \mathbb{C}_g$.

There holds: $\{\mathcal{S}$ is a complementary observability subspace$\} \Leftrightarrow \{\mathcal{S} \in \underline{\mathcal{S}}$ and $\exists L$, $\mathcal{K}_1 \supset \mathcal{K} := \mathrm{Ker}\, C$ such that $\langle \mathcal{K}_1 | A^L \rangle = \mathcal{S}\} \Leftrightarrow \{\mathcal{S} \in \underline{\mathcal{S}}$ and $\Sigma(A^L, -, C)(\mathrm{mod}\, \mathcal{S})$ is observable for any $L \in \underline{\mathcal{L}}(\mathcal{S})\}$.

These statements follow immediately from duality. Indeed, let $\underline{\mathcal{N}}$ and $\underline{\mathcal{S}}_g$ denote all complementary observability and detectability subspaces associated with a given pair (A, C). It is easily seen that $A|\mathrm{Ker}\, C$-subspaces behave dually to $A^T(\mathrm{mod}\, \mathrm{Im} \cdot C^T)$-subspaces: $\mathcal{S} \in \underline{\mathcal{S}}$ (resp. $\underline{\mathcal{N}}$, $\underline{\mathcal{S}}_g$) relative to (A, C), iff $\mathcal{S}^\perp \in \underline{\mathcal{V}}$ (resp. $\underline{\mathcal{R}}$, $\underline{\mathcal{V}}_g$) relative to (A^T, C^T). In particular $\underline{\mathcal{S}}$, $\underline{\mathcal{S}}_g$ and $\underline{\mathcal{N}}$ are closed under subspace intersection and thus there exist infimal elements of all elements of $\underline{\mathcal{S}}$, $\underline{\mathcal{S}}_g$, and $\underline{\mathcal{N}}$ containing a given subspace $\mathcal{K}$ of $\mathcal{X}$. These subspaces will be denoted by $\mathcal{S}^*_{\mathcal{K}}$, $\mathcal{S}^*_{g,\mathcal{K}}$, and $\mathcal{N}^*_{\mathcal{K}}$ respectively.

In order to compute $\mathcal{S}^*_{\mathcal{L}}$ and $\mathcal{N}^*_{\mathcal{L}}$, it suffices to dualize the algorithms given before. Let $\mathcal{K} := \mathrm{Ker}\, C$, and consider the following algorithms.

Algorithm (ISA)$'$:

$$\mathcal{S}^{\mu+1}_{\mathcal{L}} := \mathcal{L} + A(\mathcal{S}^{\mu}_{\mathcal{L}} \cap \mathcal{K}); \qquad \mathcal{S}^0 = \{0\}.$$

Algorithm (ACSA)$'$:

$$\mathcal{N}^{\mu+1}_{\mathcal{L}} := \mathcal{L} + (A^{-1}\mathcal{N}^{\mu}_{\mathcal{L}}) \cap \mathcal{K}; \qquad \mathcal{N}^0 = \mathcal{X}$$

The sequence $\mathcal{S}^{\mu}_{\mathcal{L}}$ reaches (strictly increasingly) its limit $\mathcal{S}^{\infty}_{\mathcal{L}} = \mathcal{S}^{n-\dim \mathcal{L}+1}_{\mathcal{L}}$ and $\mathcal{N}^{\mu}_{\mathcal{L}}$ reaches (strictly decreasingly) its limit $\mathcal{N}^{\infty}_{\mathcal{L}} = \mathcal{N}^{n-\dim \mathcal{L}}_{\mathcal{L}}$. Furthermore,

$$\mathcal{S}^*_{\mathcal{L}} = \mathcal{S}^{\infty}_{\mathcal{L}} \quad \text{and} \quad \mathcal{N}^*_{\mathcal{L}} = \mathcal{S}^{\infty}_{\mathcal{L}} + \mathcal{N}^{\infty}_{\mathcal{L}} = \mathcal{N}^{\infty}_{\mathcal{S}^{\infty}_{\mathcal{L}}}.$$

Computing a corresponding output injection matrix L such that

$$A^L \mathcal{S}^*_{\mathcal{L}} \subset \mathcal{S}^*_{\mathcal{L}}$$

requires solving a set of linear equations. Finding an L such that

$$A^L \mathcal{N}^*_{\mathcal{L}} \subset \mathcal{N}^*_{\mathcal{L}} \quad \text{and} \quad \chi_{A^L(\mathrm{mod}\, \mathcal{N}^*_{\mathcal{L}})} = p,$$

requires a standard pole placement computation.

Before introducing the disturbance decoupled estimation problem, we give a simple but very useful result concerning the role of dynamic extensions of linear systems.

Let Σ: $\dot{x} = Ax + Bu$, $y = Cx$ be given. We will call the system Σ^e: $\dot{x} = Ax + Bu$, $\dot{w} = v$, considered as a system with input (u, v) and output (y, w), denoted as Σ^e: $\dot{x}^e = A^e x^e + B^e u^e$, $y^e = C^e x^e$, with $\mathscr{X}^e = \mathscr{X} \oplus \mathscr{W}$, etc., an *extension* of Σ. The dimension of $\mathscr{W}$ is called the *dimension* of this extension. We will denote by P the canonical projection $x^e = (x, w) \overset{P}{\mapsto} x$. It is important to note that static feedback around Σ^e corresponds to dynamic feedback around Σ, and that any (finite dimensional) feedback processor around Σ may be visualized in this way. We have the following simple relations between invariant subspaces of Σ and Σ^e.

PROPOSITION 3. *Let Σ^e be an extension of Σ. Then,*

$$\{\mathscr{V}^e \in \underline{\mathscr{V}}^e\} \Leftrightarrow \{P\mathscr{V}^e \in \underline{\mathscr{V}}\}, \qquad \{\mathscr{S}^e \in \underline{\mathscr{S}}^e\} \Leftrightarrow \{\mathscr{S}^e \cap \mathscr{X} \in \underline{\mathscr{S}}\},$$

$$\{\mathscr{V}^e_g \in \underline{\mathscr{V}}^e_g\} \Leftrightarrow \{P\mathscr{V}^e_g \in \underline{\mathscr{V}}_g\} \quad \textit{and} \quad \{\mathscr{S}^e_g \in \underline{\mathscr{S}}^e_g\} \Leftrightarrow \{\mathscr{S}^e_g \cap \mathscr{X} \in \underline{\mathscr{S}}_g\},$$

$$\{\mathscr{R}^e \in \underline{\mathscr{R}}^e\} \Leftrightarrow \{P\mathscr{R}^e \in \underline{\mathscr{R}}\}, \qquad \{\mathscr{N}^e \in \underline{\mathscr{N}}^e\} \Leftrightarrow \{\mathscr{N}^e \cap \mathscr{X} \in \underline{\mathscr{N}}\}.$$

Consider now the plant Σ: $\dot{x} = Ax + Bu + Gd$, with observation (y, u) where $y = Cx$, and the output to be estimated $z = Hx$. The disturbance decoupled estimation problem DDEP is the problem of constructing a data processor, (an *observer*) Σ_p: $\dot{w} = Fw + Ey + Ku$; $\hat{z} = Mw + Ny + Su$, such that the resulting estimation error $e := z - \hat{z}$ depends only on the initial conditions and not on the disturbance d or on the input u. The resulting signal flow graph is then as shown in Fig. 4.

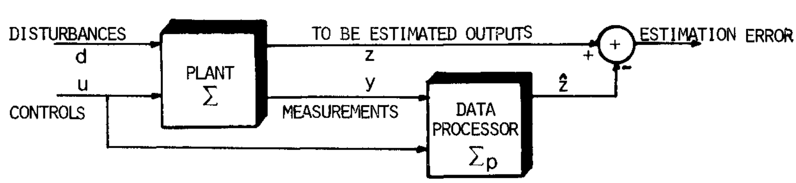


FIG. 4

We emphasize again that, as in the disturbance decoupling problem, the disturbance may also depend on the state through an unknown function or on unknown parameters.

Since in a disturbance decoupled observer the transfer function $d, u \mapsto e$ is zero, all signals $e(\cdot)$, obtainable by varying the initial conditions $x(0)$, $w(0)$, are exactly those obtainable by varying the initial conditions $v(0)$ as the output of a system of the form $\dot{v} = Pv$, $z - \hat{z} = Qv$, for some P, Q. If (P, Q) is observable (which we may always assume to be the case) then we will call $\sigma(P)$ the spectrum (or poles) of the error dynamics of the observer. Note that in an input decoupled observer $x(0) = 0$ and $w(0) = 0$ together imply $e(t) = 0$ for all t (i.e., we have the possibility of perfect tracking of the to be estimated signal by means of the observed signal).

The following proposition treats the disturbance decoupled estimation problem DDEP. This refers to the possibility of finding a disturbance decoupled observer. The problems DDEPS and DDEPPP add the stability or pole placement requirement to the estimation error dynamics.

PROPOSITION 4. *Consider the system $\dot{x} = Ax + Bu + Gd$, with observation (y, u) where $y = Cx$, and the to be estimated output $z = Hx$. Consider an observer of the form $\dot{w} = Fw + Ey + Ru$, $\hat{z} = Mw + Ny + Su$. Then:*

(i) DDEP. *There exists a disturbance decoupled observer, iff* $\mathcal{S}^*_{\mathrm{Im}\,G} \cap \mathrm{Ker}\, C \subset \mathrm{Ker}\, H$.

(ii) DDEPS. *There exists a disturbance decoupled observer with error spectrum* $\subset \mathbb{C}_g$, *iff* $\mathcal{S}^*_{g,\,\mathrm{Im}\,G} \cap \mathrm{Ker}\, C \subset \mathrm{Ker}\, H$.

(iii) DDEPPP. *For any* $\mathbb{C}_g$ *there exists a disturbance decoupled observer with error spectrum contained in* $\mathbb{C}_g$, *iff* $\mathcal{N}^*_{\mathrm{Im}\,G} \cap \mathrm{Ker}\, C \subset \mathrm{Ker}\, H$.

Proof. Claims (i) and (ii) are essentially proven, by duality arguments, in [2] (see also the references of this paper). For completeness, we include a short proof.

I *Necessity.* Let $e := z - \hat{z}$, and consider the dynamics of $x^e = (x, w) \in \mathcal{X} \oplus \mathcal{W}$, written as $\dot{x}^e = A^e x^e + G^e(u, d)$, $e = H^e x^e + D^e(u, d)$. This must have zero transfer function $(u, d) \mapsto e$. Equivalently, $D^e = 0$ and $\langle A^e | \mathrm{Im}\, G^e \rangle \subset \langle \mathrm{Ker}\, H^e | A^e \rangle =: \mathcal{S}^e$. Obviously, $\mathcal{S}^e$ is A^e-invariant and $\mathrm{Im}\, B^e \subset \mathcal{S}^e \subset \mathrm{Ker}\, C^e$. Now Σ^e: $\dot{x} = Ax + Bu + Gd$; $\dot{w} = Fw + Ey + Ku$ is clearly obtainable from an extension of Σ by extended output injection. Hence any A^e-invariant subspace belongs to $\underline{\mathcal{S}}^e$: $\mathcal{S}^e \in \underline{\mathcal{S}}^e$ and, from Proposition 3, $\mathcal{S}^e \cap \mathcal{X} =: \mathcal{S} \in \underline{\mathcal{S}}$, which implies $\mathrm{Im}\, G = \mathcal{X} \cap \mathrm{Im}\, G^e \subset \mathcal{S} \subset \mathrm{Ker}\, H^e = \mathrm{Ker}\,(H - NC)$, which yields $\mathcal{S} \cap \mathrm{Ker}\, C \subset \mathrm{Ker}\, H$. This proves (i). To prove (ii) it suffices to note that the spectrum of the dynamics e equals the spectrum of $A^e (\mathrm{mod}\, \mathcal{S}^e)$. Hence $\mathcal{S}^e \in \underline{\mathcal{S}}^e_g$, and thus $\mathcal{S} \in \underline{\mathcal{S}}_g$ (see Proposition 3), if DDEPS is solvable, whereas the condition for DDEPPP follows directly from the fact that $\mathcal{S}^*_{g,\,\mathrm{Im}\,G} = \mathcal{N}^*_{\mathrm{Im}\,G}$, whenever $\mathbb{C}_g \cap \sigma(A^L(\mathrm{mod}\, \mathcal{S}^*_{\mathrm{Im}\,G}) | \mathcal{N}^*_{\mathrm{Im}\,G}(\mathrm{mod}\, \mathcal{S}^*_{\mathrm{Im}\,G})) = \varnothing$, for any $L \in \underline{L}(\mathcal{S}^*_{\mathrm{Im}\,G})$.

II *Sufficiency.* Assume $\mathcal{S} \in \underline{\mathcal{S}}$, $\mathrm{Im}\, G \subset \mathcal{S}$, and $\mathcal{S} \cap \mathrm{Ker}\, C \subset \mathrm{Ker}\, H$. By this last inclusion there exists M, N such that $Hx = Mx(\mathrm{mod}\, \mathcal{S}) + NC$. Let $L \in \underline{L}(\mathcal{S})$ and consider the observer $\dot{w} = A^L(\mathrm{mod}\, \mathcal{S})\; w - L(\mathrm{mod}\, \mathcal{S})y + B(\mathrm{mod}\, \mathcal{L})\; u$, $\hat{z} = Mw + Ny$, with $\mathcal{W} \cong \mathcal{X}(\mathrm{mod}\, \mathcal{S})$, and M, N such that $Hx = Mx(\mathrm{mod}\, \mathcal{S}) + NCx$. A simple calculation then shows that the following equation holds,

$$\frac{d}{dt} x(\mathrm{mod}\, \mathcal{S}) = A^L(\mathrm{mod}\, \mathcal{S}) x(\mathrm{mod}\, \mathcal{S}) - L(\mathrm{mod}\, \mathcal{S})y + B(\mathrm{mod}\, \mathcal{S})u.$$

Thus $e := z - \hat{z}$, is governed by $\dot{r} = A^L(\mathrm{mod}\, \mathcal{S})r$, $e = Nr$, with $r := x(\mathrm{mod}\, \mathcal{S}) - w$. Hence the transfer function $(u, d) \mapsto e$ is zero, which yields (i). If $\mathcal{S} \in \underline{\mathcal{S}}_{g,\,\mathrm{Im}\,G}$, or $\mathcal{S} \in \underline{\mathcal{N}}_{\mathrm{Im}\,G}$, then this reasoning yields (ii) and (iii). □

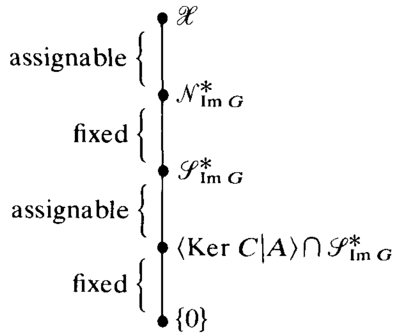
Remarks.

1. In some applications it may be desired that the observations should in any case be "filtered" before being used in $\hat{z}$. This requirement is translated into the constraints $N = 0$, $S = 0$. The results of Proposition 4 then need to be modified, respectively, to:

(i) DDEP′. $\mathcal{S}^*_{\mathrm{Im}\,G} \subset \mathrm{Ker}\, H$,
(ii) DDEPS′. $\mathcal{S}^*_{g,\,\mathrm{Im}\,G} \subset \mathrm{Ker}\, H$,
(iii) DDEPPP′. $\mathcal{N}^*_{\mathrm{Im}\,G} \subset \mathrm{Ker}\, H$.

2. The estimate given on the order of the observer given in the above proposition is, in general, conservative. In fact, the minimal order estimator design is the dual of the minimal dynamic cover problem, and is not solved at this point. However, it is easily seen from the above proposition that if the to be estimated output is the state, then the dimension of the required observer is at least n-Rank C. The proposition hence also shows in what sense the "Luenberger observer" is minimal. In fact, the order of the observer which achieves pole placement needs, assuming (A, C) to be observable, only be n-Rank C, whereas the above proposition would predict n. This is due to the result described in [1, Lemma 3.5, Th. 3.3]. The procedure described there may actually be generalized to the situation at hand, but we will not go into that here.

3. The situation with the spectrum of conditioned invariant subspaces may be illustrated [11] as follows:



4. If we assume all elements of (A, B, C, G, H) to be arbitrary, then DDEP is generically solvable iff

$$\# \text{ measurements} \geqq \# \text{ disturbances}.$$

This condition also holds for DDEP′ provided we add the a priori requirement $HG = 0$. DDEPP instead requires

$$\# \text{ measurements} > \# \text{ disturbances}$$

while DDEPP′ again needs the a priori assumption $HG = 0$.

5. DDPM. In this section we will give the main result of this paper: the disturbance decoupling problem with measurements and stability or pole placement requirements.

DEFINITION. Consider the system $\dot{x} = Ax + Bu$, $y = Cx$. The subspace $\mathscr{L} \subset \mathscr{X}$ is said to be an (A, B, C)-*invariant* subspace if there exists K such that $(A + BKC)\,\mathscr{L} \subset \mathscr{L}$.

We will denote all (A, B, C)-invariant subspaces by $\underline{\mathscr{L}}$. The following proposition is easily seen.

PROPOSITION 5. $\underline{\mathscr{L}} = \underline{\mathscr{V}} \cap \underline{\mathscr{S}}$.

In fact, if $\mathscr{L} \in \underline{\mathscr{L}}$, then it is a matter of solving a set of linear equations to compute a suitable $\mathscr{K}$ for it.

The following elegant result of [3] shows how one can produce (A, B, C)-invariant subspaces by extension.

PROPOSITION 6. *Let $\mathscr{V} \in \underline{\mathscr{V}}$ and $\mathscr{S} \in \underline{\mathscr{S}}$, with $\mathscr{S} \subset \mathscr{V}$. Then there exist an extension of Σ of dimension $\leqq \dim \mathscr{V} - \dim \mathscr{S}$ and an $\mathscr{L}^e \in \underline{\mathscr{L}}^e$, such that $\mathscr{V} = P\mathscr{L}^e$, and $\mathscr{S} = \mathscr{L}^e \cap \mathscr{X}$.*

Proof. The idea behind this proof is shown in Fig. 5. Take $\mathscr{W} \cong \mathscr{V} (\operatorname{mod} \mathscr{S})$, i.e., $\dim \mathscr{W} = \dim \mathscr{V} - \dim \mathscr{S}$, and $\mathscr{X}^e := \mathscr{X} \oplus \mathscr{W}$. Let $\mathscr{V}'$ be such that $\mathscr{V} = \mathscr{S} \oplus \mathscr{V}'$, and $\mathscr{Z} \subset \mathscr{V}' \oplus \mathscr{W}$, such that $\mathscr{Z} \cap \mathscr{V}' = \mathscr{Z} \cap \mathscr{W} = \{0\}$ and $\dim \mathscr{Z} = \dim \mathscr{V}' = \dim \mathscr{W}$. Now $\mathscr{L}^e := \mathscr{Z} \oplus \mathscr{S}$, will have the required properties (see Fig. 5). □

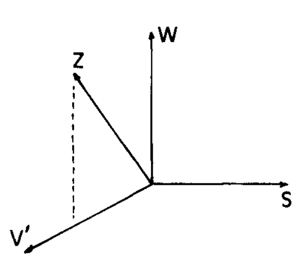


FIG. 5

Remark. The above proposition in effect shows how one can produce an $(A, B, C)^e$-invariant subspace from a pair $\mathcal{S} \subset \mathcal{V}$. Actually this result also solves the following problem. Consider $\dot{x} = Ax + Bu$, $y = Cx$. Let $\mathcal{V} \subset \mathcal{X}$ and suppose that we would like to make $\mathcal{V}$ invariant by feedback from y. Clearly for this $\mathcal{V}$ needs to be $A(\text{mod } \mathcal{B})$-invariant. A systematic procedure for achieving such a feedback law is given in Proposition 6. First choose an $A|\text{Ker } C$-invariant subspace $\mathcal{S} \subset \mathcal{V}$, taking $\mathcal{S} = \{0\}$ shows that it is always possible to achieve this.

Now let $\mathcal{L}^e$ be such that $P\mathcal{L}^e = \mathcal{V}$, and $\mathcal{L}^e \cap \mathcal{X} = \mathcal{S}$. Then $\mathcal{L}^e \in \underline{\mathcal{L}}^e$, and, hence, there exists K^e (defining a dynamic feedback law) such that $(A^e + B^e K^e C^e)\, \mathcal{L}^e \subset \mathcal{L}^e$. The ensuing closed loop system will have the property that if $x^e(0) \in \mathcal{L}^e$, then $x^e(t) \in \mathcal{L}^e$, i.e., $Px^e(t) \in \mathcal{V}$ for all t, as desired. Actually this procedure may be viewed in terms of separation, with an observer used to estimate the feedback law $u = Fx$, with $F \in \underline{F}(\mathcal{V})$.

We continue with a lemma which is an interesting generalization of well-known results about stabilizability and pole placement by output feedback.

LEMMA. *Let Σ: $\dot{x} = Ax + Bu$, $y = Cx$ be given, and let $\mathcal{L}$ be an A-invariant subspace of $\mathcal{X}$.*

I. *Consider the system $\Sigma|\mathcal{L}$ and assume that it is stabilizable and detectable. Then there exists an extension Σ_1^e of Σ and a static feedback law K_1^e around Σ_1^e such that, with $A_{1,cl} := A^e + B^e K_1^e C^e$,*

(i) $\mathcal{L} \oplus \mathcal{W}_1$ is $A_{1,cl}$-invariant,
(ii) $\sigma(A_{1,cl}|(\mathcal{L} \oplus \mathcal{W}_1)) \subset \mathbb{C}_g$,
(iii) $\sigma(A_{1,cl} = \sigma(A_{1,cl})|(\mathcal{L} \oplus \mathcal{W}_1)) \cup \sigma(A(\text{mod } \mathcal{L}))$.

This can always be achieved with an extension of dimension of at most $\gamma_1 := \min(\kappa_{\Sigma|L}, \nu_{\Sigma|L}) - 1 \leqq \min(\kappa_\Sigma, \nu_\Sigma) - 1$. Moreover, if $\Sigma|\mathcal{L}$ is minimal, then given any real polynomial p_1 of degree $\geqq \dim \mathcal{L} + \gamma_1$ one can in fact achieve this with the characteristic polynomial of $A_{1,cl}|(\mathcal{L} \oplus \mathcal{W}_1)$ equal to p_1.

II. *Consider the system $\Sigma(\text{mod } \mathcal{L})$, and assume that it is stabilizable and detectable. Then there exists an extension Σ_2^e of Σ and a static feedback law K_2^e around Σ_2^e such that, with $A_{2,cl} := A^e + B^e K_2^e C^e$:*

(i) *$\mathcal{L}$ is $A_{2,cl}$-invariant,*
(ii) $\sigma(A_{2,cl}(\text{mod } \mathcal{L})) \subset \mathbb{C}_g$,
(iii) $\sigma(A_{2,cl}) = \sigma(A|\mathcal{L}) \dot{\cup} \sigma(A_{2,cl}(\text{mod } \mathcal{L}))$.

This can always be achieved with an extension of dimension of at most $\gamma_2 := \min(\kappa_{\Sigma(\text{mod } \mathcal{L})}, \nu_{\Sigma(\text{mod } \mathcal{L})}) - 1 \leqq \min(\kappa_\Sigma, \nu_\Sigma) - 1$. Moreover, if Σ $(\text{mod } \mathcal{L})$ is minimal, then given any real polynomial p_2 of degree $\geqq n\text{-}\dim \mathcal{L} + \gamma_2$, one can in fact achieve this with the characteristic polynomial of $A_{2,cl}(\text{mod } \mathcal{L})$ equal to p_2.

Proof. In an suitable basis with $\mathcal{X} \cong \mathcal{L} \oplus \mathcal{X}(\text{mod } \mathcal{L})$, $\mathcal{U} \cong \mathcal{B}^{-1}\mathcal{L} \oplus \mathcal{U}(\text{mod } \mathcal{B}^{-1}\mathcal{L})$, $\mathcal{Y} \cong C\mathcal{L} \oplus \mathcal{Y}(\text{mod } C\mathcal{L})$, Σ may be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

In this representation, $\Sigma|\mathcal{L} \cong \{A_{11}, B_{11}, C_{11}\}$ and $\Sigma(\text{mod } \mathcal{L}) \cong \{A_{22}, B_{22}, C_{22}\}$. In order to prove the lemma it suffices to synthesize a Brasch–Pearson stabilization or a placement compensator (see [1, § 3.8]) from y_1 to u_1 for (i), or y_2 to u_2 for (ii).

It remains to be shown that $\min(\kappa_{\Sigma|\mathcal{L}}, \nu_{\Sigma|\mathcal{L}})$, $\min(\kappa_{\Sigma(\text{mod } \mathcal{L})}, \nu_{\Sigma(\text{mod } \mathcal{L})}) \leqq \min(\kappa_\Sigma, \nu_\Sigma)$. This however is due to the fact that $\kappa_{\Sigma|\mathcal{L}} \leqq \kappa_\Sigma$ (this follows from the results in [14]) and that $\nu_{\Sigma|\mathcal{L}} \leqq \nu_\Sigma$, which is easily derived from first principles. Dually, $\kappa_{\Sigma(\text{mod } \mathcal{L})} \leqq \kappa_\Sigma$ and $\nu_{\Sigma|\mathcal{L}} \leqq \nu_\Sigma$. $\square$

The above lemma shows under what conditions stabilization or pole placement by feedback from y to u can be done in a decentralized fashion by feedback from y_1 to u_1 and from u_2 to u_2 without destroying the special subsystem structure induced by the A-invariant subspace $\mathscr{L}$

We are now in a position to state and prove our main result.

THEOREM. *Consider the system* $\Sigma: \dot{x} = Ax + Bu + Gd$, $y = Cx$, $z = Hx$, *and the feedback processor* Σ_f: $\dot{w} = Fw + Ey$, $u = Mw + Ny$. *Let* Σ_{cl} *be the resulting closed loop system with*

$$A_{cl} := \left[\begin{array}{c|c} A + BNC & BM \\ \hline EC & F \end{array}\right].$$

Then

(i) DDPM ([2], [3]). *There exists* Σ_f *such that the transfer function* $d \mapsto z$ *in* Σ_{cl} *is zero iff* $\mathscr{S}^*_{\mathrm{Im}\,G} \subset \mathscr{V}^*_{\mathrm{Ker}\,H}$. *Moreover the required* $\dim \mathscr{W} \leqq \dim \mathscr{V}^*_{\mathrm{Ker}\,H} - \dim \mathscr{S}^*_{\mathrm{Im}\,G}$.

(ii) DDPMS. *There exists* Σ_f *such that the transfer function* $d \mapsto z$ *in* Σ_{cl} *is zero and* $\sigma(A_{cl}) \subset \mathbb{C}_g$ *iff* Σ *is stabilizable and detectable and* $\mathscr{S}^*_{g,\,\mathrm{Im}\,G} \subset \mathscr{V}^*_{g,\,\mathrm{Ker}\,H}$. *Moreover the required* $\dim \mathscr{W} \leqq \dim \mathscr{V}^*_{g,\,\mathrm{Ker}\,H} - \dim \mathscr{S}^*_{g,\,\mathrm{Im}\,G} - 2(\min(\kappa_\Sigma, \nu_\Sigma) - 1)$.

(iii) DDPMPP. *For any* $\mathbb{C}_g$ *there exists* Σ_f *such that the transfer function* $d \mapsto z$ *in* Σ_{cl} *is zero and* $\sigma(A_{cl}) \subset \mathbb{C}_g$ *iff* Σ *is minimal and* $\mathscr{N}^*_{\mathrm{Im}\,G} \subset R^*_{\mathrm{Ker}\,H}$. *Moreover, the required* $\dim \mathscr{W} \leqq \dim \mathscr{R}^*_{\mathrm{Ker}\,H} - \dim \mathscr{N}^*_{\mathrm{Im}\,G} - 2(\min(\kappa_\Sigma, \nu_\Sigma) - 1)$.

Proof.

I. *Necessity.* Assume that a required Σ_f exists. Consider the extension Σ^e on which static feedback results in a Σ_{cl} with zero transfer function $d \mapsto z$, and write it as Σ_{cl}: $\dot{x}^e = A_{cl}x^e + G^e d$, $z = H^e x^e$. Hence, $\langle A_{cl} | \mathrm{Im}\, G^e \rangle \subset \langle \mathrm{Ker}\, H^e | A_{cl} \rangle =: \mathscr{L}^e$. Obviously $\mathscr{L}^e$ is A_{cl}-invariant and hence, as shown in the proof of Proposition 4, $\mathscr{L}^e \in \underline{\mathscr{L}}^e = \underline{\mathscr{S}}^e \cap \underline{\mathscr{V}}^e$. Furthermore, $\mathrm{Im}\, G^e \subset \mathscr{L}^e \subset \mathrm{Ker}\, H^e$. Hence, $\mathrm{Im}\, G \subset \mathscr{X} \cap \mathrm{Im}\, G^e \subset \mathscr{L}^e \cap \mathscr{X} =: \mathscr{S} \subset \mathscr{V} := P\mathscr{L}^e \subset P\, \mathrm{Ker}\, H^e = \mathrm{Ker}\, H$. From Proposition 3, it follows that $\mathscr{V} \in \underline{\mathscr{V}}$ and $\mathscr{S} \in \underline{\mathscr{S}}$, as desired.

The above reasoning also shows the solvability of DDPMS. Indeed, when A_{cl} is stable then $A_{cl}|\mathscr{L}^e$ is stable as well; thus $\mathscr{L}^e \in \underline{\mathscr{V}}^e_g \cap \underline{\mathscr{S}}^e_g$, which by the above reasoning shows that there exist $\mathscr{V}_g \in \underline{\mathscr{V}}_g$ and $\mathscr{S}_g \in \underline{\mathscr{S}}_g$ such that $\mathrm{Im}\, G \subset \mathscr{S}_g \subset \mathscr{V}_g \subset \mathrm{Ker}\, H$. That stabilizability and detectability of Σ is also a necessary condition follows from general principles.

Consider now DDPMPP. If $\sigma((A_F | \mathscr{V}^*_{\mathrm{Ker}\,H})(\mathrm{mod}\, \mathscr{R}^*_{\mathrm{Ker}\,H})) \cap \mathbb{C}_g = \varnothing$ for $F \in \underline{F}(\mathscr{V}^*_{\mathrm{Ker}\,H})$, then $\mathscr{V}^*_{g,\mathrm{Ker}\,H} = \mathscr{R}^*_{\mathrm{Ker}\,H}$ and, dually, if $\sigma(A^L(\mathrm{mod}\, \mathscr{S}^*_{\mathrm{Im}\,G}) | \mathscr{N}^*_{\mathrm{Im}\,G}(\mathrm{mod}\, \mathscr{S}^*_{\mathrm{Im}\,G})) \cap \mathbb{C}_g = \varnothing$, for $L \in \underline{L}(\mathscr{S}^*_{\mathrm{Im}\,G})$, then $\mathscr{S}^*_{g,\mathrm{Im}\,G} = \mathscr{N}^*_{\mathrm{Im}\,G}$. Hence, there exist plenty of $\mathbb{C}_g$'s such that $\mathscr{V}^*_{g,\mathrm{Ker}\,H} = \mathscr{R}^*_{\mathrm{Ker}\,H}$ and $\mathscr{S}^*_{g,\mathrm{Im}\,G} = \mathscr{N}^*_{\mathrm{Im}\,G}$, which yields $\mathscr{N}^*_{\mathrm{Im}\,G} \subset \mathscr{R}^*_{\mathrm{Ker}\,H}$, since if DDPMPP is solvable, then DDPMS is solvable for those $\mathbb{C}_g$'s, as required. That minimality of Σ is also a necessary condition follows again from general principles.

II. *Sufficiency.* This part of the proof is constructive and the procedure may be divided into three parts.

Step 1 (*Disturbance decoupling*). Since $\mathscr{S}^*_{\mathrm{Im}\,G} \subset \mathscr{V}^*_{\mathrm{Ker}\,H}$ there exists, by Proposition 7, a first extension of Σ, Σ^e_1 of dimension $\leqq \dim V^*_{\mathrm{Ker}\,H} - \dim S^*_{\mathrm{Im}\,G}$ and an $(A, B, C)^e$-invariant subspace L^e_1 such that $\mathscr{S}^*_{\mathrm{Im}\,G} = \mathscr{L}^e_1 \cap X \subset P\mathscr{L}^e_1 = \mathscr{V}^*_{\mathrm{Ker}\,H}$. Write Σ^e_1 as $\dot{x}^e_1 = A^e_1 x^e_1 + B^e_1 u^e_1 + G^e d$, $y^e_1 = C^e_1 x_1$, $z = H^e_1 x^e_1$. Hence $\mathrm{Im}\, G^e_1 \subset \mathscr{L}^e_1 \subset \mathrm{Ker}\, H^e_1$. Thus there exists K^e_1 such that $\mathscr{L}^e_1$ is $A_{1,cl}(:= A^e_1 + B^e_1 K^e_1 C^e_1)$-invariant. Since $\langle A_{1,cl} | \mathrm{Im}\, G^e_1 \rangle \subset \mathscr{L}^e_1 \subset \langle \mathrm{Ker}\, H^e_1 | A_{1,cl} \rangle$ this yields the solution to DDPM. The resulting closed system is

$$\Sigma^e_{1,cl}: \dot{x}^e_1 = A_{1,cl} x^e_1 + B^e_1 u^e_1 + G^e_1 d,\ y^e_1 = C^e_1 x^e_1,\ z = H^e_1 x^e_1.$$

It is disturbance decoupled but enjoys no further stability properties as yet.

We now consider DDPMS. If Σ is stabilizable and detectable, so are Σ_1^e and $\Sigma_{1,cl}^e$. Let $\mathscr{L}_1^e$ be constructed as in the previous paragraph starting from $\mathscr{S}_{g,\operatorname{Im} G}^* \subset \mathscr{V}_{g,\operatorname{Ker} H}^*$. Hence $\mathscr{L}_1^e$ is $A_{1,cl}$-invariant, $\operatorname{Im} G_1^e \subset \mathscr{L}_1^e \subset \operatorname{Ker} H_1^e$, and $\mathscr{L}_1^e \in \mathscr{S}_{1,g}^e \cap \mathscr{V}_{1,g}^e$. Consider $\Sigma_{1,cl}^e|\mathscr{L}_1^e$ and $\Sigma_{1,cl}^e (\operatorname{mod} \mathscr{L}_1^e)$. Now, $\Sigma_{1,cl}^e|\mathscr{L}_1^e$ is stabilizable and detectable; stabilizable because $\mathscr{L}_1^e \in \mathscr{V}_{1,g}^e$, and detectable because Σ_1^e is. Dually, $\Sigma_{1,cl}^e$ (mod $\mathscr{L}_1^e$) is stabilizable and detectable; stabilizable because $\Sigma_{1,cl}^e$ is and detectable because $\mathscr{L}_1^e \in \mathscr{S}_{1,g}$.

Using these properties of $\Sigma_{1,cl}^e|\mathscr{L}_1^e$ and $\Sigma_{1,cl}^e (\operatorname{mod} \mathscr{L}_1^e)$ it is now possible to carry out the stabilization steps by a decentralized procedure, by first putting feedback around $\Sigma_{1,cl}^e|\Sigma_1^e$ (we will call this the *disturbance loop stabilization*) and then putting feedback around $\Sigma_{1,cl}^e (\operatorname{mod} \mathscr{L}_1^e)$ (we will call this the *controlled output loop stabilization*).

Step 2 (*Disturbance loop stabilization*). Let us now use procedure (i) of the lemma on $\Sigma_{1,cl}^e|\mathscr{L}^e$. This yields a new extension Σ_2^e and a feedback such that $\mathscr{L}_2^e := \mathscr{L}_1^e \oplus \mathscr{W}_2$ is $A_{2,cl}$-invariant, $\sigma(A_{2,cl}|\mathscr{L}_2^e) \subset \mathbb{C}_g$, and $\sigma(A_{2,cl}(\operatorname{mod} \mathscr{L}_2^e)) = \sigma(A_{1,cl}(\operatorname{mod} \mathscr{L}_2^e)) \subset \mathbb{C}_g$. *Furthermore*, $\operatorname{Im} G_2^e \subset \mathscr{L}_2^e \subset \operatorname{Ker} H_2^e$ remains satisfied, which still yields a disturbance decoupled system.

Step 3 (*Controlled output stabilization*). Let us now use procedure (ii) of the lemma on $\Sigma_{2,cl}^e(\operatorname{mod} \mathscr{L}_2^e)$. (Note that $\Sigma_{2,cl}^e(\operatorname{mod} \mathscr{L}_2^e) = \Sigma_{1,cl}^e(\operatorname{mod} \mathscr{L}_1^e)$.) This yields an extension Σ_3^e and a feedback such that $\mathscr{L}_3^e = \mathscr{L}_2^e$ remains $A_{3,cl}$-invariant, which implies $\operatorname{Im} G_3^e \subset \mathscr{L}_3^e \subset \operatorname{Ker} H_3^e$ and hence, the DDPM conditions will remain satisfied. Furthermore, $\sigma(A_{3,cl}(\operatorname{mod} \mathscr{L}_3^e)) \subset \mathbb{C}_g$ and $\sigma((A_{3,cl})|\mathscr{L}_3^e) = \sigma((A_{2,cl})|\mathscr{L}_2^e) \subset \mathbb{C}_g$, which yields DDPMS.

We still need to show the estimate on the required order of the extension. This follows from the estimates

$$\kappa_{\Sigma_{1,cl}^e}(\operatorname{mod} \mathscr{L}_1^e) \leqq \kappa_{\Sigma_{1,cl}^e} = \kappa_{\Sigma_1^e} = \kappa_\Sigma,$$

and

$$\kappa_{\Sigma_{2,cl}^e|\mathscr{L}_2^e} = \kappa_{\Sigma_{1,cl}^e|\mathscr{L}_1^e} \leqq \kappa_{\Sigma_{1,cl}^e} = \kappa_{\Sigma_1^e} = \kappa_\Sigma,$$

with similar estimates for the observability indices.

Turning now to DDPMPP we see that the procedure sketched above will also work with an arbitrary $\mathbb{C}_g$ provided $\mathscr{N}_{\operatorname{Im} G}^* \subset \mathscr{R}_{\operatorname{Ker} H}^*$, since $\Sigma_{1,cl}^e$, as constructed in Step 1, will then be such that $\Sigma_{1,cl}^e(\operatorname{mod} \mathscr{L}_1^e)$ and $\Sigma_{1,cl}^e|\mathscr{L}_1^e$ are both minimal, and hence Steps 2 and 3 can be done with pole placement.

This ends the sketchy proof of the theorem. $\square$

Remarks. 1. DDPM is solvable only if $HG = 0$. However, in this subclass we have generic solvability if and only if

$$\# \text{ controls} \geqq \# \text{ controlled outputs},$$

$$\# \text{ observations} \geqq \# \text{ disturbances}.$$

For DDPMPP this condition becomes:

$$\# \text{ controls} > \# \text{ controlled outputs},$$

$$\# \text{ observations} > \# \text{ disturbances},$$

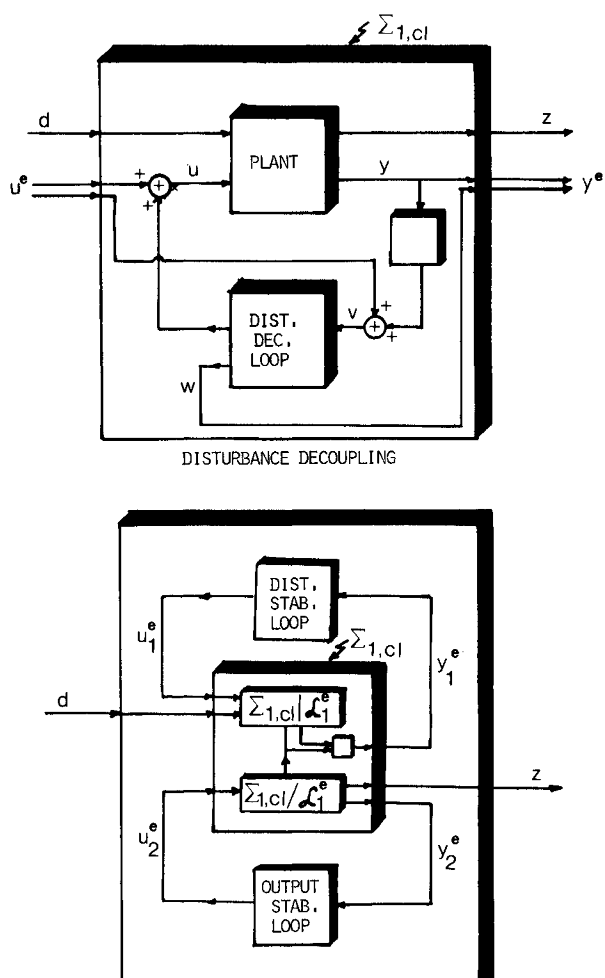
Finally, if feedforward is allowed (i.e., d is measured and available in the feedback processor) then we have solvability of DDPM, (resp. DDPMPP) iff we have it for DDP, (resp. DDPPP).

2. Note that the estimates for the dimension of the feedback processors as given in the theorem and the lemma are conservative, and may in any specific situation be improvable by analyzing the controllability and observability indices of $\Sigma|\mathscr{L}$ and $\Sigma(\operatorname{mod} \mathscr{L})$. Of course, the *minimal* order required, or generically required, is not known and will be a complex combination of the minimal cover and the minimal order stabilizing compensator design (research) problems.

3. We have been referring to Step 2 in the proof of our theorem as *disturbance loop stabilization* because it stabilizes the loop which is influenced by external disturbances (even though it does not influence the to be controlled outputs). Step 3 is called *controlled output stabilization* because it stabilizes the loop which influences the to be controlled outputs (even though it is not influenced by the disturbances).

The total design procedure with the disturbance decoupling loop and the two stabilization loops has an appealing hierarchical structure. This structure may be made more elegant yet by viewing all three control loops in terms of a separation philosophy, with the observer elements driven by the estimation errors and having their own internal control feedback. It seems appropriate to mention at this point that, as shown in [3], in a closed loop configuration it is in general not possible to distinguish *observer error dynamic modes* and *state feedback controlled modes.*

The signal flow graph of the controller may be visualized as shown in Fig. 6.



STABILIZATION

FIG. 6

Altogether this results in a complex, but nevertheless logically structured and, from a cybernetic point of view appealing, synthesis. Even though the order of the feedback control compensator may be up to three times the dynamic order of the plant, the resulting feedback controller could be implemented on a microprocessor for moderately complicated plants.

4. The synthesis procedure explained in the proof of the theorem can obviously be made into a computer-aided design algorithm. In the case of DDPMPP one would proceed as follows:

Data. A, B, C, G, H, (which must satisfy $HG = 0$), and the desired $\mathbb{C}_g$ (or the desired symmetric set of poles $\lambda_1, \lambda_2, \cdots, \lambda_N$, or the desired characteristic polynomial p of degree N. The synthesis will work as long as N is large enough and as long as a factorizability condition on p, which comes out of the structure of the controller, is satisfied).

Verify whether $m > l$ and $p > q$. If so, proceed with confidence (see Remark 1). If not, count on luck due to special structure of the system matrices.

Step 1. Compute $\mathcal{N}^*_{\mathrm{Im}\, G}$ and $\mathcal{R}^*_{\mathrm{Ker}\, H}$ using, e.g., the linear algorithms given above. If $\mathcal{N}^*_{\mathrm{Im}\, G} \subset \mathcal{R}^*_{\mathrm{Ker}\, H}$, proceed. Otherwise, look for some other control system design approach, e.g., an LQG approach.

Step 2. Solve DDPM by computing $\mathcal{L}^e_1$ and K^e_1, using the ideas in the proofs of Proposition 6 and Step 1 of the theorem.

Step 3. Design Brasch–Pearson compensators for $\Sigma^e_1(\mathrm{mod}\, \mathcal{L}^e_1)$ and $\Sigma^e_1|\mathcal{L}^e_1$.

Obviously, in order to implement such procedures into good working high level computer-aided design packages, a lot of numerical work remains to be done [15]. However, it seems very important that such packages be developed, and the failure of control theorists to give adequate attention to such efforts undoubtedly contributes to the widely advertised gap between control theory and practice.

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