Diverse novel analytical and semi-analytical wave solutions of the generalized (2+1)-dimensional shallow water waves model

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ABSTRACT

This article studies the generalized (2 + 1)-dimensional shallow water equation by applying two recent analytical schemes (the extended simplest equation method and the modified Kudryashov method) for constructing abundant novel solitary wave solutions. These solutions describe the bidirectional propagating water wave surface. Some obtained solutions are sketched in two- and three-dimensional and contour plots for demonstrating the dynamical behavior of these waves along shallow water. The accuracy of the obtained solutions and employed analytical schemes is investigated using the evaluated solutions to calculate the initial condition, and then the well-known variational iterational (VI) method is applied. The VI method is one of the most accurate semi-analytical solutions, and it can be applied for high derivative order. The used schemes' performance shows their effectiveness and power and their ability to handle many nonlinear evolution equations.

I. INTRODUCTION

In recent years, the shallow water (SW) wave phenomenon has emerged as a particularly attractive candidate for studying and investigating the dynamical and physical behavior of the bidirectional propagating water wave surface and the flow below a pressure surface in a fluid.[1–3] A set of hyperbolic nonlinear partial differential equations is formulated to demonstrate the shallow water wave attitude.[4,5] These hyperbolic equations in the unidirectional form are derived by Adhémar Jean Claude Barré de Saint-Venant as a model of transient open-channel flow and surface runoff.[6,7] Consequently, these equations are called Saint-Venant equations that can be seen as a contraction of the two-dimensional (2-D) shallow water equations.[8,9] In addition, the famous Navier–Stokes equations, also considered a member of shallow water wave equations, are employed when the horizontal length scale is somewhat higher than the vertical length scale and mass conservation ensures that the vertical velocity scale of the fluid is small when compared to the horizontal speed scale.[10–12] Moreover, the (2 + 1)-dimensional Kadomtsev–Petviashvili–Benjamin–Bona–Mahony equation belongs to this hyperbolic set.[13–16] The shallow water wave's applications are distinct in various fields such as electromagnetic theory, astrophysics, electrochemistry, fluid dynamics, plasma physics, acoustics, and cosmology.[17–19] Many mathematical techniques have been formulated to evaluate analytical, semi-analytical, and numerical solutions of these equations such as B-spline schemes, Khater methods, the auxiliary equation method, the exponential expansion method, the well-known $\left(\frac{\Psi'}{\Psi}\right)$-expansion methods, the Kudryashov method, the direct algebraic equation method, the Adomian decomposition method, the iteration method, and the sech-tanh expansion method.[20–24] All these schemes have been employed for many nonlinear evolution equations. Many computational, semi-analytical, and numerical solutions have been constructed, but unfortunately, one solution cannot be applied for all nonlinear evolution equations; hence, the search for a unified method still continues. The great revolution in computer technology is the most helpful tool in this study; however, this technology is only used to discover new techniques, and no one has used it to check the accuracy of the schemes derived already.[25,26]

This article studies the (2 + 1)-dimensional SW equation, which is given by[27–29]

$$\mathcal{S}_{xxxt} + s_1\,\mathcal{S}_x\,\mathcal{S}_{xt} + s_2\,\mathcal{S}_t\,\mathcal{S}_{xx} - \mathcal{S}_{xt} - s_3\,\mathcal{S}_{xx} = 0, \tag{1}$$

where $s_i(i = 0, 1, 2, 3)$ are the arbitrary constants to be evaluated and $\mathcal{S} = \mathcal{S}(x, y, t)$ is a function of space and time, which describes a bidirectional propagating water wave surface. Employing the wave transformation $\mathcal{S}(x, y, t) = \mathcal{Q}(\psi)$, $\psi = x + y + \lambda\,t$, where λ is the wave velocity, and integrating the result once with zero constants of the integration convert Eq. (1) into the following ordinary differential equation:

$$\mathcal{Q}''' + \frac{s_1 + s_2}{2}\left(\mathcal{Q}'\right)^2 - \frac{\lambda + s_3}{\lambda}\,\mathcal{Q}' = 0. \tag{2}$$

Using the homogeneous balance principles and the auxiliary equations for the extended simplest equation (ESE) and modified Kudryashov (MK) methods,[30–33] that is, $\mathcal{F}'(\psi) = l_1 + l_2\,\mathcal{F}(\psi) + l_3\,\mathcal{F}(\psi)^2$ & $\mathfrak{Y}'(\psi) = \ln(k)\big(\mathfrak{Y}(\psi)^2 - \mathfrak{Y}(\psi)\big)$, where l_1, l_2, l_3, and k are the arbitrary constants to be constructed later, in Eq. (2) gives $n = 1$. Thus, the general solutions of Eq. (2) are formulated in the following forms:

$$\mathcal{Q} = \begin{cases} \sum_{i=-n}^{n} a_i\,\mathfrak{F}(\psi)^i = a_1\,\mathcal{F}(\psi) + \dfrac{a_{-1}}{\mathcal{F}(\psi)} + a_0, \\ \sum_{i=0}^{n} a_i\,\mathfrak{Y}(\psi)^i = a_1\,\mathfrak{Y}(\psi) + a_0, \end{cases} \tag{3}$$

where a_i, $(i = -2, -1, 0, 1, 2)$ are the arbitrary constants.

The remaining sections of this article are organized as follows: Section II applies the above-mentioned analytical and semi-analytical schemes to the nonlinear (2 + 1)-dimensional SW equation for explaining the accuracy of each analytically employed scheme. Section III discusses the obtained solutions and the shown figures and tables. Section IV gives the conclusion of the whole paper.

II. ACCURACY OF COMPUTATIONAL SOLUTIONS

Applying the ESE and MK methods to Eq. (2) to construct the computational solutions of the (2 + 1)-dimensional SW equation then using these solutions to evaluate the initial and boundary conditions that allow applying the VI method are carried out as follows.

A. ESE method's solutions

Employing Eq. (3) in the ESE method's framework gets the values of the above-mentioned parameters as follows:

For family I,

$$a_{-1} \to \frac{12 l_1}{s_1 + s_2}, \quad a_1 \to 0, \quad \lambda \to \frac{s_3}{l_2^2 - 4 l_1 l_3 - 1}.$$

For family II,

$$a_{-1} \to 0, \quad a_1 \to -\frac{12 l_3}{s_1 + s_2}, \quad \lambda \to \frac{s_3}{l_2^2 - 4 l_1 l_3 - 1}.$$

Thus, the computational solutions of the (2 + 1)-dimensional SW equation are constructed as follows:

For $l_2 = 0$, $l_1 l_3 > 0$, we get

$$\mathcal{S}_{\mathrm{I},1}(x, y, t) = a_0 + \frac{12\sqrt{l_1 l_3}\cot\left(\sqrt{l_1 l_3}\left(\eta + \frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right)\right)}{s_1 + s_2}, \tag{4}$$

$$\mathcal{S}_{\mathrm{I},2}(x, y, t) = a_0 + \frac{12\sqrt{l_1 l_3}\tan\left(\sqrt{l_1 l_3}\left(\eta + \frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right)\right)}{s_1 + s_2}, \tag{5}$$

$$\mathcal{S}_{\mathrm{II},1}(x, y, t) = a_0 - \frac{12\sqrt{l_1 l_3}\tan\left(\sqrt{l_1 l_3}\left(\eta + \frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right)\right)}{s_1 + s_2}, \tag{6}$$

$$\mathcal{S}_{\mathrm{II},2}(x, y, t) = a_0 - \frac{12\sqrt{l_1 l_3}\cot\left(\sqrt{l_1 l_3}\left(\eta + \frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right)\right)}{s_1 + s_2}. \tag{7}$$

For $l_2 = 0$, $l_1 l_3 < 0$, we get

$$\mathcal{S}_{\mathrm{I},3}(x, y, t) = a_0 - \frac{12\sqrt{-l_1 l_3}\coth\left(\sqrt{-l_1 l_3}\left(\frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right) \mp \frac{\log(\eta)}{2}\right)}{s_1 + s_2}, \tag{8}$$

$$\mathcal{S}_{\mathrm{I},4}(x, y, t) = a_0 - \frac{12\sqrt{-l_1 l_3}\tanh\left(\sqrt{-l_1 l_3}\left(\frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right) \mp \frac{\log(\eta)}{2}\right)}{s_1 + s_2}, \tag{9}$$

$$\mathcal{S}_{\mathrm{II},3}(x, y, t) = a_0 - \frac{12\sqrt{-l_1 l_3}\tanh\left(\sqrt{-l_1 l_3}\left(\frac{s_3 t}{-4 l_1 l_3 - 1} + x + y\right) \mp \frac{\log(\eta)}{2}\right)}{s_1 + s_2}, \tag{10}$$

$$\mathcal{S}_{\mathrm{II},4}(x,y,t)=a_0-\frac{12\sqrt{-l_1l_3}\coth\left(\sqrt{-l_1l_3}\left(\frac{s_3t}{-4l_1l_3-1}+x+y\right)\mp\frac{\log(\eta)}{2}\right)}{s_1+s_2}. \tag{11}$$

For $l_1=0, l_2>0$, we get

$$\mathcal{S}_{\mathrm{II},5}(x,y,t)=a_0+\frac{1}{s_1+s_2}\left(12l_2\left(\frac{1}{l_3e^{l_2\left(\eta+\frac{s_3t}{l_2^2-1}+x+y\right)}-1}+1\right)\right). \tag{12}$$

For $l_1=0, l_2<0$, we get

$$\mathcal{S}_{\mathrm{II},6}(x,y,t)=a_0+\frac{1}{s_1+s_2}\left(12l_3\left(1-\frac{1}{l_3e^{l_2\left(\eta+\frac{s_3t}{l_2^2-1}+x+y\right)}+1}\right)\right). \tag{13}$$

For $4\,l_1\,l_3>l_2^2$, we get

$$\mathcal{S}_{\mathrm{I},5}(x,y,t)=a_0-\frac{24l_1l_3}{(s_1+s_2)\left(l_2-\sqrt{4l_1l_3-l_2^2}\tan\left(\frac{1}{2}\sqrt{4l_1l_3-l_2^2}\left(\eta+\frac{s_3t}{l_2^2-4l_1l_3-1}+x+y\right)\right)\right)}, \tag{14}$$

$$\mathcal{S}_{\mathrm{I},6}(x,y,t)=a_0-\frac{24l_1l_3}{(s_1+s_2)\left(l_2-\sqrt{4l_1l_3-l_2^2}\cot\left(\frac{1}{2}\sqrt{4l_1l_3-l_2^2}\left(\eta+\frac{s_3t}{l_2^2-4l_1l_3-1}+x+y\right)\right)\right)}, \tag{15}$$

$$\mathcal{S}_{\mathrm{II},7}(x,y,t)=a_0-\frac{6\sqrt{4l_1l_3-l_2^2}\tan\left(\frac{1}{2}\sqrt{4l_1l_3-l_2^2}\left(\eta+\frac{s_3t}{l_2^2-4l_1l_3-1}+x+y\right)\right)}{s_1+s_2}+\frac{6l_2}{s_1+s_2}, \tag{16}$$

$$\mathcal{S}_{\mathrm{II},8}(x,y,t)=a_0-\frac{6\sqrt{4l_1l_3-l_2^2}\cot\left(\frac{1}{2}\sqrt{4l_1l_3-l_2^2}\left(\eta+\frac{s_3t}{l_2^2-4l_1l_3-1}+x+y\right)\right)}{s_1+s_2}+\frac{6l_2}{s_1+s_2}. \tag{17}$$

1. Semi-analytical solutions

Applying the variational iteration method[34] to Eq. (1) with the initial condition $\mathcal{S}_I(x,y,0)=2-\tanh(x+y)$ & $\mathcal{S}_{II}(x,y,0)=2-\coth(x+y)$ according to Eqs. (9) and (11) gives the following solutions:

$$\mathcal{S}_{\mathrm{I},1}(x,y,t)=2-\tanh(x+y)\left(6t\,\mathrm{sech}^2(x+y)+1\right), \tag{18}$$

$$\mathcal{S}_{\mathrm{I},2}(x,y,t)=2-\tanh(x+y)\big(12t\,\mathrm{sech}^2(x+y)\times\left(9t\,\mathrm{sech}^2(x+y)-3t+1\right)+1\big), \tag{19}$$

$$\mathcal{S}_{\mathrm{I},3}(x,y,t)=2-\tanh(x+y)\big(18t\,\mathrm{sech}^2(x+y)\times\left(8t^2+6t\,\mathrm{sech}^2(x+y)\left(30t\,\mathrm{sech}^2(x+y)-20t+3\right)-6t+1\right)+1\big), \tag{20}$$

$$\mathcal{S}_{\mathrm{II},1}(x,y,t)=\coth(x+y)\left(6t\,\mathrm{csch}^2(x+y)-1\right)+2, \tag{21}$$

$$\mathcal{S}_{\mathrm{II},2}(x,y,t)=2-\coth(x+y)\big(12t\,\mathrm{csch}^2(x+y)\times\left(9t\,\mathrm{csch}^2(x+y)+3t-1\right)+1\big), \tag{22}$$

$$\mathcal{S}_{\mathrm{II},3}(x,y,t)=\coth(x+y)\big(18t\,\mathrm{csch}^2(x+y)\times\left(8t^2+6t\,\mathrm{csch}^2(x+y)\left(30t\,\mathrm{csch}^2(x+y)+20t-3\right)-6t+1\right)-1\big)+2. \tag{23}$$

B. MK method's solutions

Employing Eq. (3) in the MK method's framework gets the values of the above-mentioned parameters as follows:

$$a_1\to-\frac{12\,\log(k)}{s_1+s_2},\quad \lambda\to\frac{s_3}{\log^2(k)-1}.$$

Thus, the computational solutions of the (2 + 1)-dimensional SW equation are constructed as follows:

$$\mathcal{S}(x,y,t) = a_0 - \frac{12\log(k)}{(s_1+s_2)\left(1 \pm k^{\frac{s_3 t}{\log^2(k)-1}+x+y}\right)}. \tag{24}$$

1. Semi-analytical solutions

Applying the variational iteration method to Eq. (1) with the initial condition $\mathcal{S}(x,y,0) = 4 - \frac{12\log(10)}{10^{x+y}+1}$ based on Eq. (24) gives the following solutions:

$$\mathcal{S}_1(x,y,t) = \frac{4\left(\left(10^{x+y}+1\right)^2\left(10^{x+y}+1-3\log(10)\right) - 3t\log^3(10)10^{x+y}\left(10^{x+y}-1\right)\right)}{\left(10^{x+y}+1\right)^3}, \tag{25}$$

$$\mathcal{S}_2(x,y,t) = 2\left(-\frac{3t\log^3(10)10^{x+y}\left(10^{x+y}-1\right)\left(t\log^2(10)\left(-10^{x+y+1}+100^{x+y}+1\right)+4\left(10^{x+y}+1\right)^2\right)}{\left(10^{x+y}+1\right)^5} - \frac{6\log(10)}{10^{x+y}+1}+2\right), \tag{26}$$

$$\begin{aligned}\mathcal{S}_3(x,y,t) = \frac{1}{\left(10^{x+y}+1\right)^7}\Big(2\Big(t\log^3(10)10^{x+y}\Big(-t^2\log^4(10)\Big(57\ 10^{x+y} - 57\ 10^{4(x+y)} + 10^{5(x+y)} \\ - 151\ 2^{2x+2y+1}25^{x+y} + 151\ 2^{3x+3y+1}125^{x+y} - 1\Big) - 9t\log^2(10)\Big(11\ 10^{x+y} - 11\ 100^{x+y} \\ + 1000^{x+y} - 1\Big)\left(10^{x+y}+1\right)^2 - 18\left(10^{x+y}-1\right)\left(10^{x+y}+1\right)^4\Big) + 2\left(10^{x+y}+1\right)^6\left(10^{x+y}+1-3\log(10)\right)\Big)\Big).\end{aligned} \tag{27}$$

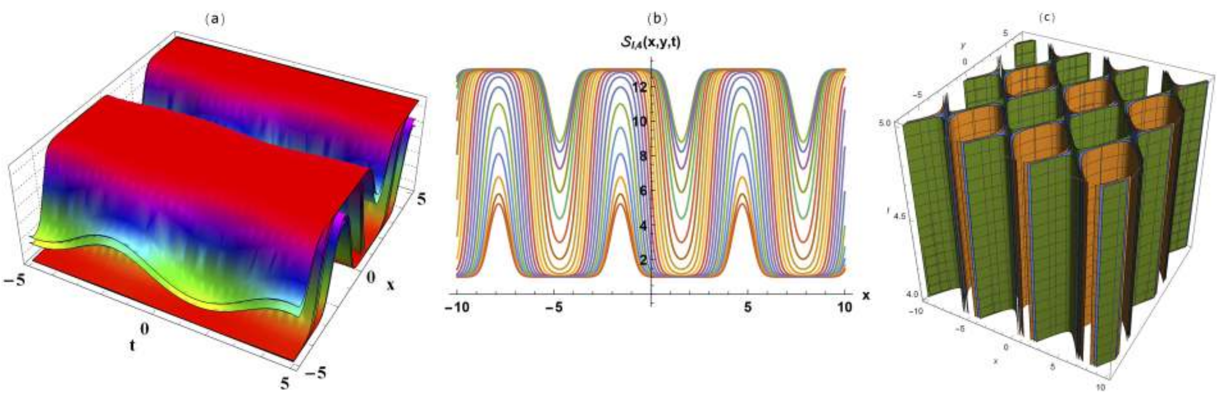


FIG. 1. Solitary wave solutions, Eq. (9), in (a) three-dimensional, (b) two-dimensional, and (c) contour plots.

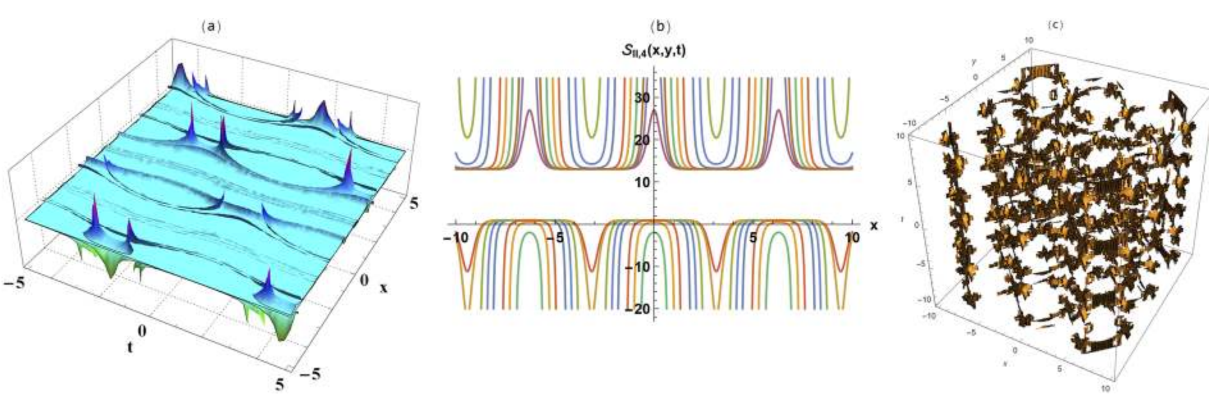


FIG. 2. Solitary wave solutions, Eq. (11), in (a) three-dimensional, (b) two-dimensional, and (c) contour plots.

TABLE I. Absolute value of error for Eq. (1) based on ESE [Eq. (9)] and VI methods.

Value of x	t = 1	t = 3	t = 5	t = 7	t = 9	t = 11	t = 13	t = 15	t = 17	t = 19
0	4.45209×10^{-7}	2.44865×10^{-5}	0.000126885	0.000364626	0.000794698	0.001474087	0.00245978	0.003808764	0.005578025	0.0078245 5
1	6.02525×10^{-8}	3.31389×10^{-6}	1.7172×10^{-5}	4.93468×10^{-5}	0.000107551	0.000199496	0.000332895	0.00051546	0.000754904	0.001058938
2	8.15429×10^{-9}	4.48486×10^{-7}	2.32397×10^{-6}	6.67836×10^{-6}	1.45554×10^{-5}	2.69989×10^{-5}	4.50525×10^{-5}	6.976×10^{-5}	0.000102165	0.000143312
3	1.10356×10^{-9}	6.0696×10^{-8}	3.14516×10^{-7}	9.03818×10^{-7}	1.96986×10^{-6}	3.6539×10^{-6}	6.09719×10^{-6}	9.44098×10^{-6}	1.38265×10^{-5}	1.93951×10^{-5}
4	1.49351×10^{-10}	8.21431×10^{-9}	4.2565×10^{-8}	1.22319×10^{-7}	2.66592×10^{-7}	4.94501×10^{-7}	8.25165×10^{-7}	1.2777×10^{-6}	1.87122×10^{-6}	2.62484×10^{-6}
5	2.02125×10^{-11}	1.11169×10^{-9}	5.76055×10^{-9}	1.6554×10^{-8}	3.60793×10^{-8}	6.69235×10^{-8}	1.11674×10^{-7}	1.72918×10^{-7}	2.53242×10^{-7}	3.55234×10^{-7}
6	2.73537×10^{-12}	1.5045×10^{-10}	7.79606×10^{-10}	2.24034×10^{-9}	4.8828×10^{-9}	9.05711×10^{-9}	1.51134×10^{-8}	2.34019×10^{-8}	3.42726×10^{-8}	4.80757×10^{-8}
7	3.70037×10^{-13}	2.03612×10^{-11}	1.05508×10^{-10}	3.03197×10^{-10}	6.60814×10^{-10}	1.22575×10^{-9}	2.04538×10^{-9}	3.1671×10^{-9}	4.63829×10^{-9}	6.50634×10^{-9}
8	5.01821×10^{-14}	2.75557×10^{-12}	1.4279×10^{-11}	4.10332×10^{-11}	8.94316×10^{-11}	1.65887×10^{-10}	2.76812×10^{-10}	4.2862×10^{-10}	6.27724×10^{-10}	8.80537×10^{-10}
9	6.77236×10^{-15}	3.72924×10^{-13}	1.93234×10^{-12}	5.55322×10^{-12}	1.21031×10^{-11}	2.24502×10^{-11}	3.74624×10^{-11}	5.80073×10^{-11}	8.4953×10^{-11}	1.19168×10^{-10}
10	8.88178×10^{-16}	5.04041×10^{-14}	2.61569×10^{-13}	7.51621×10^{-13}	1.63802×10^{-12}	3.03824×10^{-12}	5.06994×10^{-12}	7.85039×10^{-12}	1.14972×10^{-11}	1.61275×10^{-11}
11	2.22045×10^{-16}	6.88338×10^{-15}	3.53051×10^{-14}	1.01696×10^{-13}	2.21601×10^{-13}	4.11227×10^{-13}	6.86118×10^{-13}	1.06248×10^{-12}	1.55587×10^{-12}	2.1827×10^{-12}
12	0	8.88178×10^{-16}	4.88498×10^{-15}	1.37668×10^{-14}	2.9976×10^{-14}	5.57332×10^{-14}	9.28146×10^{-14}	1.43885×10^{-13}	2.10498×10^{-13}	2.95319×10^{-13}
13	0	2.22045×10^{-16}	6.66134×10^{-16}	1.77636×10^{-15}	3.9968×10^{-15}	7.54952×10^{-15}	1.26565×10^{-14}	1.95399×10^{-14}	2.84217×10^{-14}	3.9968×10^{-14}
14	0	0	0	2.22045×10^{-16}	4.44089×10^{-16}	1.11022×10^{-15}	1.77636×10^{-15}	2.66454×10^{-15}	3.77476×10^{-15}	5.32907×10^{-15}
15	0	0	0	0	0	2.22045×10^{-16}	2.22045×10^{-16}	4.44089×10^{-16}	4.44089×10^{-16}	6.66134×10^{-16}

TABLE II. Absolute value of error for Eq. (1) based on ESE [Eq. (11)] and VI methods.

Value of x	t = 1	t = 3	t = 5	t = 7	t = 9	t = 11	t = 13	t = 15	t = 17	t = 19
0	4.4521×10^{-7}	2.4487×10^{-5}	0.00012688	0.00036463	0.0007947	0.00147409	0.00245978	0.00380877	0.005578027	0.007824552
1	6.0253×10^{-8}	3.3139×10^{-6}	1.7172×10^{-5}	4.9347×10^{-5}	0.00010755	0.0001995	0.0003329	0.00051546	0.000754904	0.001058938
2	8.1543×10^{-9}	4.4849×10^{-7}	2.324×10^{-6}	6.6784×10^{-6}	1.4555×10^{-5}	2.6999×10^{-5}	4.5052×10^{-5}	6.976×10^{-5}	0.000102165	0.000143312
3	1.1036×10^{-9}	6.0696×10^{-8}	3.1452×10^{-7}	9.0382×10^{-7}	1.9699×10^{-6}	3.6539×10^{-6}	6.0972×10^{-6}	9.441×10^{-6}	1.38265×10^{-5}	1.93951×10^{-5}
4	1.4935×10^{-10}	8.2143×10^{-9}	4.2565×10^{-8}	1.2232×10^{-7}	2.6659×10^{-7}	4.945×10^{-7}	8.2516×10^{-7}	1.2777×10^{-6}	1.87122×10^{-6}	2.62484×10^{-6}
5	2.0212×10^{-11}	1.1117×10^{-9}	5.7606×10^{-9}	1.6554×10^{-8}	3.6079×10^{-8}	6.6923×10^{-8}	1.1167×10^{-7}	1.7292×10^{-7}	2.53242×10^{-7}	3.55234×10^{-7}
6	2.7355×10^{-12}	1.5045×10^{-10}	7.7961×10^{-10}	2.2403×10^{-9}	4.8828×10^{-9}	9.0571×10^{-9}	1.5113×10^{-8}	2.3402×10^{-8}	3.42726×10^{-8}	4.80757×10^{-8}
7	3.7026×10^{-13}	2.0361×10^{-11}	1.0551×10^{-10}	3.032×10^{-10}	6.6081×10^{-10}	1.2257×10^{-9}	2.0454×10^{-9}	3.1671×10^{-9}	4.63829×10^{-9}	6.50634×10^{-9}
8	5.0071×10^{-14}	2.7556×10^{-12}	1.4279×10^{-11}	4.1033×10^{-11}	8.9432×10^{-11}	1.6589×10^{-10}	2.7681×10^{-10}	4.2862×10^{-10}	6.27724×10^{-10}	8.80537×10^{-10}
9	6.7724×10^{-15}	3.7292×10^{-13}	1.9325×10^{-12}	5.5532×10^{-12}	1.2103×10^{-11}	2.245×10^{-11}	3.7462×10^{-11}	5.8007×10^{-11}	8.49533×10^{-11}	1.19168×10^{-10}
10	8.8818×10^{-16}	5.0515×10^{-14}	2.6157×10^{-13}	7.5151×10^{-13}	1.638×10^{-12}	3.0383×10^{-12}	5.0699×10^{-12}	7.8505×10^{-12}	1.14971×10^{-11}	1.61277×10^{-11}
11	1.1102×10^{-16}	6.8834×10^{-15}	3.5416×10^{-14}	1.017×10^{-13}	2.2171×10^{-13}	4.1123×10^{-13}	6.8612×10^{-13}	1.0625×10^{-12}	1.55598×10^{-12}	2.18259×10^{-12}
12	0	8.8818×10^{-16}	4.774×10^{-15}	1.3767×10^{-14}	2.9976×10^{-14}	5.5622×10^{-14}	9.2815×10^{-14}	1.4377×10^{-13}	2.10609×10^{-13}	2.9543×10^{-13}
13	0	1.1102×10^{-16}	6.6613×10^{-16}	1.8874×10^{-15}	4.1078×10^{-15}	7.5495×10^{-15}	1.2546×10^{-14}	1.9429×10^{-14}	2.85327×10^{-14}	3.9968×10^{-14}
14	0	0	1.1102×10^{-16}	2.2204×10^{-16}	5.5511×10^{-16}	9.992×10^{-16}	1.6653×10^{-15}	2.6645×10^{-15}	3.88578×10^{-15}	5.44009×10^{-15}
15	0	0	0	0	1.1102×10^{-16}	1.1102×10^{-16}	2.2204×10^{-16}	3.3307×10^{-16}	5.55112×10^{-16}	7.77156×10^{-16}

TABLE III. Absolute value of error for Eq. (1) based on KM [Eq. (24)] and VI methods.

Value of x	t = 1	t = 3	t = 5	t = 7	t = 9	t = 11	t = 13	t = 15	t = 17	t = 19
0	2.2909×10^{-7}	3.0335×10^{-6}	1.1712×10^{-5}	2.9558×10^{-5}	5.9867×10^{-5}	0.000 105 93	0.000 171 05	0.000 258 51	0.000 371 617	0.000 513 655
1	2.2909×10^{-8}	3.0335×10^{-7}	1.1712×10^{-6}	2.9558×10^{-6}	5.9867×10^{-6}	1.0593×10^{-5}	1.7105×10^{-5}	2.5851×10^{-5}	3.71617×10^{-5}	5.13655×10^{-5}
2	2.2909×10^{-9}	3.0335×10^{-8}	1.1712×10^{-7}	2.9558×10^{-7}	5.9867×10^{-7}	1.0593×10^{-6}	1.7105×10^{-6}	2.5851×10^{-6}	3.71617×10^{-6}	5.13655×10^{-6}
3	2.2909×10^{-10}	3.0335×10^{-9}	1.1712×10^{-8}	2.9558×10^{-8}	5.9867×10^{-8}	1.0593×10^{-7}	1.7105×10^{-7}	2.5851×10^{-7}	3.71617×10^{-7}	5.13655×10^{-7}
4	2.2909×10^{-11}	3.0335×10^{-10}	1.1712×10^{-9}	2.9558×10^{-9}	5.9867×10^{-9}	1.0593×10^{-8}	1.7105×10^{-8}	2.5851×10^{-8}	3.71617×10^{-8}	5.13655×10^{-8}
5	2.2911×10^{-12}	3.0335×10^{-11}	1.1712×10^{-10}	2.9558×10^{-10}	5.9867×10^{-10}	1.0593×10^{-9}	1.7105×10^{-9}	2.5851×10^{-9}	3.71617×10^{-9}	5.13655×10^{-9}
6	2.2871×10^{-13}	3.0327×10^{-12}	1.1712×10^{-11}	2.9558×10^{-11}	5.9866×10^{-11}	1.0593×10^{-10}	1.7105×10^{-10}	2.5851×10^{-10}	3.71616×10^{-10}	5.13655×10^{-10}
7	2.2649×10^{-14}	3.0287×10^{-13}	1.1706×10^{-12}	2.9554×10^{-12}	5.9863×10^{-12}	1.0593×10^{-11}	1.7105×10^{-11}	2.5851×10^{-11}	3.71614×10^{-11}	5.13651×10^{-11}
8	2.2204×10^{-15}	2.9754×10^{-14}	1.1724×10^{-13}	2.9576×10^{-13}	5.9863×10^{-13}	1.0592×10^{-12}	1.7102×10^{-12}	2.585×10^{-12}	3.71614×10^{-12}	5.13634×10^{-12}
9	0	2.6645×10^{-15}	1.1102×10^{-14}	2.931×10^{-14}	5.9508×10^{-14}	1.0569×10^{-13}	1.7053×10^{-13}	2.5846×10^{-13}	3.71259×10^{-13}	5.12923×10^{-13}
10	0	0	8.8818×10^{-16}	2.2204×10^{-15}	5.7732×10^{-15}	1.0658×10^{-14}	1.6875×10^{-14}	2.6201×10^{-14}	3.68594×10^{-14}	5.15143×10^{-14}
11	0	0	0	0	8.8818×10^{-16}	8.8818×10^{-16}	1.7764×10^{-15}	3.1086×10^{-15}	3.55271×10^{-15}	5.32907×10^{-15}
12	0	0	0	0	0	0	0	0	4.44089×10^{-16}	4.44089×10^{-16}
13	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.8818×10^{-16}	-8.8818×10^{-16}
14	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0
20	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.8818×10^{-16}	-8.8818×10^{-16}
21	0	0	0	0	0	0	0	0	0	0
22	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.882×10^{-16}	-8.8818×10^{-16}	-8.8818×10^{-16}
23	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.44089×10^{-16}	4.44089×10^{-16}
24	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.44089×10^{-16}	4.44089×10^{-16}
25	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.44089×10^{-16}	4.44089×10^{-16}
26	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0
28	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}	4.44089×10^{-16}	4.44089×10^{-16}
29	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0

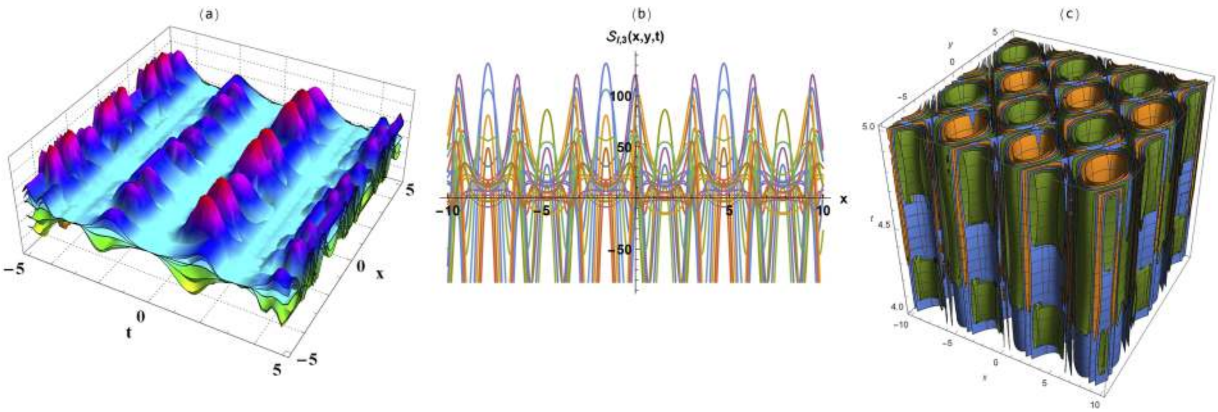


FIG. 3. Variational iteration method solutions in (a) three-dimensional, (b) two-dimensional, and (c) contour plots for Eq. (23).

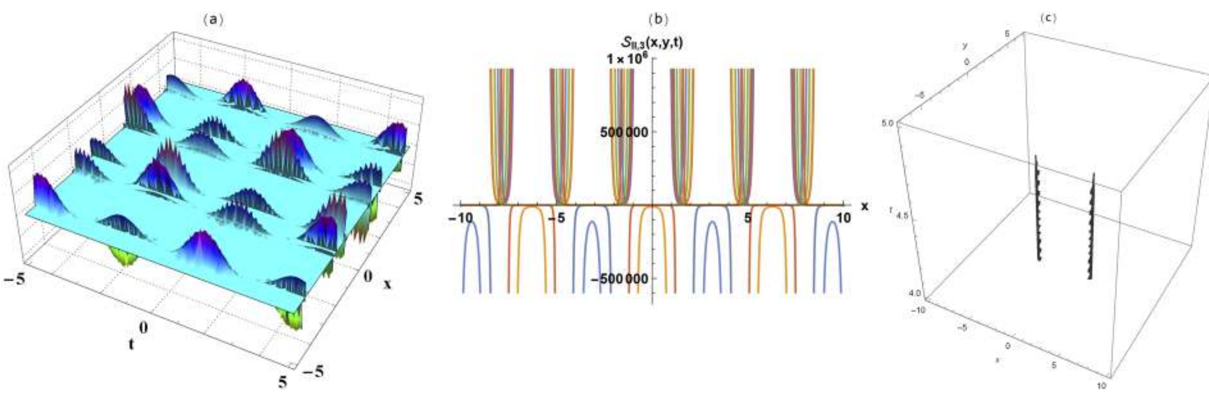


FIG. 4. Variational iteration method solutions in (a) three-dimensional, (b) two-dimensional, and (c) contour plots for Eq. (23).

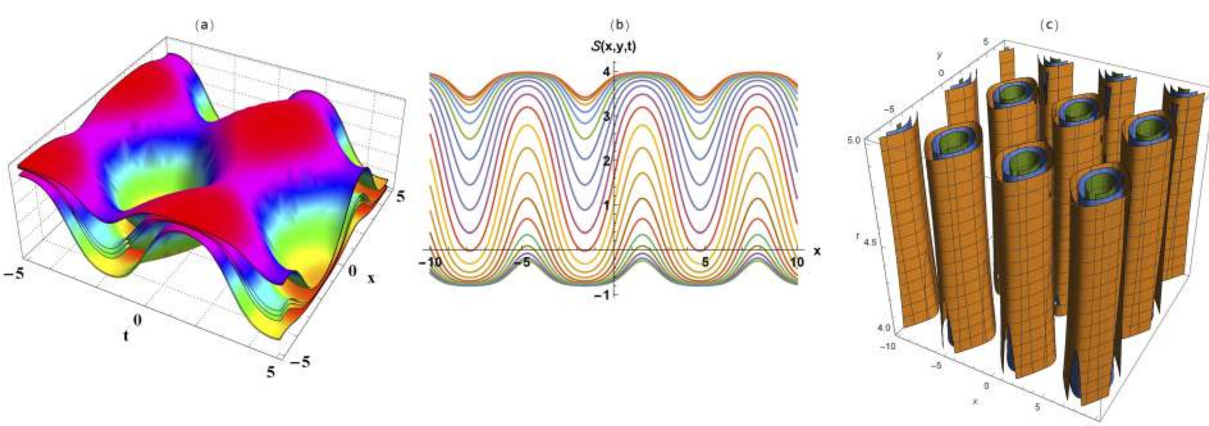


FIG. 5. Solitary wave solutions Eq. (24) in (a) three-dimensional, (b) two-dimensional, and (c) contour plots.

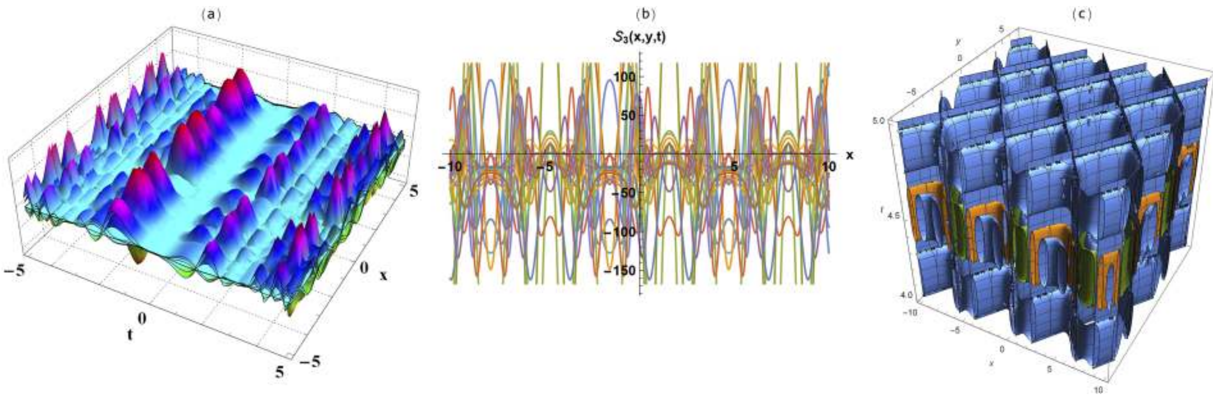
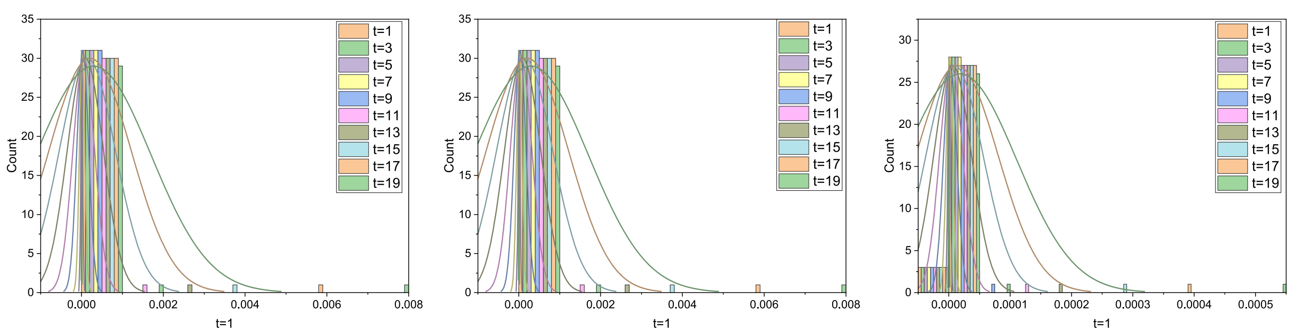


FIG. 6. Variational iteration method solutions in (a) three, (b) two, and (c) contour plots for Eq. (27).



(a) (b) (c)

FIG. 7. Absolute error along ESM and variational iteration methods (a) for Eq. (9) and (b) for Eq. (11) while (c) shows the absolute error along MK and variational iteration methods.

III. RESULTS' INTERPRETATION

This section discusses the obtained results of this research paper. This research applied two recent computational schemes (ESE and MK methods) to the (2 + 1)-dimensional SW equation and constructed abundant novel analytical solutions that show the flow's dynamical behavior through shallow water waves. Figures 1, 2, and 5 show periodic kink, singular, and cone waves in two- and three-dimensional and contour plots of Eqs. (9), (11), and (24), respectively, when $[a_0 = 7, \eta = 1, l_1 = 3, l_3 = -3, s_1 = 18, s_2 = -12, s_3 = 4$ & $a_0 = 7, \eta = 1, l_1 = 3, l_3 = -3, s_1 = 18, s_2 = -12, s_3 = 4$ & $a_0 = 4, k = 5, s_1 = 3, s_2 = 1, s_3 = 2]$. However, the main goal of this paper is not just obtaining a novel solution of the shallow water wave model, but it also aims to determine the accuracy of both employed schemes by applying the AD and VI methods. Thus, these two semi-analytical schemes have been employed based on the evaluated analytical solutions. The accuracy of each of the ESE and MK methods has been explained through Tables I–III and Figs. 3, 4 and 6 while the matching between the computational and semi-analytical solutions is illustrated in Fig. 7. Thus, it has been demonstrated that the obtained solution via the MK method is more accurate than that obtained by the ESE method.

IV. CONCLUSION

This article has successfully applied three computational and semi-analytical schemes to the (2 + 1)-dimensional SW equation that is used as a shallow water wave model. Many novel computational solutions have been obtained, and some of them have been demonstrated by two- and three-dimensional and contour plots. The accuracy of the obtained results and the used computational schemes has been explained. The matching between computational and numerical solutions has been illustrated by two-dimensional plots.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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