Divide-and-conquer: A proportional, minimal-envy cake-cutting algorithm

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DIVIDE-AND-CONQUER: A PROPORTIONAL, MINIMAL-ENVY CAKE-CUTTING ALGORITHM*

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Abstract. We analyze a class of proportional cake-cutting algorithms that use a minimal number of cuts ($n-1$ if there are n players) to divide a cake that the players value along one dimension. While these algorithms may not produce an envy-free or efficient allocation—as these terms are used in the fair-division literature—one, divide-and-conquer (D&C), minimizes the maximum number of players that any single player can envy. It works by asking $n \geq 2$ players successively to place marks on a cake—valued along a line—that divide it into equal halves (when n is even) or nearly equal halves (when n is odd), then halves of these halves, and so on. Among other properties, D&C ensures players of at least $1/n$ shares, as they each value the cake, if and only if they are truthful. However, D&C may not allow players to obtain proportional, connected pieces if they have unequal entitlements. Possible applications of D&C to land division are briefly discussed.

Key words. cake-cutting, fair division, proportional algorithm, binary tree, minimal envy

AMS subject classifications. 91B32

1. Introduction. A cake is a metaphor for a heterogeneous good, whose parts each of n players may value differently. A *proportional division* of a cake is one that gives each player, as it values the cake, at least a $1/n$ portion, which we call a *proportional share*.

We represent a cake by the interval $[0, 1]$, over which each player's preference is given by a probability density function with a continuous cumulative distribution function. There exist several algorithms for cutting this cake into pieces such that each player receives a proportional share, but we know of only one algorithm, due to Dubins and Spanier (1961), that does so using only $n-1$ cuts (the minimal number), which are assumed to cut the interval at points in $(0, 1)$. However, this algorithm, which we will describe later, requires a knife to move continuously across a cake and players to make cuts by calling "stop." By contrast, a discrete algorithm specifies when and what kinds of cuts will be made that do not depend on the continuous movement of knives.[1]

Whether discrete or continuous, almost all the proportional algorithms have a serious limitation: They restrict at least one player to receiving exactly $1/n$ of the cake. By contrast, the class of minimal-cut, proportional algorithms that we analyze

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[1] A discrete algorithm due to Banach and Knaster (Brams and Taylor 1996, pp. 35-36), called "last diminisher," gives a proportional allocation similar to that of Dubins and Spanier (1961), but it involves the trimming of pieces to give players exactly proportional pieces that in general requires more that $n-1$ cuts. However, if the trimmings are not actually cut but only indicated by marks, then the Banach-Knaster algorithm can be interpreted as a discrete version of the Dubins-Spanier algorithm. The equitability procedure (EP) (Brams, Jones, and Klamler, 2006), which ensures that each player receives exactly the same amount in its eyes, uses only $n-1$ cuts. However, EP requires players to provide a referee, who makes the cuts, with complete information about their valuations of the cake, whereas the proportional algorithm discussed here does not require the help of such a third party. For a description of different proportional algorithms, see Brams and Taylor (1996) and Robertson and Webb (1998).

carry no such restriction—they allow all players to receive at least $1/n$ of a cake, and generally more.

Because proportional algorithms do not guarantee a player a most-valued piece, some players may envy others for receiving what they perceive to be more valuable pieces. We call a proportional algorithm *envy-free* if there are no examples in which applying it results in at least one player envying another, and this envy is not mutual (so it can be eliminated by a trade of pieces). An allocation is Pareto inefficient if there exists another allocation in which at least one player does better (receiving more according to her preferences) and all other players do at least as well. Consequently, we call a proportional algorithm *efficient* (Pareto-optimal) if there are no examples in which applying it results in an inefficient allocation when the cake is divided with $n-1$ cuts.

The proportional algorithms we present here put an upper bound on the number of envies (to be defined) that all players may have without the possibility of making trades that would reduce this number. We show that one of these algorithms, which has been called divide-and-conquer (D&C), minimizes the maximum number of players that any single player may envy.[2] D&C also minimizes the maximum number of rounds on which players must place "marks" (to be defined) on a cake.

D&C is a variation on *divide-and-choose*, the well-known 2-player cake-cutting procedure in which one player cuts a cake into two pieces, and the other player chooses one piece.[3] We substitute the stronger "conquer" for "choose" to emphasize that n players can, in general, do better than $1/n$ shares under D&C.

Besides not being envy-free, D&C may not give an efficient allocation using $n-1$ cuts. By contrast, an envy-free allocation that uses $n-1$ cuts is always efficient (Gale, 1993; Brams and Taylor, 1996, pp. 150-151). If $n = 3$, there are two known algorithms—one that uses two moving knives (Barbanel and Brams, 2004) and the other that uses four (Stromquist, 1980)—that yield an envy-free allocation. While there is no known minimal-cut algorithm that yields an envy-free, efficient allocation for $n > 3$, there is a discrete algorithm that gives an approximate envy-free, efficient division (Su, 1999).

On the positive side, D&C is relatively simple to apply: It does not require that the players know the valuations of the other players, nor does it require a referee to implement it, although such a person could be helpful. Also, D&C is *truth-inducing*: It guarantees players at least $1/n$ shares if and only if they are truthful. Players, therefore, have good reason not to try to manipulate D&C. Should they try to gain an edge over other players, they may only succeed in hurting themselves and not obtaining a proportional share.

The paper proceeds as follows. In section 2 we describe D&C first with an example and then formally define it by giving six rules of play.

In section 3, we count the maximum number of envies under D&C, beginning with a 7-player example. For a class of proportional algorithms that includes D&C, we show that all give the same maximum sum of envies of all players, $(n-1)(n-2)/2$,

[2]The computational complexity of D&C and related cake-cutting algorithms is analyzed in, among other places, Even and Paz (1984) and Busch, Magdon-Ismail, and Krishnamoorthy (2005). A somewhat different definition of D&C from the one given in section 2 is proposed in Robertson and Webb (1998, pp. 25-28), wherein "cuts" are used for what we later call "marks." Because the Robertson-Webb algorithm cuts at some players' marks—rather than in between them—it gives fewer players more-than-$1/n$ shares than does D&C.

[3]What is divided need not be a cake but could, for example, be separate items that the divider puts into two piles, one of which the chooser selects.

but D&C minimizes the maximum number of players that any player may envy.

In section 4, we show with a 3-player example that D&C is not envy-free. A different 3-player example establishes that D&C is not efficient. For the latter example, we show that there are efficient allocations that are envy-free or equitable (each player receives exactly the same amount in its eyes), but they are quite different from any D&C allocation.

In section 5, we show that D&C is truth-inducing, but it may not allow players to obtain proportional, connected pieces if they have unequal entitlements (e.g., one player is entitled to 2/3 of the cake, the other to 1/3). In such a case, we introduce fictitious players, or clones, who together can obtain proportional but disconnected pieces. Curiously, one clone may envy another clone (if clones can envy each other).

In the absence of an envy-free, efficient cake-cutting algorithm, we conclude in section 6 that D&C is a compelling minimal-cut algorithm that ensures proportionality while limiting the number of players that any player may envy. Coupled with its economy and practicality, it seems applicable to the division of land and other divisible goods among a finite number of players.

2. Divide-and-Conquer (D&C). As noted earlier, a *cake* is a one-dimensional heterogeneous good, represented by the unit interval $[0, 1]$. Each of n players has a personal valuation of portions of the cake, characterized by a probability density function with a continuous cumulative distribution function. This implies that players' preferences are finitely additive and nonatomic.

Finite additivity ensures that the value of a finite number of disjoint pieces is equal to the value of their union, so that there are no complementarities between subpieces. *Nonatomic measures* imply that a single cut, which defines the border of a piece, has no area and so contains no value. We also assume that the measure of a player may be zero for some subinterval, in which case where it places a mark, and where a cut is made, in the subinterval does not affect the value it receives. Clearly, if all players have zero measure for some subinterval, it is worthless to everybody and so can be shrunk to a single point.

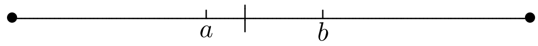
D&C is an example of an *m-proportional algorithm.* Under such a procedure, if there are n players, a first cut divides the cake into two pieces such that m players are to divide the piece to the left that each values at m/n or more of the entire cake, and $n - m$ players are to divide the piece to the right that each values at $(n - m)/n$ or more, where m is an integer satisfying $1 \leq m < n$. Subsequent cuts are made in a similar manner to divide and subdivide these pieces into proportional shares, with the process terminating after $n - 1$ cuts so that each player receives exactly one piece.

To introduce D&C, it is useful to begin with a simple example. It illustrates how D&C can be defined recursively, starting with $n = 2$ players and moving up to $n = 5$. With each step of the algorithm we associate a fraction λ that depends upon the number of players. Each player $(A, B, C, \dots)$ is asked to place a mark $(a, b, c, \dots)$ at its λ point such that the region to the left of this mark has λ value of the total, with $1 - \lambda$ remaining to the right:[4]

$n = 2$. Players A and B independently (i.e., unaware of the marks of each other)

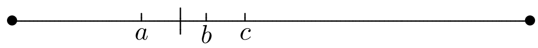
[4] The use of marks in cake division is discussed in Shishido and Zeng (1999). As an example of an algorithm that uses marks, Lucas's "method of markers" requires that players mark $1/n$ points across a one-dimensional cake (Brams and Taylor, 1996, pp. 57-62), which asks more of players than D&C, as we will show. Although this algorithm ensures that each player receives a proportional share, it may leave pieces of cake unassigned and offers no way to award them, or parts thereof, to the players.

put marks, a and b, at their 1/2 points. Without loss of generality, assume $a \leq b$. A cut is made in the open interval (a, b)—or at a if $a = b$[5]—indicated by the vertical bar $|$. The left and right endpoints are assumed to be 0 and 1, respectively.



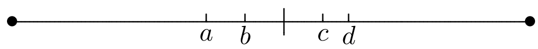
Each player gets a piece that includes its mark, except when the marks are the same (in which case one player is given a piece that includes the mark): Player A gets $[0, |]$ that includes a, and player B gets $(|, 1]$ that includes b.

$n = 3$. Players A, B, and C independently put their marks at their 1/3 points, and a first cut is made in the open interval between the 1st and 2nd marks—or at a if $a = b$—as shown:



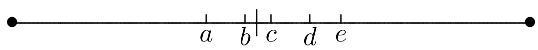
Player A gets $[0, |]$ that includes its mark; the $n = 2$ procedure is then applied to $(|, 1]$ for players B and C.

$n = 4$. Players A, B, C, and D independently put marks at their 1/2 points, and a first cut is made in the open interval between the 2nd and 3rd marks—or at b if $b = c$—as shown:



The $n = 2$ procedure is applied to $[0, |]$ for players A and B, and to $(|, 1]$ for players C and D.

$n = 5$. Players A, B, C, D, and E independently put marks at their 2/5 points, and a first cut is made in the open interval between the 2nd and 3rd marks—or at b if $b = c$—as shown:



The $n = 2$ procedure is applied to $[0, |]$ for players A and B, and the $n = 3$ procedure is applied to $(|, 1]$ for players C, D, and E.

Clearly, each player receives a proportional share by getting a piece that it values at $1/n$ or more. We next define D&C formally by specifying its rules of play for $n \geq 2$ players.

1. Each player independently places a mark at a point such that
   (i) if n is even, 1/2 the cake lies to the left and 1/2 to the right;
   (ii) if n is odd, $[(n-1)/2]/n$ proportion of the cake lies to the left, and $[(n+1)/2]/n$ proportion lies to the right.
2. The cake is cut in the open interval defined by the $(n/2)$th and the $(n/2+1)$st marks in case (i), and in the open interval defined by the $[(n-1)/2]$th and the $[(n+1)/2]$st marks in case (ii). If these marks coincide, the cake is cut at this point.
3. If $n = 2$ in case (i), stop. If $n = 3$ in case (ii), cut the subpiece containing 2 marks according to case (i), and stop.

[5]With this exception, however, which still gives each player a proportional share (exactly 1/2 the cake), D&C gives each player more than 1/2, as each values it, as long as neither player has measure zero from its mark to the cutpoint.

4. If $n \geq 4$, cut the subpiece on the left and the subpiece on the right of | according to rules 1 and 2, changing n in each case to the number of players that made marks on the left and the right subpieces.
5. Apply rule 4 repeatedly to the smaller and smaller subpieces that remain after cuts are made. When subpieces are reached where $n = 2$ or $n = 3$, apply rule 1.
6. After all cuts are made, assign pieces to the players, which include their marks, so as to give each player a proportional share.[6] If there is mutual envy or envy cycles (to be discussed in section 3), have players make trades that eliminate them.

The sequence of cuts under D&C can be described by a binary tree. Each subpiece of the cake is divided in two, according to (i) or (ii), in successive rounds until there are n individual pieces that can be assigned to each player. This requires $n - 1$ cuts.

The depth (or height) of the binary tree is the number of rounds that are needed before each player receives an individual piece. This number is $\lceil \log_2(n) \rceil$—where $\lceil k \rceil$ is the ceiling function of k and denotes k rounded up to the next integer if k is not an integer—because on each round the cake is divided into two subpieces. Each subpiece contains the same number of marks on each side of | if n is even; if n is odd, the numbers differ by 1.

To illustrate the successive division of a cake, assume $n = 7$, so the depth of the tree is $\lceil \log_2(7) \rceil = \lceil 2.81 \rceil = 3$ (see Figure 2.1). On the 1st round, the division, at 3/7, is into a left subpiece containing 3 marks and a right subpiece containing 4 marks. The first cut is shown by a boldface "1."

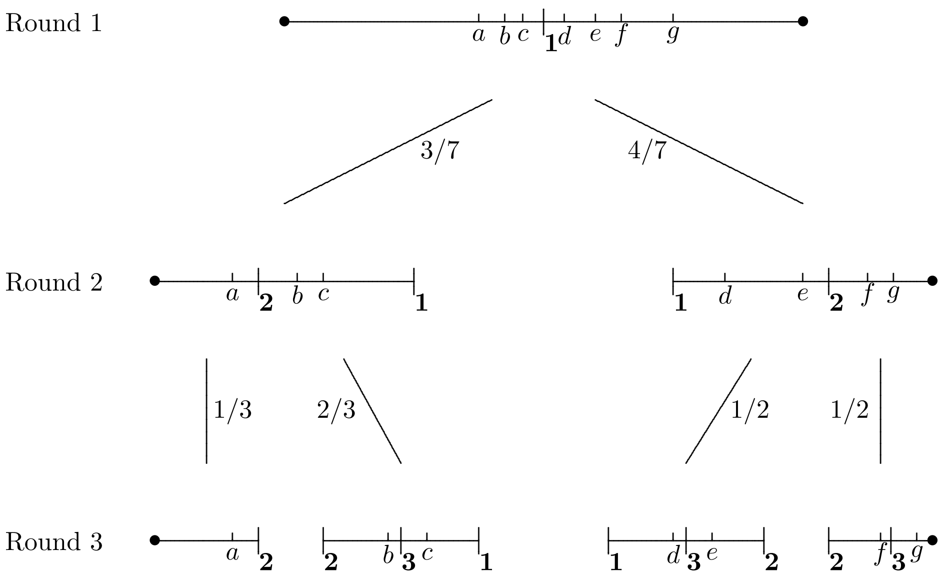


FIG. 2.1. *Binary tree illustrating the division of a cake into 7 pieces.*

On the 2nd round, the division of the left subpiece at 1/3 (of 3/7) is into two subsubpieces containing 1 and 2 marks each; the division of the right subpiece at 1/2 (of

[6] If two or more players make the same mark or one mark is at a cutpoint, the assignment of the cake at this point does not matter since it has measure zero. We include the marks for definiteness in stating the algorithm.

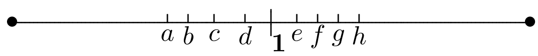
4/7) is into two sub-subpieces containing 2 marks each. The 2nd-round cuts are shown by two boldface “2’s”; the 1st-round cut is now separated, becoming an endpoint for two subpieces. On the 3rd round, each of the three sub-subpieces containing 2 marks each (the fourth contains 1 mark) is divided. The three 3rd-round cuts are shown by three boldface “3’s”; the 1st and 2nd-round cuts now become endpoints. Thereby the cake is cut into a total of 7 individual pieces. Note that $a, b, \ldots, g$ represent different players’ marks on each round.

3. Maximum Number of Envies. To count the maximum number of players that all players may envy under D&C, which we call the *envies of players*, we count only envies that cannot be alleviated by trades. If there are 4 or more players, D&C does not preclude mutual envy and thus a trade that would give the traders preferred pieces.

In the case of 4 players, for example, assume the first cut gives players A and B the left portion of a cake, and players C and D the right portion. Then after the left and right portions are divided, it is possible that A would envy C and C would envy A. In that case, by rule 6 of D&C, A and C would trade their pieces to eliminate their mutual envy. On the other hand, if there were only three players (A, B, and C), it is possible that after the first cut is made that gives A the left piece (see section 2), A will envy B or C (but not both) after the right piece is divided, but B and C will not envy A or each other.

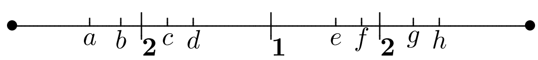
Trades that benefit the traders when $n \geq 4$ are precluded if each player I’s envy of some player J is strictly one-way, which we indicate by $I \rhd J$. Hence, there is no mutual envy—so $I \rhd J$ and $J \rhd I$ cannot both be true—which would allow players I and J, by trading pieces, to eliminate their envy of each other. In counting envies next, we preclude not only mutual envy but also envy cycles, whereby, for example, $I \rhd J$, $J \rhd K$, and $K \rhd I$, in which case a three-way trade would rid the players of envy.

To illustrate how we count envies without the possibility of trades, assume there is a set of 8 players, $\{A, B, C, D, E, F, G, H\}$. Assume the 1/2 points of players $A - D$ lie to the left of the 1/2 points of players $E - H$, so the first cut under D&C, indicated by $|_{\mathbf{1}}$, is made between the 1/2 points of players D and E:



Now apply D&C to divide the cake into 8 individual pieces. It is possible that the 4 players to the left of $|_{\mathbf{1}}$ may envy up to 3 of the 4 players to the right (say, E, F, and G) if player H’s piece is sufficiently small in the eyes of players $A - D$. (A player on the left cannot envy all 4 players on the right, because not all 4 on the right can each receive more than 1/8 in the eyes of a left player.) By the same token, player H may envy players B, C, and D on the left if player A’s piece is sufficiently small in the eyes of player H. Thus, 5 players (A, B, C, D, H) may envy up to 3 other players across $|_{\mathbf{1}}$ without the possibility of trades (total possible envies: 15). (If there were additional envies across $|_{\mathbf{1}}$, there would be trades the players could make that would eliminate them.)

More envies are possible. On the 2nd round, assume the cuts on the left and the right, both indicated by $|_{\mathbf{2}}$, are as follows:



To the left of $|_{\mathbf{1}}$, players A and B may envy up to 1 player (say, C) across $|_{\mathbf{2}}$ on the left, and player D may envy up to 1 player (say, B) in the other direction. To the right of $|_{\mathbf{1}}$, players E and F may envy player G across $|_{\mathbf{2}}$ on the right, and player H may envy player F in the other direction. Thus, 6 players (A, B, D, E, F, H) may envy up to 1 other player across each of the two $|_{\mathbf{2}}$'s without the possibility of trades (total possible envies: 6).

Across the four cuts $|_{\mathbf{3}}$ made on the 3rd round that divide the cake into 8 individual pieces, no new envies are created. For example, the cut $|_{\mathbf{3}}$ dividing the pieces that A and B receive does not cause one of these players to envy the other. Altogether, we have

$$A, B \rhd C, E, F, G; \quad C \rhd E, F, G; \quad D \rhd B, E, F, G; \quad E, F \rhd G; \quad H \rhd B, C, D, F.$$

Thus, 4 players (A, B, D, H) may envy up to 4 others, 1 player (C) may envy up to 3 others, 2 players (E, F) may envy up to 1 other, and 1 player (G) may envy no others, making for a total of 21 one-way envies, or an average of $21/8 = 2.625$ envies per player without the possibility of trades.

What if we used an m-proportional algorithm with a different m from D&C? In the 8-player example, assume $m = 2$ for the first cut, which produces a division at $1/4$. Two players then must divide what they consider to be at least $1/4$ of the cake lying to the left of the cut that separates their two marks from the other six marks; and six players must divide that portion to the right of this cut, which each of the six considers to be at least $3/4$ of the cake. Now the two players on the left must make marks at the $1/2$ point of their piece, whereas the six players on the right may make marks at any one of the five possible divisions, $\{1/6, 1/3, 1/2, 2/3, 5/6\}$, of their piece. Further division of the right piece will be required to give each of the six players individual pieces. We next count how many envies are possible.

Theorem 3.1. *For $n \geq 2$, the maximum number of envies, with no possibility of trades, that all players may have under an m-proportional algorithm is*

$$T(n) = (n-1)(n-2)/2.$$

Proof. We use induction, making the base case $n = 2$. It satisfies the formula $(n-1)(n-2)/2$, because there are zero envies when there are two players. Under the strong induction hypothesis, assume the formula holds for any number of players less than n.

Suppose there are n players. Under an m-proportional algorithm that assumes some m, the cake is cut initially so that m players are to divide the piece to the left that each values at m/n or more of the entire cake, and $n - m$ players are to divide the piece to the right that each values at $(n-m)/n$ or more. Applying the induction hypothesis, the maximum number of envies on the left is $(m-1)(m-2)/2$, and the maximum number of envies on the right is $(n-m-1)(n-m-2)/2$. Summing these numbers gives the maximum number of possible envies on both the left and right sides of the initial cut:

$$\begin{aligned} &\frac{(m^2 - 3m + 2) + (n^2 - nm - 2n - mn + m^2 + 2m - n + m + 2)}{2} \\ &= \frac{2m^2 - 2mn - 3n + n^2 + 4}{2}. \end{aligned} \tag{3.1}$$

Next we count the maximum number of envies of (i) the right players by the left players and of (ii) the left players by the right players without the possibility of trades. Because each player must receive a proportional piece, each of the m left players can envy a maximum of $(n-m-1)$ right players. Thus in case (i), there are a maximum of $(m)(n-m-1)$ envies caused by the initial cut if the envies of all the players on the left leave one player unenvied on the right. This right player, in turn, may envy a maximum of $(m-1)$ players on the left. This construction of players' possible envies prohibits two-way envy as well as envy cycles across stages, because rightward envy never changes to leftward envy, or vice versa, across stages.[7]

Altogether, the maximum number of possible envies caused by the initial cut is

$$(m)(n-m-1)+(1)(m-1)=mn-m^2-1. \tag{3.2}$$

Adding (3.1) and (3.2), we obtain the maximum total (T) number of envies without the possibility of trades:

$$T(n,m)=(m^2-mn-3n/2+n^2/2+2)+(mn-m^2-1)=n^2/2-3n/2+1. \tag{3.3}$$

But (3.3) is $(n-1)(n-2)/2$, the $(n-2)$nd triangular number, which validates the formula for n players. □

Theorem 3.1 establishes that we can drop the argument m in $T(n,m)$ and write the maximum total number of envies of a proportional cake-cutting algorithm as $T(n)$. Because one of the two factors in the numerator of $(n-1)(n-2)/2$ must be even, the numerator must also be even, rendering $T(n)$ always an integer.

If every m-proportional algorithm gives the same maximum total number of envies, what is special about D&C?

THEOREM 3.2. *Assume $n \geq 2$ and let $k=\lfloor log_2(n) \rfloor$. Under D&C, the maximum number of envies of an individual player, with no possibility of trades, is*

$$I_{\mathrm{D\&C}}(n)=n-k-1$$

where $k+1$ is the depth of the binary tree. This number is minimal among all m-proportional algorithms.

Proof. We prove this result by induction and by using an appropriate recursive formulation. For $2m$ players, the first cut divides the cake into left and right pieces so that m players value the left piece to be worth at least half of the cake while m players value the right piece to be worth at least half of the cake. Hence, a player on the left (resp., right) can envy at most $m-1$ players on the right (resp., left), and at most $I_{\mathrm{D\&C}}(m)$ players on the left (resp., right) from future cuts under D&C. It follows that $I_{\mathrm{D\&C}}(2m)=m-1+I_{\mathrm{D\&C}}(m)$.

A similar argument can be used for $2m+1$ players. In particular, the first cut under D&C divides the cake into a left piece that m players think is worth at least $m/(2m+1)$ of the cake, and a right piece that $m+1$ players think is worth at least $(m+1)/(2m+1)$ of the cake. A player on the left may envy at most m players on the right and $I_{\mathrm{D\&C}}(m)$ on the left, so $m+I_{\mathrm{D\&C}}(m)$. By comparison, a player on the right may envy at most $m-1$ players on the left and $I_{\mathrm{D\&C}}(m+1)$ players

[7] As an example, observe that the envies of players A, B, and C in our earlier example are only rightward; although player D has leftward envy of player B, it is not reciprocated. Likewise, players E and F have only rightward envies, whereas player H has only leftward envies. Analogous calculations of the maximum number of leftward envies, or of a mixture of leftward and rightward envies, gives the same maximum we give in (3.2) below.

on the right for a maximum of $m - 1 + I_{\text{D\&C}}(m+1)$ envies; this number is less than $m + I_{\text{D\&C}}(m)$—because $I_{\text{D\&C}}(m+1)$ cannot be greater than $I_{\text{D\&C}}(m)+1$—so $I_{\text{D\&C}}(2m+1) = m + I_{\text{D\&C}}(m)$.

Recall from the earlier discussion of D&C that $I_{\text{D\&C}}(2) = 0$ and $I_{\text{D\&C}}(3) = 1$. To proceed by induction, the base cases of $I_{\text{D\&C}}(2) = 2 - \lfloor \log_2(2) \rfloor - 1 = 0$ and $I_{\text{D\&C}}(3) = 3 - \lfloor \log_2(3) \rfloor - 1 = 1$ are satisfied. Assume that $I_{\text{D\&C}}(n) = n - \lfloor \log_2(n) \rfloor - 1$ for all $n < 2m$. It follows that

$$I_{\text{D\&C}}(2m) = m-1+I_{\text{D\&C}}(m) = m-1+m-\lfloor \log_2(m) \rfloor - 1 = 2m-1-(\lfloor \log_2(m) \rfloor + 1)$$

and

$$I_{\text{D\&C}}(2m+1) = m + I_{\text{D\&C}}(m) = m + m - \lfloor \log_2(m) \rfloor - 1 = 2m - (\lfloor \log_2(m) \rfloor + 1).$$

The result follows because $\log_2(2m) = \log_2(2) + \log_2(m)$ and $\lfloor \log_2(2m+1) \rfloor = \lfloor \log_2(2) + \log_2(m+1/2) \rfloor = 1 + \lfloor \log_2(m+1/2) \rfloor = 1 + \lfloor \log_2(m) \rfloor$.

That $I_{\text{D\&C}}(n)$ minimizes the maximum number of envies of a player without the possibility of trades among m-proportional procedures follows from the fact that cuts under D&C are made at the (approximate) halfway points on each round. Without loss of generality, take any integer $m < \lfloor n/2 \rfloor$. There can be at most $n - m - 1$ envies to the right in the first round. If, in addition, we use D&C for the m players to the left, a player can envy at most $I_{\text{D\&C}}(m) + n - m - 1$ other players. Using the previous formula, $I_{\text{D\&C}}(m) = m - \lfloor \log_2(m) \rfloor - 1$. Hence, the maximum number of envies is $m - \lfloor \log_2(m) \rfloor - 1 + n - m - 1 = n - \lfloor \log_2(m) \rfloor - 2$. Hence $I_{\text{D\&C}}(n) = n - \lfloor \log_2(n) \rfloor - 1$ is at least as small as any procedure using marks different from approximate halves in the first round whenever $\lfloor \log_2(m) \rfloor + 2 \leq \lfloor \log_2(n) \rfloor + 1$ or $\lfloor \log_2(m) \rfloor + 1 \leq \lfloor \log_2(n) \rfloor$. Because $m < \lfloor n/2 \rfloor$, this is true. Applying the same argument to any division of the m players on the left proves the theorem.[8] □

It is useful to compare the values of $T(n)$ and $I_{\text{D\&C}}(n)$, where $k = \lfloor \log_2(n) \rfloor$ in Table 3.1, and give their formulas as well. (We also include $I_{\text{DS}}(n)$, which will be defined shortly and whose values will be compared with those of $I_{\text{D\&C}}(n)$.) Observe that whereas $T(n)$ increases rapidly with n, $I_{\text{D\&C}}(n)$ increases more slowly than n.

n	2	3	4	5	6	7	8	9	10	Formula
$T(n)$	0	1	3	6	10	15	21	28	36	$(n-1)(n-2)/2$
$I_{\text{D\&C}}(n)$	0	1	1	2	3	4	4	5	6	$n-k-1$
$I_{\text{DS}}(n)$	0	1	2	3	4	5	6	7	8	$n-2$

TABLE 3.1

Total possible envies for n players under an m-proportional algorithm ($T(n)$), and the maximum number of individual envies under D&C ($I_{\text{D\&C}}(n)$) and under the Dubins-Spanier algorithm ($I_{\text{DS}}(n)$), where $k = \lfloor \log_2(n) \rfloor$.

Desirably, $I_{\text{D\&C}}(n)$ is less than the maximum number of envies, $I_{\text{DS}}(n)$, that a player may have under the well-known moving-knife algorithm of Dubins and Spanier

[8] After we derived the formula for $I_{\text{D\&C}}(n)$, we found the numerical sequence shown in Table 3.1, which is A083058 in the *Online Encyclopedia of Integer Sequences*. The recursion we give is one of several different representations that generate the sequence, which have been used previously in divide-and-conquer algorithms in computer science.

(DS) (1961), which also uses $n-1$ cuts.[9] Under the DS algorithm, a referee moves a knife slowly across a cake from left to right. A risk-averse player that has not yet received a piece calls "stop," and makes a cut, when the knife reaches a point that gives this player exactly $1/(n-m)$ of the cake rightward of the last point at which a previously served player called stop—or, for the first player to call stop, from the left edge—where m is the number of those previously served players.

Under DS, the first player to call stop may envy as many as all the other players except one, or $n-2$ other players, so $I_{\mathrm{DS}}(n) = n-2$. This is the maximum number of envies of an individual player. The second player to call stop may envy as many as $n-3$ other players, ..., and the nth player, which receives the cake from the point at which the $(n-1)$st player called stop up to the right edge, will envy no other players. In each case, no player envies a previously served player. Altogether, the n players may envy as many as

$$T_{\mathrm{DS}}(n) = \sum_{i=0}^{n-2} i = (n-1)(n-2)/2.$$

other players under DS, which duplicates $T(n)$.[10] Obviously, no trades are possible, because envy goes only from left to right.

As Table 3.1 shows, $I_{\mathrm{DS}}(n)$ and $I_{\mathrm{D\&C}}(n)$ are the same for $n \leq 3$. However, a player can have up to $\lfloor \log_2(n) - 1 \rfloor$ more envies under DS than under D&C for $n > 3$. In particular, $I_{\mathrm{DS}}(n)$ jumps ahead of $I_{\mathrm{D\&C}}(n)$ at $n = 4$, slowly increasing its lead in envies as n increases. Consequently, D&C in general does better than DS in preventing any player from being too aggrieved.

Because DS requires that a referee continuously move a knife across a cake, it would seem less practical than an m-proportional algorithm, like D&C, which requires each player to make at most $\lfloor \log_2(n) \rfloor$ (n even) or $\lfloor \log_2(n+1) \rfloor$ (n odd) marks. While players under D&C would not necessarily need a referee to record their marks and keep them secret from other players, a trustworthy referee may facilitate this process.

4. Envy-Freeness and Efficiency. In this section we show that D&C, except when $n = 2$, may not yield an envy-free or efficient allocation.

THEOREM 4.1. *For $n \geq 3$, there exist probability density functions of the players such that D&C, applied from either the left or the right edge of the cake, does not produce an envy-free allocation.*

Proof. Assume that players A and B have piecewise linear value functions over the cake that are symmetric and V-shaped:[11]

$$v_A(x) = \begin{cases} -4x+2 & \text{for } x \in [0,1/2] \\ 4x-2 & \text{for } x \in (1/2,1] \end{cases} \text{ and } v_B(x) = \begin{cases} -2x+3/2 & \text{for } x \in [0,1/2] \\ 2x-1/2 & \text{for } x \in (1/2,1] \end{cases}.$$

Whereas both functions have maxima at $x = 0$ and $x = 1$ and a minimum at $x = 1/2$, A's function is steeper (higher maximum, lower minimum) than B's, as

[9]Moving-knife algorithms are discussed in, among other places, Brams, Taylor, and Zwicker (1995), Brams and Taylor (1996), and Robertson and Webb (1998). For nonconstructive results on cake-cutting, which address the existence but not the construction of fair divisions that satisfy different properties, see Barbanel (2005). For results on pie-cutting, see Brams, Jones, and Klamler (2008), Barbanel, Brams, and Stromquist (2009), and Barbanel and Brams (to appear, 2011).

[10]An m-proportional algorithm in which $m = 1$ is arguably better than DS, because it does not limit players to exactly $1/n$ pieces.

[11]This example, with a similar figure, was used for a different purpose in Brams, Jones, and Klamler (2006).

illustrated in Figure 4.1. In addition, suppose that a third player, C, has a uniform value function, $v_C(x) = 1$, for $x \in [0, 1]$.

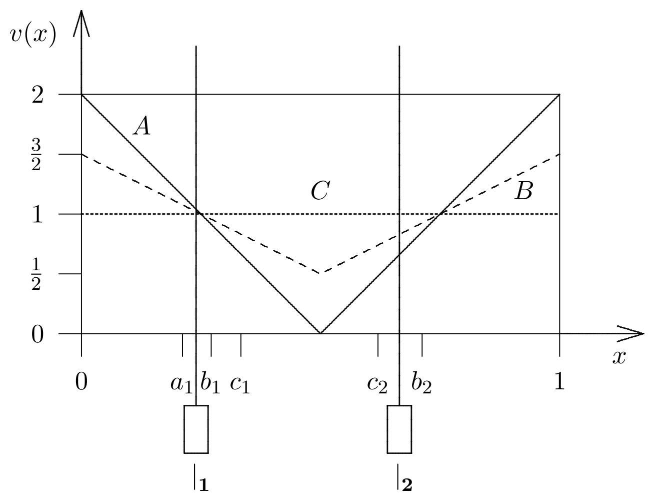


FIG. 4.1. *Impossibility of an envy-free division for three players under D&C*

We show in Figure 4.1 player A, B, and C's 1/3 marks (a_1, b_1, and c_1), where

$$a_1 = 1/2 - \sqrt{3}/6 \approx 0.211 \qquad b_1 = 3/4 - \sqrt{33}/12 \approx 0.271 \qquad c_1 = 1/3.$$

The first cut under D&C is between a_1 and b_1, which we denote by $|_{\mathbf{1}}$. Player A receives the piece to the left of this cut.

Players B and C, which value the remaining piece at more than 2/3, place 1/2 marks, which we denote by b_2 and c_2, on this remainder. Let $y = |_{\mathbf{1}}$. To ensure that players B and C value the middle piece and the right piece equally, b_2 and c_2 must satisfy the following two equations:

$$\int_y^{b_2} v_B(x)dx = \int_{b_2}^1 v_B(x)dx \text{ and } \int_y^{c_2} v_C(x)dx = \int_{c_2}^1 v_C(x)dx.$$

Solving these equations for b_2 and c_2, we obtain

$$b_2 = \frac{1 + \sqrt{2y - 2y^2}}{2} \text{ and } c_2 = \frac{y+1}{2}.$$

The second cut, $|_{\mathbf{2}}$, is made between b_2 and c_2.

As functions of y, $b_2 > c_2$ for all positive $y < 2/3$. Because y cannot exceed $b_1 \approx 0.271$, it follows that $b_2 > c_2$, as shown in Figure 4.1.

We now show that the right piece that B receives is a larger interval (along the horizontal axis) than the left piece that A receives, so A will envy B because, in A's eyes, B receives more cake. Consider the position of b_2. It must be farther from 1 than b_1 is from 0 in order to give B more than 1/3. But because the length of A's piece on the left is less than b_1, A must think that the length of B's piece on the right is greater. Hence, A will envy B no matter where $|_{\mathbf{1}}$ and $|_{\mathbf{2}}$ lie in their respective intervals.[12]

[12]In the Figure 4.1 example, the equitability procedure (EP) mentioned in note 2 also causes A

If D&C is applied from the right so that A receives the right piece, A will envy B for the piece that B receives on the left for analogous reasons. This example can be generalized to more than 3 players by assuming that the additional players have, like C, uniform distributions. If any of these players (say, D) gets an endpiece, A and B will envy D because they value the endpieces more. Hence, D must receive a middle piece while A and B receive the left and right pieces, creating a situation in which A envies B, whether or not D&C is applied from the left edge or the right edge of the cake. $\square$

We turn next to analyzing the efficiency of allocations under D&C.

THEOREM 4.2. *For $n \geq 3$, there exist probability density functions of the players such that D&C, applied from either the left or the right edge of the cake, does not produce an efficient allocation.*

Proof. Assume that players A and B have the piecewise uniform value functions over different 1/3's of the cake, whereas player C's function is uniform over the entire cake:[13]

$$v_A(x) = \begin{cases} 1.03 & \text{on } [0,1/3) \\ 0.73 & \text{on } [1/3,2/3] \\ 1.24 & \text{on } (2/3,1] \end{cases}, \quad v_B(x) = \begin{cases} 0.49 & \text{on } [0,1/3) \\ 1.24 & \text{on } [1/3,2/3] \\ 1.27 & \text{on } (2/3,1] \end{cases}, \text{ and}$$
$$v_C(x) = 1 \text{ on } [0,1].$$

To begin with, assume that D&C is applied from the left, in which case A obtains the left piece, C the middle piece, and B the right piece, which we call order ACB. Recall that $D\&C$ leaves open where a cut between the marks specified by the players is to be made. Consequently, one can determine the cuts most favorable to each player and, from these, the maximum value each player can obtain under D&C when it is applied from the left.

To illustrate, players A, C, and B would place their 1/3 marks at $a_1 = 100/309 \approx 0.324$, $c_1 = 1/3$, and $b_1 = 175/372 \approx 0.470$, respectively. Thus, the 1st cut will be made between a_1 and c_1. The 1st cut most favorable to A, who gets the first piece, is at c_1, giving A a maximum of $103/300 \approx 0.343$. By contrast, the 1st cut most favorable to C and B is at a_1. By applying D&C to the cake that remains in the interval $[a_1, 1]$, one can determine the maxima for C and B in an analogous manner. The results are as follows for the most favorable cuts for each player, and the maxima they receive from them, starting from the left:

- A: Make the 1st cut at c_1, giving A the portion $[0, c_1]$ that it values at $103/300 \approx 0.343$.
- C: Make the 1st cut at a_1 and the 2nd cut at $b_2 = 26243/39243 \approx 0.669$, giving C the portion $[a_1, b_2]$ that it values at $13543/39243 \approx 0.345$.
- B: Make the 1st cut at a_1 and the 2nd cut at $c_2 = 409/618 \approx 0.662$, giving B the portion $[c_2, 1]$ that it values at $13267/30900 \approx 0.429$.

When D&C is applied from the right, B obtains the right piece, C the middle piece, and A the left piece, so the order is BCA (i.e., ACB from the left, the same

to envy B by giving all three players exactly the same value (0.393); it gives a left cutpoint of 0.269 and a right cutpoint of 0.662 (Brams, Jones, and Klamler, 2006, p. 1318), which fall in the D&C intervals of $[a_1, b_1]$ and $[c_2, b_2]$. Whereas EP always gives an efficient allocation, however, D&C may not, as we next show.

[13]Note that while the probability density functions for players A and B are not continuous, their cumulative distribution functions are.

as earlier). But now define 1st cuts from the right, with $b_1 = 281/381 \approx 0.738$, $a_1 = 68/93 \approx 0.731$, and $c_1 = 2/3$. The results are as follows:

- B: Make the 1st cut at a_1, giving B the portion $[a_1, 1]$ that it values at $127/372 \approx 0.341$.
- C: Make the 1st cut at b_1 and the 2nd cut at $a_2 = 12850/39243 \approx 0.327$, giving C the portion $[a_2, b_1]$ that it values at $16093/39243 \approx 0.410$.
- A: Make the 1st cut at b_1 and the 2nd cut at $c_2 = 281/762 \approx 0.3688$, giving A the portion $[0, c_2]$ that it values at $28133/76200 \approx 0.3692$. (The fourth decimals are used here to indicate that c_2 and A's maximum portion are not the same.)

Whether D&C is applied from the left or from the right, we next show that there are cuts that give all three players more than their maxima, which cannot be realized simultaneously under D&C. But unlike D&C, players receive pieces from the left in the order CBA instead of ACB.[14] More specifically,

1. If cuts are made at 0.346 and 0.693, C obtains 0.346 from the left piece, B obtains $43107/100000 \approx 0.431$ from the middle piece, and A obtains $9517/25000 \approx 0.381$ from the right piece, which exceed the three players' maxima when D&C is applied from the left.
2. If cuts are made at 0.411 and 0.691, C obtains 0.411 from the left piece, B obtains $34793/100000 \approx 0.348$ from the middle piece, and A obtains $9579/25000 \approx 0.383$ from the right piece, which exceed the three players' maxima when D&C is applied from the right.

In sum, there are allocations that Pareto-dominate (see note 15) the D&C allocations, whether D&C is applied from the left or from the right, rendering the D&C allocations inefficient.

This example can readily be extended to 4 or more players. For example, if there are 4 players, assume that the 4th player has virtually all its value concentrated in the 2nd quarter of a 4-way division of the cake into equal-length portions, whereas the other three players have almost all their values distributed in the 1st, 3rd, and 4th quarters in the same manner given in the 3-player example. Then the D&C allocations will be as in the 3-player example, except that the 4th player will obtain the 2nd piece from the left. But as in the 3-player example, there will be an allocation that Pareto-dominates each of the D&C allocations, giving the 4th player the 2nd piece from the left and changing the order for the other 3 players from ACB to CBA. □

The allocations of (1) and (2) above are efficient, but they are not envy-free. In the case of (1), C envies B for obtaining what it thinks is 0.347, which exceeds its allocation of 0.346. In the case of (2), B envies A for obtaining what it thinks is $39243/10000 \approx 0.392$, which exceeds its allocation of 0.348. But there is no mutual envy or an envy cycle, which would render trades possible that would lead to a still more efficient allocation.

If neither D&C nor an allocation that Pareto-dominates it is envy-free, can one be assured that there always is an efficient, envy-free allocation? The answer is "yes," though there is only an approximate n-player algorithm for finding such an allocation

[14] If each player receives a different 1/3 equal-length piece, order CBA maximizes the sum (1.16) of the values that the three players can receive, whereas the D&C order ACB can sum to no more than 1.117 (=0.343+0.345+0.429) from the left and can sum to no more than 1.120 (=0.341+0.410+0.369) from the right. The extra "wiggle room" of CBA—the $1.16 - 1.117 = 0.043$ from the left and $1.16 - 1.120 = 0.040$ from the right—is what enables us to find CBA allocations that *Pareto-dominate* (i.e., give more to at least one player and not less to any other player) the ACB allocations of D&C.

(Su, 1999). In the 3-player case, however, there are two moving-knife procedures that yield exact envy-free allocations (Stromquist, 1980; Brams and Barbanel, 2004), which are always efficient in the minimal-cut case (Gale, 1993; Brams and Taylor, 1996, pp. 150-151).

To illustrate, we apply the "squeezing procedure" of Brams and Barbanel (2004) to the previous 3-player example, wherein player C is the squeezer.[15] This produces cuts at the marks of 0.336 and 0.672, yielding the following efficient, envy-free allocation:

- C obtains the left piece, which it values at 0.336 (same as the middle piece).
- B obtains the middle piece, which it values at $521/1250 \approx 0.417$ (same as the right piece).
- A obtains the right piece, which it values at $1271/3125 \approx 0.407$.

In general, there will be an infinite number of envy-free allocations if the players' cumulative value functions are continuous.

By contrast, there is usually a unique equitable allocation (see notes 2 and 13), which in our example gives each player an allocation of approximately 0.383. C obtains the left piece (1st cut at $385/10119 \approx 0.383$), B obtains the middle piece (2nd cut at $6994/10119 \approx 0.691$), and A obtains the right piece. This allocation, however, is not envy-free: B envies A for getting a piece that it thinks is worth $15875/40476 \approx 0.392$.

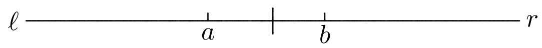
To recapitulate, D&C allocations may be neither envy-free nor efficient. While there are efficient allocations that are envy-free or equitable, they may not be both.[16]

In the next section, we consider two other properties of D&C. The first—that it is truth-inducing—is satisfied, and the second—that players' proportional shares reflect their possibly unequal entitlements—is in general impossible to satisfy unless clones are allowed and players can receive disconnected pieces.

5. Truthfulness and Entitlements. Recall from section 1 that an algorithm is truth-inducing if it guarantees players at least $1/n$ shares if and only if they are truthful.

Theorem 5.1. *D&C is truth-inducing.*

Proof. Under D&C, a player, say A, will need in some round to divide a portion of the cake between itself and one or more other players. For simplicity, assume the division is between A and B. Assume that the boundaries of the portion that A and B must divide are ℓ (for left) and r (for right), and their truthful $1/2$ points for this portion are a and b. Then cutting the cake at | gives each player more than $1/2$ of the portion:



But if player A should report that its $1/2$ point is either to the left or to the right of a, it risks getting less than $1/2$ the portion if, respectively, (i) | is to the left of a

[15] The idea is that player C, using two moving knives, continuously increases its left and middle $1/3$ pieces equally so as to diminish its right $1/3$ piece until one of A and B, both of which initially prefer the right piece to the other two, calls "stop" when this diminished piece ties one of the other two (now enlarged) pieces. The first player to call stop will be B, when the middle piece ties with the right piece for it (at the cutpoints given in the text), so B gets this piece, C gets the left piece, and A gets the right piece. Because of the two ties (see text) that this procedure creates, each player thinks it receives at least a tied-for-largest piece and so is not envious of anybody else.

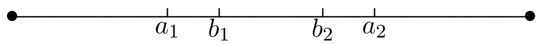
[16] That envy-free and equitable allocations can be different is illustrated by the previous example; that these two properties cannot always be satisfied simultaneously is proved in Brams, Jones, and Klamler (2006) using the Figure 4.1 example.

or (*ii*) $|$ is to the right of b, which is possible when a is to the right of b. Thus, any misrepresentation by A (or B) may give it less than a proportional share, whereas we showed in section 2 that D&C guarantees the players proportional shares if they are truthful. If the division is not between A and B but between A and more than one other player, the players will indicate different cutpoints, but for analogous reasons misrepresentation of a player's truthful cutpoint may deprive it of a proportional share. ∎

Finally, we consider a situation in which players are not equally entitled to portions of a cake. If the players have different entitlements, which we assume to be rational numbers, then we have the following impossibility result.

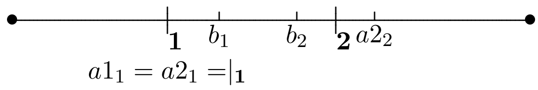
THEOREM 5.2. *If n players do not have equal entitlements, then it is not always possible for D&C to produce a proportional allocation in $n-1$ cuts.*

Proof. Assume that player A is entitled to 2/3 of the cake and player B to 1/3. Then if their 1/3 and 2/3 points are a_1 and a_2 and b_1 and b_2, respectively, as shown below,



then it is not hard to see that no single cut can give *both* players at least their entitlements. (For example, a cut just to the right of a_2, in which player A gets the cake to the left and player B the cake to the right, gives A more than 2/3 but B less than 1/3.) For $n > 2$, assume that all the *other* players have uniform distributions but have only arbitrarily small entitlements. By forcing a first cut in (b_1, b_2), player A cannot get at least 2/3. ∎

But now suppose that player A divides into two players with identical preferences, clones $A1$ and $A2$; they, like B, are each entitled to 1/3 each. Assume the application of D&C results in the following cuts between the 1/3 and 2/3 points (because $A1$ and $A2$ are identical, we indicate their 1/3 cuts $a1_1$ and $a2_1$ as being equal) and a 2/3 cut only for $A2$, $a2_2$):



Now $A1$ gets $[0, |_{\mathbf{1}}]$, B gets $(|_{\mathbf{1}}, |_{\mathbf{2}})$, and $A2$ gets $[|_{\mathbf{2}}, 1]$, giving each player at least 1/3. But because $a1_1$ is identical to $|_{\mathbf{1}}$—compared with $a2_2$ being to the right of $|_{\mathbf{2}}$—the piece that $A2$ gets is worth more to both clones than the piece that $A1$ gets. Consequently, $A1$ will envy $A2$ because it gets a larger proportional share (if clones can envy each other).

As this example illustrates, a more entitled player like A can create clones of itself and thereby ensure proportional shares for its clones, given that they have the same entitlements as all the other players. But as this example also illustrates, the clones together may not get a single piece but, rather, two nonadjacent pieces. Furthermore, if one of the clones gets a larger proportional share than the other, this may lead to this clone's being envied by the other clone.[17]

6. Conclusions. D&C is not perfect—its allocations may not be envy-free or efficient—but there is no exact algorithm, using $n-1$ cuts, that guarantees these

[17] For a discussion of the use of clones to define envy-freeness in terms of entitlements, see Brams and Taylor (1996, pp. 48-49, 152-153) and Robertson and Webb (1998, pp. 35-36).

properties for more than three players. This is perhaps a price one pays for its simplicity. It has, however, several positive features:

1. It requires n players to make at most $\lceil \log_2(n) \rceil$ (n even) or $\lceil \log_2(n+1) \rceil$ (n odd) marks on a cake.
2. It minimizes the number of rounds on which players must indicate their marks, minimizing the depth of the binary tree.
3. It requires the minimal $n-1$ cuts to divide the cake into n pieces, which matches the Dubins-Spanier (DS) moving-knife algorithm.
4. While giving the same upper bound as DS, and other m-proportional cake-cutting algorithms that use a different m, on the total number of envies of all players (without the possibility of trades), it gives a lower upper bound on the number of players that any individual player may envy.
5. Its proportional shares are generally greater than $1/n$, which DS and some other proportional algorithms do not guarantee for all players.
6. It is truth-inducing, guaranteeing truthful players proportional shares whatever the choices of the other players.
7. It is applicable to players with unequal entitlements, but this may require the creation of fictitious players, or clones, that get disconnected pieces, and one may envy the other.

We think D&C would not be difficult to apply to a divisible good like land if it is feasible to divide it with parallel, vertical cuts. Unlike DS, it does not require players to make instantaneous decisions about when to stop a moving knife sweeping across the land, which is likely to make people quite anxious about when to call stop. D&C players, by comparison, make decisions about how to divide smaller and smaller parcels of the land—exactly in half if there is an even number of players, approximately in half if there is an odd number—without being under extreme time pressure.

Although players can implement D&C on their own, it may be helpful to use a referee to record players' marks unbeknownst to the other players. Of course, this could also be done by a computer if there were safeguards to ensure that the players' marks, when submitted, cannot be read or inferred by the other players.

A question that we think is worth exploring further is exactly where the cuts between players' marks should be made. For example, should the goal be to give players *equitable shares*, so that every player gets the same proportion greater than $1/n$? Although the equitability procedure (EP) is an efficient, truth-inducing procedure for doing this (Brams, Jones, and Klamler, 2006), EP is not envy-free. Additionally, it requires much more information from the players than does D&C as well as possibly complex calculations by a referee.

EP generally gives a unique division, whereas an envy-free division is generally not unique. An "ideal" division might be an envy-free division that is as close as possible to being equitable, but there is no general n-person algorithm that yields even an envy-free division, much less one that is as equitable as possible. Approximate procedures of the kind Su (1999) discusses, however, might be feasible.

We conclude that D&C is a parsimonious and practical algorithm for dividing a divisible good like land. The question of complicating it to allow for envy-free or equitable shares requires further study.

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