Divisible modules over integral domains

Alberto Facchini

1. Introduction

The aim of this paper is to describe an equivalence between the full subcategory of Mod-R whose objects are all the divisible modules over an integral domain R and a suitable full subcategory of modules over the endomorphism ring E of a fixed divisible module ∂. This equivalence corresponds to the similar equivalences for torsion divisible abelian groups due to Harrison [6] and for torsion h-divisible modules over an integral domain due to Matlis [7], [8] and [9].

Let R denote a commutative integral domain with 1 (not a field) and let ∂_R denote the divisible right R-module defined by L. Fuchs in [3] (see § 2 for the exact definition of ∂_R). The module ∂_R has interesting properties that are shown in [3], in [4, § VI.3] and in §§ 2 and 3 of this paper. For instance, if E is the endomorphism ring of ∂_R and ∂ is viewed as a left E-module ${}_E\partial$, then $\mathrm{End}\,({}_E\partial)\cong R$ and ${}_E\partial\cong E/I$ for a suitable projective principal left ideal I of E. Moreover, ∂ has flat and projective dimensions equal to one both as a right R-module and a left E-module, and this implies that the class $\mathscr{F}$ of all right E-modules M such that $\mathrm{Tor}_1^E(M, \partial)=0$ is the torsion-free class for a (non-hereditary) torsion theory $(\mathscr{T}, \mathscr{F})$ in Mod-E. This torsion theory is generated by the cyclic right E-module $\mathrm{Ext}_R^1({}_E\partial_R, R)$, and a right E-module M_E is a torsion-free module in this torsion theory (we say that M_E is *I-torsion-free*) if and only if the canonical homomorphism $M\otimes_E I\to M\otimes_E E\cong M$ induced by the embedding $I\to E$ is a monomorphism. Dually, we say that a module M_E is an *I-divisible* module if the canonical homomorphism $M\otimes_E I\to M$ is an epimorphism, and that a right E-module N_E is *I-reduced* if it is cogenerated by the right E-module $\partial^*=\mathrm{Hom}_R(\partial, C)$, where C is the minimal injective cogenerator in Mod-R. It is easy to show that a module M_E is I-divisible if and only if $\mathrm{Hom}\,(M, N)=0$ for every I-reduced E-module N_E.

Now define a right E-module M to be an *I-cotorsion module* if it is I-reduced and $\mathrm{Ext}_E^1(N, M)=0$ for every I-divisible I-torsion-free right E-module N. The main result of this paper is the proof of the following theorem: the functors $\mathrm{Hom}_R(\partial, -)$: Mod-$R \to$ Mod-E and $-\otimes_E \partial$: Mod-$E \to$ Mod-R induce an equivalence between the full subcategory of Mod-R whose objects are the divisible R-modules and the full subcategory of Mod-E whose objects are the I-cotorsion E-modules. This generalizes the corresponding results of Harrison for torsion divisible abelian groups [6] and of Matlis for torsion h-divisible R-modules ([7] and [9]). In our equivalence the injective R-modules correspond to the I-reduced I-pure-injective E-modules. Here I-pure-injective means injective relatively to the *I-pure* exact sequences, that is, the sequences $0 \to M' \to M \to M'' \to 0$ of right E-modules for which the sequence $0 \to M' \otimes_E \partial \to M \otimes_E \partial \to M'' \otimes_E \partial \to 0$ is exact. (This extends the corresponding result due to Warfield for Matlis' equivalence between torsion h-divisible modules and torsion-free cotorsion modules, see [4, Th. V.1.8].) Our I-purity is a purity in the sense of Warfield [14].

Finally, we prove that I-cotorsion E-modules are exactly the right E-modules of ∂^*-dominant dimension $\geqq 2$, that is, the modules M_E for which there exists an exact sequence $0 \to M \to \partial^{*X} \to \partial^{*Y}$ with ∂^{*X} and ∂^{*Y} suitable direct products of copies of ∂^*.

For technical reasons (proof of Lemma 2.2) the way we define the R-module ∂ is a little different from the way Fuchs defines it in [3] and [4]. The difference is that our generators are the k-tuples $(r_1, \ldots, r_k)$ of non-zero elements r_i of R, and Fuchs' generators are the k-tuples $(r_1, \ldots, r_k)$ of non-zero and *non-invertible* elements r_i of R. Fuchs' results in [3] and [4] hold with this small modification as well.

2. The R-module ∂_R and its endomorphism ring E

In this paper R will be an integral domain and we will assume that it is not a field. We will denote the field of fractions of R by Q.

Let ∂ be the right R-module generated by the set $\mathcal{G}$ of all k-tuples $(r_1, \ldots, r_k)$ of non-zero elements r_i of R, for $k \geqq 0$, with defining relations

$$(r_1, \ldots, r_k) r_k = (r_1, \ldots, r_{k-1}), \quad k \geqq 1.$$

The right R-module ∂ is obviously divisible, that is, $\partial r = \partial$ for every $r \in R$, $r \neq 0$. The length of $(r_1, \ldots, r_k)$ is defined to be k, and the unique generator $w = \emptyset$ in $\mathcal{G}$ of length 0 generates a submodule wR of ∂ isomorphic to R [4, § VI.3]. Note that for every $x \in \partial$ there exists $r \in R$, $r \neq 0$, such that $xr \in wR$ (possibly $xr = 0$).

The fundamental property of ∂ is the following one:

Proposition 2.1 [4, Lemma VI.3.2]. *Let D be a divisible right R-module and $a \in D$. Then there exists a homomorphism f: $\partial \to D$ with $f(w) = a$.*

Let ∂_n be the submodule of ∂ generated by the elements of $\mathcal{G}$ of length $\leqq n$, so that in particular $\partial_0 = wR$.

Lemma 2.2. *Fix a nonnegative integer n and an element a of R, $a \neq 0$ and $a \neq 1$. Then the correspondence $\mathcal{G} \to \partial$ defined by*

$$(r_1, \ldots, r_k) \in \mathcal{G} \mapsto \begin{cases} 0 & \text{if } k \leqq n \\ (r_1, \ldots, r_n, 1, r_{n+1}, \ldots, r_k) - (r_1, \ldots, r_n, a, r_{n+1}, \ldots, r_k)a & \text{if } k > n \end{cases}$$

extends to an endomorphism of ∂ whose kernel is ∂_n and whose image is a direct summand of ∂.

Proof. It is easy to show that the defining relations of ∂ are preserved by the correspondence; for instance, when $k = n+1$, the relation $(r_1, \ldots, r_k)r_k = (r_1, \ldots, r_{k-1})$ is preserved because $[(r_1, \ldots, r_n, 1, r_{n+1}) - (r_1, \ldots, r_n, a, r_{n+1})a]r_k = (r_1, \ldots, r_n, 1) - (r_1, \ldots, r_n, a)a = (r_1, \ldots, r_n) - (r_1, \ldots, r_n) = 0$. Therefore the correspondence extends to an endomorphism φ of ∂. Note that $\partial_n \subset \ker \varphi$ because $\varphi(r_1, \ldots, r_n) = 0$ for every $(r_1, \ldots, r_n)$. In particular $\varphi = \varphi' \circ \pi$ where $\pi\colon \partial \to \partial/\partial_n$ is the canonical projection and $\varphi'\colon \partial/\partial_n \to \partial$ is a homomorphism.

Now consider the correspondence $\mathcal{G} \to \partial/\partial_n$ defined by

$$(r_1, \ldots, r_k) \in \mathcal{G} \mapsto \begin{cases} \partial_n & \text{if } k \leqq n+1 \\ \partial_n & \text{if } k > n+1 \text{ and } r_{n+1} \neq 1 \\ (r_1, \ldots, r_n, \widehat{r_{n+1}}, r_{n+2}, \ldots, r_k) + \partial_n & \text{if } k > n+1 \text{ and } r_{n+1} = 1, \end{cases}$$

where $(r_1, \ldots, r_n, \widehat{r_{n+1}}, r_{n+2}, \ldots, r_k)$ denotes the $(k-1)$-tuple in which r_{n+1} has been deleted. The defining relations of ∂ are preserved by this correspondence as well; for instance, when $k = n+2$ and $r_{n+1} = 1$, the relation $(r_1, \ldots, r_k)r_k = (r_1, \ldots, r_{k-1})$ is preserved because $[(r_1, \ldots, r_n, \widehat{r_{n+1}}, r_k) + \partial_n]r_k = (r_1, \ldots, r_n) + \partial_n = \partial_n$. Therefore this correspondence also extends to a homomorphism $\psi\colon \partial \to \partial/\partial_n$.

The composed homomorphism $\psi\varphi\colon \partial \to \partial/\partial_n$ is defined by $\psi\varphi(r_1, \ldots, r_k) = \partial_n$ if $k \leqq n$ and $\psi\varphi(r_1, \ldots, r_k) = \psi[(r_1, \ldots, r_n, 1, r_{n+1}, \ldots, r_k) - (r_1, \ldots, r_n, a, r_{n+1}, \ldots, r_k)a] = (r_1, \ldots, r_n, r_{n+1}, \ldots, r_k) + \partial_n$ if $k > n$, i.e., $\psi\varphi\colon \partial \to \partial/\partial_n$ is the canonical projection π. Therefore $\pi = \psi\varphi = \psi\varphi'\pi$, hence $\psi\varphi'$ is the identity of ∂/∂_n, so that φ' is injective and $\partial = \varphi'(\partial/\partial_n) \oplus \ker \psi$. Since φ' is injective, $\ker \varphi = \ker(\varphi'\pi) = \ker \pi = \partial_n$. Moreover $\varphi(\partial) = \varphi'(\partial/\partial_n)$ is a direct summand of ∂.

Fix the following notations:

— E is the endomorphism ring $\operatorname{End}(\partial_R)$ of the R-module ∂_R;

— φ is a fixed R-endomorphism of ∂ (i.e., $\varphi \in E$) with $\ker \varphi = wR$ and $\varphi(\partial)$ a direct summand of ∂ (it exists by Lemma 2.2);

— ε is a fixed idempotent R-endomorphism of ∂ (i.e., $\varepsilon \in E$ and $\varepsilon^2 = \varepsilon$) with $\varepsilon(\partial) = \varphi(\partial)$;

— I is the left ideal $\{f \in E \mid f(w) = 0\}$ of E;

— J is the two sided ideal $\{f \in E \mid f(\partial) \subset t(\partial)\}$ of E, where $t(\partial)$ denotes the torsion submodule of ∂.

Since R is a commutative ring and ∂_R is a faithful module, the ring R is a subring of the center $Z(E)$ of E. In the next theorem we prove that R is equal to $Z(E)$.

Theorem 2.3. *The integral domain R is the center of $E = \mathrm{End}(\partial_R)$.*

Proof. It is sufficient to show that if f belongs to the center of E then there exists $r \in R$ such that $f(x) = xr$ for every $x \in \partial$. If f is in the center of E and φ denotes the endomorphism defined before the statement of this theorem, then $\varphi f(w) = f\varphi(w) = f(0) = 0$, so that $f(w) \in \ker \varphi = wR$; hence there exists $r \in R$ with $f(w) = wr$. If $x \in \partial$, then there is a homomorphism $g \colon \partial \to \partial$ with $g(w) = x$ by Proposition 2.1, and $f(x) = f(g(w)) = g(f(w)) = g(wr) = g(w)r = xr$. This concludes the proof of the theorem.

If $\alpha \colon \partial \to Q$ is the R-module homomorphism defined by $\alpha(r_1, \ldots, r_k) = (r_1 \ldots r_k)^{-1}$ for $k \geqq 1$ and $\alpha(w) = 1$, then $\ker \alpha$ is the torsion submodule $t(\partial)$ of ∂. This is easily seen, because $t(\partial) \subset \ker \alpha$ since Q is torsion-free, and if $x \in \ker \alpha$ and $r \in R$, $r \neq 0$, is such that $xr \in wR$, $xr = ws$ say, then $0 = \alpha(xr) = \alpha(ws) = \alpha(w)s = s$; therefore $xr = 0$ and $x \in t(\partial)$. In particular $\partial / t(\partial) \cong Q$.

If we apply the functor $\mathrm{Hom}_R(\partial, -)$ to the exact sequence $0 \to t(\partial) \to \partial \xrightarrow{\alpha} Q \to 0$, we obtain the exact sequence $0 \to J \to E \to \mathrm{Hom}_R(\partial, Q) \to \mathrm{Ext}^1_R(\partial, t(\partial))$. But $\mathrm{Hom}_R(\partial, Q) \cong \mathrm{Hom}_R(\partial / t(\partial), Q) \cong \mathrm{Hom}_R(Q, Q) \cong Q$ and $\mathrm{Ext}^1_R(\partial, t(\partial)) = 0$ because $t(\partial)$ is a divisible R-module [4, Prop. VI.3.4]. Hence $E/J \cong Q$ and J is an ideal of E maximal among the two sided ideals of E.

Note that the left annihilator of φ, $l(\varphi) = \{g \in E \mid g\varphi = 0\}$, is $E(1-\varepsilon)$. In fact, $(1-\varepsilon)\varphi = 0$ because $\varepsilon(\partial) = \varphi(\partial)$, so that $E(1-\varepsilon) \subset l(\varphi)$. And if $g \in l(\varphi)$, then $g\varphi = 0$, i.e., $\ker g \supset \varphi(\partial) = \varepsilon(\partial)$; it follows that $g\varepsilon = 0$ and $g = g - g\varepsilon = g(1-\varepsilon) \in E(1-\varepsilon)$. The right annihilator of φ, $r(\varphi) = \{g \in E \mid \varphi g = 0\}$, is 0, because if $\varphi g = 0$, then $g(\partial) \subset \ker \varphi = wR$. Since $g(\partial)$ is a divisible module, it must be the zero submodule of wR, i.e., $g = 0$.

Theorem 2.4. *If B_R is any right R-module and $f \colon \partial \to B$ is a homomorphism such that $f(w) = 0$, then there exists $g \colon \partial \to B$ such that $f = g\varphi$. In particular, $I = \{f \in E \mid f(w) = 0\}$ is the left principal ideal $E\varphi$ generated by φ and is a projective ideal of E isomorphic to $E\varepsilon$.*

Proof. Since $\ker \varphi = wR$ and $\varphi(\partial)$ is a direct summand of ∂, there exists $\psi \colon \partial \to \partial / wR$ such that $\psi\varphi$ is the canonical projection $\pi \colon \partial \to \partial / wR$ (this had been also shown in the proof of Lemma 2.2). Since $f\colon \partial \to B$ annihilates w, f can be written as $f=f'\pi$ for a suitable $f'\colon \partial/wR \to B$ induced by f. If $g=f'\psi\colon \partial \to B$, then $f=f'\pi=f'\psi\varphi=g\varphi$. This proves the first assertion.

In particular, $I=\{f\in E \mid f(w)=0\}\subset\{g\varphi \mid g\in \mathrm{Hom}_R(\partial, \partial)\}=E\varphi$, so that $I=E\varphi$, the other inclusion being trivial.

Finally, since $l(\varphi)=E(1-\varepsilon)=l(\varepsilon)$, the ideal $I=E\varphi\cong E\varepsilon$ is projective.

3. The E-modules ${}_E\partial$ and ∂_E°

Since $E=\mathrm{End}(\partial_R)$, the module ∂ can be viewed as a left E-module, and $R=\mathrm{End}({}_E\partial)$ by Theorem 2.3. In this section we shall study the E-module ${}_E\partial$.

Lemma 3.1. *The left E-module ${}_E\partial$ is isomorphic to E/I.*

Proof. Consider the mapping $E\to\partial$ defined by $f\mapsto f(w)$ for every $f\in E$. Obviously it is a left E-module homomorphism. It is surjective by proposition 2.1 and its kernel is I.

Fuchs [4, Lemma VI.3.1] has proved that the projective dimension of ∂_R, $\mathrm{proj.dim}\,\partial_R$, is equal to one (this can also be shown by proving that the relations $(r_1, \ldots, r_k)r_k-(r_1, \ldots, r_{k-1})$ generate a free submodule H of the module F freely generated by $\mathscr{G}$); since ∂_R is not flat (every flat R-module is torsion-free, and ∂_R is not torsion-free) and $\mathrm{proj.dim}\,\partial_R\geqq \mathrm{flat.dim}\,\partial_R$, where $\mathrm{flat.dim}\,\partial_R$ is the flat dimension of ∂_R, it follows that $\mathrm{flat.dim}\,\partial_R=\mathrm{proj.dim}\,\partial_R=1$. This holds for the module ${}_E\partial$ too.

Corollary 3.2. $\mathrm{flat.dim}\,{}_E\partial=\mathrm{proj.dim}\,{}_E\partial=1$.

Proof. By Lemma 3.1 and Theorem 2.4 $\mathrm{proj.dim}\,{}_E\partial\leqq 1$. If $\mathrm{proj.dim}\,{}_E\partial<1$, then ${}_E\partial$ is projective, so that $I=E\varphi$ is a direct summand of E, i.e., $E\varphi=E\beta$ for an idempotent $\beta\in E$. Then $wR=\cap\{\ker f \mid f\in E\varphi\}=\cap\{\ker f \mid f\in E\beta\}=\ker\beta$ is a direct summand of the divisible module ∂_R, contradiction, because wR is not divisible. This proves that $\mathrm{proj.dim}\,{}_E\partial=1$. Moreover $\mathrm{flat.dim}\,{}_E\partial\leqq\mathrm{proj.dim}\,{}_E\partial=1$, and ${}_E\partial$ is not flat, because ${}_E\partial$ is finitely presented by Lemma 3.1 and Theorem 2.4 and every finitely presented flat module is projective [13, Cor. I.11.5]. Therefore $\mathrm{flat.dim}\,{}_E\partial=1$.

By Corollary 3.2 $\mathrm{Tor}_n^E(-, {}_E\partial)=\mathrm{Ext}_E^n({}_E\partial, -)=0$ for $n\geqq 2$. In the sequel we need the exact formulas for the functors $\mathrm{Tor}_1^E(-, {}_E\partial)$ and $\mathrm{Ext}_E^1({}_E\partial, -)$ that are calculated in the next corollary.

Corollary 3.3. *If M_E is any right E-module, then* $\mathrm{Tor}_1^E(M, \partial)\cong(0:_M\varphi)\varepsilon$, *where* $(0:_M\varphi)=\{x\in M \mid x\varphi=0\}$.

If ${}_E N$ is any left E-module, then $\mathrm{Ext}^1_E(\partial, N) \cong \varepsilon N/\varphi N$.

Proof. Consider the exact sequence $0 \to I \to E \to \partial \to 0$. By applying the functor $M \otimes_E -$, we obtain that the sequence $0 \to \mathrm{Tor}^E_1(M, \partial) \to M \otimes I \to M \otimes E$ is exact. Since $I = E\varphi \cong E\varepsilon$ and $M \otimes_E E \cong M$, it follows that $\mathrm{Tor}^E_1(M, \partial)$ is isomorphic to the kernel of the abelian group homomorphism $M\varepsilon \to M$ defined by $x\varepsilon \mapsto x\varphi$ for every $x \in M$. It follows that $\mathrm{Tor}^E_1(M, \partial) \cong (0 :_M \varphi)\varepsilon$. Similarly for $\mathrm{Ext}^1_E(\partial, N)$.

Note that since $\mathrm{proj.dim}\, \partial_R = 1$, the torsion submodule $t(\partial_R)$ of ∂_R is isomorphic to a submodule of $K^{(X)}$, where $K = Q/R$ and $K^{(X)}$ is a direct sum of copies of K. Namely, if M_R is any module with $\mathrm{proj.dim}\, M_R = 1$, fix a free resolution $0 \to R^{(X)} \to R^{(Y)} \to M \to 0$ of M (this is possible by [10, page 90, Ex. 3]) and apply the functor $- \otimes_R K$ to this sequence. Then the sequence $\mathrm{Tor}^1_R(R^{(Y)}, K) \to \mathrm{Tor}^1_R(M, K) \to R^{(X)} \otimes K \to R^{(Y)} \otimes K$ can be rewritten as $0 \to t(M) \to K^{(X)} \to K^{(Y)}$ by [8, page 10].

Since $\mathrm{proj.dim}\, \partial_R = 1$, it follows that $\mathrm{Ext}^n_R(\partial, -) = 0$ for $n \geq 2$. Consider $\mathrm{Ext}^1_R(\partial, R)$. Since $\mathrm{Ext}^1_R(-, R)$ is a contravariant functor, every R-homomorphism $f: \partial \to \partial$ induces an R-homomorphism $\mathrm{Ext}^1_R(f, R): \mathrm{Ext}^1_R(\partial, R) \to \mathrm{Ext}^1_R(\partial, R)$, so that $\mathrm{Ext}^1_R(\partial, R)$ is a *right* E-module.

Theorem 3.4. *The right E-module* $\mathrm{Ext}^1_R(\partial, R)$ *is isomorphic to* $\varepsilon E/\varphi E$.

Proof. Let C be the image of the endomorphism $1-\varepsilon$ of ∂, so that $\partial = \varepsilon(\partial) \oplus (1-\varepsilon)(\partial) = \varphi(\partial) \oplus C$. Consider the exact sequence of R-modules

$$S\colon 0 \to R \xrightarrow{\alpha} \partial \oplus C \xrightarrow{\beta} \partial \to 0,$$

where $\alpha(r) = (wr, 0)$ for every $r \in R$ and $\beta(x, y) = \varphi(x) + y$ for every $(x, y) \in \partial \oplus C$. Let $\bar{S}$ be the image of the extension S into $\mathrm{Ext}^1_R(\partial, R)$. In order to prove the theorem it is sufficient to show that $\Phi\colon \varepsilon E \to \mathrm{Ext}^1_R(\partial, R)$ defined by $\Phi(\varepsilon f) = \bar{S}f$ for every $f \in E$ is a well defined surjective E-homomorphism with kernel φE.

If $f \in E$ and $\varepsilon f = 0$, then $f(\partial) \subset \ker \varepsilon = C$, so that it is possible to define a homomorphism $g\colon R \oplus \partial \to \partial \oplus C$ by setting $g(r, x) = (wr, f(x))$ for every $(r, x) \in R \oplus \partial$. If Z denotes the trivial extension, the diagram

$$\begin{array}{rcccccccc} Z\colon & 0 & \to & R & \to & R\oplus\partial & \to & \partial & \to 0 \\ & & & \| & & \downarrow g & & \downarrow f & \\ S\colon & 0 & \to & R & \xrightarrow{\alpha} & \partial\oplus C & \xrightarrow{\beta} & \partial & \to 0 \end{array}$$

commutes. This shows that $\bar{S}f$ is zero in $\mathrm{Ext}^1_R(\partial, R)$ and proves that Φ is a well defined homomorphism of right E-modules.

Now we shall show that Φ is surjective. Let

$$T\colon 0 \to R \xrightarrow{\gamma} A \xrightarrow{\delta} \partial \to 0$$

be any extension and $\bar{T}$ its image into $\mathrm{Ext}^1_R(\partial, R)$. Since $\mathrm{Ext}^1_R(\partial, \partial) = 0$ [4, Prop. VI.3.4], the R-homomorphism $\gamma^*\colon \mathrm{Hom}_R(A, \partial) \to \mathrm{Hom}_R(R, \partial)$ is surjective. Hence there exists $\chi\in\mathrm{Hom}_R(A,\partial)$ such that $(\gamma^*(\chi))(1)=w$, that is, $\chi(\gamma(1))=w$. Define $h\colon A\to\partial\oplus C$ by $h(a)=(\chi(a),0)$ for every $a\in A$. Then $h(\gamma(1))=(\chi(\gamma(1)),0)=(w,0)=\alpha(1)$, so that $h\gamma=\alpha$ and the left-hand square in the diagram

$$\begin{array}{llllllllll} T\colon & 0 & \to & R & \xrightarrow{\gamma} & A & \xrightarrow{\delta} & \partial & \to & 0 \\ & & & \| & & \downarrow h & & \downarrow f & & \\ S\colon & 0 & \to & R & \xrightarrow{\alpha} & \partial\oplus C & \xrightarrow{\beta} & \partial & \to & 0 \end{array}$$

commutes; it follows that there exists an $f\in E$ making the right-hand square commute. Then $\bar{S}f=\bar{T}$, and Φ is surjective.

In order to prove that $\ker\Phi=\varphi E$, fix an $f\in E$, so that $\varepsilon f\in\varepsilon E$. Then $\varepsilon f\in\ker\Phi$ if and only if $\bar{S}f=\bar{Z}$, i.e., if and only if there exists a homomorphism $g\colon R\oplus\partial\to\partial\oplus C$ making the diagram

$$\begin{array}{llllllllll} Z\colon & 0 & \to & R & \to & R\oplus\partial & \to & \partial & \to & 0 \\ & & & \| & & \downarrow g & & \downarrow f & & \\ S\colon & 0 & \to & R & \xrightarrow{\alpha} & \partial\oplus C & \xrightarrow{\beta} & \partial & \to & 0 \end{array}$$

commute. This means that $g(r,0)=(wr,0)$ and $\beta g(r,x)=f(x)$ for every $r\in R$ and $x\in\partial$. Since the homomorphisms $g\colon R\oplus\partial\to\partial\oplus C$ such that $g(r,0)=(wr,0)$ for every $r\in R$ are exactly of the form $g(r,x)=(wr+h(x),l(x))$ for suitable $h\colon\partial\to\partial$ and $l\colon\partial\to C$, it follows that $\varepsilon f\in\ker\Phi$ if and only if there exists $h\colon\partial\to\partial$ and $l\colon\partial\to C$ such that $f(x)=\beta g(r,x)=\beta(wr+h(x),l(x))=\varphi(wr+h(x))+l(x)=(\varphi h+l)(x)$, i.e., $f-\varphi h=l$. But $C=(1-\varepsilon)(\partial)=\ker\varepsilon$, so that $\varepsilon f\in\ker\Phi$ if and only if $\varepsilon(f-\varphi h)=0$ for some $h\colon\partial\to\partial$, i.e., $\varepsilon f=\varepsilon\varphi h=\varphi h\in\varphi E$. This proves that $\ker\Phi=\varphi E$.

We shall often need the right E-module $\mathrm{Ext}^1_R(\partial,R)$ in the sequel, and we shall denote it by ∂°. Hence $\partial^\circ=\mathrm{Ext}^1_R(\partial,R)\cong\varepsilon E/\varphi E$ as a right E-module. There are other "presentations" of the module ∂°. For instance the right E-modules ∂° and $\mathrm{Ext}^1_E(\partial,E)$ are isomorphic right E-modules by Corollary 3.3. Moreover the functor $\mathrm{Hom}_R({}_E\partial_R,-)$ applied to the exact sequence of R-modules $0\to wR\to\partial\to\partial/wR\to 0$ gives the exact sequence of right E-modules $0\to E\to\mathrm{Hom}_R(\partial,\partial/wR)\to\mathrm{Ext}^1_R(\partial,wR)\to\mathrm{Ext}^1_R(\partial,\partial)$. The last module is zero by [4, Prop. VI.3.4], so that the right E-modules $\partial^\circ\cong\mathrm{Ext}^1_R(\partial,wR)$ and $\mathrm{Hom}_R(\partial,\partial/wR)/E$ are isomorphic.

Furthermore, the functor $\mathrm{Hom}_R({}_E\partial_R,-)$ applied to the exact sequence of R-modules $0\to R\to Q\to K\to 0$ gives the exact sequence of right E-modules $0\to\mathrm{Hom}_R(\partial,Q)\to\mathrm{Hom}_R(\partial,K)\to\mathrm{Ext}^1_R(\partial,R)\to\mathrm{Ext}^1_R(\partial,Q)$. The last module is zero by [4, Prop. VI.3.4], and the first module is $\mathrm{Hom}_R(\partial,Q)\cong\mathrm{Hom}_R(\partial/t(\partial),Q)\cong Q$ by the remarks after proposition 2.3. Therefore $\partial^\circ\cong\mathrm{Hom}_R(\partial,K)/Q$ as E-modules.

If we are only interested in the structure of ∂° as an R-module, there is one more "presentation" of ∂°: the functor $\mathrm{Hom}_R(-,R)$ applied to the exact sequence $0\to H\to F\to\partial\to 0$ (where F is the R-module freely generated by $\mathscr{G}$ and H is the free submodule of F generated by the relations) gives $0\to\mathrm{Hom}_R(F,R)\to\mathrm{Hom}_R(H,R)\to$

$\mathrm{Ext}^1_R(\partial, R) \to 0$, which is a presentation of ∂° as a quotient of two R-modules isomorphic to direct products of copies of R.

Corollary 3.5. $\mathrm{flat.dim}\,\partial^\circ_E = \mathrm{proj.dim}\,\partial^\circ_E = 1$.

Proof. Since $r(\varphi)=0$, it follows that $\varphi E \cong E$ is projective, so that $\partial^\circ \cong \varepsilon E/\varphi E$ has projective dimension $\leqq 1$. Hence $1 \geqq \mathrm{proj.dim}\,\partial^\circ \geqq \mathrm{flat.dim}\,\partial^\circ$. It remains to prove that $\varepsilon E/\varphi E$ is not flat. But $\varepsilon + \varphi E \in \varepsilon E/\varphi E$ is annihilated by φ (because $\varepsilon\varphi = \varphi$) so that it belongs to $(0:\varphi)\varepsilon \cong \mathrm{Tor}^E_1(\partial^\circ, \partial)$ (Corollary 3.3). Thus $\mathrm{Tor}^E_1(\partial^\circ, \partial) \neq 0$ and ∂° is not flat.

Theorem 3.6. $\mathrm{End}(\partial^\circ_E) \cong R$.

Proof. First of all observe that ∂° is a torsion-free R-module, because if $r \in R$ and $r \neq 0$, the functor $\mathrm{Hom}_R(\partial, -)$ applied to the exact sequence $0 \to R \xrightarrow{r} R \to R/rR \to 0$ gives the exact sequence $\mathrm{Hom}_R(\partial, R/rR) \to \partial^\circ \xrightarrow{r} \partial^\circ$. The first module is zero because ∂ is divisible and R/rR is torsion of bounded order. Hence the multiplication by r is an injective endomorphism of ∂°, and ∂° is a torsion-free R-module.

Since $\partial^\circ \cong \varepsilon E/\varphi E \cong E/(\varphi E + (1-\varepsilon)E)$ is a cyclic E-module, it follows that $\mathrm{End}_E(\partial^\circ) \cong U/(\varphi E + (1-\varepsilon)E)$, where U is the subring $\{f \in E \mid f(\varphi E + (1-\varepsilon)E) \subset \varphi E + (1-\varepsilon)E\}$ of E (for instance see [10, page 24]). Similarly, since $\partial \cong E/E\varphi$, the ring $\mathrm{End}_E(\partial)$ is isomorphic to $V/E\varphi$, where $V = \{g \in E \mid E\varphi g \subset E\varphi\}$. But $\mathrm{End}_E(\partial)$ is canonically isomorphic to R (Theorem 2.3), and thus $V = R + E\varphi$.

Now we prove that $U = R + \varphi E + (1-\varepsilon)E$. The inclusion $U \supset R + \varphi E + (1-\varepsilon)E$ is trivial. Conversely, if $f \in U$, that is, $f \in E$ and $f(\varphi E + (1-\varepsilon)E) \subset \varphi E + (1-\varepsilon)E$, then $\varepsilon f \varphi \in \varepsilon(\varphi E + (1-\varepsilon)E) = \varphi E$. Therefore $\varepsilon f \varphi = \varphi g$ for some $g \in E$. In particular $E\varphi g = E\varepsilon f \varphi \subset E\varphi$, that is, $g \in V = R + E\varphi$. Hence $g = r + h\varphi$ for some $r \in R$ and $h \in E$, and $\varepsilon f \varphi = \varphi g = \varphi(r + h\varphi) = (r + \varphi h)\varphi$. Then $(\varepsilon f - r - \varphi h)\varphi = 0$, and since $l(\varphi) = E(1-\varepsilon) = l(\varepsilon)$ (§ 2), we have $(\varepsilon f - r - \varphi h)\varepsilon = 0$, so that $\varepsilon f \varepsilon = r\varepsilon - \varphi h \varepsilon = r - (1-\varepsilon)r - \varphi h \varepsilon \in R + (1-\varepsilon)E + \varphi E$. Moreover $f \in U$ implies $f(1-\varepsilon) \in \varphi E + (1-\varepsilon)E$, so that $f = f(1-\varepsilon) + (1-\varepsilon)f\varepsilon + \varepsilon f \varepsilon \in (\varphi E + (1-\varepsilon)E) + (1-\varepsilon)E + (R + (1-\varepsilon)E + \varphi E) = R + \varphi E + (1-\varepsilon)E$. This proves that $U = R + \varphi E + (1-\varepsilon)E$.

It follows that

$$\begin{aligned}\mathrm{End}_E(\partial^\circ) &\cong U/(\varphi E + (1-\varepsilon)E) = (R + \varphi E + (1-\varepsilon)E)/(\varphi E + (1-\varepsilon)E)\\ &\cong R/(R \cap (\varphi E + (1-\varepsilon)E)),\end{aligned}$$

i.e., every element of $\mathrm{End}_E(\partial^\circ)$ is induced by the multiplication by an element of R. But ∂° is a torsion-free R-module, so that $\mathrm{End}_E(\partial^\circ) \cong R$.

4. The functors $\mathrm{Hom}_R(\partial, -)$ and $-\otimes_E\partial$

Consider the two functors $\mathrm{Hom}_R({}_E\partial_R, -)$: Mod-$R\to$Mod-$E$ and $-\otimes_E\partial_R$: Mod-$E\to$ Mod-R. Then $\mathrm{Hom}_R({}_E\partial_R, -)$ is the right adjoint of $\otimes_E\partial_R$, for each $M\in$Mod-E there is a canonical E-module homomorphism

$$\eta_M\colon M\to \mathrm{Hom}_R(\partial, M\otimes_E\partial)$$

defined by $\eta_M(m)(x)=m\otimes x$ for every $m\in M$ and $x\in\partial$ (the unit of the adjunction), and for each $A\in$Mod-R there is a canonical R-module homomorphism $\varepsilon_A\colon \mathrm{Hom}_R(\partial, A)\otimes_E\partial\to A$ defined by $\varepsilon_A(f\otimes x)=f(x)$ for every $f\in\mathrm{Hom}_R(\partial, A)$ and $x\in\partial$ (the counit of the adjunction).

Note that if M_E is any E-module, the R-module $M\otimes_E\partial$ is divisible (because ∂_R is divisible and $-\otimes_E\partial_R$ is right exact). Hence $-\otimes_E\partial$ is a functor of Mod-E into the full subcategory $\mathscr{D}_R$ of Mod-R whose objects are the divisible R-modules.

Theorem 4.1. *Let A_R be a right R-module. Then $\varepsilon_A\colon \mathrm{Hom}_R(\partial, A)\otimes_E\partial\to A$ is an isomorphism if and only if A is a divisible R-module.*

Proof. If ε_A is an isomorphism and $F_E\to\mathrm{Hom}_R(\partial, A)$ is a surjective E-homomorphism of a free E-module F_E onto $\mathrm{Hom}_R(\partial, A)$, then $F\otimes_E\partial\to\mathrm{Hom}_R(\partial, A)\otimes\partial$ is a surjective R-homomorphism of the R-module $F\otimes_E\partial$ onto $\mathrm{Hom}_R(\partial, A)\otimes\partial\cong A$. Hence A, homomorphic image of the divisible R-module $F\otimes_E\partial$, is divisible.

Conversely, suppose A_R divisible and apply the functor $\mathrm{Hom}_R(\partial, A)\otimes_E-$ to the exact sequence $0\to E\varphi\to E\to\partial\to 0$, where the first homomorphism is the inclusion and the second is defined by $1\mapsto w$ (Theorem 2.4 and Lemma 3.1). The first homomorphism in the obtained sequence

$$\mathrm{Hom}_R(\partial, A)\otimes_E E\varphi\to\mathrm{Hom}_R(\partial, A)\to\mathrm{Hom}_R(\partial, A)\otimes_E\partial\to 0$$

is induced by the multiplication, so that its image is $\{g\varphi \mid g\in\mathrm{Hom}_R(\partial, A)\}$, which is equal to $B=\{f\mid f\in\mathrm{Hom}_R(\partial, A), f(w)=0\}$ by Theorem 2.4.

The homomorphism $\chi\colon\mathrm{Hom}_R(\partial, A)\to A$ defined by $\chi(f)=f(w)$ for every $f\in\mathrm{Hom}_R(\partial, A)$ is surjective by proposition 2.1 because A is divisible, and has B as its kernel. Moreover the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(\partial, A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(\partial, A)\otimes_E\partial \arrow[r] \arrow[d, "\varepsilon_A"] & 0 \\
0 \arrow[r] & B \arrow[r] & \mathrm{Hom}_R(\partial, A) \arrow[r, "\chi"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes, because $\chi(f)=f(w)=\varepsilon_A(f\otimes w)$ for every $f\in\mathrm{Hom}_R(\partial, A)$. It follows that ε_A is an isomorphism.

If $\mathscr{D}_R$ denotes the full subcategory of Mod-R whose objects are the divisible modules, the functor $\mathrm{Hom}_R(\partial, -)\colon\mathscr{D}_R\to$Mod-$E$ is full and faithful by Theorem 4.1

[11, prop. 5.2], so that $\mathscr{D}_R$ is equivalent to the full subcategory $\mathscr{I}_E$ of Mod-E whose objects are the E-modules isomorphic to $\mathrm{Hom}_R(\partial, A)$ for some $A \in$ Mod-R.

In the next sections we shall study and characterize the right E-modules isomorphic to $\mathrm{Hom}_R(\partial, A)$ for some $A \in$ Mod-R. In order to do this we shall often need the following result.

Proposition 4.2. *For every R-module A_R,* $\mathrm{Tor}_1^E(\mathrm{Hom}_R(\partial, A), {}_E\partial)=0$.

Proof. By Corollary 3.3 we must show that $(0:\varphi)\varepsilon=0$, where $(0:\varphi)=\{f \in \mathrm{Hom}_R(\partial_R, A) \mid f\varphi=0\}$. Now $f\varphi=0$ if and only if $\varphi(\partial)\subset \ker f$. But $\varphi(\partial)=\varepsilon(\partial)$. Hence if $f \in (0:\varphi)$, then $\varepsilon(\partial)\subset \ker f$, so that $f\varepsilon=0$. This concludes the proof of the proposition.

Theorem 4.3. *Let $\mathscr{I}$ be the class of all right E-modules isomorphic to* $\mathrm{Hom}_R(\partial, A)$ *for some right R-module A. Let $0\to L_E\to M_E\to N_E\to 0$ be a short exact sequence of right E-modules.*

(i) *If $L, N \in \mathscr{I}$, then $M \in \mathscr{I}$.*
(ii) *If $M, N \in \mathscr{I}$, then $L \in \mathscr{I}$.*
(iii) *If $L, M \in \mathscr{I}$ and* $\mathrm{Tor}_1^E(N, \partial)=0$, *then $N \in \mathscr{I}$.*

Proof. In all of the three cases $\mathrm{Tor}_1^E(N, \partial)=0$ by proposition 4.2. Hence the functor $-\otimes_E\partial$ applied to the sequence of the statement of the theorem gives the exact sequence $0\to L\otimes\partial\to M\otimes\partial\to N\otimes\partial\to 0$. The functor $\mathrm{Hom}_E(\partial, -)$ applied to this sequence and the naturality of the transformation η give the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\eta_L"] & M \arrow[r] \arrow[d, "\eta_M"] & N \arrow[r] \arrow[d, "\eta_N"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(\partial, L\otimes_E\partial) \arrow[r] & \mathrm{Hom}_R(\partial, M\otimes_E\partial) \arrow[r] & \mathrm{Hom}_R(\partial, N\otimes_E\partial) \arrow[r] & 0.
\end{tikzcd}$$

The second row in this diagram is exact because $\mathrm{Ext}_R^1(\partial, L\otimes_E\partial)=0$ by [4, Prop. VI.3.4]. Hence if two of the mappings η_L, η_M, η_N are isomorphisms, so is the third. It remains to prove that for a module P_E the mapping $\eta_P: P\to \mathrm{Hom}_R(\partial, P\otimes\partial)$ is an isomorphism if and only if $P \in \mathscr{I}$. But if $P \in \mathscr{I}$, then the functors $-\otimes_E\partial$ and $\mathrm{Hom}_R(\partial, -)$ give an equivalence $\mathscr{D}\to\mathscr{I}$, so that η_P is an isomorphism. And if $P\cong \mathrm{Hom}_R(\partial, P\otimes\partial)$, then $P\cong \mathrm{Hom}_R(\partial, A) \in \mathscr{I}$ with $A=P\otimes\partial$.

The hypothesis $\mathrm{Tor}_1^E(N, \partial)=0$ in part (iii) of Theorem 4.3 cannot be eliminated as the following example shows: set $L=M=E$ and let r be any non-zero and non-invertible element of R. Since $E=\mathrm{Hom}_R(\partial, \partial)$ is a torsion-free R-module (because ∂ is divisible), the multiplication by r gives an exact sequence $0\to E\to E\to E/Er\to 0$ of E-modules. In this sequence the first two modules are in $\mathscr{I}$ and the third E-module E/Er is torsion of bounded order as an R-module. But $E\neq Er$, otherwise r would be invertible in E, that is, $1=fr$ for some $f\in E$, contradiction, because the multiplication by r is not an injective mapping $\partial\to\partial$. Hence $E/Er\neq 0$ is not a torsion-free R-module, and in particular $E/Er\notin\mathscr{I}$ (every module in $\mathscr{I}$ is torsion-free as an R-module).

5. The torsion theory $(\mathscr{T},\mathscr{F})$ and its cotorsion theory

In this section S is an arbitrary associative ring with identity and $I=S\varphi$ is a projective principal *left* ideal of S.

If M_S is any right S-module, the inclusion $I\to S$ induces a homomorphism $M\otimes_S I\to M$, and we say that M is *I-torsion-free* if this mapping $M\otimes_S I\to M$ is injective, and say that M is *I-divisible* if it is surjective. Note that the definition of I-divisible module is obtained by dualizing the definition of I-torsion-free module. Moreover M I-divisible simply means $M\varphi=M$.

Denote the class of all I-torsion-free right S-modules by $\mathscr{F}$.

Lemma 5.1. *If S is an algebra over a commutative ring R, C is an injective cogenerator in* Mod-R, *$(S/I)^*$ is the right S-module* $\mathrm{Hom}_R(S/I, C)$, *and M is a right S-module, then*

(i) *M is I-torsion-free if and only if* $\mathrm{Tor}_1^S(M, S/I)=0$, *if and only if*

$$\mathrm{Ext}_S^1(M,(S/I)^*) = 0;$$

(ii) *M is I-divisible if and only if* $M\otimes_S(S/I)=0$.

Proof. From the exact sequence $0\to I\to S\to S/I\to 0$ we obtain the exact sequence $0\to\mathrm{Tor}_1^S(M, S/I)\to M\otimes_S I\to M\to M\otimes_S(S/I)\to 0$. Hence M is I-torsion-free if and only if $\mathrm{Tor}_1^S(M, S/I)=0$, and M is I-divisible if and only if $M\otimes_S(S/I)=0$. Moreover $\mathrm{Hom}_R(\mathrm{Tor}_1^S(M, S/I), C)\cong\mathrm{Ext}_S^1(M,(S/I)^*)$, so that $\mathrm{Tor}_1^S(M, S/I)=0$ if and only if $\mathrm{Ext}_S^1(M,(S/I)^*)=0$.

Proposition 5.2. *The class $\mathscr{F}$ is the torsion-free class for a torsion theory $(\mathscr{T},\mathscr{F})$.*

Proof. We must show that $\mathscr{F}$ is closed under submodules, products and extensions [13, Prop. VI.2.2]. Since I is projective, the flat dimension of S/I is ≤ 1, so that $\mathrm{Tor}_2^S(-, S/I)=0$. In particular the functor $\mathrm{Tor}_1^S(-, S/I)$ is left exact. Hence if $\mathrm{Tor}_1^S(M, S/I)=0$, then $\mathrm{Tor}_1^S(N, S/I)=0$ for every submodule N of M. Therefore $\mathscr{F}$ is closed under submodules. Moreover if $N\leq M$, $\mathrm{Tor}_1^S(N, S/I)=0$ and $\mathrm{Tor}_1^S(M/N, S/I)=0$, then $\mathrm{Tor}_1^S(M, S/I)=0$, that is, $\mathscr{F}$ is closed under extensions. Finally, since I is a projective principal ideal, I is a finitely presented module, so that if $\{M_\lambda|\lambda\in\Lambda\}\subset\mathscr{F}$ is a family of S-modules, $\prod_\lambda(M_\lambda\otimes I)$ and $(\prod_\lambda M_\lambda)\otimes I$ are canonically isomorphic [13, Lemma I.13.2]. Then the mapping $(\prod_\lambda M_\lambda)\otimes I\cong\prod_\lambda(M_\lambda\otimes I)\to\prod_\lambda M_\lambda$ is injective, and $\mathscr{F}$ is closed under products.

In the statement of Proposition 5.2 the torsion class $\mathscr{T}$ consists of all right S-modules T with $\mathrm{Hom}_S(T, M)=0$ for all $M \in \mathscr{F}$. Note that S_S is an I-torsion-free module. Moreover the torsion theory $(\mathscr{T}, \mathscr{F})$ is not hereditary in general. Our torsion theory $(\mathscr{T}, \mathscr{F})$ generalizes the p-torsion theory of abelian groups, where p is a prime. In fact, it is easy to see that for $S=\mathbf{Z}$ and $I=p\mathbf{Z}$ the I-torsion-free, I-divisible and I-torsion modules are exactly the p-torsion-free, p-divisible and p-torsion abelian groups respectively.

Proposition 5.3. *Let φ be a generator of the projective principal left ideal I of S, so that the left annihilator $l(\varphi)$ of φ is equal to $S(1-\varepsilon)$ for an idempotent $\varepsilon \in S$. Then the torsion theory $(\mathscr{T}, \mathscr{F})$ is generated by the right S-module $\varepsilon S/\varphi S$.*

Proof. In order to prove that the torsion theory $(\mathscr{T}, \mathscr{F})$ is generated by $\varepsilon S/\varphi S$, we must prove that a right S-module F belongs to $\mathscr{F}$ if and only if $\mathrm{Hom}_S(\varepsilon S/\varphi S, F)=0$.

Suppose $F \in \mathscr{F}$ and fix an $f \in \mathrm{Hom}_S(\varepsilon S/\varphi S, F)$. Set $x=f(\varepsilon+\varphi S) \in F$. Then $x\varepsilon=f(\varepsilon+\varphi S)\varepsilon=f(\varepsilon+\varphi S)=x$ and $x\varphi=f(\varepsilon+\varphi S)\varphi=f(\varepsilon\varphi+\varphi S)=f(\varphi+\varphi S)=0$. Consider the element $x\otimes\varphi \in F\otimes I$. Since $x\varphi=0$ and the mapping $F\otimes I \to F$ is injective because $F\in\mathscr{F}$, it follows that $x\otimes\varphi=0$. Apply the functor $F\otimes -$ to the exact sequence $0\to S(1-\varepsilon)\to S\to I\to 0$, where the first homomorphism is the inclusion and the second homomorphism is defined by $1\mapsto\varphi$. Then the sequence $0\to F\otimes_S S(1-\varepsilon)\to F\otimes_S S\to F\otimes_S I\to 0$ is exact because I is projective, hence flat. The last sequence can be rewritten as $0\to F(1-\varepsilon)\to F\to F\otimes_S I\to 0$ where the first homomorphism is the inclusion and the second homomorphism maps x into $x\otimes\varphi$. Since $x\otimes\varphi=0$, it follows that $x\in F(1-\varepsilon)$, so that $x\varepsilon=0$. In particular $f(\varepsilon+\varphi S)= x=x\varepsilon=0$ and $f\colon \varepsilon S/\varphi S\to F$ is the zero homomorphism. This proves that $\mathrm{Hom}_S(\varepsilon S/\varphi S, F)=0$.

Conversely, suppose that $\mathrm{Hom}_S(\varepsilon S/\varphi S, F)=0$. We must prove that $F\otimes I\to F$ is injective. Since $I=S\varphi$, every element in $F\otimes I$ can be written as $x\otimes\varphi$, $x\in F$. Suppose $x\otimes\varphi$ is in the kernel of $F\otimes I\to F$, i.e., $x\varphi=0$. The mapping $f\colon \varepsilon S/\varphi S\to F$ defined by $f(\varepsilon s+\varphi S)=x\varepsilon s$ is a well defined homomorphism, because if $\varepsilon s\in\varphi S$, then $x\varepsilon s\in x\varphi S=\{0\}$. It follows that f must be zero, hence $x\varepsilon=0$. Then $x\otimes\varphi= x\otimes\varepsilon\varphi=x\varepsilon\otimes\varphi=0$. This proves that $F\in\mathscr{F}$.

Our concept of I-divisibility differs from the concept of divisibility in [13, § VI.9], because our I-torsion-free modules and I-divisible modules are both right S-modules.

Define a right S-module M to be *I-reduced* if it is cogenerated by $(S/I)^*$, that is, if it is isomorphic to a submodule of a direct product of copies of $(S/I)^*$. Here $(S/I)^*=\mathrm{Hom}_R(S/I, C)$, where R is a commutative ring such that S is an R-algebra and C is an injective cogenerator of Mod-R. Therefore M_S is I-reduced if and only if for every $x \in M$, $x \neq 0$, there exists $\vartheta_x\colon M \to (S/I)^*$ such that $\vartheta_x(x) \neq 0$. Since $\mathrm{Hom}_S(M, (S/I)^*) \cong \mathrm{Hom}_R(M \otimes_S (S/I), C) \cong \mathrm{Hom}_R(M/MI, C)$, this happens if and only if for every $x \in M$, $x \neq 0$, xS is not contained in MI. Therefore a right S-module M is I-reduced if and only if MI does not contain nonzero right S-submodules of M.

Note that a module N_S is I-divisible if and only if $\mathrm{Hom}_S(N, M) = 0$ for every I-reduced S-module M_S. In fact, $\mathrm{Hom}_S(N, M) = 0$ for every I-reduced S-module M_S if and only if $\mathrm{Hom}_S(N, (S/I)^*) = 0$. This happens if and only if $N \otimes (S/I) = 0$, that is, if and only if N is I-divisible (Lemma 5.1 (ii)).

We conclude this section with a last definition. We say that a right S-module M is an *I-cotorsion* module if it is I-reduced and $\mathrm{Ext}_S^1(N, M) = 0$ for every I-divisible I-torsion-free right S-module N. I-cotorsion modules will be studied in § 7.

6. Purity

In this section S is an arbitrary (associative) ring with identity and $I = S\varphi$ is a fixed projective principal left ideal of S. We say that a short exact sequence $0 \to M' \to M \to M'' \to 0$ of right S-modules is *I-pure* if one of the equivalent conditions of next lemma holds.

Lemma 6.1. *The following properties of a short exact sequence $0 \to M' \to M \to M'' \to 0$ of right S-modules are equivalent:*

(a) *The short exact sequence $0 \to \mathrm{Hom}_S(S/\varphi S, M') \to \mathrm{Hom}_S(S/\varphi S, M) \to \mathrm{Hom}_S(S/\varphi S, M'') \to 0$ is exact.*

(b) *The short exact sequence $0 \to M' \otimes S/S\varphi \to M \otimes S/S\varphi \to M'' \otimes S/S\varphi \to 0$ is exact.*

(c) *$M'\varphi = M' \cap M\varphi$.*

Under these equivalent conditions we shall also say that M' is an *I-pure submodule* of M. The proof of this lemma is analogous to the proof of [14, Prop. 2 and 3]. Our purity is a particular case of Warfield's $\mathscr{S}$-purity [14] with $\mathscr{S} = \{S/\varphi S, S\}$. (See also [12].) It would also be possible to apply Gruson's and Jensen's idea developed in [5] to the study of I-purity: if $\mathcal{O} = \{S, S/S\varphi\}$ is viewed as a full subcategory of S-Mod and $D(S)$ is the category of additive functors of $\mathcal{O}$ into the category of abelian groups $\mathscr{Ab}$, then the functor $M \mapsto M \otimes_S -$ of Mod-S into $D(S)$ is the left adjoint to the functor $F \mapsto F(S)$ of $D(S)$ into Mod-S and is an equivalence of Mod-S onto a full subcategory of $D(S)$; in this equivalence short exact sequences of $D(S)$ correspond to I-pure short exact sequences of Mod-S, and the injective objects in $D(S)$ correspond to the I-pure-injective S-modules. See also [2]. We shall not need this remark in the sequel.

Note that if M is an I-torsion-free S-module, that is, $M\in\mathscr{F}$, then a submodule M' of M is I-pure in M if and only if M/M' is I-torsion-free. This can be seen from the exact sequence $\mathrm{Tor}_1^S(M, S/S\varphi)\to\mathrm{Tor}_1^S(M/M', S/S\varphi)\to M'\otimes S/S\varphi\to M\otimes S/S\varphi$, where $\mathrm{Tor}_1^S(M, S/S\varphi)=0$ because $M\in\mathscr{F}$ (Lemma 5.1), so that $M'\otimes S/S\varphi\to M\otimes S/S\varphi$ is injective if and only if $\mathrm{Tor}_1^S(M/M', S/S\varphi)=0$.

The theory developed in [12] applies to our notion of I-purity. If $\mathscr{E}$ is the class of I-pure short exact sequences of S-modules, then $\mathscr{E}$ is a *flatly generated, proper* class [12, § 3], closed under direct limits and *projectively closed* [12, Prop. 3.1 and 2.2]. For every right S-module M'' there is an I-pure exact sequence $0\to M'\to M\to M''\to 0$ with M I-pure-projective (i.e., M $\mathscr{E}$-projective). Moreover a module M is I-pure-projective if and only if it is isomorphic to a direct summand of a direct sum of copies of S_S and $S/\varphi S$. These statements follow immediately from [12, Prop. 2.3]. I-pure-injective modules (that is, $\mathscr{E}$-injectives) are characterized as the direct summands of direct products of copies of $\mathrm{Hom}_R(S, C)$ and $\mathrm{Hom}_R(S/S\varphi, C)$; here R is any commutative ring such that S is an R-algebra, and C is an injective cogenerator in Mod-R [12, Prop. 3.3]. Finally, every module has a suitably defined I-pure-injective envelope [12, Prop. 4.5], and I-pure-injective modules are directs summands of every module which contains them as I-pure submodules.

7. The equivalences

Now we apply the theory developed in §§ 5 and 6 to the study of the functors $\mathrm{Hom}_R({}_E\partial_R, -)$: Mod-$R\to$Mod-$E$ and $-\otimes_E\partial_R$: Mod-$E\to$Mod-R introduced in § 4.

As in the first four sections R is an integral domain, ∂_R is the R-module of § 2, E is its endomorphism ring $\mathrm{End}(\partial_R)$, φ is an endomorphism of ∂_R whose kernel is wR and image is a direct summand of ∂_R. The left ideal $I=E\varphi$ of E is a projective principal ideal by Theorem 2.4, so that the theory developed in § 5 can be applied. Let C be the minimal injective cogenerator in Mod-R and $\partial^*=\mathrm{Hom}_R(\partial, C)$. There is a torsion theory $(\mathscr{T}, \mathscr{F})$ for Mod-E where the I-torsion-free class $\mathscr{F}$ consists of the right E-modules M with $\mathrm{Tor}_1^E(M, \partial)=0$, or, equivalently, with $\mathrm{Ext}_E^1(M, \partial^*)=0$ (Lemmas 3.1 and 5.1). The class of I-divisible E-modules consists of the right E-modules M with $M\otimes_E\partial=0$. The torsion theory $(\mathscr{T}, \mathscr{F})$ is generated by the right E-module $\partial^\circ=\mathrm{Ext}_R^1(\partial, R)$ (Proposition 5.3 and Theorem 3.4) and E_E is a torsion-free E-module in the torsion theory $(\mathscr{T}, \mathscr{F})$.

The I-reduced E-modules are the right E-modules cogenerated by ∂^*; and a module M_E is I-reduced if and only if MI does not contain nonzero right E-submodules of M.

Theorem 7.1. *Let R be an integral domain and A a right R-module. Then* $\mathrm{Hom}_R(\partial, A)$ *is an I-cotorsion E-module.*

Proof. Since C is an injective cogenerator in Mod-R, $A \leqq C^X$ for some set X, so that $\mathrm{Hom}_R(\partial, A) \leqq \mathrm{Hom}_R(\partial, C^X) \cong (\partial^*)^X$; hence $\mathrm{Hom}_R(\partial, A)$ is cogenerated by ∂^*, that is, it is I-reduced.

Now let N_E be an I-divisible I-torsion-free E-module and let D be an injective R-module containing A. Then the functor $\mathrm{Hom}_R(\partial, -)$ applied to the exact sequence $0 \to A \to D \to D/A \to 0$ gives an exact sequence $0 \to \mathrm{Hom}_R(\partial, A) \to \mathrm{Hom}_R(\partial, D) \to P \to 0$ for a suitable E-submodule P of $\mathrm{Hom}_R(\partial, D/A)$. Apply the functor $\mathrm{Hom}_E(N, -)$ to this sequence and obtain the exact sequence $\mathrm{Hom}_E(N, P) \to \mathrm{Ext}^1_E(N, \mathrm{Hom}_R(\partial, A)) \to \mathrm{Ext}^1_E(N, \mathrm{Hom}_R(\partial, D))$. But

$$\mathrm{Hom}_E(N, P) \leqq \mathrm{Hom}_E(N, \mathrm{Hom}_R(\partial, D/A)) \cong \mathrm{Hom}_R(N \otimes_E \partial, D/A) = 0$$

because $N \otimes_E \partial = 0$ since N is I-divisible. Moreover $\mathrm{Tor}^E_1(N, \partial) = 0$ (because N is I-torsion-free) and D is injective, and thus

$$\mathrm{Ext}^1_E(N, \mathrm{Hom}_R(\partial, D)) \cong \mathrm{Hom}_R(\mathrm{Tor}^E_1(N, \partial), D) = 0.$$

Therefore $\mathrm{Ext}^1_E(N, \mathrm{Hom}_R(\partial, A)) = 0$ and $\mathrm{Hom}_R(\partial, A)$ is I-cotorsion.

Note that $E/\varphi E \cong ((1-\varepsilon)E \oplus \varepsilon E)/\varphi E \cong (1-\varepsilon)E \oplus (\varepsilon E/\varphi E) \cong (1-\varepsilon)E \oplus \partial^\circ$ (Theorem 3.4), so that $E/\varphi E$ is projective relatively to an exact sequence of right E-modules if and only if ∂° is projective relatively to that exact sequence. It follows that an exact sequence $0 \to M' \to M \to M'' \to 0$ of right E-modules is I-pure, that is, $M'I = M' \cap MI$, if and only if $0 \to M' \otimes_E \partial \to M \otimes_E \partial \to M'' \otimes_E \partial \to 0$ is exact, if and only if $0 \to \mathrm{Hom}_E(\partial^\circ, M') \to \mathrm{Hom}_E(\partial^\circ, M) \to \mathrm{Hom}_E(\partial^\circ, M'') \to 0$ is exact. Moreover, if C is the minimal injective cogenerator in Mod-R and ∂^* is the right E-module $\mathrm{Hom}_R(\partial, C)$ then $0 \to M' \to M \to M'' \to 0$ is I-pure if and only if $0 \to \mathrm{Hom}_E(M'', \partial^*) \to \mathrm{Hom}_E(M, \partial^*) \to \mathrm{Hom}_E(M', \partial^*) \to 0$ is exact.

By the general theory developed in § 6, the I-pure-projective E-modules are exactly the direct summands of direct sums of copies of E_E and ∂°, and the I-pure-injective E-modules are exactly the direct summands of direct products of copies of $\mathrm{Hom}_R(E, C)$ and $\mathrm{Hom}_R(\partial, C) = \partial^*$.

Theorem 7.2. *Let M be a right E-module and let* $\eta_M \colon M \to \mathrm{Hom}_R(\partial, M \otimes_E \partial)$ *be the canonical homomorphism. Then:*

(a) $\ker \eta_M$ *is the largest E-submodule of M contained in MI.*

(b) *The image of η_M is an I-pure submodule of* $\mathrm{Hom}_R(\partial, M \otimes_E \partial)$.

(c) $\mathrm{coker}\, \eta_M$ *is an I-torsion-free I-divisible E-module.*

Proof. (a) Since $\partial \cong E/I$, the R-module $M \otimes_E \partial$ is isomorphic to M/MI, so that $x \in M$ is in the kernel of η_M if and only if $xe \in MI$ for every $e \in E$, that is, if and only if $xE \subset MI$. In particular $\ker \eta_M$ is an E-submodule of M contained in MI. And if N is any E-submodule of M contained in MI, then $xE \subset MI$ for every $x \in N$, that is, $x \in \ker \eta_M$ for every $x \in N$. This proves that $N \subset \ker \eta_M$.

(b) By Theorem 2.4 $\mathrm{Hom}_R(\partial, M \otimes_E \partial)I = \{f \in \mathrm{Hom}_R(\partial, M \otimes_E \partial) \mid f(w)=0\}$. Therefore $\eta_M(M) \cap \mathrm{Hom}_R(\partial, M \otimes_E \partial)I = \{\eta_M(x) \mid x \in M,\ \eta_M(x)(w)=0\} = \{\eta_M(x) \mid x \in M,\ x \otimes w$ is the zero element of $M \otimes \partial\}$. Since the homomorphism $\partial \to E/I$, $w \mapsto 1+I$ is an isomorphism of E-modules (Lemma 3.1), it follows that $M \otimes \partial \cong M \otimes E/I \cong M/MI$, and $x \otimes w = 0$ if and only if $x \in MI$. Hence $\eta_M(M) \cap \mathrm{Hom}_R(\partial, M \otimes_E \partial)I = \{\eta_M(x) \mid x \in MI\} = \eta_M(MI) = \eta_M(M)I$.

(c) Suppose that η_M is injective (by Part (a) this happens if and only if M is I-reduced). Under this hypothesis consider the exact sequence

$$0 \to M \to \mathrm{Hom}_R(\partial, M \otimes_E \partial) \to \operatorname{coker} \eta_M \to 0.$$

This sequence is I-pure by Part (b) and $\mathrm{Hom}_R(\partial, M \otimes_E \partial)$ is I-torsion-free by Proposition 4.2. Therefore $\operatorname{coker} \eta_M$ is I-torsion-free.

Now apply the functor $- \otimes_E \partial$ to the above I-pure exact sequence and obtain the exact sequence $0 \to M \otimes \partial \to \mathrm{Hom}_R(\partial, M \otimes_E \partial) \otimes_E \partial \to \operatorname{coker} \eta_M \otimes_E \partial \to 0$. The homomorphism $\eta_M \otimes \partial \colon M \otimes \partial \to \mathrm{Hom}_R(\partial, M \otimes_E \partial) \otimes_E \partial$ is equal to $\varepsilon^{-1}_{M \otimes \partial}$ (where ε is the counit of the adjunction and $\varepsilon_{M \otimes \partial}$ is an isomorphism by Theorem 4.1) because if $x \in M$ and $y \in \partial$ then $\eta_M \otimes \partial(x \otimes y) = f_x \otimes y$, where $f_x \in \mathrm{Hom}_R(\partial, M \otimes_E \partial)$ and $f_x(z) = x \otimes z$ for every $z \in \partial$. Therefore $\varepsilon_{M \otimes \partial}(\eta_M \otimes \partial(x \otimes y)) = \varepsilon_{M \otimes \partial}(f_x \otimes y) = f_x(y) = x \otimes y$, i.e., $\eta_M \otimes \partial(x \otimes y) = \varepsilon^{-1}_{M \otimes \partial}(x \otimes y)$ and $\eta_M \otimes \partial = \varepsilon^{-1}_{M \otimes \partial}$. Hence $\eta_M \otimes \partial$ is an isomorphism, and the exactness of the above sequence gives $(\operatorname{coker} \eta_M) \otimes_E \partial = 0$, i.e., $\operatorname{coker} \eta_M$ is I-divisible.

This proves Part (c) under the additional hypothesis that η_M is injective. In the general case the naturality of η applied to the canonical projection $\pi \colon M \to M/\ker \eta_M$ gives the equality $\eta_{M/\ker\eta} \cdot \pi = \mathrm{Hom}(\partial, \pi \otimes \partial) \cdot \eta_M$. But $\pi \otimes \partial \colon M \otimes \partial \to (M/\ker \eta_M) \otimes \partial$ is an isomorphism because

$$(M/\ker \eta_M) \otimes \partial \cong (M/\ker \eta_M) \otimes (E/I) \cong (M/\ker \eta_M)/(M/\ker \eta_M)I$$

$$\cong M/(\ker \eta_M + MI) \cong M/MI \cong M \otimes (E/I) \cong M \otimes \partial.$$

Therefore $\mathrm{Hom}(\partial, \pi \otimes \partial)$ is an isomorphism and

$$\operatorname{coker} \eta_M \cong \operatorname{coker}(\mathrm{Hom}(\partial, \pi \otimes \partial) \cdot \eta_M) = \operatorname{coker}(\eta_{M/\ker\eta} \cdot \pi) = \operatorname{coker} \eta_{M/\ker\eta}.$$

Now $M/\ker \eta$ is I-reduced by Part (a), so that $\operatorname{coker} \eta_M \cong \operatorname{coker} \eta_{M/\ker\eta}$ is I-torsion-free and I-divisible by the previous case.

As a corollary to Theorem 7.2 it must be noted that every I-reduced E-module is I-torsion-free. This holds because if M_E is I-reduced, then η_M is injective (Theorem 7.2(a)) and $\mathrm{Hom}_R(\partial, M\otimes\partial)$ is I-torsion-free (Proposition 4.2), so that M is I-torsion-free too. Nevertheless this fact does not hold for an arbitrary ring S (take $S=\mathbf{Z}$, $I=2\mathbf{Z}$ and M any abelian group with $2M=0$, so that M is I-reduced and is not I-torsion-free).

Theorem 7.3. *Let M be a right E-module. Then $\eta_M\colon M\to\mathrm{Hom}_R(\partial, M\otimes_E\partial)$ is an isomorphism if and only if M is I-cotorsion.*

Proof. If $M\cong\mathrm{Hom}_R(\partial, M\otimes\partial)$, M is I-cotorsion by Theorem 7.1. Conversely, if M is I-cotorsion, the homomorphism η_M is injective by Theorem 7.2(a) and the exact sequence $0\to M\to\mathrm{Hom}_R(\partial, M\otimes_E\partial)\to\operatorname{coker}\eta_M\to 0$ splits because

$$\mathrm{Ext}_E^1(\operatorname{coker}\eta_M, M) = 0$$

($\operatorname{coker}\eta_M$ is I-torsion-free and I-divisible by Theorem 7.2(c)). Hence $\operatorname{coker}\eta_M$ is isomorphic to a submodule of $\mathrm{Hom}_R(\partial, M\otimes_E\partial)$. But $\operatorname{coker}\eta_M$ is I-divisible, and $\mathrm{Hom}_R(\partial, M\otimes_E\partial)$ is I-reduced. Therefore $\operatorname{coker}\eta_M=0$ and η_M is an isomorphism.

Theorem 7.3 has the following corollary: if M is any right E-module, every E-homomorphism from M into an I-cotorsion module N_E can be uniquely factored over $\eta_M\colon M\to\mathrm{Hom}_R(\partial, M\otimes_E\partial)$. Hence $\mathrm{Hom}_R(\partial, M\otimes_E\partial)$ is a sort of "I-cotorsion completion" of M. The factorization of $f\colon M\to N$ is $f=(\eta_N^{-1}\cdot\mathrm{Hom}_R(\partial, f\otimes\partial))\cdot\eta_M$ (this equality is given by the naturality of the transformation η). The uniqueness of the factorization is proved as follows: if $f=f_1\cdot\eta_M=f_2\cdot\eta_M$, then $(f_1-f_2)\cdot\eta_M=0$, so that $f_1-f_2\colon\mathrm{Hom}_R(\partial, M\otimes_E\partial)\to N$ induces a mapping $\operatorname{coker}\eta_M\to N$. But $\operatorname{coker}\eta_M$ is I-divisible (Theorem 7.2(c)) and N is I-reduced, so that this mapping is zero. Hence $f_1-f_2=0$. This proves the corollary.

It must be remarked that our "I-cotorsion completion" $\mathrm{Hom}_R(\partial, -\otimes_E\partial)$ is substantially different from the cotorsion hull in a hereditary torsion theory developed in [1], since our torsion theory $(\mathscr{T}, \mathscr{F})$ is not hereditary.

Theorem 7.4. *If R is an integral domain and $E=\mathrm{End}(\partial_R)$, the functors $\mathrm{Hom}_R(\partial, -)\colon\mathscr{D}_R\to\mathscr{C}_E$ and $-\otimes_E\partial\colon\mathscr{C}_E\to\mathscr{D}_R$ give an equivalence between the full subcategory $\mathscr{D}_R$ of divisible R-modules and the full subcategory $\mathscr{C}_E$ of* Mod-*E whose objects are the I-cotorsion E-modules. In this equivalence injective R-modules correspond to I-reduced I-pure-injective E-modules.*

Proof. By Theorems 4.1 and 7.3 $\mathrm{Hom}_R(\partial, -)$ and $-\otimes_E\partial$ give an equivalence between the categories $\mathscr{D}_R$ and $\mathscr{C}_E$. Let us prove that if B_R is an injective right R-module then $\mathrm{Hom}_R(\partial, B)$ is an I-pure-injective E-module. If B_R is injective, then B is isomorphic to a direct summand of C^X, where C is a minimal injective cogenerator in Mod-R. Then $\mathrm{Hom}_R(\partial, B)$ is isomorphic to a direct summand in $\mathrm{Hom}_R(\partial, C^X)\cong$

$\mathrm{Hom}_R(\partial, C)^X = \partial^{*X}$. By the remark immediately above Theorem 7.2, $\mathrm{Hom}_R(\partial, B)$ is an I-pure-injective E-module.

Conversely, if M_E is an I-reduced, I-pure-injective E-module, then $\eta_M\colon M \to \mathrm{Hom}_R(\partial, M \otimes_E \partial)$ is an I-pure monomorphism (Theorem 7.2). Let D be an injective R-module containing $M \otimes \partial$, so that $\mathrm{Hom}_R(\partial, M \otimes \partial) \leqq \mathrm{Hom}_R(\partial, D)$. The submodule $\mathrm{Hom}_R(\partial, M \otimes \partial)$ is I-pure in $\mathrm{Hom}_R(\partial, D)$, because $\mathrm{Hom}_R(\partial, D)I = \{f \in \mathrm{Hom}_R(\partial, D) \mid f(w) = 0\}$ by Theorem 2.4, so that $\mathrm{Hom}_R(\partial, D)I \cap \mathrm{Hom}_R(\partial, M \otimes \partial) = \{f \in \mathrm{Hom}_R(\partial, M \otimes \partial) \mid f(w) = 0\} = \mathrm{Hom}_R(\partial, M \otimes \partial)I$ by Theorem 2.4 again. Therefore M is isomorphic to an I-pure submodule of $\mathrm{Hom}_R(\partial, D)$. Since M is I-pure-injective, M is isomorphic to a direct summand of $\mathrm{Hom}_R(\partial, D)$. Then $M \otimes \partial$ is isomorphic to a direct summand of $\mathrm{Hom}_R(\partial, D) \otimes \partial \cong D$. This proves that $M \otimes \partial$ is an injective R-module.

Thus we have seen that the class we had denoted by $\mathcal{I}$ in Theorem 4.3, i.e., the image of the functor $\mathrm{Hom}_R(\partial, -)\colon \text{Mod-}R \to \text{Mod-}E$, is exactly the class $\mathcal{C}_E$ of I-cotorsion E-modules. There is a further characterization of these modules: they are exactly the right E-modules of ∂^*-dominant dimension $\geqq 2$, that is, the right E-modules M for which there exists an exact sequence $0 \to M \to \partial^{*X} \to \partial^{*Y}$ for suitable direct powers ∂^{*X} and ∂^{*Y} of the E-module ∂^*. In order to see this, note that if M is an I-cotorsion E-module, then there is an exact sequence of R-modules $0 \to M \otimes_E \partial \to C^X \to C^Y$ because C is an injective cogenerator in Mod-R, so that by applying the left exact functor $\mathrm{Hom}_R(\partial, -)$ to this sequence one obtains an exact sequence $0 \to M \cong \mathrm{Hom}_R(\partial, M \otimes \partial) \to \partial^{*X} \to \partial^{*Y}$. Conversely, if M has ∂^*-dominant dimension $\geqq 2$, from the exact sequence $0 \to M \to \partial^{*X} \to \partial^{*Y}$ we obtain that M is cogenerated by ∂^* (i.e., it is I-reduced) and that there is an exact sequence $0 \to M \to \partial^{*X} \to N \to 0$ with $N \leqq \partial^{*Y}$. If F is any I-divisible I-torsion-free E-module then the sequence $\mathrm{Hom}_E(F, N) \to \mathrm{Ext}^1_E(F, M) \to \mathrm{Ext}^1_E(F, \partial^{*X})$ is exact, $\mathrm{Hom}_E(F, N) = 0$ (because F is I-divisible and N is I-reduced), and $\mathrm{Ext}^1_E(F, \partial^{*X}) = 0$ (because $\partial^{*X} \cong \mathrm{Hom}_R(\partial, C^X)$ is in $\mathcal{I}$, i.e., it is I-cotorsion). Therefore $\mathrm{Ext}^1_E(F, M) = 0$ and M is I-cotorsion.

References

1. Bueso, J. L. and Torrecillas, B., On the cotorsion hull in torsion theory, *Comm. Algebra* **13** (1985), 1627—1642.
2. Facchini, A., Relative injectivity and pure-injective modules over Prüfer rings, *J. Algebra* **110** (1987), 380—406.
3. Fuchs, L., On divisible modules over domains, in: *Abelian groups and modules,* Proc. of the Udine Conference, CISM Courses and Lectures 287, Springer-Verlag, Wien—New York, 1984. 341—356.
4. Fuchs, L. and Salce, L., *Modules over valuation domains,* Lecture Notes in Pure and Applied Mathematics **96**, Marcel Dekker, New York—Basel, 1985.

5. GRUSON, L. and JENSEN, C. U., Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, *Séminaire d'Algèbre P. Dubreil et M.-P. Malliavin,* Lecture Notes in Mathematics **867,** Springer-Verlag, Berlin—Heidelberg—New York, 1981, 234—294.
6. HARRISON, D., Infinite abelian groups and homological methods, *Ann. of Math.* **69** (1959), 366—391.
7. MATLIS, E., Cotorsion modules, *Mem. Amer. Math. Soc.* **49** (1964).
8. MATLIS, E., *Torsion-free modules,* Chicago Lectures in Mathematics, Univ. of Chicago Press, Chicago—London, 1972.
9. MATLIS, E., *1-dimensional Cohen—Macaulay rings,* Lecture Notes in Mathematics **327,** Springer-Verlag, Berlin—Heidelberg—New York, 1973.
10. PIERCE, R. S., *Associative algebras,* Graduate Texts in Mathematics **88,** Springer-Verlag, New York—Heidelberg—Berlin, 1982.
11. POPESCU, N., *Abelian categories with applications to rings and modules,* Academic Press, London—New York, 1973.
12. STENSTRÖM, B., Pure submodules, *Ark. Mat.* **7** (1967), 159—171.
13. STENSTRÖM, B., *Rings of quotients,* Springer-Verlag, Berlin—Heidelberg—New York, 1975.
14. WARFIELD, R. B., Purity and algebraic compactness for modules, *Pacific J. Math.* **28** (1969), 699—719.

Received August 21, 1986

A. Facchini
Universita' di Udine
Istituto di Matematica
Informatica e Sistemistica
33100 Udine
Italy