DIVISION IN DOUGLAS ALGEBRAS

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1. Introduction. The purpose here is to get some understanding of how Douglas algebras fit into L^∞. I use the following notation and terminology: H^∞ denotes the space of boundary functions for bounded holomorphic functions in the open unit disk D. Then $H^\infty \subseteq L^\infty(d\lambda)$ where λ is Lebesgue measure on ∂D. L^∞ is given the essential supremum norm $\|\cdot\|_\infty$. The term Douglas algebra refers to a closed subalgebra of L^∞ that contains H^∞. The smallest such algebra, next to H^∞ itself, is $H^\infty + C$, where C denotes the continuous functions on ∂D. Let A stand for the generic Douglas algebra and consider the following question: Given $\varphi \in A$ and $f \in L^\infty$, how can we tell if $\varphi f \in A$? If φ and f both have constant modulus and $\bar{f} \in A$, this is the situation studied by Guillory and Sarason in [4] for $A = H^\infty + C$. Here I consider the more general Douglas algebra *and* the more general $f \in L^\infty$. The sufficient condition obtained depends only on the values of the Poisson extensions of φ and f to the open unit disk and seems to be new even in the case $A = H^\infty + C$.

The techniques used here are quite different from those used in [4] and developed out of a somewhat heretical attempt on my part (some would say misguided) to rid the [4] results of their dependence on the Carleson–Ziskind–Marshall construction of Carleson measures (see [1] and [7]). Whether or not the attempt was misguided, I believe the result is a new and useful technique that yields a somewhat stronger result in this case. The technique developed from a result in [6] originally obtained for application to Bergman spaces. The link needed to apply it to the present situation is a refinement of a result of Ken Hoffman [5] on factoring Blaschke products.

2. A special case. The main result of [4] is the following.

THEOREM A. *If* $\varphi, \psi \in H^\infty + C$ *and* $|\psi| = 1$ a.e. *then a necessary and sufficient condition that* $\bar{\psi}^n \varphi \in H^\infty + C$ *for all* n *is that* $\lim_{|z|\to 1}[\varphi(z)(1 - |\psi(z)|)] = 0$ *where* $\varphi(z)$, $\psi(z)$ *denote the harmonic extensions of* φ, ψ *to* D.

The generalization of this to be proved here is the following.

THEOREM 1. *Let* A *be a Douglas algebra and let* $\varphi \in A$ *and* $f \in L^\infty$ *with* $|f| = 1$ a.e. *Suppose for every* $\epsilon > 0$ *there is a Blaschke product* b_ϵ *which is invertible in* A *such that* $|\varphi(z)|(1 - |f(z)|) < \epsilon$ *on the set* $\{z \in D : |b_\epsilon(z)| > 1 - \epsilon\}$. *Then* $f\varphi \in A$.

In the case $A = H^\infty + C$ the b_ϵ's may be taken to be z^n and then the condition in Theorem 1 reduces to that in Theorem A.

A similar result, with A replaced by QC was proved by T. Wolff [10, embedded in Lemma II.2] using quite different methods.

In order to get an idea of the techniques needed for Theorem 1, I will sketch here an outline of a proof of Theorem A different from that used by Guillory and Sarason. I will make two simplifying assumptions; first, that $\varphi, \psi \in H^\infty$ and second, that

Received June 5, 1981. Revision received August 7, 1981.
Michigan Math. J. 29 (1982).

$\varphi = \varphi_1 \varphi_2$ where $\|\varphi_i\|_\infty \leqslant C\|\varphi\|_\infty$ and φ_i satisfies the same condition

$$\lim_{|z| \to 1} \varphi_i(z)(1 - |\psi(z)|) = 0, \qquad i = 1, 2.$$

The first assumption is unimportant, the second is crucial and it is its elimination that gives rise to Theorem 1. The symbol K in what follows denotes a constant which may differ from one occurrence to the next.

Using the well-known duality between L^∞/H^∞ and $H^1_0 = \{f \in H^1 : \int f\,d\lambda = 0\}$ it suffices in one direction to prove that $\lim_{k\to\infty} \sup_h |\int \bar{\psi}^n \varphi z^k h\,d\lambda| = 0$, the sup being taken over all $h \in H^1$ with L^1-norm equal to 1. (Here use is made of the fact that $\bigcup_n \bar{z}^n H^\infty$ is dense in $H^\infty + C$.) If $|\varphi(z)|(1 - |\psi(z)|)$ tends to 0 as $|z|$ tends to 1 the same is true with ψ replaced by ψ^n. Thus it will be enough to do the case $n = 1$. Now

$$\int \bar{\psi}\varphi z^k h\,d\lambda = 2 \iint_{|z|<1} \operatorname{grad} \bar{\psi} \cdot \operatorname{grad}(\varphi z^k h) \log \frac{1}{|z|}\,dm$$

where m is 2-dimensional Lebesgue measure and $\operatorname{grad} = (\partial/\partial x, \partial/\partial y)$. Computing the gradients we need to estimate the size of

$$I = 4 \iint_{|z|<1} \overline{\psi'(z)}(\varphi(z) z^k h(z))' \log \frac{1}{|z|}\,dm.$$

Now write $\varphi = \varphi_1 \varphi_2$ as assumed and $h = h_1 h_2$ where $h_i \in H^2$ $\|h_i\|^2_{L^2(\lambda)} = \|h\|_{L^1(\lambda)}$, $i = 1, 2$. For convenience let $k = 2n$, an even integer. Then $I = I_1 + I_2$ where

$$I_1 = 4 \iint_{|z|<1} \bar{\psi}'(\varphi_1 z^n h_1)' \varphi_2 z^n h_2 \log \frac{1}{|z|}\,dm$$

and I_2 is the same with 1 and 2 interchanged. By the Cauchy-Schwartz inequality

$$\left|\frac{I_1}{4}\right|^2 \leqslant \iint_{|z|<1} |\psi'|^2 |\varphi_2|^2 |z^{2n} h_2^2| \log \frac{1}{|z|}\,dm$$
$$\cdot \iint_{|z|<1} |(\varphi_1 z^n h_1)'|^2 \log \frac{1}{|z|}\,dm = I_3 \cdot I_4$$

so $I_4 = \|\varphi_1 z^n h_1\|^2_{L^2(\lambda)} \leqslant \|\varphi_1\|^2_\infty \|h\|_{L^1(\lambda)}$ by the Littlewood-Paley identity. Given $\epsilon > 0$, $I_3 \leqslant \iint_A + \iint_B + \iint_C$ where $A = \{z \in D : |z| < r\}$, $B = \{z \in D \backslash A : 1 - |\psi| < \epsilon\}$ and $C = \{z \in D \backslash A : |\varphi_2| < \epsilon\}$ for some choice of $r < 1$. Standard estimates show $\iint_A \leqslant K r^{2n} \|h_2\|^2_{L^2(\lambda)}$ where K depends only on r and the norms $\|\varphi_2\|_\infty$, $\|\psi\|_\infty$. Also,

$$\iint_C \leqslant \epsilon^2 \iint |\psi'|^2 |z^{2n} h_2^2| \log \frac{1}{|z|}\,dm \leqslant \epsilon^2 K \|\psi\|_\infty \|h_2\|^2_{L^2(\lambda)}$$

by a result originally due to Fefferman and Stein [3] in the upper half-plane. A proof in the case of the disk can be found in [9, Theorems on p. 39 and p. 5]. See also [2]. Finally, a result of Chang's [2, Lemma 5] gives $\iint_B \leqslant K\epsilon \|\psi\|_\infty \|\varphi_2\|_\infty \|h_2\|^2_{L^2(\lambda)}$.

Consequently $|I_1| < K(\epsilon+\epsilon^2+r^{2n})^{1/2}\|h\|_{L^1(\lambda)}$ and $|I_2|$ can be similarly estimated. Since $\epsilon>0$ is arbitrary and n may be chosen so r^n is less than ϵ we see that $\operatorname{dist}(\bar{\psi}\varphi, H^\infty + C) = 0$ as desired.

For the converse no special assumptions are needed; the reader is referred to [4].

The important feature of this proof is that throughout the argument, the fact that $\psi \in H^\infty$ is irrelevant. Thus, under the special assumption $\varphi = \varphi_1\varphi_2$, it actually proves Theorem 1 in the case $A = H^\infty + C$. (Altering the proof to handle an arbitrary $f \in L^\infty$ in place of $\bar{\psi}$ amounts to replacing $\bar{\psi}'$ by $\partial f/\partial \bar{z}$.) In the next section the factorization assumption $\varphi = \varphi_1\varphi_2$ is examined.

3. Factoring Blaschke products. In [4], Hoffman obtains the following result (Theorem 5.2):

THEOREM B. *Let $\sigma > 0$. There exists constants a, b > 0 such that if C is a Blaschke product with zero sequence Γ satisfying $|\gamma| > \frac{1}{2}$ all $\gamma \in \Gamma$ and if $K_\sigma = \{z \in D$: $\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| > \sigma$ all $\gamma \in \Gamma\}$, then there exist Blaschke products A and B with $C = AB$ such that $a|B(z)|^{1/b} \leqslant |A(z)| \leqslant (1/a)|B(z)|^b$ for $z \in K_\sigma$.*

Hoffman actually gets constants depending on r_0 as well as σ, when $|\gamma| > r_0 > 0$ all $\gamma \in \Gamma$. It is not necessary here to consider any r_0 other than $\frac{1}{2}$. It is easily seen that $|B(z)| \leqslant |(1/a)C(z)|^{1/(b+1)}$ if $z \in K_\sigma$ with a similar inequality for $A(z)$. It is of course impossible to extend these inequalities over the whole disk (after all, C may have zeros that A and B do not both have) but if we imagine it could be done then we could obtain the $\varphi = \varphi_1\varphi_2$ factorization of the previous section: Simply write $\varphi = C \cdot F$ where C is a Blaschke product and F has no zeros. Then let $\varphi_1 = AF^{1/2}$, $\varphi_2 = BF^{1/2}$. This argument can actually be saved because it is not really necessary that A and B be dominated *everywhere* by a power of C but merely on a large enough set to ensure the validity of inequalities analogous to those on integrals in Section II. If K_σ were large enough our troubles would be over and in fact this is the case precisely when $\inf\left|\frac{\gamma-\gamma'}{1-\bar{\gamma}\gamma'}\right| > 0$, where the infimum is over the distinct pairs of zeros of C. The general case is handled by the following.

THEOREM 2. *There exist positive constants δ, α and β such that if C is a Blaschke product and $\epsilon > 0$ there exist sets G_1 and G_2 and a factorization $C = C_1C_2$ satisfying, for $i = 1, 2$:*

(i) *$m(G_i \cap \Delta) > \delta m(\Delta \cap D)$ for all disks Δ whose centers lie on ∂D, and*

(ii) *$|C_i(z)| < \alpha\epsilon^\beta$ for all $z \in G_i \cap \{z : |C(z)| < \epsilon\}$.*

Condition (ii) is the domination needed, and condition (i) gives the proper meaning for "a large enough set."

Before beginning the proof let us establish some notation: $\rho(z,\omega) = \left|\frac{z-\omega}{1-\bar{\omega}z}\right|$ is the pseudo-hyperbolic "distance" function on D. It satisfies $\rho(z,\omega) = \rho(M(z), M(\omega))$ for any Möbius transformation M from D onto D. Let $\mathcal{S} = \{S\}$ be the partition of D into what will be called "dyadic squares" i.e. all sets S of the form

$$S = S(j,k) = \{z = re^{i\theta} : 2^{-j} < 1-r \leqslant 2^{-j+1},\ k2^{-j-1}\pi \leqslant \theta < (k+1)2^{-j-1}\pi\}$$
$$j = 1,2,3,\ldots, \qquad k = 0,1,\ldots 2^{j+2}.$$

Some key properties of such a partition: Except for the eight squares containing 0 they are disjoint, their union is D, and there are constants K_1 and K_2 such that $0 < K_1 < \sup\{\rho(z,\omega) : z,\omega \in S\} < K_2 < 1$ with K_i independent of $S \in \mathcal{S}$.

Proof of Theorem 2. Suppose z, ω_1 and ω_2 are three points in D with $\rho(z,\omega_1) > a$ $i = 1,2$ and $\rho(\omega_1,\omega_2) < b < 1$. Then there is a constant c depending only on a and b such that $\rho(z,\omega_1) < \rho(z,\omega_2)^c$. This is really quite obvious if $z = 0$ and the Möbius invariance of ρ easily reduces it to that case. This observation implies the following: For $S \in \mathcal{S}$, let z_S be its center, i.e. $z_S = (1-3\cdot 2^{-j-1})\exp(i(k+\frac{1}{2})2^{-j-1})$ if $S = S(j,k)$. Let $\Delta(S) = \{z : \rho(z_S, z) < \eta\}$ where η is chosen so small that $\operatorname{dist}_\rho(\Delta(S), D-S) > \frac{1}{2}K_1$. Then if $\gamma_1,\ldots,\gamma_n,\mu_1,\ldots,\mu_n$ all belong to one dyadic square other than S we have

$$\prod_{i=1}^n \left| -\frac{z-\gamma_i}{1-\bar{\gamma}_i z}\right| < \prod_{i=1}^n \left|\frac{z-\mu_i}{1-\bar{\mu}_i z}\right|^c, \qquad z \in \Delta(S)$$

where c depends only on K_1 and K_2.

Thus if S contains no zeros of the Blaschke product C and if all other dyadic squares contain an even number of zeros then $G_i \cap S$ could be defined to be $\Delta(S)$ and any factorization of C which divided the zeros in a dyadic square equally would satisfy (ii) within $\Delta(S)$, with $\alpha = 1$, $b = c/(1+c)$. Moreover, even if S contains zeros of C but $\Pi_{\gamma \in \Gamma\setminus S}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| < \epsilon$ on $\Delta(S)$, the same choice of $G_i \cap S$, α and β will satisfy (ii). Finally, if S contains zeros of C and $\Pi_{\gamma\notin S}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| > \epsilon$ somewhere on $\Delta(S)$ but $\Pi_{\gamma\notin S}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| < \epsilon^{1/2}$ on at least $\frac{1}{4}$ of the area of $\Delta(S)$, then $G_i \cap S$ can be taken to be $\{z \in \Delta(S) : \Pi_{\gamma\notin S}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| < \epsilon^{1/2}\}$. Then $\alpha = 1$, $\beta = c/(2+2c)$ will satisfy (ii) in S with, again, any factorization of C that divides the zeros in any one square equally. Thus, to define $G_i \cap S$ it may be assumed that $\Pi_{\gamma\notin S}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| > \epsilon^{1/2}$ on at least $\frac{3}{4}$ of the area of $\Delta(S)$ provided it is assumed that the number of zeros of C in each dyadic square is even. (This assumption will be in force for all but the last two paragraphs of this proof.) It may also be assumed that $|C(z)| < \epsilon$ on at least $\frac{3}{4}$ of the area of $\Delta(S)$. (Otherwise one could take $G_i \cap S$ to be $\Delta(S) \cap \{z : |C(z)| > \epsilon\}$ and then (ii) is vacuously satisfied in S.) With these reductions we can assume $\Pi_{\gamma\in S}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right| < \epsilon^{1/2}$ on at least half of $\Delta(S)$. The problem has been reduced to a factorization of a finite Blaschke product. (Note that at each stage of the reduction $m(G_1 \cap S) > \frac{1}{4}m(\Delta(S)) > (K/4)m(S)$. If this property is established for every $S \in \mathcal{S}$ then (i) will also follow.)

We are now required to find a factorization of $\prod_{\gamma\in S}\frac{z-\gamma}{1-\bar{\gamma}z}$ which divides the zeros equally into two sets N_1 and N_2 such that $\Pi_1(z)\equiv\prod_{\gamma\in N_1}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right|$ and $\Pi_2(z)\equiv\prod_{\gamma\in N_2}\left|\frac{z-\gamma}{1-\bar{\gamma}z}\right|$ satisfy $m(\{z\in\Delta(S):\Pi_i(z)<K\epsilon^{1/4}\})>Km(\Delta(S))$. Since $\{z\in\Delta(S):\Pi_1(z)\Pi_2(z)<\epsilon^{1/2}\}$ covers over half of $\Delta(S)$, there is some choice of N_1 and N_2 such that

(1) $m(z\in\Delta(S):\Pi_1(z)<\epsilon^{1/4})\geqslant\frac{1}{4}m(\Delta(S))$

(2) $m(z\in\Delta(S):\Pi_1(z)<\epsilon^{1/4})\geqslant m(z\in\Delta(S):\Pi_2(z)<\epsilon^{1/4})$

(3) Some exchange of a single point $\gamma_1\in N_1$ for a single point $\gamma_2\in N_2$ will reverse the inequality in (2); i.e., if $M_i(z)=\left|\frac{z-\gamma_i}{1-\bar{\gamma}_i z}\right|$, $i=1,2$, then

$$m(\Pi_1M_2M_1^{-1}<\epsilon^{1/4})\leqslant m(\Pi_2M_1M_2^{-1}<\epsilon^{1/4}) \tag{2'}$$

(All sets here are subsets of $\Delta(S)$.)

Now

$$\begin{aligned}\{\Pi_1<\epsilon^{1/4}\}&\subseteq\{\Pi_1M_1^{-1}<K\epsilon^{1/4}\}\cup\{M_1<1/K\}\\&\subseteq\{\Pi_1M_1^{-1}M_2<K\epsilon^{1/4}\}\cup\{M_1<1/K\}\end{aligned}$$

and $\{M_1<1/K\}=\{z:\rho(z,\gamma_1)<1/K\}$. By taking $K>1$ large enough we can make the area of $\{M_1<1/K\}$ less than $\frac{1}{8}m(\Delta(S))$ so that (1) implies

$$m(\{z\in\Delta(S):\Pi_1(z)M_1^{-1}(z)M_2(z)<K\epsilon^{1/4}\})>\tfrac{1}{8}m(\Delta(S))$$

and (2′) implies the same inequality on $\Pi_2M_1M_2^{-1}$. If $G_1\cap S$ is defined to be $\{z\in\Delta(S):\Pi_1(z)M_1(z)^{-1}M_2(z)<K\epsilon^{1/4}\}$ and $G_2\cap S$ is defined to be $\{z\in\Delta(S):\Pi_2(z)M_1(z)M_2(z)^{-1}<K\epsilon^{1/4}\}$ and $\prod_{\gamma\in S}\frac{z-\gamma}{1-\bar{\gamma}z}$ is factored corresponding to the sets $N_1\cup\{\gamma_2\}\backslash\{\gamma_1\}$ and $N_2\cup\{\gamma_1\}\backslash\{\gamma_2\}$, then (i) and (ii) are satisfied in S.

To summarize: In this last case factor $\prod_{\gamma\in S}\frac{z-\gamma}{1-\bar{\gamma}z}$ as above. In all other cases factor it arbitrarily by dividing the zeros equally between factors. Define G_i, $i=1,2$, by defining them in each $S\in\mathcal{S}$ as above. The resulting G_1,G_2 and $C=C_1C_2$ will satisfy (i) and (ii) with appropriate α and β.

Now to remove the condition that each S contain an even number of zeros of C: Let $C=AB$ where A has at most one zero in each $S\in\mathcal{S}$ and B has an even number in each S. Factor $B=B_1B_2$ as before with ϵ replaced by $\epsilon^{1/2}$ obtaining $G_i(B)$. Let σ be small enough that $m[\{z:\rho(z,a)<\sigma$ for some a in the zero set of $A\}\cap\Delta(S)]<(1/16)m(\Delta(S))$.

If K_σ is defined as in Theorem B relative to the zero set of A then $G_i\equiv K_\sigma\cap G_i(B)$ will satisfy (i) for some $\delta>0$. Factor A according to Theorem B into A_1A_2. Then $C_1=A_1B_1$, $C_2=A_2B_2$ is the required factorization of C. □

The reason G_1 and G_2 are required to satisfy (i) is the following result which is the main theorem of [6, p. 2; see also p. 10].

THEOREM C. *Let $p>0$, $\alpha>-1$. The following conditions on a measurable set $G\subseteq D$ are equivalent.*

(1) *There is a constant $K>0$ such that*

$$\iint_D |g(z)|^p(1-|z|)^\alpha\,dm(z)\leqslant K\iint_G |g(z)|^p(1-|z|)^\alpha\,dm(z)$$

for all g analytic in D for which the left side is finite.

(2) *There is a constant $\delta>0$ such that $m(G\cap\Delta)>\delta m(D\cap\Delta)$ for all disks Δ whose centers lie on ∂D.*

This will be used in the next section with $\alpha=1$ ($1-|z|$ is replaced by the equivalent weight $\log(1/|z|)$), $p=2$, and g of the form $(\partial\bar f/\partial z)h$ where f is harmonic and $h\in H^2$.

4. Proof of Theorem 1. For the case $A=H^\infty+C$, alter the given proof as follows: First, let $\epsilon>0$ be given, write $\varphi=CF$ where C is a Blaschke product and F has no zeros in D. Factor C as in Theorem 2 obtaining C_1,C_2 and sets $G_1^\epsilon, G_2^\epsilon$. Let $\varphi_i=C_iF^{1/2}$. Follow the proof as before until consideration of

$$I_3=\iint\limits_{|z|<1}\left|\frac{\partial f}{\partial\bar z}\right|^2|\varphi_2|^2|z^{2n}h_2^2|\log\frac{1}{|z|}\,dm$$

is reached. Here we employ Theorem C to obtain

$$I_3\leqslant K\iint\limits_{G_2^\epsilon}\left|\frac{\partial f}{\partial\bar z}\right|^2|\varphi_2|^2|z^{2n}h_2^2|\log\frac{1}{|z|}\,dm$$

and continue as before, replacing all integrals by integrals over subsets of G_2^ϵ. The result is $|I_1|<K\epsilon^p\|h\|_{L^1(\lambda)}$ for some power $p>0$. And this case is finished. (The assumption $\varphi\in H^\infty$ still has to be removed but this will be done for general Douglas algebras.)

Before continuing the proof a reminder of some facts about Douglas algebras and their maximal ideal spaces seems appropriate. (The material in Sarason's lecture notes [9] would be sufficient background.) According to the Chang-Marshall theorem ([2] and [7]) any Douglas algebra A is generated as a closed algebra by H^∞ and conjugates of Blaschke products, i.e. A is the smallest closed subalgebra of L^∞ containing H^∞ and $I(A)\equiv\{\bar b: b$ is a Blaschke product and $\bar b\in A\}$. The maximal ideal space of A, $M(A)$, is the set $\{x\in M(H^\infty):|b(x)|=1$ for all $\bar b\in I(A)\}$. The Corona Theorem of Carleson [1] states that the unit disk D is a dense subset of $M(H^\infty)$. Every function in L^∞ extends to a continuous function on $M(H^\infty)$ whose restriction to D is the Poisson extension. Since products of elements of $I(A)$ lie in $I(A)$ we see that any open set in $M(H^\infty)$ that contains $M(A)$ must contain $\{x\in M(H^\infty):|b(x)|>1-\epsilon\}$ for some $\bar b\in I(A)$ and some $\epsilon>0$. In fact $M(A)$ is in the closure in $M(H^\infty)$ of the set $\{z\in D:|b(z)|>1-\epsilon\}$. The Chang-Marshall theorem implies that if $\varphi\in A$ then φ is a uniform limit of functions of the form $\bar bg$ with $\bar b\in I(A)$, $g\subset H^\infty$.

To finish the proof of Theorem 1 it suffices, for given $\epsilon > 0$, to pick b_0 and g as above with $\|\varphi - \bar{b}_0 g\|_\infty < \epsilon/8$ and show $\operatorname{dist}(f\bar{b}_0 g, A) < K\epsilon^p$ under the assumptions of the Theorem.

Now, $\operatorname{dist}(f\bar{b}_0 g, A) = \inf_{\bar{b} \in I(A)} \operatorname{dist}(bf\bar{b}_0 g, H^\infty)$ where again the Chang-Marshall Theorem is used. The infimum is dominated by an infimum over only those b's containing b_0 as a factor so it suffices to prove $\operatorname{dist}(fg, A) < K\epsilon^p$. To do this it must be verified that f and g satisfy the same hypotheses as f and φ, at least for the current choice of ϵ. By hypothesis there is a $\bar{b}_\epsilon \in I(A)$ such that $|\varphi(z)|(1 - |f(z)|) < \epsilon^2/4$ on the set where $|b_\epsilon(z)| > 1 - \epsilon$. Since the Poisson extensions of φ, g and b_0 are continuous on $M(H^\infty)$ there is a neighborhood U of $M(A)$ such that in $U \cap D$ we have simultaneously $|b_\epsilon(z)| > 1 - \epsilon$, $|b_0(z)| > 1 - \epsilon$ and $||\varphi(z)| - |g(z)\overline{b_0(z)}|| < \epsilon/4$. Thus in $U \cap D$ $||\varphi(z)| - |g(z)|| < \epsilon/2$. Now $U \supseteq \{z \in D: |b_1(z)| > 1 - \epsilon\}$ for some $\bar{b}_1 \in I(A)$. On this latter set it is easily seen that $|g(z)|(1 - |f(z)|) < \epsilon$. Now $\operatorname{dist}(fg, A) = \inf_{\bar{b} \in I(A)} \operatorname{dist}(bfg, H^\infty) = \inf_{\bar{b} \in I(A)} \sup_h |\int bfgh\, d\lambda|$ where the supremum is taken over $h \in H^1$ with $\|h\|_{L^1(\lambda)} \leqslant 1$ and the infimum may just as well be taken over all $\bar{b} \in I(A)$ of the special form $b = b_1^{2n}$.

$$\int bfgh\, d\lambda = 4 \iint\limits_{|z|<1} \frac{\partial f}{\partial \bar{z}} \cdot (bgh)' \log \frac{1}{|z|}\, dm$$

Now factor h into $h_1 h_2$ with $h_i \in H^2$ $\|h_i\|_{L^2}^2 = \|h\|_{L^1}$ and g into $g_1 g_2$ as was done for φ at the beginning of this section. Then

$$\iint\limits_{|z|<1} \frac{\partial f}{\partial \bar{z}} (bgh)' \log \frac{1}{|z|}\, dm = I_1 + I_2$$

where

$$I_1 = \iint\limits_{|z|<1} \frac{\partial f}{\partial \bar{z}} \cdot (b_1^n g_1 h_1)'(b_1^n g_2 h_2) \log \frac{1}{|z|}\, dm$$

with a similar formula for I_2.

Now $|I_1|^2 \leqslant I_3 \cdot I_4$ where

$$I_4 = \iint\limits_{|z|<1} |(b_1^n g_1 h_1)'|^2 \log \frac{1}{|z|}\, dm \leqslant K\|h\|_{L^1}$$

and

$$I_3 = \iint\limits_{|z|<1} \left|\frac{\partial f}{\partial \bar{z}}\right|^2 |b_1|^{2n} |g_2|^2 |h_2|^2 \log \frac{1}{|z|}\, dm.$$

The same arguments work here as did for $H^\infty + C$: $|z| < 1$ is covered by the sets $\{|b_1| < 1 - \epsilon\}$, $\{|g_2| < \epsilon^{1/2}\}$ and $\{1 - |f| < \epsilon^{1/2}\}$, from which estimates

$$|I_1| \leqslant K((1-\epsilon)^{2n} + \epsilon^p + \epsilon^{1/2})\|h\|_{L^1}$$

are obtained. This yields $\operatorname{dist}(fg, A) < K\epsilon^p$, whence $\operatorname{dist}(f\varphi, A) = 0$ since $\epsilon > 0$ is arbitrary. This completes the proof of Theorem 1. □

5. Remarks. I believe the techniques used in [4] would yield Theorem A for general Douglas algebras, but I do not see any way they can be modified to apply to the general $f \in L^\infty$. Guillory and Sarason obtain a much more precise result to the effect that if $\varphi, \psi \in H^\infty + C$, $|\varphi(e^{i\theta})| = |\psi(e^{i\theta})| = 1$ a.e. (λ), and $|\varphi(x)| \leqslant |\psi(x)|$ when $x \in M(H^\infty + C)$, then $\bar{\psi}\varphi^N \in H^\infty + C$ for some integer $N > 1$. I do not believe any result like this is possible for the general L^∞ function.

It surprised me somewhat to find that results like those in [6], originally obtained to answer some natural questions in the context of Bergman spaces, would have applications to problems about Douglas algebras. But perhaps I should have been alerted by the opposite phenomenon occurring in work of McDonald and Sundberg [8].

Finally, I would be interested to know if the G_1 and G_2 in Theorem 2 can be chosen independent of ϵ so that one could obtain the more aesthetic inequality $|C_i| \leqslant \alpha |C|^\beta$ on G_i in place of condition (ii) of that theorem. A closely related, if not inseparable, question is whether the factorization can be made independent of ϵ.

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