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Divisors of Mersenne Numbers

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Abstract. We add to the heuristic and empirical evidence for a conjecture of Gillies about the distribution of the prime divisors of Mersenne numbers. We list some large prime divisors of Mersenne numbers M_p in the range $17000 < p < 10^5$.

1. Introduction. In 1964, Gillies [6] made the following conjecture about the distribution of prime divisors of Mersenne numbers $M_p = 2^p - 1$:

CONJECTURE. *If $A < B \leqslant \sqrt{M_p}$, as B/A and $M_p \to \infty$, the number of prime divisors of M_p in the interval $[A, B]$ is Poisson distributed with mean* $\approx \log((\log B)/\log(\max(A, 2p)))$.

He noted that his conjecture would imply that

(i) The number of Mersenne primes $\leqslant x$ is about $(2/\log 2)\log\log x$.

(ii) The expected number of Mersenne primes M_p with p between x and $2x$ is 2.

(iii) The probability that M_p is prime is about $2\log 2p/p\log 2$.

He supported his conjecture with a heuristic argument and empirical data. Ehrman [5] sharpened Gillies' conjecture slightly and supplied more empirical evidence. The present paper strengthens the heuristic argument and adds to the empirical data in support of the conjecture.

Consequence (iii) follows from the conjecture by taking $A = 2p$ and $B = M_p^{1/2}$. The first two consequences follow easily from the third. Lenstra [8] has objected that one is not entitled to take B as large as $M_p^{1/2}$ in the conjecture because similar reasoning leads to a contradiction with the prime number theorem. We discuss Lenstra's objection.

The paper concludes with a table of large prime divisors of some Mersenne numbers and a table of some primes between 50000 and 100000 for which no prime divisors of M_p are known.

2. The Heuristic Argument. It is well known that all divisors of M_p have the form $q = 2kp + 1$, where $k \equiv 0$ or $-p \pmod 4$. How often is such a q prime? When q is prime, what are its chances of dividing M_p? The first question is answered heuristically by the Bateman-Horn conjecture [1] which is consistent with the prime number

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theorem and which is believed by many mathematicians. According to that conjecture, for each k the number of $p \leqslant x$ for which both p and $2kp + 1$ are prime is asymptotically

$$2 \prod_{\substack{q \text{ odd} \\ \text{prime}}} \left(1 - \frac{1}{(q-1)^2}\right) \cdot \prod_{\substack{q|2k \\ q \text{ odd prime}}} \frac{q-1}{q-2} \cdot \frac{x}{(\log x)\log(2kx)}.$$

(See also (7) of [11] and compare with [3], [4] and [10].) Write C_2 for the first product and $f(2k)$ for the second one. Thus, if we are given that p is prime, then for fixed k the probability that $2kp + 1$ is also prime is about $2C_2 f(2k)/\log(2kp)$.

Now suppose p is prime, k is a positive integer, $q = 2kp + 1$ is prime, and $k \equiv 0$ or $-p \pmod 4$. Shanks and Kravitz [11] present this good heuristic argument that $q \mid M_p$ with probability $1/k$: Let g be a primitive root of q. The congruence satisfied by k insures that $2kp + 1 \equiv \pm 1 \pmod 8$. Hence, 2 is a quadratic residue modulo q and $g^{2s} \equiv 2 \pmod q$ for some s. Now $2kp + 1 \mid M_p$ if and only if 2 is a $(2k)$-ic residue of $2kp + 1$, that is, if and only if $2k \mid 2s$. It is natural to assume that $k \mid s$ with probability $1/k$. There is empirical evidence for this, too. For example, there are 4783 primes $p \equiv 1 \pmod 4$ with $p < 100000$. For 1037 of these p is $6p + 1$ also prime and for 350 of these p does $6p + 1$ divide M_p, and $350/1037 = 0.34$.

Combining the apparent answers to our two questions yields this estimate for the expected number $F_p(A, B)$ of prime divisors of M_p between A and B:

$$F_p(A, B) \approx \sum_k 2C_2 f(2k)/(k \log(2kp)), \tag{1}$$

where the sum extends over all integers k with $k \equiv 0$ or $-p \pmod 4$ and $A < 2kp + 1 \leqslant B$. Suppose next that A and $B - A$ are large. Let q be an odd prime for which $8pq^2 < B - A$. Then q divides about $1/q$ of the k's in the sum in (1). For precisely these k's the product $f(2k)$ includes the factor $(q-1)/(q-2)$. Thus, the average contribution of q to all $f(2k)$ in (1) is

$$\frac{1}{q} \cdot \frac{q-1}{q-2} + \left(1 - \frac{1}{q}\right) \cdot 1 = \left(1 - \frac{1}{(q-1)^2}\right)^{-1}. \tag{2}$$

For each odd prime $q < ((B-A)/(8p))^{1/2}$, remove the factor $(q-1)/(q-2)$ from each $f(2k)$ in which it appears, and insert the factor (2) into each term of (1) instead. Since A and $B - A$ are large, the denominators of (1) change very slowly and little net change is made in (1). Now the product of the factors (2) over all primes $q < ((B-A)/(8p))^{1/2}$ is essentially $1/C_2$, the error being by a factor of about $\exp(-((8p)/(B-A))^{1/2})$, which is very close to 1 provided $B - A$ is large. In summary, if we change $C_2 f(2k)$ to 1 in (1), it makes very little difference. After that, we may change the factor of 2 in (1) to 1 if we drop the congruence condition on k. Hence (1) becomes

$$F_p(A, B) \approx \sum_{\substack{k \\ A < 2kp+1 \leqslant B}} \frac{1}{k \log(2kp)} \approx \log((\log B)/\log A), \tag{3}$$

which is part of Gillies' conjecture.

If we allow A or $B - A$ to be small, then $F_p(A, B)$ is not approximately Poisson distributed with the mean of Gillies' conjecture. For nearby integers j and k, the numbers $2jp + 1$ and $2kp + 1$ may have different probabilities of dividing M_p because of the fluctuation possible in $f(2k)$. Shanks and Kravitz [11] have studied these probabilities in detail. However, we do have $1 \leqslant f(2k) = O(\log\log k)$ (see page 117 of [7]) so that the fluctuations are not very great.

The possible values of k in (1) are $3, 4, 7, 8, 11, 12, \ldots$ if $p \equiv 1 \pmod 4$ and $1, 4, 5, 8, 9, 12, \ldots$ if $p \equiv 3 \pmod 4$. Hence, the possible divisors $2kp + 1$ of M_p are slightly smaller on the average and therefore more likely to divide M_p if $p \equiv 3 \pmod 4$ than if $p \equiv 1 \pmod 4$. Thus, M_p has a better chance of being prime if $p \equiv 1 \pmod 4$ than if $p \equiv 3 \pmod 4$. In fact 16 of the known Mersenne primes have $p \equiv 1 \pmod 4$ while 10 of them have $p \equiv 3 \pmod 4$. (See the list in [12].) All Mersenne primes discovered in the last 19 years (those with $5000 < p < 50000$) have $p \equiv 1 \pmod 4$. This evidence is suggestive but not statistically significant.

The only property of the Poisson distribution which Gillies used to deduce the three consequences from his conjecture was that if the mean is m, then the probability of the value 0 is e^{-m}. In our case, the probability that M_p is prime is about

$$\prod_k \left(1 - \frac{2C_2 f(2k)}{k \log(2kp)}\right), \tag{4}$$

where k runs over $2p + 1 \leqslant 2kp + 1 \leqslant M_p^{1/2}$ and $k \equiv 0$ or $-p \pmod 4$. The logarithm of (4) is about

$$\sum_k \frac{-2C_2 f(2k)}{k \log(2kp)}.$$

If we use the approximation (3) for $F_p(A, B)$, we find that the probability that M_p is prime is about

$$\frac{\log ap}{\log\left(M_p^{1/2}\right)} \approx \frac{2 \log ap}{p \log 2}, \tag{5}$$

where $a = 2$ if $p \equiv 3 \pmod 4$ and $a = 6$ if $p \equiv 1 \pmod 4$, which is Ehrman's [5] sharpened form of Gillies' third consequence. The first two consequences follow easily from either version of the third.

It is well known that the reasoning we used in (4) leads to this contradiction with the prime number theorem: we would say that the probability that a large integer x is prime is about

$$\prod_{\substack{p \text{ prime} \\ p \leqslant x^{1/2}}} \left(1 - \frac{1}{p}\right) \approx \frac{\mu}{\log(x^{1/2})} = \frac{2\mu}{\log x},$$

where $\mu = e^{-\gamma} \approx 0.5614594836$, and γ is Euler's constant. But the probability should be $1/\log x$, and $2\mu > 1$. This is Lenstra's [8] complaint. It is almost as well known (see [10] and 22.20 of [13]) that the correct answer is obtained in this simple problem if we replace the exponent $1/2$ by μ.

Should we make the same change in Gillies' argument? If we let k in (4) run over $ap + 1 \leqslant 2kp + 1 \leqslant M_p^\mu$, the three consequences become:

(I) The number of Mersenne primes $\leqslant x$ is about $(e^\gamma/\log 2)\log\log x$.

(II) The expected number of Mersenne primes M_p with p between x and $2x$ is e^γ.

(III) The probability that M_p is prime is about $e^\gamma \log ap/p \log 2$.

The first consequences are easiest to compare and are equivalent to the respective third consequences. Let $M(x)$ denote the number of Mersenne primes $\leqslant x$. Consequences (I) and (i) predict that the ratio $M(x)/\log\log x$ is approximately $e^\gamma/\log 2 = 2.5695$ and $2/\log 2 = 2.8854$, respectively. This ratio decreases slowly between Mersenne primes and jumps up from $(m - 1)/\log\log M_p$ to $m/\log\log M_p$ at the mth Mersenne prime M_p. The following table gives these two values for the five largest known Mersenne primes M_p.

m	p	$\frac{m-1}{\log\log M_p}$	$\frac{m}{\log\log M_p}$
23	11213	2.46	2.57
24	19937	2.41	2.52
25	21701	2.50	2.60
26	23209	2.58	2.68
27	44497	2.52	2.61

Although this data is too meager to be statistically significant, it suggests a clear preference for (I) over (i). We believe that (I) is correct because (a) replacing $1/2$ by μ works for the prime number theorem and (b) the limited empirical evidence agrees with (I). It would be desirable to have a plausible heuristic explanation for why the fudge factor μ works for the prime number theorem. Lenstra and Pomerance have been led independently to (I).

3. The Empirical Evidence. Using a computer, we found all primes p and q in the intervals $20000 < p < 10^5$, $q < 2^{34}$, for which $q \mid M_p$. We used this data to test Gillies' conjecture by calculating statistics similar to those of Ehrman [5] for $10^5 < p < 3 \cdot 10^5$, $q < 2^{31}$. Primes p were grouped in 80 intervals defined by

$$20000 + 1000i < p < 21000 + 1000i$$

for $i = 0(1)79$. Primes $p \equiv 1$ and $3 \pmod 4$ were considered separately. A *sample* consists of the primes in one of the 80 intervals and in a fixed residue class modulo 4.

Consider a sample of size N. Let T be the total number of prime divisors $q < 2^{34}$ of M_p for p in the sample. We computed the sample mean $\bar{x} = T/N$ and the sample variance

$$s^2 = N^{-1} \sum_{n=1}^{6} n^2 K_n - (\bar{x})^2,$$

where K_n is the number of M_p with exactly n prime divisors $< 2^{34}$. (Six was the greatest number of divisors we found for any M_p.) According to (3), the expected value for the mean m is the average of $\log((\log 2^{34})/\log ap)$, with a as in (5), taken over all p in the sample. We computed m and the two statistics

$$t = (N-1)^{1/2}(\bar{x} - m)/s$$

and

$$\chi^2 = \frac{(Ne^{-m} - K_0)^2}{Ne^{-m}} + \frac{(Nme^{-m} - K_1)^2}{Nme^{-m}} + \frac{(N(1 - e^{-m} - me^{-m}) - K_2 - K_3 - K_4 - K_5 - K_6)^2}{N(1 - e^{-m} - me^{-m})}$$

for each sample. If Gillies' conjecture were true, then for large N, t should have a standard normal distribution and χ^2 should have a chi-square distribution with 2 degrees of freedom. To test whether this was so we tabulated the number of values of t and χ^2 in 8 ranges of equal probability, just as Ehrman [5] did. These values are shown in Tables 1 and 2, together with Ehrman's data. We performed a chi-square test with 7 degrees of freedom on the numbers in each column of these tables. The agreement between the expected and observed distributions of t and χ^2 was not as good for our data as for Ehrman's data. One reason for this is that we have smaller sample sizes N. However, the chi-square statistics for the first two columns of Table 1 are nearly large enough for us to reject at the 5% level the hypothesis that t has a standard normal distribution. Another aspect of the difficulty is seen in the large mean value of t. In deriving (3) we assumed that both A and $B - A$ were large. Now we have used (3) with a small A. To determine the effect of the small A, we repeated all of the preceding statistical analysis with $m = \log((\log 2^{34})/\log 2^{24})$ and the divisors q restricted to the interval $(2^{24}, 2^{34})$. The results are given in Tables 1 and 2.

TABLE 1
Observed distribution of t
The expected number of values in each range is 10

Upper limit on t	$0 < q < 2^{34}$		$2^{24} < q < 2^{34}$		Ehrman
	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$	
-1.15	7	4	12	2	5
-.674	5	4	10	10	11
-.319	5	9	7	9	7
0.0	7	10	6	12	10
+.319	13	15	10	13	13
+.674	15	13	13	10	8
+1.15	13	10	12	11	12
∞	15	15	10	13	14
chi-square	13.6	13.2	4.4	8.8	5.8
mean t	+.321	+.335	-.043	+.234	+.247

TABLE 2

Observed distribution of χ^2

The expected number of values in each range is 10

Upper limit on χ^2	$0 < q < 2^{34}$		$2^{24} < q < 2^{34}$		
	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$	Ehrman
0.266	6	14	5	10	10
0.576	10	11	8	9	12
0.940	6	8	12	13	9
1.386	9	7	15	10	10
1.962	10	9	13	3	8
2.772	11	12	8	14	8
4.158	17	8	6	9	14
∞	11	11	13	12	9
chi-square	8.5	4.0	9.6	8.0	3.0
mean χ^2	2.305	2.142	2.318	2.009	1.947

Both the chi-square and the mean t in Table 1 were smaller for the restricted q's. This confirms our earlier statement that $F_p(A, B)$ is not approximately Poisson distributed with the mean of Gillies' conjecture when A is small, while it is when A and $B - A$ are large.

It is well known [7, Theorem 2.5] that

$$\prod_{\substack{p \text{ prime} \\ p \leqslant y}} \left(1 - \frac{1}{p}\right)$$

is the correct probability that a large integer x has no prime divisor $\leqslant y$, *provided* $\log y = o(\log x)$ *as* $x \to \infty$. The analog of this for Mersenne numbers is Gillies' conjecture with $\log B = o(p)$ as $p \to \infty$. The empirical evidence just discussed supports only this restricted conjecture. It does not suggest, nor do we believe, Gillies' conjecture for B as large as $M_p^{1/2}$.

4. The Other Tables. In Table 3 we list all pairs p, k which we found for which $20000 < p < 10^5$, p and $2pk + 1$ are prime, $2pk + 1 > 2^{31}$, and $2pk + 1$ divides M_p. We do not list the divisors $< 2^{31}$ because they are too numerous and may be calculated easily. On the other hand, we do list some divisors $> 2^{34}$. For $20000 < p < 50000$ we searched for divisors of M_p up to 2^{35} and when none had been found we went a little further. Table 3 also gives five divisors $2pk + 1 > 2 \cdot 10^{10}$ for $17000 < p < 20000$, which do not appear in [2].

TABLE 3

Pairs p, k for which $2kp + 1$ divides M_p

17851,784760	19081,649599	19681,541559	19759,730296	19763,570493
20021,618583	20021,696628	20113,762227	20359,140216	20369,140520
20369,453787	20441,84988	20479,635145	20627,104784	20641,54395
20641,54911	20663,86532	20939,160021	20939,756841	20983,65613
20983,179513	21089,74607	21107,60469	21143,856548	21179,64201
21179,362772	21313,320331	21377,1272195	21391,272828	21401,348288
21557,661587	21817,599787	21929,118371	21937,163820	21943,94436
21943,607928	22063,312656	22093,190835	22171,343605	22273,105800
22349,722256	22433,541803	22447,300468	22483,113676	22501,67260
22531,149253	22531,473481	22531,520208	22751,110409	22769,171564
22817,1397364	22907,147604	22937,387264	23027,185140	23173,794300
23197,112320	23327,536973	23557,73544	23609,410431	23957,182844
23977,131355	23993,95551	24097,54960	24107,110545	24373,431087
24413,193552	24469,1633587	24697,111687	24851,650484	24979,1596801
25013,142288	25037,559767	25057,691223	25171,56829	25367,573348
25561,386579	25579,135332	25643,353116	25703,86017	25771,122549
25799,84477	25841,64071	25873,51267	25873,316467	25873,641111
25951,269121	26003,219948	26053,935756	26153,208875	26209,72647
26293,45176	26339,158001	26431,684689	26479,104076	26501,114340
26539,242937	26561,219615	26591,389605	26647,126972	26839,107436
26993,922416	27011,197005	27077,180403	27107,155712	27127,143432
27197,99024	27239,55320	27367,275412	27427,305720	27427,471500
27481,160848	27653,161667	27737,477040	27779,189772	27803,66748
27817,171972	27967,71225	28001,137643	28097,286708	28123,71472
28219,1635692	28283,66673	28297,285179	28309,122907	28309,432500
28403,67936	28477,181784	28607,45240	28723,105092	28729,80787
28793,525168	29059,171516	29101,693920	29123,1041108	29137,376464
29167,572829	29201,242851	29269,192567	29311,53120	29363,129016
29759,51904	29837,135900	30011,616468	30089,427999	30109,125939
30313,96392	30319,1411745	30391,221289	30467,164373	30469,221831
30493,899220	30677,1299288	30839,52785	30841,35336	30881,346236
30941,482875	30949,265895	31121,170059	31219,42932	31219,58749
31481,470568	31489,475499	31567,125648	31573,290628	31627,313184
31667,47973	31687,91773	31687,213612	31699,830457	31769,93687
31873,56928	31883,631392	31963,581441	32059,151604	32159,86356
32257,66059	32299,532944	32303,35341	32303,515892	32323,228116
32327,229425	32377,1424235	32441,35928	32467,1347072	32479,35177
32479,42185	32491,362069	32531,387709	32563,57513	32569,1021011
32579,176724	32713,189612	32779,41829	32831,556428	32843,53265
32993,1297648	33023,33556	33029,276367	33071,154509	33349,362291
33349,380636	33353,42012	33353,449547	33413,71032	33563,58101
33589,93375	33589,145547	33703,33848	33857,52804	33863,37653
33863,88581	34123,281117	34127,69745	34127,314064	34147,135072
34159,244236	34211,675621	34337,498744	34351,168564	34457,47724
34471,168441	34591,253793	34673,122532	34739,239880	34883,107116
34897,113472	35107,1013985	35111,183309	35267,271129	35393,36727
35419,448845	35597,291420	35863,89400	35879,136704	35897,880399
35951,54409	35983,1111697	36007,43428	36241,252975	36277,89312
36293,93015	36319,138900	36389,329095	36469,1279991	36583,30840
36607,368729	36697,487868	36899,82417	36973,64191	36997,175515
37097,302340	37361,171844	37369,330839	37463,162220	37567,528273
37633,191456	37649,139491	37663,137076	37781,150903	37813,140268
37957,246332	38053,998736	38119,136964	38329,91740	38393,72856
38449,93936	38449,209439	38543,83125	38543,259645	38669,223372
38833,130911	38839,284657	38861,568804	38933,177768	38953,123891
39181,70596	39181,96768	39191,65373	39209,203316	39233,1282996
39251,32241	39293,53568	39367,97104	39607,218420	39623,44221
39679,968609	39799,56861	39827,109572	39847,175524	40031,33733

TABLE 3 (*continued*)

40063,144813	40099,75177	40237,380259	40433,122343	40493,1141348
40577,37492	40597,290655	40637,176584	40693,214052	40697,214248
40699,245732	40787,213592	40813,1002836	40849,50319	41011,46709
41039,353185	41057,323283	41141,174163	41183,184053	41227,124689
41243,158101	41381,1003644	41389,913584	41621,46363	41651,222013
41809,48696	41903,44353	41953,428816	42013,64667	42061,529928
42101,335920	42139,131529	42197,85107	42359,74020	42491,196021
42499,1202361	42643,53472	42697,717623	42853,599495	42979,268704
43003,67745	43261,109724	43397,139435	43451,533353	43753,60476
43783,85376	43801,44615	43889,167715	43891,78416	43963,1022433
43969,80244	44021,24799	44101,26768	44189,150079	44279,74484
44531,78829	44543,42741	44711,138160	44819,434076	44893,33996
44909,33660	44971,337893	45119,57144	45131,129985	45139,506156
45281,469515	45289,112247	45337,47643	45341,52224	45439,933300
45503,123825	45541,187575	45751,598724	45833,251191	45853,29367
45971,33816	45979,40716	46021,42723	46147,125928	46199,133845
46237,54480	46601,129375	46649,24795	46703,36036	46727,76840
46747,402285	46771,169641	46811,54793	46831,37196	46877,81808
46877,145332	46877,201724	46889,55467	47051,165085	47111,181153
47119,706385	47149,111939	47207,796237	47221,234536	47237,161140
47279,30561	47303,378996	47309,55572	47317,45684	47351,67704
47353,444536	47441,66528	47521,74351	47743,31521	47837,400083
47939,31621	48109,177240	48179,37269	48179,43329	48187,648252
48337,126332	48397,41004	48407,1112308	48463,179105	48491,85201
48731,41004	48847,50204	48847,187712	48953,97896	48989,53820
49003,176673	49009,68555	49081,268671	49121,236655	49157,30267
49169,189391	49201,76304	49391,102648	49411,558368	49429,331071
49547,35224	49549,47907	49597,278804	49627,27449	49633,279972
49669,51276	49739,71164	49853,207868	49943,955812	50023,28193
50033,31840	50101,64584	50131,108329	50177,37524	50441,32451
50503,33036	50581,23520	50647,153009	50789,89464	50833,137583
50857,112763	50873,116623	51169,68336	51197,37980	51473,130972
51481,48543	51511,30644	51647,20932	51817,83319	51827,155652
51829,53951	51859,50801	52027,39420	52163,157660	52223,63705
52237,29855	52237,41388	52253,58455	52289,102136	52543,98652
52667,79693	52697,42523	52697,58183	52807,56940	52813,159551
52973,39100	53047,125672	53089,127995	53117,70063	53173,25436
53239,43320	53299,31917	53309,133812	53381,146016	53401,102960
53437,21408	53479,51129	53551,22385	53551,75224	53657,69223
53899,23592	53951,44848	53987,22665	54163,42252	54287,62428
54413,78352	54539,113781	54581,67539	54623,33757	54767,28065
54787,53333	54973,30972	54979,49256	55147,31589	55229,86887
55343,32752	55381,54896	55411,24221	55547,66633	55579,99252
55733,21775	55967,39585	56039,123876	56377,92063	56519,49504
56633,74631	56659,19061	56957,78124	56963,31981	56989,31836
57221,115531	57241,25575	57269,63267	57331,84401	57367,80405
57601,26468	57679,77201	57793,29043	57859,26156	57977,48523
58027,42432	58031,48364	58147,132140	58363,28880	58393,57375
58439,21049	58441,20999	58693,75548	59023,23148	59063,134293
59219,56652	59233,43976	59419,82512	59611,96348	59617,61208
59659,26405	59699,41176	59863,46452	59999,36921	60101,131871
60317,23355	60539,111645	60607,58889	60703,92220	60917,39604
61099,31397	61169,75904	61211,29304	61291,18984	61357,55772
61409,52711	61483,35993	61511,46249	61547,47437	61657,82403
61949,61132	62119,17457	62131,97893	62143,87368	62299,75237
62351,35965	62459,20712	62617,21984	62761,133568	63031,105933
63059,42552	63067,58917	63299,91489	63331,103064	63391,30156
63521,48760	63743,111841	63839,66804	64171,21273	64187,36192

TABLE 3 (*continued*)

64231,33473	64301,84288	64783,55196	64817,92968	64901,77388
64927,66105	65101,57008	65101,131280	65239,117101	65327,122589
65419,32477	65713,53195	65809,21155	65843,33025	65921,18579
65951,126808	66041,60804	66271,63365	66347,90360	66463,44672
66553,93095	66601,70131	66713,68080	66721,98355	66949,68276
67169,44475	67189,53396	67481,100939	67867,36725	68071,52229
68443,77525	68699,18697	68881,58184	69073,16716	69149,31527
69233,22315	69497,18879	69677,22932	69829,35087	69833,83680
69857,19119	69859,16557	69877,71592	70201,28508	70229,45412
70423,51512	70429,83580	70501,101840	70717,15687	70729,29027
70957,100523	70999,16065	71287,27117	71287,59732	71339,77556
71389,103976	71471,15609	71837,92272	71843,15381	71867,35257
71881,18635	71993,47200	71999,36157	72481,31139	72661,103620
73009,67655	73063,66837	73379,44629	73589,103932	73757,80904
73867,65753	73973,14568	74177,58308	74177,65019	74779,22509
74959,42969	75079,19880	75083,73737	75167,33049	75217,93548
75277,57087	75391,83385	75533,35935	75797,20763	75821,38620
75979,49356	76123,24428	76379,84816	76481,90100	76493,48687
76631,60588	76753,30288	77003,26556	77137,23520	77171,30069
77267,49173	77369,77119	77419,18360	78203,17968	78283,23552
78317,37164	78439,95189	78539,65464	78713,46003	78853,56432
78919,24092	79193,71652	79229,57400	79319,27720	79399,39672
79549,45495	79693,27780	79867,41849	80141,40515	80231,14628
80309,65824	80387,61264	80447,72793	80603,41553	80677,18860
80929,37047	81017,71148	81031,14724	81049,26772	81119,31024
81131,31149	81131,80509	81233,20076	81281,36543	81401,17340
81547,59469	81619,38412	81727,22709	82009,15807	82153,30575
82207,26048	82421,27444	82483,30972	82487,40929	82567,16205
82567,54009	82727,59833	82913,55495	83063,38848	83267,16353
83269,13095	83269,43764	83299,46265	83311,70269	83407,22092
83537,43935	83719,16961	83773,87707	84067,18552	84307,16368
84319,21149	84437,25024	84731,12865	84751,39965	85081,97215
85193,95115	85199,24117	85297,74232	85597,27648	85711,29720
85817,35172	85889,44871	85999,46740	86077,76559	86131,29393
86291,84309	86323,14216	86357,42108	86423,23512	86453,72600
86677,18315	86711,71193	86861,27024	87133,75708	87281,38256
87491,36540	87517,73292	87523,55757	87587,21780	87751,45345
88499,23296	88513,27908	88799,49869	88969,87051	89083,18060
89209,38876	89237,16984	89371,30093	89809,16536	89819,17784
89983,34013	90401,51991	90703,42848	90833,72000	90847,12900
90847,27092	91019,15316	91099,17061	91121,59460	91291,66540
91331,14101	91393,60716	91573,27611	91573,38087	91703,68541
91781,89259	91807,29169	91951,16008	92077,13583	92119,42597
92233,75492	92237,74392	92269,24567	92353,20736	92413,20747
92581,16836	92699,16357	92821,15000	92861,13680	92893,21756
93001,27863	93059,17245	93133,12627	93257,12684	93419,81421
93427,15597	93563,18153	93761,24520	93851,48804	93901,58463
93971,78748	94057,19415	94153,75696	94229,22635	94253,54435
94261,63356	94421,80319	94541,11724	94543,16773	94561,54096
94603,45441	94651,36564	94849,22919	94999,44712	95027,15417
95107,56300	95131,61149	95177,23835	95203,17888	95393,42072
95479,54281	95701,43991	95737,84623	95881,18423	95891,34993
95929,52284	95987,20433	96017,36748	96043,61392	96149,22152
96259,22001	96259,51201	96329,78336	96451,16184	96769,20396
96851,29569	97021,18080	97157,11520	97177,64883	97187,23649
97549,33719	97613,27783	97771,14625	97859,25752	97883,26817
98009,86799	98017,16719	98297,23524	98347,32792	98467,85344
98533,29075	98627,69424	98893,14715	98963,13060	99367,33420
99371,16284	99623,16428			

TABLE 4

Primes p for which no divisor of M_p is known

50069,087,111,119,123,153,221,227,231,261,263,273,287,329,341,359
50383,417,461,513,543,551,599,683,723,741,753,767,821,839,929,951
50957,969,989,993,001,031,043,047,059,061,071,151,193,217,229,257
51263,307,341,347,349,407,421,427,431,437,439,449,479,487,517,551
51563,577,581,599,607,613,637,679,691,749,767,853,869,871,899,907
51913,929,941,949,971,973,991,009,021,051,081,177,183,267,301;313
52321,363,369,391,457,489,501,517,529,541,561,579,609,639,673,711
52721,747,757,769,859,889,901,903,963,967,999,003,017,077,101,113
53129,147,189,197,201,231,267,279,323,327,407,453,503,507,527,549
53569,593,611,623,629,653,681,777,813,819,857,881,887,917,939,959
53993,001,013,049,059,139,151,167,269,277,293,311,319,323,331,347
54361,371,377,401,403,409,419,421,437,449,493,497,499,517,547,563
54583,617,629,647,673,709,713,727,751,773,833,851,869,881,907,919
54941,949,983,021,057,061,079,117,127,163,201,213,243,259,313,331
55337,351,399,469,487,501,511,529,589,609,667,681,697,763,787,807
55817,819,823,829,837,849,889,903,009,041,053,101,113,131,149,197
56207,237,239,267,269,299,311,333,359,401,417,431,443,453,473,477
56489,509,527,533,543,591,597,611,629,681,687,711,731,737,747,767
56779,807,813,827,843,873,893,897,909,911,921,941,041,059,073,089
57139,143,149,163,179,191,193,223,259,283,287,301,349,383,389,397
57413,457,487,503,557,559,587,593,637,641,697,709,713,719,737,773
57803,829,847,853,881,901,917,923,943,973,991,043,057,099,109,111
58193,199,207,217,367,369,379,391,403,453,477,481,537,543,549,613
58631,687,699,711,727,733,741,757,763,771,789,889,897,907,913,937
58943,011,051,053,083,107,113,149,159,167,183,197,207,209,239,243
59263,333,357,377,387,393,443,467,471,473,497,557,567,581,627,629
59651,671,693,729,747,753,771,779,797,887,929,957,971,013,029,089
60091,103,107,149,161,167,169,209,257,259,271,289,293,337,353,413
60427,443,449,493,497,521,601,611,617,623,637,649,661,679,727,733
60737,757,763,811,821,869,889,899,901,953,007,027,031,043,051,057
61091,121,151,223,253,297,339,363,379,417,463,469,471,487,493,519
61543,553,583,603,631,643,667,673,681,687,729,781,813,819,837,843
61861,879,909,927,933,967,979,987,003,017,047,053,129,141,207,233
62273,303,311,327,347,383,483,501,533,539,549,597,633,653,659,683
62687,723,731,773,801,827,869,903,927,929,939,983,987,029,113,127
63149,197,199,211,241,277,313,337,353,377,397,443,467,487,527,533
63541,559,589,599,601,617,647,649,659,667,689,691,697,709,781,809
63823,841,853,857,901,907,929,949,977,007,013,019,063,067,091,109
64217,223,237,279,319,327,373,381,399,403,433,453,483,489,499,553
64579,591,601,609,613,621,633,661,679,717,747,781,811,849,877,879
64937,969,003,011,027,029,053,071,089,119,123,129,141,179,203,213
65257,267,269,287,293,309,323,371,393,413,423,447,479,519,521,537
65539,543,557,563,629,633,647,687,699,701,717,719,729,731,761,777
65831,839,929,957,993,029,037,047,083,089,103,107,109,137,161,169
66173,179,221,239,343,359,361,377,413,449,457,467,491,499,509,523
66541,617,643,653,683,733,751,797,841,853,863,883,889,919,923,931
66943,947,973,033,121,129,141,181,211,213,217,219,231,247,273,289
67307,343,369,421,427,433,453,477,537,547,601,607,619,631,651,679
67733,751,757,763,777,783,789,807,843,853,901,927,931,933,939,957
67961,967,979,987,993,023,059,087,099,141,161,207,209,213,219,227

TABLE 4 (*continued*)

68239,261,281,311,329,371,473,477,483,491,501,507,521,531,581,597
68659,669,683,687,711,713,743,749,777,791,813,821,863,879,891,899
68903,927,947,993,011,019,029,031,061,067,143,151,191,193,197,239
69257,317,337,341,371,383,427,493,499,557,653,691,697,709,739,761
69763,767,821,847,899,929,931,003,009,019,051,061,099,111,117,123
70139,141,157,177,181,183,207,237,249,297,321,381,457,459,487,529
70537,583,607,619,663,667,687,753,769,783,823,841,843,849,867,877
70879,913,949,951,969,991,011,039,069,081,119,129,143,147,167,191
71209,233,249,257,261,263,293,327,329,333,353,411,419,437,443,453
71473,479,483,549,551,563,569,593,633,663,693,699,707,711,719,741
71789,807,821,849,861,887,899,909,917,933,941,963,983,987,043,047
72077,091,109,161,169,221,229,269,271,277,307,337,353,379,421,431
72461,467,469,493,559,643,649,673,679,689,701,707,727,733,739,797
72817,859,883,889,893,923,931,949,953,997,019,037,043,079,091,121
73133,181,237,243,277,303,309,327,331,361,369,417,421,433,471,483
73517,561,583,607,609,637,643,673,679,699,709,783,819,847,859,883
73897,907,939,999,017,021,047,071,131,149,159,167,189,201,209,231
74257,279,287,297,317,323,377,381,383,441,449,453,471,489,531,551
74573,597,611,623,653,687,717,747,857,861,869,891,903,923,933,941
75011,029,193,209,223,227,239,289,307,323,329,377,401,403,407,431
75527,539,553,557,571,577,619,629,653,659,689,703,709,731,773,787
75793,869,883,931,937,989,991,997,003,039,099,129,147,159,231,243
76253,261,343,367,387,403,421,441,471,507,537,541,543,603,607,649
76697,733,777,801,829,837,847,873,949,963,991,029,041,047,101,141
77191,239,249,263,269,279,291,317,339,383,417,477,489,509,527,557
77563,569,573,621,647,687,689,713,719,723,731,743,747,761,801,849
77863,893,899,969,999,041,049,079,179,193,229,233,241,259,301,347
78401,437,467,479,487,497,517,541,553,569,571,577,583,593,607,643
78649,691,697,721,737,787,797,823,857,877,901,929,977,989,031,039
79043,063,103,133,139,147,153,181,241,273,279,309,333,349,357,393
79427,433,451,531,537,579,589,609,621,627,631,657,669,687,699,757
79777,801,813,817,823,843,847,861,901,907,943,979,051,107,149,153
80167,177,207,233,239,263,287,341,363,407,429,449,513,567,599,611
80627,629,657,669,671,713,737,777,779,783,803,809,849,863,909,911
80917,923,933,989,047,163,173,181,197,203,239,283,293,299,307,331
81371,373,409,421,439,457,509,533,553,637,647,649,667,671,677,689
81701,737,749,769,869,901,919,929,937,943,967,971,973,003,007,013
82021,031,037,039,051,073,141,163,171,189,219,241,261,267,339,373
82387,463,493,507,529,531,549,559,571,601,619,633,657,699,721,723
82759,781,793,813,837,883,889,891,939,023,059,089,093,101,117,177
83203,207,219,227,231,233,273,383,389,443,449,561,563,591,597,609
83621,663,717,737,761,777,791,813,833,869,873,891,911,921,987,053
84127,137,143,163,179,181,191,199,347,377,389,391,407,457,467,509
84521,559,589,631,649,697,701,713,719,737,761,811,857,859,869,871
84913,967,009,027,037,049,091,093,121,147,159,201,259,303,313,331
85333,363,369,381,447,451,469,549,577,619,627,661,667,703,717,733
85781,819,831,843,847,909,933,011,083,113,117,161,197,209,239,243
86249,269,311,351,371,389,399,441,467,491,509,531,561,573,579,587
86599,627,629,689,693,719,743,813,851,923,927,951,959,993,011,013
87037,041,049,103,121,149,151,179,187,211,221,251,253,257,317,359

TABLE 4 (*continued*)

87403,407,443,481,547,553,557,559,589,623,631,641,649,671,679,691
87697,719,739,743,767,811,877,887,931,943,973,977,001,007,019,069
88117,169,177,223,237,241,261,321,339,379,397,423,427,469,493,609
88651,661,663,667,681,721,741,747,771,789,793,817,819,843,867,873
88883,897,937,951,017,041,057,069,101,107,137,203,213,227,231,261
89269,273,293,303,317,387,393,413,431,443,449,477,491,501,519,527
89533,561,563,567,591,599,603,611,627,633,653,659,669,671,681,689
89767,783,797,821,833,839,891,899,909,939,959,963,977,001,017,019
90023,031,053,059,067,071,073,089,107,121,149,163,173,187,199,203
90217,227,247,263,281,289,313,371,379,397,403,407,437,439,473,511
90533,547,583,631,641,647,679,709,731,749,787,823,841,863,887,911
90917,931,997,009,081,097,127,153,193,237,243,249,253,297,309,367
91369,373,387,411,423,459,493,513,529,621,711,801,811,813,823,841
91909,921,939,943,957,967,009,033,041,051,083,143,153,173,177,179
92221,227,311,317,377,381,387,399,401,419,431,503,507,557,567,569
92593,623,627,639,641,647,657,671,681,707,717,737,761,779,791,809
92849,857,863,867,899,921,927,941,959,993,077,083,097,103,131,139
93179,187,229,251,263,281,283,287,307,329,337,377,383,407,463,491
93493,553,557,581,601,607,629,637,701,703,739,809,811,871,887,889
93913,923,937,941,949,967,997,009,049,109,111,121,201,207,273,307
94309,321,327,331,349,351,379,397,433,439,441,447,529,583,613,621
94649,723,727,777,793,811,819,823,837,873,889,933,003,021,071,087
95089,101,143,153,189,191,233,239,261,311,317,327,369,383,401,413
95441,443,461,467,471,483,507,527,531,549,597,617,621,633,717,731
95747,783,803,813,869,911,923,947,971,989,013,059,079,097,137,157
96167,181,199,221,223,263,269,293,323,331,337,353,377,401,419,431
96457,461,469,479,487,493,497,517,587,671,703,739,749,757,763,797
96799,823,847,893,907,911,931,959,973,997,001,003,073,103,127,169
97231,301,327,367,369,373,379,381,387,423,429,453,459,463,499,547
97579,607,609,651,673,687,711,777,787,789,813,841,847,849,861,919
97927,961,973,011,041,047,081,143,179,207,221,251,269,299,321,323
98327,369,389,411,443,479,543,561,563,597,663,717,729,737,849,869
98873,887,899,909,911,927,929,947,953,993,041,053,089,103,109,119
99131,133,137,149,173,223,233,257,277,317,349,409,431,497,523,529
99577,581,611,679,689,707,719,721,761,767,793,809,823,829,833,877
99881,901,923,929,971,989

Several years ago when we made available a list of divisors of M_p for $17000 < p < 50000$, Noll and Nickel [9] and Slowinski [12] were inspired to search for Mersenne primes within this range and found three new ones. To provide further inspiration we present in Table 4 the 2166 primes $50000 < p < 10^5$ for which M_p has no divisor $< 2^{34}$. The first prime in each row is written in full; only the low-order three digits of the other primes are shown. According to the second consequence (II) we should expect that M_p is prime for about 1.78 of the primes in Table 4.

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