# Divisors of the Euler and Carmichael functions 

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1. Introduction. Two of the most studied functions in the theory of numbers are Euler's totient function $\phi(n)$ and Carmichael's function $\lambda(n)$, the first giving the order of the group $(\mathbb{Z} / n \mathbb{Z})^{*}$ of reduced residues modulo $n$, and the latter giving the maximum order of any element of $(\mathbb{Z} / n \mathbb{Z})^{*}$. The distribution of $\phi(n)$ and $\lambda(n)$ has been investigated from a variety of perspectives. In particular, many interesting properties of these functions require knowledge of the distribution of prime factors of $\phi(n)$ and $\lambda(n)$ (see e.g., [3]-[7], [12], [19]).

The distribution of all of the divisors of $\phi(n)$ and $\lambda(n)$ has thus far received little attention, perhaps due to the complicated way in which prime factors interact to form divisors. From results about the normal number of prime factors of $\phi(n)$ and $\lambda(n)$ [5], one deduces immediately that $\tau(\phi(n))$ and $\tau(\lambda(n))$ are each about $\exp \left\{\frac{\log 2}{2}(\log \log n)^{2}\right\}$ for almost all $n$. However, the determination of the average size of $\tau(\phi(n))$ and of $\tau(\lambda(n))$ is more complex, and has been studied recently by Luca and Pomerance [13].

In this note we investigate problems about localization of divisors of $\phi(n)$ and $\lambda(n)$. Our results have application to the structure of $(\mathbb{Z} / n \mathbb{Z})^{*}$, since the set of divisors of $\lambda(n)$ is precisely the set of orders of elements of $(\mathbb{Z} / n \mathbb{Z})^{*}$. We say that a positive integer $m$ has u-dense divisors (for short, $m$ is $u$-dense) if whenever $1 \leq y<m$, there is a divisor of $m$ in the interval ( $y, u y$ ]. The distribution of $u$-dense numbers for general $u$ has been investigated by Tenenbaum ([17], [18]) and Saias ([14], [15]). According to Théorème 1 of [14], the number of $u$-dense integers $m \leq x$ is $\asymp(x \log u) / \log x$, uniformly for $2 \leq u \leq x$. In particular, the number of 2 -dense integers

[^0]$m \leq x$ is $\asymp x / \log x$, that is, the 2 -dense integers are about as sparse as the primes.

By contrast, we show that 2-dense values of $\phi(n)$ and $\lambda(n)$ are very common.

Theorem 1. If $x$ is sufficiently large, then for $\gg x$ integers $n \leq x$, both $\phi(n)$ and $\lambda(n)$ are 2-dense.

There are relatively simple heuristic reasons for believing Theorem 1. Recall that

$$
\begin{aligned}
\phi\left(p_{1}^{e_{1}} \cdots p_{k}^{e_{k}}\right) & =p_{1}^{e_{1}-1}\left(p_{1}-1\right) \cdots p_{k}^{e_{k}-1}\left(p_{k}-1\right) \\
\lambda\left(p_{1}^{e_{1}} \cdots p_{k}^{e_{k}}\right) & =\operatorname{lcm}\left[\lambda\left(p_{1}^{e_{1}}\right), \ldots, \lambda\left(p_{k}^{e_{k}}\right)\right]
\end{aligned}
$$

where $\lambda\left(p_{i}^{e_{i}}\right)=\phi\left(p_{i}^{e_{i}}\right)$ if $p_{i}$ is odd or if $p_{i}=2$ and $e_{i} \leq 2$, and $\lambda\left(2^{e}\right)=2^{e-2}$ for $e \geq 3$. In particular, $\phi(n)$ and $\lambda(n)$ have the same prime factors. We recall that

- most of these prime factors are factors of shifted primes $p-1$ where $p \mid n$,
- for most primes $p, p-1$ has about $\log \log p$ prime factors [3],
- for most integers $n$, we have $\omega(n)$ about $\log \log n$ and, writing these distinct prime factors as $p_{1}(n)<\cdots<p_{\omega(n)}(n)$, we have for all $k$ except for the smallest ones $\log \log p_{k}(n) \approx k$ (see, e.g., Theorem 10 of [10]),
- $\log \phi(n) \sim \log n$.

With these four facts, we find that for most values of $n$,

$$
\begin{aligned}
\Omega(\phi(n)) & \approx \sum_{p \mid n} \Omega(p-1) \approx \sum_{p \mid n} \log \log p \\
& \approx \sum_{k \leq \log \log n} k \sim(\log \log n)^{2} / 2 \sim(\log \log \phi(n))^{2} / 2
\end{aligned}
$$

See [5] for a more precise result about the normal behavior of $\Omega(\phi(n))$. On the other hand, for most values of $m, \Omega(m)$ is about $\log \log m$. So, usually $\phi(n)$ has far more divisors than a typical integer of its size.

We therefore expect the divisors of $\phi(n)$, especially the smaller divisors, to be "very dense" for most $n$, and the same should be true of small divisors of $\lambda(n)$. On the other hand, there are a large proportion of $n$ for which the divisors of $\phi(n)$ and $\lambda(n)$ are not very dense. To state our next result, we define $\theta$ to be the supremum of real numbers $c$ so that there are $\gg x / \log x$ primes $p \leq x$ with $p-1$ having a prime factor $>p^{c}$. Many papers have been written on bounding $\theta$, and the current record is $\theta \geq 0.677$ and is due to Baker and Harman [1].

Theorem 2. Let $0<c<2 \theta-1$. If $x$ is sufficiently large, then for $>_{c} x$ of the integers $n \leq x$, neither $\phi(n)$ nor $\lambda(n)$ is $x^{c}$-dense.

It is conjectured that $\theta=1$, and this would imply the conclusion of Theorem 2 for any $c<1$.

If $u<2$, there are no $u$-dense integers $m>1$. However, it is possible that the divisors of a given integer in some long interval do have consecutive ratios which are $\leq u$. We say that an integer $n$ has $u$-dense divisors in a set $I$ (for short, $n$ is $u$-dense in a set $I$ ) if for every $y \in I$, the interval ( $y, u y$ ] contains a divisor of $n$. The following makes precise what we claimed earlier about the "very dense" nature of the small divisors of $\phi(n)$ and $\lambda(n)$.

Theorem 3. For every positive integer $h$ and $0<\delta<1$, there is a constant $c=c(h, \delta)>0$ so that if $x$ is sufficiently large, then for more than (1$\delta) x$ of the integers $n \leq x, \phi(n)$ and $\lambda(n)$ are both $(1+1 / h)$-dense in $\left[h, x^{c}\right]$.

Notice that the left endpoint $h$ of the interval cannot be replaced by $h-1$, since if $h-1 \leq a<h /(1+1 / h)$, there are no integers in $(a, a(1+1 / h)]$. In addition, if we assume that $\theta=1$, then we cannot take $c$ independent of $\delta$ in light of Theorem 2.

Using Theorem 3, we prove a more general version of Theorem 1.
Theorem 4. For every positive integer $h$, there are $>_{h} x$ integers $n \leq x$ such that $\phi(n)$ is $(1+1 / h)$-dense in $[h, \phi(n) /(h+1))$ and $\lambda(n)$ is $(1+1 / h)$ dense in $[h, \lambda(n) /(h+1))$.

We also record a limiting case of Theorem 3.
Corollary 1. Suppose $g(x)$ is a positive function decreasing monotonically to 0 and let $h$ be a positive integer. Almost all $n \leq x$ have the property that $\phi(n)$ and $\lambda(n)$ are $(1+1 / h)$-dense in $\left[h, x^{g(x)}\right]$.

Analogous to the problems studied in [9], [8], [16], we can study the distribution of integers with $\phi(n)$ having a divisor in a single interval. Let

$$
B(x, y, z)=|\{n \leq x: \exists d \mid \phi(n), y<d \leq z\}|
$$

An almost immediate corollary of Theorems $1-3$ is the following result in the special case $z=2 y$.

Corollary 2.
(i) Uniformly for $1 \leq y \leq x / 2$, we have $B(x, y, 2 y) \gg x$.
(ii) Fix $1-\theta<c<1 / 2$. Then, uniformly for $x^{c} \leq y \leq x^{1-c}$, we have $x-B(x, y, 2 y) \gg x$.
(iii) Let $g(x) \rightarrow 0$ monotonically. Then, for $1 \leq y \leq x^{g(x)}$, we have $B(x, y, 2 y) \sim x$.
We leave as an open problem the determination of the order of magnitude of $B(x, y, z)$ for all $x, y, z$.

We note that easy modifications of our proofs give the same results for the sum of divisors function $\sigma(n)$ in place of $\phi(n)$, since $\sigma(p)=p+1$ for primes $p$.

The authors would like to thank Igor Shparlinski for posing the question to study the divisors of $\phi(n)$.
2. Preliminaries. Throughout this paper, the letters $p$ and $q$, with or without subscripts, will always denote primes. Constants implied by the $O$ and $\ll$ symbols are absolute, unless dependence on a parameter is indicated by a subscript. All constants are effectively computable as well. We denote by $P^{+}(m)$ the largest prime factor of $m$, with the convention that $P^{+}(1)=0$.

Our key lemma, presented below, says roughly that the small prime factors of $\phi(n)$ are quite dense.

Lemma 2.1. For some large constant $C$, if $C / \log x \leq g \leq 1 / 10$ and $1 /(g \log x) \leq \varepsilon \leq 1 / 4$, then the number of $n \leq x$ for which $\phi(n)$ does not have a prime divisor in $\left(x^{g}, x^{g(1+\varepsilon)}\right]$ is $<g^{\varepsilon / 2} \log (1 / g) x$.

Proof. First, we claim that for large $x$ and $w \geq x^{6 g}$,

$$
\begin{align*}
& \mid\left\{p \leq w: p-1 \text { has no prime factor in }\left(x^{g}, x^{g(1+\varepsilon)}\right]\right\} \mid  \tag{2.1}\\
& \qquad \leq\left(1-\frac{2 \varepsilon}{3}\right) \frac{w}{\log w} .
\end{align*}
$$

Let $\pi(w ; r, a)$ be the number of primes $p \leq w$ which satisfy $p \equiv a(\bmod r)$. For positive integer $r$, write

$$
\pi(w ; r, 1)=\frac{\operatorname{li}(w)}{\phi(r)}+E(w ; r),
$$

where

$$
\operatorname{li}(w)=\int_{2}^{w} \frac{d t}{\log t} .
$$

Using the Bombieri-Vinogradov theorem ([2, Ch. 28]) and the Mertens estimates, the number of primes $p \leq w$ such that $p-1$ has a prime factor in $\left(x^{g}, x^{g(1+\varepsilon)}\right]$ is

$$
\begin{aligned}
& \geq \sum_{x^{g}<q \leq x^{(1+\varepsilon) g}} \pi(w ; q, 1)-\sum_{x^{g}<q_{1}<q_{2} \leq x^{(1+\varepsilon) g}} \pi\left(w ; q_{1} q_{2}, 1\right) \\
& =\sum_{q}\left(\frac{\operatorname{li}(w)}{q-1}+E(w ; q)\right)-\sum_{q_{1}, q_{2}}\left(\frac{\operatorname{li}(w)}{\left(q_{1}-1\right)\left(q_{2}-1\right)}+E\left(w ; q_{1} q_{2}\right)\right) \\
& =\operatorname{li}(w)\left[\log (1+\varepsilon)-\frac{1}{2} \log ^{2}(1+\varepsilon)+O\left(\frac{1}{\log ^{2} x^{g}}\right)\right]+O\left(\frac{w}{\log ^{3} w}\right) \\
& \geq \frac{3 \varepsilon}{4} \frac{w}{\log w} .
\end{aligned}
$$

For the last step, we used the fact that $w \geq x^{6 g} \geq e^{6 C}$ and $C$ is sufficiently large. This proves (2.1).

Consider $x / \log x<n \leq x$ such that $\phi(n)$ does not have a prime divisor in $\left(x^{g}, x^{g(1+\varepsilon)}\right]$. We can write $n=q_{1}^{\alpha_{1}} \cdots q_{k}^{\alpha_{k}} m$, where $q_{1}>\cdots>q_{k}>x^{6 g}$, $\alpha_{i} \geq 1$ for $1 \leq i \leq k$ and $P^{+}(m) \leq x^{6 g}$. Then $q_{1}, \ldots, q_{k} \in T$, the set of primes $p$ such that $p-1$ does not have a prime factor in $\left(x^{g}, x^{g(1+\varepsilon)}\right]$. By (2.1) and partial summation,

$$
\begin{aligned}
\sum_{\substack{x^{6 g}<q \leq x \\
q \in T}} \sum_{a \geq 1} \frac{1}{q^{a}} & \leq\left(1-\frac{2 \varepsilon}{3}\right)\left(\log \frac{1}{6 g}+\frac{1}{\log x}\right)+\sum_{q>e^{6 C}} \frac{1}{q(q-1)} \\
& \leq\left(1-\frac{\varepsilon}{2}\right) \log \frac{1}{6 g}
\end{aligned}
$$

for sufficiently large $x$. By Theorem 7 of [10], for some positive constant $c_{0}$ and uniformly in $x \geq z, y \geq 2$, the number of integers $n \leq x$ divisible by a number $m>z$ with $P^{+}(m) \leq y$ is $\ll x \exp \left\{-c_{0} \frac{\log z}{\log y}\right\}$. Consequently, the number of $n$ with $m>x^{1 / 3}$ is $\ll x e^{-c_{0} / 18 g} \ll g x$. For other $n$, we may assume that $m \leq x^{1 / 3}$, and thus $k \geq 1$. Again by Theorem 7 of [10], the number of $n$ with $q_{1} \leq \log ^{10} x$ is $\ll x / \log x \ll g x$. For remaining $n$, we have $q_{1}^{\alpha_{1}-1} \cdots q_{k}^{\alpha_{k}-1} \leq \log ^{2} x$, for otherwise, $q_{1}^{\left\lfloor\alpha_{1} / 2\right\rfloor} \cdots q_{k}^{\left\lfloor\alpha_{k} / 2\right\rfloor} \geq$ $q_{1}^{\left(\alpha_{1}-1\right) / 2} \cdots q_{k}^{\left(\alpha_{k}-1\right) / 2}>\log x$ and the number of $n$ divisible by $d^{2}$ for some $d>\log x$ is $O(x / \log x)$. Hence $q_{1} \cdots q_{k} \geq x^{1 / 2}$. In particular, $q_{1} \geq \max \left(x^{1 /(2 k)}\right.$, $\left.\log ^{10} x\right)$ and $\alpha_{1}=1$. Given $q_{2}^{\alpha_{2}}, \ldots, q_{k}^{\alpha_{k}}$, and $m$, the number of $q_{1}$ is, by the Chebyshev estimates for primes,

$$
\ll \frac{x}{q_{2}^{\alpha_{2}} \cdots q_{k}^{\alpha_{k}} m \log \left(x /\left(q_{2}^{\alpha_{2}} \cdots q_{k}^{\alpha_{k}} m\right)\right)} \ll \frac{k x}{\log x} \frac{1}{q_{2}^{\alpha_{2}} \cdots q_{k}^{\alpha_{k}} m} .
$$

Given $q_{2}^{\alpha_{2}}, \ldots, q_{k}^{\alpha_{k}}$,

$$
\sum_{P^{+}(m) \leq x^{6 g}} \frac{1}{m} \ll \log \left(x^{6 g}\right)=6 g \log x .
$$

With fixed $k$, we have

$$
\begin{aligned}
\sum_{q_{2}, \ldots, q_{k} \in T} \sum_{\alpha_{2}, \ldots, \alpha_{k} \geq 1} \frac{1}{q_{2}^{\alpha_{2}} \cdots q_{k}^{\alpha_{k}}} & \leq \frac{1}{(k-1)!}\left(\sum_{\substack{x^{6 g}<q \leq x \\
q \in T}} \sum_{a \geq 1} \frac{1}{q^{a}}\right)^{k-1} \\
& \leq \frac{\left(\left(1-\frac{\varepsilon}{2}\right) \log \frac{1}{6 g}\right)^{k-1}}{(k-1)!}
\end{aligned}
$$

The total number of such $n$ is

$$
\begin{aligned}
& \ll g x+g x \sum_{1 \leq k \leq 1 /(6 g)} \frac{k}{(k-1)!}\left(\left(1-\frac{\varepsilon}{2}\right) \log \frac{1}{6 g}\right)^{k-1} \\
& \ll g x+g x\left(\log \frac{1}{6 g}\right) \sum_{j=0}^{\infty} \frac{\left(\left(1-\frac{\varepsilon}{2}\right) \log \frac{1}{6 g}\right)^{j}}{j!} \\
& =g x+g x\left(\log \frac{1}{6 g}\right)\left(\frac{1}{6 g}\right)^{1-\varepsilon / 2} \ll g^{\varepsilon / 2}\left(\log \frac{1}{g}\right) x
\end{aligned}
$$

This completes the proof.
REmARKS. Since $\phi(n)$ and $\lambda(n)$ have the same prime factors, Lemma 2.1 holds with $\phi$ replaced by $\lambda$. With a finer analysis, it is possible to remove the factor $\log (1 / g)$ appearing in the conclusion of Lemma 2.1. Also, if $\varepsilon$ is fixed, then $g^{\varepsilon / 2} \log (1 / g)<_{\varepsilon} g^{\varepsilon / 3}$, an inequality we shall use in the application of Lemma 2.1.

We next give a method of constructing integers which are dense in an interval.

LEmma 2.2. Suppose that $h$ is a positive integer, $y \geq h$, and $D$ is $(1+$ $1 / h)$-dense in $[h, y]$. Suppose also that $m=D m_{1} \cdots m_{k}$, where for $1 \leq j \leq k$, $m_{j} \leq(y / h) m_{1} \cdots m_{j-1}$. Then $m$ is $(1+1 / h)$-dense in $\left[h, m_{1} \cdots m_{k} y\right]$.

Proof. By hypothesis, the assertion holds for $k=0$. Suppose the assertion is true for $k=l, m$ satisfies the hypotheses with $k=l+1$, and put $m^{\prime}=D m_{1} \cdots m_{l}$. Then $m^{\prime}$ is $(1+1 / h)$-dense in $\left[h, m_{1} \cdots m_{l} y\right]$. Multiplying the divisors of $m^{\prime}$ by $m_{l+1}$, we find that $m$ is also $(1+1 / h)$-dense in $\left[m_{l+1} h, m_{1} \cdots m_{l+1} y\right]$. Our assumption about $m_{l+1}$ implies that $m$ is $(1+1 / h)$-dense in $\left[h, m_{1} \cdots m_{l+1} y\right]$, as desired.

Lemma 2.3. Given any positive integer $D, n$ is divisible by a prime $q \equiv 1$ $(\bmod D)$ for almost all $n$.

Proof. By a theorem of Landau [11], the number of $n \leq x$ which have no prime factor $q \equiv 1(\bmod D)$ is asymptotic to $c(D) x(\log x)^{-1 / \phi(D)}$ for some constant $c=c(D)$.

Luca and Pomerance [12] have recently proven a stronger statement, namely that for some constant $c_{1}$, for almost all integers $n, \phi(n)$ is divisible by every prime power $\leq c_{1} \frac{\log \log n}{\log \log \log n}$.

## 3. Proof of the theorems

Proof of Theorem 3. Fix $h$ and $\delta$, and let $y$ be sufficiently large, depending on $h$, and such that $y>h^{5}$. Let $D$ be the product of all prime powers $\leq y$. Let $\varepsilon=1 / 4$ and let $Y=(y / h)^{4 / 5}$. Let $C$ be the constant in Lemma 2.1.

Consider the intervals $I_{j}=\left(Y^{(5 / 4)^{j-1}}, Y^{(5 / 4)^{j}}\right](1 \leq j \leq J)$, where $Y \geq$ $e^{C}$. Fix $c$ so that $0<c \leq 1 / 20$, let $x$ be sufficiently large, and take $J$ so that $Y^{(5 / 4)^{J-2}}<x^{c} \leq Y^{(5 / 4)^{J-1}}$. Then $Y^{(5 / 4)^{J}}<\left(Y^{(5 / 4)^{J-2}}\right)^{2}<x^{2 c} \leq x^{1 / 10}$. By Lemma 2.1, if $y$ is large enough, then the number of integers $n \leq x$ for which $\phi(n)$ does not have prime factors in $I_{j}$ is

$$
\ll\left(\frac{\log Y^{(5 / 4)^{j-1}}}{\log x}\right)^{1 / 12} x
$$

Summing over $j$, we find that $\phi(n)$ has a prime factor in every interval $I_{j}$ for all $n \leq x$ except for a set of size

$$
\ll\left(\frac{\log Y^{(5 / 4)^{J}}}{\log x}\right)^{1 / 12} x<(2 c)^{1 / 12} x
$$

If $c$ is small enough, for at least $(1-\delta / 2) x$ of the integers $n \leq x, \phi(n)$ has a prime factor in every interval $I_{j}$. By Lemma 2.3 , for at least $(1-\delta) x$ integers $n \leq x, \phi(n)$ is divisible by a prime $q \equiv 1\left(\bmod D^{3}\right)$ and has a prime factor in every interval $I_{j}$. For each such $n$, let $p_{1}, \ldots, p_{J}$ be primes dividing $\phi(n)$ and such that $p_{j} \in I_{j}$ for $1 \leq j \leq J$. By hypothesis, $p_{3}>(y / h)^{5 / 4}>y$, hence $p_{j} \nmid D$ for $j \geq 3$. Since $D^{3}|(q-1)| \lambda(n) \mid \phi(n)$, we see that $\lambda(n)$ and $\phi(n)$ are each divisible by $D p_{1} \cdots p_{J}$. By definition, $D$ is divisible by every positive integer $\leq y$, hence $D$ is $(1+1 / h)$-dense in $[h, y]$. Also, $p_{1} \leq Y^{5 / 4}=y / h$, and for $j \geq 2$,

$$
p_{j} \leq Y^{(5 / 4)^{j}} \leq Y^{5 / 4} \prod_{1 \leq i \leq j-1} Y^{(5 / 4)^{i-1}} \leq(y / h) p_{1} \cdots p_{j-1}
$$

By Lemma 2.2, $\phi(n)$ and $\lambda(n)$ are $(1+1 / h)$-dense in $\left[h, p_{1} \cdots p_{J} y\right]$. Since $p_{J}>Y^{(5 / 4)^{J-1}} \geq x^{c}$, this concludes the proof.

Proof of Theorems 1 and 4. In view of Theorem 3, there is a positive integer $k$ so that when $z$ is large enough, for more than half of the positive integers $d \leq z, \phi(d)$ and $\lambda(d)$ are $(1+1 / h)$-dense in $\left[h, z^{1 / k}\right]$. Put $\varepsilon=$ $1 /\left(5 k^{2}\right)$, let $x$ be sufficiently large and $x^{1 / 2}<d \leq x^{1 / 2+\varepsilon}$, where $\phi(d)$ is $(1+1 / h)$-dense in $\left[h, x^{1 /(2 k)}\right]$. Consider distinct primes $p_{1}, \ldots, p_{k} \in I:=$ $\left[x^{1 /(2 k)-2 \varepsilon}, x^{1 /(2 k)-\varepsilon}\right]$ which do not divide $d$. Note that

$$
\begin{equation*}
x^{1-2 k \varepsilon} \leq d p_{1} \cdots p_{k} \leq x^{1-(k-1) \varepsilon} \tag{3.1}
\end{equation*}
$$

Let $q$ be a prime not dividing $d p_{1} \cdots p_{k}$ and satisfying

$$
\begin{equation*}
\frac{1}{2} \frac{x}{d p_{1} \cdots p_{k}}<q \leq \frac{x}{d p_{1} \cdots p_{k}} \tag{3.2}
\end{equation*}
$$

so that by (3.1) and the definition of $\varepsilon$,

$$
\begin{equation*}
x^{1 /(6 k)} \leq q \leq x^{2 /(5 k)} \tag{3.3}
\end{equation*}
$$

We claim that for all such numbers $n=d p_{1} \cdots p_{k} q$ satisfying the additional hypothesis

$$
\begin{equation*}
\lambda(n) \geq x^{1-\varepsilon} \tag{3.4}
\end{equation*}
$$

$\phi(n)$ is $(1+1 / h)$-dense in $[h, \phi(n) /(h+1))$ and $\lambda(n)$ is $(1+1 / h)$-dense in $[h, \lambda(n) /(h+1))$. Let $y=x^{1 /(2 k)}$. Note that $\phi(n)=\phi(d)\left(p_{1}-1\right) \cdots\left(p_{k}-1\right)$ $\cdot(q-1), \phi(d)$ is $(1+1 / h)$-dense in $[h, y], p_{i}-1 \leq x^{1 /(2 k)-\varepsilon}<(y / h)(1 \leq i \leq k)$ and $q \leq(y / h)$. By Lemma 2.2 with $D=\phi(d), m_{i}=p_{i}-1(1 \leq i \leq k)$ and $m_{k+1}=q-1, \phi(n)$ is $(1+1 / h)$-dense in $[h, w]$, where $w=y\left(p_{1}-1\right) \cdots\left(p_{k}-1\right)$ - $(q-1)$. By (3.1) and (3.2),

$$
w \geq 2^{-k-1} y p_{1} \cdots p_{k} q \geq 2^{-k-2} \frac{x^{1+1 /(2 k)}}{d} \geq h \sqrt{x}
$$

But $\phi(n)$ is also $(1+1 / h)$-dense in $[\phi(n) / w, \phi(n) /(h+1))$ since $d \mid m \Leftrightarrow$ $(m / d) \mid m$, and consequently $\phi(n)$ is $(1+1 / h)$-dense in $[h, \phi(n) /(h+1))$.

The argument for $\lambda(n)$ is similar, except that now

$$
\lambda(n)=\lambda(d) \frac{q-1}{f} \prod_{i=1}^{k} \frac{p_{i}-1}{f_{i}}
$$

where $f$ is some divisor of $q-1$ and $f_{i}$ is some divisor of $p_{i}-1(1 \leq i \leq k)$. Here we use (3.4), which implies that $f f_{1} \cdots f_{k} \leq x^{\varepsilon}$. By Lemma 2.2 with $D=\lambda(d), m_{i}=\left(p_{i}-1\right) / f_{i}(1 \leq i \leq k)$ and $m_{k+1}=(q-1) / f$, we see that $\lambda(n)$ is $(1+1 / h)$-dense in $[h, w]$, where

$$
w=y \frac{q-1}{f} \prod_{i=1}^{k} \frac{p_{i}-1}{f_{i}} \geq 2^{-k-2} \frac{x^{1+1 /(2 k)-\varepsilon}}{d} \geq h \sqrt{x} .
$$

As with $\phi(n)$, we conclude that $\lambda(n)$ is $(1+1 / h)$-dense in $[h, \lambda(n) /(h+1))$.
Notice that for the above $n$, when $h=1, \phi(n)$ is 2-dense in $[1, \phi(n) / 2)$. Since $\phi(n)$ is a divisor of itself, we conclude that $\phi(n)$ is 2 -dense in $[1, \phi(n))$ and hence 2 -dense. This conclusion also holds for $\lambda(n)$ by similar arguments.

Finally, we show that the number of such integers $n \leq x$ is $>_{h} x$. First, (3.4) holds for almost all $n$ by Theorem 2 of [6]. By the prime number theorem and (3.3), given $d, p_{1}, \ldots, p_{k}$, the number of possible primes $q$ is $>_{k} x /\left(d p_{1} \cdots p_{k} \log x\right)$. We also have

$$
\sum_{p_{1}, \ldots, p_{k} \in I} \frac{1}{p_{1} \cdots p_{k}} \gg_{k} 1
$$

and $\sum 1 / d \gg \log x$ by partial summation. Hence, there are $>_{k} x$ tuples $\left(d, p_{1}, \ldots, p_{k}, q\right)$ with product $n \in(x / 2, x]$ and with $\phi(n)$ and $\lambda(n)$ being $(1+1 / h)$-dense respectively. Given such an integer $n, n$ has at most $6 k$ prime factors $\geq x^{1 /(6 k)}$, hence the number of tuples $\left(d, p_{1}, \ldots, p_{k}, q\right)$ with product $n$ is bounded by a function of $k$. Thus the proof is complete.

Proof of Theorem 2. Suppose $0<c<2 \theta-1$, and let $\varepsilon>0$ be so small that $2 \theta-1-6 \varepsilon>c$. Consider $n=p m \leq x$, where $x^{1-2 \varepsilon}<p \leq x^{1-\varepsilon}$, and $P^{+}(p-1)>p^{\theta-\varepsilon}$. By the definition of $\theta$, there are $\gg z / \log z$ such primes $\leq z$, if $z$ is large enough. Then $\phi(n)$ and $\lambda(n)$ are each divisible by a prime $q$ with $q>x^{(1-2 \varepsilon)(\theta-\varepsilon)}>x^{\theta-3 \varepsilon}$, and therefore neither function has divisors in $\left[x^{1-\theta+3 \varepsilon}, x^{\theta-3 \varepsilon}\right]$. The number of such $n$ is, by partial summation,

$$
=\sum_{\substack{x^{1-2 \varepsilon}<p \leq x^{1-\varepsilon} \\ P^{+}(p-1)>p^{\theta-\varepsilon}}}\left\lfloor\frac{x}{p}\right\rfloor \gg \varepsilon x,
$$

and the proof is complete.
Proof of Corollary 1. Let $\delta>0$. By Theorem 3, if $x$ is sufficiently large, then for at least $(1-\delta) x$ integers $n \leq x$, both $\phi(n)$ and $\lambda(n)$ are $(1+1 / h)$ dense in $\left[h, x^{g(x)}\right]$. Since $\delta$ is arbitrary, the corollary follows.

Proof of Corollary 2. (i) The elementary inequality $\sum_{n \leq x} n / \phi(n) \ll x$ implies that

$$
|\{n \leq x: \phi(n) \leq \varepsilon n\}| \ll \varepsilon x \quad(0<\varepsilon \leq 1)
$$

Consequently, using Theorem 1, if $c$ is small enough then there are $\gg x$ integers $n \leq x$ for which $\phi(n)$ is 2 -dense and $\phi(n) \geq c x$. This proves (i) for $y \leq c x$. For a given constant $f \in[c, 1 / 2]$, it is an elementary fact that $f x<\phi(n) \leq 2 f x$ for $>_{f} x$ integers $n \leq x$. This completes the proof for the remaining $y$.
(ii) From the proof of Theorem 2, for a positive proportion of integers $n$, $\phi(n)$ has no divisors in $\left[x^{c}, x^{1-c}\right]$.
(iii) This follows immediately from Corollary 1.

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