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JCAP02(2016)036

(http://iopscience.iop.org/1475-7516/2016/02/036)

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

δN formalism from superpotential and holography

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Received September 29, 2015
Accepted January 18, 2016
Published February 16, 2016

Abstract. We consider the superpotential formalism to describe the evolution of D scalar fields during inflation, generalizing it to include the case with non-canonical kinetic terms. We provide a characterization of the attractor behaviour of the background evolution in terms of first and second slow-roll parameters (which need not be small). We find that the superpotential is useful in justifying the separate universe approximation from the gradient expansion, and also in computing the spectra of primordial perturbations around attractor solutions in the δN formalism. As an application, we consider a class of models where the background trajectories for the inflaton fields are derived from a product separable superpotential. In the perspective of the holographic inflation scenario, such models are dual to a deformed CFT boundary theory, with D mutually uncorrelated deformation operators. We compute the bulk power spectra of primordial adiabatic and entropy cosmological perturbations, and show that the results agree with the ones obtained by using conformal perturbation theory in the dual picture.

Keywords: inflation, cosmological perturbation theory

ArXiv ePrint: 1509.07339

doi:10.1088/1475-7516/2016/02/036

Contents

1 Introduction

Measurements of the temperature anisotropies and polarization of the cosmic microwave background place stringent constraints on a wide range of inflationary models. While current data are consistent with single-field inflation, multi-field scenarios arise quite naturally in most attempts to embed inflation within a broader theory, and it is therefore important to address this more generic situation.

The amplitude of primordial perturbations is often described in terms of ζ, the curvature perturbation on hypersurfaces of constant energy density. In single-field inflation this quantity is conserved on super-horizon scales, but in multi-field inflation it can evolve after Hubble crossing [1–3]. To take this evolution into account, it is convenient to use the so-called δN formalism [4–8], which gives the correlators of ζ at the end of inflation in terms of the correlators of the scalar-fields fluctuations at Hubble crossing. The relation between them is established by considering the background evolution of an ensemble of homogeneous universes with different initial conditions (for a review, see refs. [9, 10]). This approach is not restricted to the correlators of ζ, but can also be applied to entropy perturbations (see e.g. [11, 12]).

A useful tool for describing the evolution on super-horizon scales is the Hamilton-Jacobi (H-J) formalism first introduced by Salopek and Bond [5] (see also [13, 14]). This was originally developed for the case of D scalar fields ϕ^I (here and below capital latin indices $I, J, K, \ldots$ run from 1 to D) with canonical kinetic terms, but for generality here we shall consider an extension to Lagrangians of the form $P(X^{IJ}, \phi^K)$, where

$$X^{IJ} \equiv -\left(\partial_\mu \phi^I \partial^\mu \phi^J\right)/2\,. \tag{1.1}$$

The first step is to encode the background dynamics in a time independent H-J equation for a "superpotential" W. This object is just the Hubble rate expressed as a function of the field values ϕ^I,

$$W = \frac{1}{2} H\left(\phi^I, c_I\right)\,, \tag{1.2}$$

where a complete solution of the H-J equation contains an equal number of integration constants c_I, which account for the freedom in the initial values of the field momenta. As we shall see, cosmological evolution can then be seen as a "gradient flow" of $W(\phi^I)$ in field space. The purpose of this paper is to further develop the δN formalism by taking advantage of this description.

The superpotential approach is also interesting in connection with the possibility of a holographic description of inflation [15–18]. For the case of de Sitter, this idea was first considered in refs. [19, 20], by analogy with the gauge/gravity duality which holds in asymptotically AdS spaces (see also refs. [21, 22]). Recently, field theories which are dual to de Sitter have been identified for the case of higher spin gravity [23]. On the other hand, for the case of Einstein gravity, the duality remains at an exploratory stage. In the absence of a more concrete realization, a fruitful strategy has been to focus on small deformations of a generic boundary conformal field theory (CFT), with Lagrangian of the form $\mathcal{L} = \mathcal{L}_{CFT} + \sum_I g^I \mathcal{O}_I$. This setup is characterized by a few parameters, such as the central charge of the CFT, and the operator product expansion coefficients for the deformation operators $\mathcal{O}_I$. Such parametrization allows for some explicit calculations, which can be done by using conformal perturbation theory [24–26]. With the identification $g^I = \phi^I$ (up to a proportionality constant), the renormalisation group (RG) flow of the couplings g^I in the boundary theory corresponds to field evolution in the bulk, while the superpotential W plays the role of a c-function for the RG flow.

The case with a single deformation operator $\mathcal{O}$ corresponds to single field inflation, which has been extensively considered in the literature [15–18, 24, 25, 27–46]. In particular, refs. [24, 25] studied the power spectrum and bispectrum of ζ, showing agreement between the boundary and bulk calculations.[1] The four-point correlation function was also computed in ref. [33], recovering the result from the bulk calculation of ref. [47] in the slow-roll regime.

[1]Furthermore, in ref. [25], it was shown that the power spectrum of ζ is conserved at large scales, as expected in the standard cosmological perturbation theory for the case of single field models. The conservation of higher order correlation functions, however, remains an unresolved open issue.

More recently, ref. [26] extended the holographic approach to multi-field inflation by considering a CFT with D mutually uncorrelated deformation operators $\mathcal{O}_I$, and the primordial power spectra for adiabatic and entropy perturbations were computed in conformal perturbation theory. As an application of the methods presented in this paper, here we will compare the results of ref. [26] with a bulk calculation based on the δN technique.

The paper is organized as follows. In section 2 we review the superpotential formalism. We also give a characterization of the attractor behavior of background solutions in terms of the first and second slow-roll parameters. Note, however, that such parameters will not be required to be small in our discussion. In section 3 we review the separate universe approximation, on which the δN formalism is based. As we shall see, the superpotential will be very useful for computing the primordial spectra, particularly in the case when the background is an attractor. The expressions for the primordial spectra of adiabatic and entropy perturbations in terms of the superpotential are given in section 4. In section 5, we compare these results with those recently obtained in ref. [26] from the holographic point of view. In section 6, we elaborate on several explicit models of inflation with a product separable superpotential, which should be dual to a QFT with uncorrelated multi-deformation operators. Finally, we conclude in section 7.

2 Superpotential and background evolution

Consider a scalar field theory in a $(d+1)$-dimensional spacetime, with an action of the form

$$S = \int \mathrm{d}^{d+1}x\sqrt{-g}\, P\left(X^{IJ}, \phi^K\right) . \tag{2.1}$$

The corresponding energy momentum tensor is given by

$$T_{\mu\nu} = P_{IJ}\, \partial_\mu\phi^I \partial_\nu\phi^J + P g_{\mu\nu}, \tag{2.2}$$

where we use the notation

$$P_{IJ} = \frac{\partial P}{\partial X^{IJ}} . \tag{2.3}$$

Since X^{IJ} is symmetric about I and J, so is P_{IJ}. If the gradients of all fields are aligned in the same time-like direction, the energy momentum tensor (2.2) has the form of a perfect fluid, with pressure P and energy density given by

$$\rho = 2P_{IJ}X^{IJ} - P. \tag{2.4}$$

The perfect fluid form will be valid for our background solution, where all fields depend only on time. However, in general the fluid will be imperfect for a perturbed solution.

We assume a flat $(d+1)$-dimensional FRW universe described by the metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2$. The background field equations for $\phi^J(t)$ are given by

$$\left(P_{IJ}\dot\phi^J\right)^{\cdot} + dHP_{IJ}\dot\phi^J - (\partial P/\partial\phi_I) = 0 . \tag{2.5}$$

Here

$$H \equiv \frac{\dot a}{a} \tag{2.6}$$

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is the expansion rate. The Friedmann equation reads

$$H^2 = \frac{2\kappa^2}{d(d-1)}\rho = \frac{2\kappa^2}{d(d-1)}\left(2P_{IJ}X^{IJ} - P\right) , \tag{2.7}$$

and

$$\dot{H} = -\frac{\kappa^2}{d-1}P_{IJ}\dot{\phi}^I\dot{\phi}^J \tag{2.8}$$

where $\kappa^2 \equiv 8\pi G$ is the gravitational constant.

As pointed out by Salopek and Bond in ref. [5] (see also refs. [13, 48–51]), the field equations for canonical scalar fields can be recast into first-order form by introducing a superpotential. Here, we show that the same treatment applies to non-canonical scalar fields. Let us start by defining the momenta

$$\pi_I \equiv \frac{\partial P}{\partial \dot{\phi}^I} = P_{IJ}\dot{\phi}^J. \tag{2.9}$$

In general P_{IJ} can have explicit dependence in field velocities $\dot{\phi}^K$, but we assume that our Lagrangian is non-singular, so that (2.9) can be solved for the field velocities as a function of positions and momenta:

$$\dot{\phi}^I = F^I\left(\phi^J, \pi_K\right). \tag{2.10}$$

Note that the momenta π_J differ from the canonical ones $\Pi_J = \partial\mathcal{L}/\partial\dot{\phi}^J = \sqrt{-g}\,\pi_J$ by a factor of a^d, but with our choice, the relation (2.10) does not contain any explicit time dependence through the scale factor a. We may now use (2.10) in order to remove all occurrences of $\dot{\phi}^I$ in the energy density. Then, substituting

$$H = 2W\left(\phi^K\right) , \tag{2.11}$$

and

$$\pi_I = P_{IJ}\dot{\phi}^J = -\frac{2(d-1)}{\kappa^2}\frac{\partial W}{\partial \phi^I} , \tag{2.12}$$

which satisfies eq. (2.8), into the Friedmann equation (2.7), we obtain a partial differential equation for the superpotential $W(\phi^I)$:

$$W^2 = \frac{\kappa^2}{2d(d-1)}\rho\left[\phi^K, \partial W/\partial\phi^K\right]. \tag{2.13}$$

Following ref. [5], we shall call this the separated, or time independent, H-J equation.[2] The substitutions (2.11) and (2.12) are actually motivated by the form of the momentum constraint in the long wavelength limit [5]. We will discuss this constraint in the next section, where we consider cosmological perturbations [see eq. (3.12)]. Nonetheless, for the time being, we may simply think of (2.11), (2.12) and (2.13) as a convenient reformulation of the background equations of motion. Indeed, in appendix A, we show that for any solution $W(\phi^K)$ of (2.13), eqs. (2.11) and (2.12) generate a solution of the field equations (2.5) and (2.7).

A complete solution of the H-J equation $W(\phi^K, c_K)$ depending on D independent integration constants c_K, can be used for generating any background solution from the first

[2] For the case of canonical scalar fields, it is shown in [5] that the separated H-J equation can be obtained from the Einstein-Hamilton-Jacobi equation, after factoring out the volume of space in Hamilton's principal function $S \propto a^d W$.

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order equations (2.12) and (2.11). The existence of such a solution can be understood as follows. Given some initial data (ϕ_0^K, π_K^0) at $t = t_0$, the background equations of motion can be solved in order to find $H(t; \phi_0^K, \pi_K^0)$. If one of the fields has a monotonic evolution, we can use it as the time variable, and express the initial positions of the rest in terms of their positions at the time t. With this, we have $H = H(\phi^K; \pi_K^0)$. Finally, we may choose $c_K = \pi_K^0$ (or any invertible relation between c_K and π_K^0), leading to $W = H(\phi^K, c_K)/2$.

When the field space metric is non-trivial, one may wish to write the equation of motion (2.5) in a covariant form [52–54] which is manifestly independent of the choice of field space coordinates ϕ^I. As stressed, e.g., in ref. [54], while the covariance is not a physical requirement, it can be very convenient for certain purposes. Here we note that the H-J formulation, given by equations (2.12) and (2.13), involves only first partial derivatives of scalar functions in field space, and so it is automatically covariant.

2.1 Cosmological evolution as a gradient flow

A cosmological solution can be thought of as a trajectory in field space, parametrized by the e-folding number N. For each field ϕ^I, we may define the corresponding beta function β^I as the dimensionless component of the tangent vector $\mathrm{d}/\mathrm{d}N$ in the corresponding direction,

$$\frac{\mathrm{d}}{\mathrm{d}N} = \frac{1}{\kappa}\beta^I \frac{\partial}{\partial \phi^I} \,, \qquad \beta^I \equiv \kappa \frac{\mathrm{d}\phi^I}{\mathrm{d}N} \,. \tag{2.14}$$

The standard "first" slow-roll parameter can then be written in terms of the beta functions as

$$\varepsilon_1 \equiv -\frac{\dot{H}}{H^2} = \frac{\kappa^2(\rho + P)}{(d-1)H^2} = \frac{P_{IJ}\beta^I\beta^J}{(d-1)} \geq 0 \,, \tag{2.15}$$

where the last inequality holds provided that the null energy condition is satisfied. In what follows, we shall assume that P_{IJ} is non-degenerate.

In that case, P_{IJ} can be thought of as a metric, which we may use in order to raise and lower indices. Introducing

$$f \equiv -\ln W, \tag{2.16}$$

eq. (2.12) can be written as

$$\beta_I \equiv P_{IJ}\beta^J = \frac{d-1}{\kappa}\frac{\partial f}{\partial \phi^I} \,, \tag{2.17}$$

which expresses the beta functions as a gradient. From eq. (2.18)

$$\frac{\mathrm{d}f}{\mathrm{d}N} = \frac{\beta^2}{d-1} = \varepsilon_1 > 0 \,, \tag{2.18}$$

and therefore f is monotonically increasing along any cosmological trajectory. Here, we have used the notation $\beta^2 \equiv \beta_I\beta^I$. Also, since we assume that P_{IJ} is positive definite, the last inequality is strict provided that at least one of the fields is moving.

If we interpret the β^I as beta functions for couplings $g^I = \kappa\phi^I$ in the boundary theory, then f plays the role of a c-function which decreases monotonically along the RG flow (note that the RG flow proceeds from the UV to the IR, which is opposite to the direction of cosmological evolution as a function of N). In this interpretation, P_{IJ} corresponds to the Zamolodchicov metric in the space of couplings [55]. Such metric is needed in order to relate the components of the RG flow tangent vector β^I, which have the upper index, to the gradient of a c-function, whose components have the lower index. We shall further elaborate on the dual picture in section 5.

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2.2 Attractor behaviour

As mentioned above, the integration constants c_K in $W(\phi^K, c_K)$ correspond to the freedom of choosing the initial value of the momenta π_I^0 for given initial values of the fields ϕ_0^K. In the present context, the initial conditions for the long wavelength bulk evolution are implemented around the time of horizon crossing. Under a small variation of the integration constants, the gradient flow will change. It is then important to characterize whether such dependence remains at late times, or whether it decays.

First of all, it should be noted that along the field trajectory

$$\frac{\partial W}{\partial c_K} = A^K a^{-d}, \tag{2.19}$$

where A^K are constants and we remind that d is the number of spatial dimensions. For the case of scalar fields with canonical kinetic terms, this result was first derived in ref. [5]. We show in appendix B that the same result generalizes to Lagrangians with non-canonical kinetic terms. Eq. (2.19) tells us that the effect of the integration constants on the Hubble rate ($H = 2W$) decays quite rapidly with the scale factor. Of course, this does not immediately imply that the gradient flow will always have an attractor behaviour. It is not enough that the Hubble rate converges to the unperturbed value, but also the perturbed trajectories should converge to the unperturbed ones. It seems difficult to formulate a sufficient condition for such convergence in the general case.[3] Here, we shall consider a necessary condition, which is also a sufficient condition in the one-field case, or when there is effectively just an adiabatic perturbation by the end of inflation.

The idea is to consider the change in the field momentum, projected on the unperturbed trajectory. The fractional change in $\pi_I \pi^I$ under a small variation Δc_K is given by

$$\Delta \ln\left(\pi_I \pi^I\right) \equiv \delta_1 + \delta_2 = \frac{\pi^I \Delta\pi_I + \pi_I \Delta\pi^I}{\pi_J \pi^J}. \tag{2.20}$$

Here, we use the notation $\pi^I = P^{IJ}\pi_J = \dot{\phi}^I$, and δ_1 and δ_2 correspond, respectively, to the first and second terms in the right hand side of the last equality. For the background to be an attractor, we require that these go to zero as the universe expands,

$$\lim_{a\to\infty} \delta_{1,2} = 0. \tag{2.21}$$

Since the metric P_{IJ} depends on momenta, it will have dependence on c_K, and so the two terms δ_1 and δ_2 can have a somewhat different behaviour. We require that the condition (2.21) should hold separately for the two terms, so that the projection of both $\Delta\pi^I$ and $\Delta\pi_I$ on the unperturbed trajectory can be considered to be small.

Using eq. (2.12) it is straightforward to show that

$$\pi^I \Delta\pi_I = -\frac{4(d-1)}{\kappa^2} W \frac{\mathrm{d}}{\mathrm{d}N}\left(\frac{\partial W}{\partial c_K}\right)\Delta c_K. \tag{2.22}$$

[3] Attractor behaviour cannot be formulated in the true phase space, as emphasized by Remmen and Carroll [56]. Our condition here will be formulated in the (ϕ^I, π_I) space, where $\pi_I = P_{IJ}\dot{\phi}^J$. As pointed out in [56], for a single field with canonical kinetic term, this is an "effective" phase space under Hamiltonian evolution, in the sense that the Hamiltonian vector field of phase space can be uniquely mapped to a vector field in the effective phase space.

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Using (2.19) in (2.22), and (2.7), we have

$$\pi^I \Delta\pi_I = (\partial_{c_K}\rho)\,\Delta c_K \equiv \Delta\rho. \tag{2.23}$$

For later convenience, here we have used the Friedmann equation to write W^2 in terms of the energy density ρ (which depends on the c_K only through the kinetic variables X^{IJ}). Using (2.19) again in order to evaluate the derivative with respect to c_K, we have

$$\delta_1 = \frac{2d}{H\varepsilon_1 a^d}\Delta W_0, \tag{2.24}$$

where $\Delta W_0 = A^J \Delta c_J$ is the change in W at the initial time due to the variation Δc_J.

From (2.24) we find that the first condition (2.21) for the attractor behavior, concerning the behavior of δ_1, will be satisfied provided that

$$d - \varepsilon_1 + \varepsilon_2 > p \geq 0\,. \tag{2.25}$$

Here we have introduced the second slow-roll parameter

$$\varepsilon_2 \equiv \frac{\mathrm{d}\ln\varepsilon_1}{\mathrm{d}N} = \frac{\dot\varepsilon_1}{\varepsilon_1 H}\,, \tag{2.26}$$

and a positive constant p. Eq. (2.25) guarantees that δ_1 decays faster than a^{-p}. If p is small, then the approach to the attractor can be slow, requiring a time-scale of the order $\Delta N \sim p^{-1}$ e-foldings. An efficient approach to the attractor on the Hubble time-scale requires $p \gtrsim 1$.

In addition, we need to consider the behaviour of δ_2, which may place additional restrictions. Since $\pi_I\pi^I = (\rho + P)$, from (2.20) and (2.23) we immediately obtain

$$\pi_I \Delta\pi^I = (\partial_{c_K}P)\,\Delta c_K \equiv \Delta P. \tag{2.27}$$

Note that the pressure P only depends on the c_K through the kinetic variables X^{IJ}, and therefore

$$\partial_{c_K}P = P_{IJ}\,\partial_{c_K}X^{IJ}. \tag{2.28}$$

The same is true of the energy density,

$$\partial_{c_K}\rho = \rho_{IJ}\,\partial_{c_K}X^{IJ}, \tag{2.29}$$

where $\rho_{IJ} = P_{IJ} + 2X^{KL}P_{KL,IJ}$ is the symmetrized partial derivative of ρ with respect to X^{IJ}. Since in general $\rho_{IJ} \neq P_{IJ}$, δ_2 can behave differently than δ_1.

For instance if the background solution satisfies the relation

$$\Delta P = c_s^2\,\Delta\rho, \tag{2.30}$$

for some function c_s^2, then we have

$$\delta_2 = \frac{\pi_I\Delta\pi^I}{\pi_J\pi^J} = c_s^2\delta_1 = \frac{2c_s^2 d}{H\varepsilon_1 a^d}\Delta W_0. \tag{2.31}$$

The second condition in (2.21), concerning the behavior of δ_2 will then be satisfied provided that

$$d - \varepsilon_1 + \varepsilon_2 - s > p \geq 0\,. \tag{2.32}$$

This differs from (2.25) by the last term in the left hand side of the inequality, which is defined as

$$s \equiv \frac{\mathrm{d}\ln c_s^2}{\mathrm{d}N}. \tag{2.33}$$

Although (2.30) is not completely general, it does cover some important cases. For instance, if the Lagrangian is linear in X^{IJ}, then the metric $P_{IJ} = G_{IJ}(\phi^I)$ is just a function of the fields, and (2.30) is satisfied with the speed of sound $c_s^2 = 1$. In this case $\delta_1 = \delta_2$ and the conditions (2.25) and (2.32) coincide. Another interesting example where (2.30) is satisfied with a non-trivial speed of sound $c_s \neq 1$ is the case of a multi-field DBI action, which is discussed in [57].[4] Finally, eq. (2.30) also applies to generic one field models. It is easy to check that in such case, the attractor condition (2.32) is related to the absence of a growing mode for the curvature perturbation on uniform field hypersurfaces [58].

For a generic multi-field model the condition that δ_2 vanishes in the asymptotic future will be more elaborate than (2.32), and should be worked out on a case by case basis. We leave this as a subject for further research.

3 Separate universe approximation and δN formalism

The approximate homogeneity and isotropy of cosmological evolution entails a useful relation between the curvature perturbation and the differential e-folding number, which is used in the so-called δN formalism. Let us start by reviewing the conditions for the realisation of a universe with such approximate symmetries. The superpotential formalism will be useful in clarifying the conditions for the validity of this approach, particularly in the implementation of the momentum constraint.

3.1 Separate universe approximation

The δN method is based on the "separate universe assumption". This is the statement that when a characteristic (physical) scale L of fluctuations is much bigger than the Hubble length H^{-1}, i.e. $L \gg H^{-1}$, each region of the universe of Hubble size evolves as a locally homogeneous and isotropic FRW universe. This is justified if each Hubble patch is approximately homogeneous and isotropic up to corrections of order $\epsilon \ll 1$, with

$$\epsilon \equiv \frac{1}{LH}\,.$$

Then, since different Hubble patches are causally disconnected for local theories, they should evolve independently of one another.

Consider the ADM line element,

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\mathrm{d}t\right)\,, \tag{3.1}$$

where α and β^i are the lapse function and the shift vector, and γ_{ij} is the spatial metric. Here, the shift vector β^i with the index i should be distinguished from the beta function β^I

[4] More generally, when $P_{KL,IJ}$ is given by

$$P_{KL,IJ} = f_1 P_{IJ}P_{KL} + f_2\left(P_{IK}P_{JL} + P_{IL}P_{JK}\right)\,,$$

where f_1 and f_2 are functionals of ϕ^I and X^{IJ}, ΔP and $\Delta\rho$ are linearly related as in eq. (2.30). In such a case, the second attractor condition can be formulated as (2.32).

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with the index I. We parametrize γ_{ij} as

$$\gamma_{ij} = a^2 e^{2\mathcal{R}(x)} \left[e^{h(x)}\right]_{ij} , \qquad \mathrm{tr}\, h = 0 . \tag{3.2}$$

The time-like congruence orthogonal to $t = const.$ slices has a unit tangent vector given by $n^\mu = \alpha^{-1}(1, -\beta^i)$, and its expansion K is given by

$$K \equiv \nabla_\mu n^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} n^\mu\right) = \alpha^{-1} \left[d \left(H + \dot{\mathcal{R}}\right) - D_i \beta^i\right] , \tag{3.3}$$

where D_i is the covariant derivative defined with respect to the spatial metric γ_{ij}. The e-folding number may be defined as the integral of the expansion along the normal congruence:[5]

$$\mathcal{N} \equiv \frac{1}{d} \int K\, \alpha\, \mathrm{d}t . \tag{3.4}$$

In what follows, the metric is assumed to take the FRW form in the long wavelength limit $\epsilon \to 0$, and we shall use the gauge where the traceless matrix h is also transverse,

$$\partial^i h_{ij} = 0. \tag{3.5}$$

For scalar fields, the anisotropic stress $T_{ij} - (1/d)\gamma^{kl} T_{kl} \gamma_{ij}$ is of order $\mathcal{O}(\epsilon^2)$. As shown in ref. [59], using Einstein's equations, after neglecting terms of lower order in ϵ, which decay like a^{-d}, we have

$$\partial_j \beta^i = \mathcal{O}\left(\epsilon^2\right) , \tag{3.6}$$

and

$$\dot{h}_{ij} = \mathcal{O}\left(\epsilon^2\right) . \tag{3.7}$$

Using these conditions and introducing the derivative with respect to the number of e-foldings $\mathcal{N}$,

$$\partial_{\mathcal{N}} \equiv \frac{d}{K\alpha} \partial_t , \tag{3.8}$$

the Einstein equations and the field equations of the scalar fields read [60]

$$K^2 = \frac{2\kappa^2}{d(d-1)} \left(P_{IJ} K^2 \partial_{\mathcal{N}} \phi^I \partial_{\mathcal{N}} \phi^J - d^2 P\right) + \mathcal{O}\big(\epsilon^2\big) , \tag{3.9}$$

$$\partial_{\mathcal{N}} K = -\frac{\kappa^2}{d-1} K P_{IJ} \partial_{\mathcal{N}} \phi^I \partial_{\mathcal{N}} \phi^J + \mathcal{O}\big(\epsilon^2\big) , \tag{3.10}$$

$$K \partial_{\mathcal{N}} \big(P_{IJ} K \partial_{\mathcal{N}} \phi^J\big) + d K^2 P_{IJ} \partial_{\mathcal{N}} \phi^J - d^2 \left(\partial P / \partial \phi^I\right) = \mathcal{O}\big(\epsilon^2\big) . \tag{3.11}$$

At the leading order in the gradient expansion, these equations coincide with the background field equations, where K should be understood as dH for this comparison. For definiteness, in what follows we shall refer to eqs. (3.9)–(3.11), together with eqs. (3.6) and (3.7), as the separate universe approximation.

[5] Alternatively, we may define the co-moving e-folding number $\mathcal{N}_c \equiv \int (K_c/d) \mathrm{d}\tau$. Here $\mathrm{d}\tau = \sqrt{\alpha^2 - \beta_i \beta^i} \mathrm{d}t$ is the element of proper time along a co-moving worldline, and K_c is the expansion of the co-moving congruence, whose unit tangent vector is given by $n_c^\mu = (\alpha^2 - \beta^i \beta_i)^{-1/2}(1, \vec{0})$. It is straightforward to check that for $\partial_i \beta^i = \mathcal{O}(\epsilon^2)$, $\mathcal{N}_c \approx \mathcal{N}$, up to terms quadratic in the spatial gradients.

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3.2 Momentum constraint

The momentum constraint has no counterpart in the background field equations, and so it might enforce additional requirements for the validity of the separate universe approximation. The momentum constraint is given by [60]

$$\partial_i K = -\frac{\kappa^2}{d-1} K P_{IJ} \partial_{\mathcal{N}} \phi^I \partial_i \phi^J + \mathcal{O}\left(a\epsilon^3\right) . \tag{3.12}$$

A factor of a is inserted in $\mathcal{O}(a\epsilon^3)$, since here we are considering the comoving derivative of the extrinsic curvature, while in our conventions a factor of ϵ corresponds to a physical gradient. On the other hand, taking the spatial derivative of the Hamiltonian constraint (3.9) and using eqs. (3.10) and (3.11), we obtain

$$\partial_i K = -\frac{\kappa^2}{d-1} K P_{IJ} \partial_{\mathcal{N}} \phi^I \partial_i \phi^J + B_i + \mathcal{O}\left(a\epsilon^3\right) , \tag{3.13}$$

where B_i is defined as

$$B_i \equiv \frac{\kappa^2 K}{(d-1)\partial_{\mathcal{N}} \ln(e^{d\mathcal{N}} K)} \left[\partial_{\mathcal{N}} \phi^I \partial_i \left(P_{IJ} \partial_{\mathcal{N}} \phi^J\right) - \partial_{\mathcal{N}} \left(P_{IJ} \partial_{\mathcal{N}} \phi^J\right) \partial_i \phi^I\right] . \tag{3.14}$$

Here we used eq. (3.10) to rewrite the denominator on the right hand side. Comparing (3.12) and (3.13) we see that the consistency of the Hamiltonian and momentum constraint requires that

$$a^{-1} B_i = O\left(\epsilon^3\right) . \tag{3.15}$$

Sugiyama, Komatsu and Futamase [59] pointed out that the condition (3.15) is automatically satisfied under the slow-roll approximation. Here, we argue that this conclusion is not restricted to slow-roll, but follows more generally from the attractor behaviour discussed in subsection 2.2. Indeed, repeating the same argument as in the background, we can express the field equations at the leading order of the gradient expansion with the use of the superpotential as

$$P_{IJ} \partial_{\mathcal{N}} \phi^J = -\frac{d-1}{\kappa^2} \frac{\partial \ln W}{\partial \phi^I} + \mathcal{O}\left(\epsilon^2\right) , \tag{3.16}$$

where the superpotential is now related to the extrinsic curvature by $K = 2dW + \mathcal{O}(\epsilon^2)$. In the attractor regime, the dependence of the superpotential $W(\phi^K, c_K)$ on the integration constants c_K can be neglected. Then, substituting eq. (3.16) in eq. (3.14), the leading terms in the two expressions within round brackets in the left hand side of eq. (3.14) cancel each other, and we are left with $a^{-1} B_i = \mathcal{O}(\epsilon^3)$.

Beyond the attractor regime, we need to consider the dependence of W in the constants c_K. In that case, substituting eq. (3.16) in eq. (3.14) we have

$$a^{-1} B_i = -\frac{2d\, W}{\partial_{\mathcal{N}} \ln(e^{d\mathcal{N}} W)} \frac{\mathrm{d}}{\mathrm{d}\mathcal{N}} \left(\frac{\partial \ln W}{\partial c_J}\right) \left(a^{-1} \partial_i c_J\right) + O\left(\epsilon^3\right) , \tag{3.17}$$

where the total derivative with respect to $\mathcal{N}$ is taken along the dynamical trajectories at fixed c_K. From eq. (2.19),

$$\frac{\partial W}{\partial c_J} = A^J e^{-d\mathcal{N}} + O\left(\epsilon^2\right) , \tag{3.18}$$

and after some simple algebra, we have

$$a^{-1}B_i = 2d\,a^{-d}A^J\,\frac{\partial_i c_J}{a} + O(\epsilon^3)\,. \tag{3.19}$$

The first term in the right hand side contains only one spatial derivative, and so it is naively of order ϵ in the gradient expansion, rather than ϵ^3. Thus, unless the c_K are constant in space, it might seem that the momentum constraint (3.12) is inconsistent with the spatial derivative of the Hamiltonian constraint given in eq. (3.13).[6] Note, however, that the term of order ϵ is accompanied by a decaying function which scales as[7] a^{-d}, while the terms of order ϵ^3 include gradients of non-decaying contributions. In particular, for spatial dimension $d > 2$, the first term in (3.19) falls off with physical wavelength faster than ϵ^3, and hence it can be safely ignored for present purposes. For any given co-moving scale, the initial conditions for the long wavelength evolution are generated at horizon crossing, and the first term in (3.19) will be negligible soon after that.

As noted in ref. [59], the momentum constraint can always be satisfied in the gradient expansion by modifying (3.6), so that β^i includes terms of lower order in ϵ, starting at $a\beta^i = O(\epsilon^{-1})$. As mentioned before eq. (3.6), such lower order terms can be shown to decay as a^{-d} for matter whose anisotropic stress is of order ϵ^2. Using the Hamiltonian and momentum constraints at the linearized order in the flat slicing we can obtain the relation [59][8]

$$\partial_i\partial_j\beta^j + B_i = 0\,, \tag{3.20}$$

where we used that

$$\partial_t = \left(H - \frac{1}{d}\partial_i\beta^i\right)\partial_{\mathcal{N}}\,, \tag{3.21}$$

which follows from eqs. (3.3) and (3.8), and assuming a flat gauge where $\mathcal{R} = 0$. Using eq. (3.20), we can relate the term of order ϵ^{-1} in the expansion of the shift vector β^i to the first term on the right hand side of (3.19):

$$\partial_i\beta^i = -\frac{2d}{a^d}\Delta W_0 + O(\epsilon^2)\,, \tag{3.22}$$

where $\Delta W_0 \equiv A^J\Delta c_J$ is a slowly varying function of position. After this mode decays, $a^{-1}B_i$ is formally of order ϵ^3 and $\partial_i\beta^j = O(\epsilon^2)$, consistent with the separate universe approximation (3.6).

3.3 δN formalism

In the separate universe approximation, the last term in square brackets of eq. (3.3) vanishes as

$$D_i\beta^i = \mathcal{O}(\epsilon^2) \tag{3.23}$$

[6] For instance, it was stated in [5] that in single-field models the integration constant in the solution of the H-J equation should be constant in space for consistency between the Hamiltonian and momentum constraint. But as we argue here, this is not necessarily the case.

[7] For the case of canonical fields, such scale dependence of the leading term in the gradient expansion of B_i was first derived in ref. [59].

[8] When we include the second term in eq. (3.21), our equation (3.20) differs from the equation (49) in ref. [59].

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in the long wavelength limit. Inserting eq. (3.3) into eq. (3.4) and neglecting $\mathcal{O}(\epsilon^2)$ terms we obtain

$$\delta N(t_2,\, t_1; \boldsymbol{x}) \equiv \mathcal{N}(t_2,\, t_1; \boldsymbol{x}) - N(t_2,\, t_1) \simeq \mathcal{R}(t_2, \boldsymbol{x}) - \mathcal{R}(t_1,\, \boldsymbol{x})\,, \tag{3.24}$$

where N is the e-folding number for the unperturbed background evolution.

With the help of eq. (3.24), one can map the spatial distribution of the scalar fields near the time when the relevant scale crosses the horizon, $\delta\phi^I_*(\boldsymbol{x})$, to the curvature perturbation at some final space-like hypersurface Σ_e near the end of inflation, $\mathcal{R}(t_e, \boldsymbol{x})$. The spatial distribution of the scalar fields should be specified on the "initial" space-like hypersurface Σ_* where the spatial curvature vanishes,

$$\mathcal{R}(t_*, \boldsymbol{x}) = 0. \tag{3.25}$$

For the final hypersurface Σ_e we have some freedom. For instance, we may take it to be a hypersurface of constant energy density,

$$\rho(t_{\rho_e}, \boldsymbol{x}) = \rho_e = \text{const}. \tag{3.26}$$

Note that eqs. (3.25) and (3.26) define the times $t_*(\boldsymbol{x})$ and $t_{\rho_e}(\boldsymbol{x})$ as implicit functions of position. Alternatively, by a suitable choice of coordinates the hypersurfaces Σ_* and Σ_e can be made to coincide with initial and final $t = const.$ slices. The adiabatic curvature perturbation is defined as:

$$\zeta(t_{\rho_e}, \boldsymbol{x}) \equiv \mathcal{R}(t_{\rho_e}, \boldsymbol{x}). \tag{3.27}$$

Using $t_1 = t_*$ and $t_2 = t_{\rho_e}$ in eq. (3.24), from (3.25) and (3.27) we obtain the familiar relation between ζ and the differential e-folding number

$$\zeta(t_{\rho_e}, \boldsymbol{x}) = \delta N(t_{\rho_e}, t_*; \boldsymbol{x}) \equiv \delta N\big(t_{\rho_e}; \delta\phi^J_*(\boldsymbol{x})\big)\,. \tag{3.28}$$

For the last equality we used the separate universe approximation and assumed that the inhomogeneity in the e-folding number can be determined from the initial distribution of the fields $\delta\phi^J_*(\boldsymbol{x}) = \delta\phi^J(t_*,\, \boldsymbol{x})$. More precisely, the e-folding number from the initial to the final hypersurfaces should be computed by solving the Friedmann equation in each Hubble patch. In principle, this requires (ϕ^J_*, π_{J*}) as initial conditions, so the initial time derivative of the field distribution is also needed. However, as discussed in the previous section, when the trajectory is an attractor, the dependence on the initial time derivative dies off rapidly and then the initial distribution can be expressed only in terms of ϕ^I_*. For the one field case, this is discussed in detail in the following subsection.

If the inflationary trajectories in field space converge to a unique one [6, 7], then the adiabatic curvature perturbation ζ becomes subsequently constant in time, i.e. independent of the value of the density ρ_e which defines the final hypersurface. However, in general, we also need to deal with entropy modes at the final hypersurface. These can be calculated along similar lines as ζ.

In particular, choosing the uniform field slicing with $\phi^I = \phi^I_e = const.$ as the final hypersurface, which we may denote by $\Sigma_{\phi^I_e}(\phi^K_e)$, one obtains a set of D independent gauge invariant variables $\zeta^{(I)}$ which represent the curvature perturbation in the different $\delta\phi^I = 0$ slicings (see for instance [11, 12]):

$$\zeta^{(I)}(\phi^K_e,\, \boldsymbol{x}) \equiv \mathcal{R}\left(\Sigma_{\phi^I_e}, \boldsymbol{x}\right) = \delta N\left(\Sigma_{\phi^I_e},\, \Sigma_*; \boldsymbol{x}\right) \equiv \delta N^{(I)}\left(\phi^K_e; \delta\phi^J_*(\boldsymbol{x})\right). \tag{3.29}$$

Expanding $\zeta^{(I)}(t_e, \boldsymbol{x})$ in terms of the scalar field fluctuations $\delta\phi_*^J(\boldsymbol{x})$, we obtain

$$\zeta^{(I)}(t_e, \boldsymbol{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} N^{(I)}_{J_1 \dots J_n} \delta\phi_*^{J_1}(\boldsymbol{x}) \cdots \delta\phi_*^{J_n}(\boldsymbol{x}), \tag{3.30}$$

with

$$N^{(I)}_{J_1 \dots J_n} \equiv \frac{\partial^n N^{(I)}}{\partial\phi_*^{J_1} \dots \partial\phi_*^{J_n}}. \tag{3.31}$$

Such partial derivatives of $N^{(I)}$ can be determined from the solution of the equation of motion for the local FRW universe as a function of initial field values ϕ_*^J in the vicinity of the background solution.

The relative entropy perturbation S^{IJ} is defined as the difference between two $\zeta^{(I)}$s (see e.g. [12, 61]),

$$S^{IJ}(t, \boldsymbol{x}) \equiv d\left[\zeta^{(I)}(t, \boldsymbol{x}) - \zeta^{(J)}(t, \boldsymbol{x})\right], \tag{3.32}$$

which can of course be expanded in powers of $\delta\phi_*^I$ by substituting eq. (3.30).

Finally, we note that the uniform Hubble (or energy density) slicing is given by

$$0 = \frac{\delta H}{H} = \sum_{I=1}^{D} \frac{\partial \ln W}{\partial \phi^I} \delta\phi^I = -\frac{\kappa}{d-1} \sum_{I=1}^{D} \beta_I \delta\phi^I . \tag{3.33}$$

On the other hand, at linear order, the curvature perturbation $\mathcal{R}$ changes under the time shift $t \to t + \delta t$ as $\mathcal{R} \to \mathcal{R} - H\delta t$, while the field fluctuations $\delta\phi^I$ change as $\delta\phi^I \to \delta\phi^I - (\beta^I/\kappa)H\delta t$. Using these transformations and eqs. (3.27) and (3.29), we have

$$\zeta = \mathcal{R} - \kappa \sum_{I=1}^{D} \frac{\beta_I}{\beta^2} \delta\phi^I = \sum_{I=1}^{D} \frac{\beta_I \beta^I}{\beta^2} \zeta^{(I)} , \qquad \zeta^{(I)} = \mathcal{R} - \kappa \frac{\delta\phi^I}{\beta^I} . \tag{3.34}$$

Hence, at linear order we can relate the curvature perturbations ζ and $\zeta^{(I)}$ by means of a simple expression involving the β functions.

3.4 Linearized perturbations in one field models

In eq. (3.28) we have neglected the dependence of δN on the initial momenta π_*^I. Let us show more explicitly how this approximation is justified in the simple example of a one field model. First, let us show that the full linearized solution is recovered from the δN computation. The e-folding number can be written as

$$N = \int_{\phi_*}^{\phi_e} \frac{H}{\dot{\phi}} \mathrm{d}\phi. \tag{3.35}$$

In this subsection, where we focus on one field models, we omit the index for ϕ^I and π_I. The variation δN is due to the variation of the initial value of the field $\delta\phi_*$, and to the variation of the initial momentum $\delta\pi_*$. The latter corresponds to a variation of the integration constant Δc_ϕ in the solution of the H-J equation. Thus, we have

$$\delta N = -\frac{H_* \delta\phi_*}{\dot{\phi}_*} + \delta N_2, \tag{3.36}$$

where the first term is constant, and corresponds to the well known constant solution for ζ on superhorizon scales, while the second term is given by

$$\delta N_2 = \Delta c_\phi \int_{\phi_*}^{\phi_e} \frac{\dot\phi}{H} \frac{\mathrm{d}}{\mathrm{d}c_\phi}\left(\frac{H}{\dot\phi}\right) \mathrm{d}N. \tag{3.37}$$

This can be cast in the form

$$\delta N_2 = \int \left(\frac{\Delta W}{W} - \delta_2\right) \mathrm{d}N, \tag{3.38}$$

where δ_2 is defined in (2.31). Using (2.19), we have

$$\frac{\Delta W}{W} = \frac{2}{Ha^d}\Delta W_0. \tag{3.39}$$

Now, by keeping track of the last term in (3.3), which is decaying and has been neglected in the relation (3.28) between ζ and δN, we get

$$\zeta = \delta N + \frac{1}{d}\int \frac{\partial_i \beta^i}{H}\mathrm{d}N = \delta N - \int \frac{\Delta W}{W}\mathrm{d}N. \tag{3.40}$$

In the last step we have used eq. (3.22), and we have kept only the term of order ϵ^0 in the gradient expansion, neglecting the terms of order ϵ^2. Combining with eq. (3.37) we obtain

$$\zeta_2 = -\int \delta_2 \mathrm{d}N = -2d\Delta W_0 \int \frac{c_s^2}{H\varepsilon_1 a^d}\mathrm{d}N. \tag{3.41}$$

This coincides with the long wavelength solution of the standard linearized equation of motion for perturbations [58], as expected. If the background is an attractor, then δ_2 is exponentially decaying with N, and so by choosing our initial time appropriately, ζ_2 can be neglected altogether. This is the standard decaying mode of the curvature perturbation. In the single-field case, this confirms that we can neglect the dependence of δN on the initial momentum π_*.

4 Primordial spectra from superpotential

In this section we consider the spectra of the curvature and entropy perturbations, respectively ζ and S^{IJ}, at $t = t_e$ in the case of a separable product superpotential. Let us start by considering the more elementary cross spectra for the $\zeta^{(I)}$, which are defined in Fourier space as

$$\langle \zeta^{(I)}(\boldsymbol{k})\zeta^{(J)}(\boldsymbol{k}')\rangle \equiv (2\pi)^d \delta(\boldsymbol{k}+\boldsymbol{k}')\mathcal{P}_\zeta^{(IJ)}(k)\,. \tag{4.1}$$

Note that $\mathcal{P}_\zeta^{(IJ)}$ is related to the power spectrum of $\delta\phi^I$ at the time t_*,

$$\langle \delta\phi_*^I(\boldsymbol{k})\delta\phi_*^J(\boldsymbol{k}')\rangle \equiv (2\pi)^d \delta(\boldsymbol{k}+\boldsymbol{k}')\mathcal{P}_{\phi_*}^{IJ}(k)\,, \tag{4.2}$$

by

$$\mathcal{P}_\zeta^{(IJ)}(k) = N_K^{(I)} N_L^{(J)} \mathcal{P}_{\phi_*}^{KL}(k)\,, \tag{4.3}$$

where we have used eq. (3.30).

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Likewise, we can consider higher order correlation functions. The bispectrum of ζ is defined by

$$\langle\zeta^{(I)}(\boldsymbol{k}_1)\zeta^{(J)}(\boldsymbol{k}_2)\zeta^{(K)}(\boldsymbol{k}_3)\rangle \equiv (2\pi)^d\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathcal{B}_\zeta^{(IJK)}(k_1,\,k_2,\,k_3)\,. \tag{4.4}$$

This can be decomposed into two contributions,

$$\mathcal{B}_\zeta^{(IJK)}(k_1,\,k_2,\,k_3) = \mathcal{B}_{\zeta,\mathrm{sub}}^{(IJK)}(k_1,\,k_2,\,k_3) + \mathcal{B}_{\zeta,\mathrm{super}}^{(IJK)}(k_1,\,k_2,\,k_3)\,. \tag{4.5}$$

The first one corresponds to the intrinsic non-Gaussianity generated until around $t = t_*$ and is related to the 3-point function of the field perturbations at horizon crossing by

$$(2\pi)^d\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathcal{B}_{\zeta,\mathrm{sub}}^{(IJK)}(k_1,\,k_2,\,k_3) = N_{J_1}^{(I)}N_{J_2}^{(J)}N_{J_3}^{(K)}\langle\delta\phi_*^{J_1}(\boldsymbol{k}_1)\delta\phi_*^{J_2}(\boldsymbol{k}_2)\delta\phi_*^{J_3}(\boldsymbol{k}_3)\rangle\,. \tag{4.6}$$

The second one is generated by the super-horizon nonlinear evolution. Using Wick's theorem, it is given by

$$\mathcal{B}_{\zeta,\mathrm{super}}^{(IJK)}(k_1,\,k_2,\,k_3) = N_{J_1}^{(I)}N_{J_2}^{(J)}N_{J_3J_4}^{(K)}\mathcal{P}_{\delta\phi_*}^{J_1J_3}(k_1)\mathcal{P}_{\delta\phi_*}^{J_2J_4}(k_2) + (2\text{ perms})\,, \tag{4.7}$$

where we have used the symmetry of $N_{J_3J_4}^{(K)}$ with respect to the lower indices.

4.1 Separable product superpotential

Let us focus on the case where the Lagrangian in eq. (2.1) is such that the metric P_{IJ} is diagonal and each element I only depends on the kinetic term $X^I \equiv X^{II}$ and on the field ϕ^I,

$$P_{IJ} = \delta_{IJ}K_I\left(X^I,\phi^I\right)\,, \qquad X^I = -\frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I\,. \tag{4.8}$$

Moreover, we will assume that the superpotential is given by a separable product, i.e.,

$$W\left(\phi^I\right) = \prod_{I=1}^{D} W^{(I)}\left(\phi^I\right)\,. \tag{4.9}$$

By the assumptions (4.8) and (4.9), eq. (2.12) becomes

$$K_I\left(\phi^I,\pi_I\right)\frac{\mathrm{d}\phi^I}{\mathrm{d}N} = -\frac{d-1}{\kappa^2}\frac{\partial\ln W^{(I)}\left(\phi^I\right)}{\partial\phi^I}\,, \tag{4.10}$$

where in the argument of K_I we have traded the field velocities $\dot{\phi}^I$ by their expression in terms of fields and conjugate momenta, through eq. (2.10) (which is also separable in this case). Note that the momentum $\pi_I \propto \partial W/\partial\phi^I$ will actually be a function of all fields ϕ^J. Hence, for the purpose of making eq. (4.10) separable, we shall further restrict to the two following cases.

The first case is when the kinetic term is linear in X^{II} so that $K_I = K_I(\phi^I)$. In that case, we can always use a field redefinition to bring it to the canonical form

$$P\left(X^I,\phi^I\right) = \sum_{I=1}^{D} X^I - V\left(\phi^I\right)\,. \tag{4.11}$$

The second case of interest is the one field case, $D = 1$, in which case we can use eq. (2.12) to express π_ϕ as a function of ϕ, for an arbitrary $P(X, \phi)$.

In these two cases, solving for the velocity and using the definition of beta function (2.14), one obtains

$$\beta^I \equiv \kappa \frac{\mathrm{d}\phi^I}{\mathrm{d}N} = \beta^I \left(\phi^I\right). \tag{4.12}$$

Hence the evolution of ϕ^I as a function of e-folding number N can be determined without being affected by the other scalar fields. The e-folding number between the slicing $\delta\phi^I = 0$ at $t = t_e$ and the flat slicing at $t = t_*$ is then given by

$$N^{(I)}(t_e, t_*) = \int_{t_*}^{t_e} H \mathrm{d}t = \int_{\phi_*^I}^{\phi_e^I} \frac{\kappa \mathrm{d}\phi^I}{\beta^I\left(\phi^I\right)}. \tag{4.13}$$

If the trajectory is an attractor, then after the decaying mode becomes negligibly small, the variation in the e-folding number is given by

$$\delta N^{(I)} = -\frac{\kappa \delta\phi_*^I}{\beta_*^I}, \tag{4.14}$$

with $\beta_*^I \equiv \beta^I(\phi_*^I)$. From this expression we can calculate the derivatives of $N^{(I)}$ with respect to the variations in $\delta\phi_*^J$:

$$N_J^{(I)} = -\frac{\kappa}{\beta_*^I} \delta^I{}_J, \tag{4.15}$$

$$N_{J_1 J_2}^{(I)} = \frac{\kappa}{(\beta_*^I)^2} \frac{\mathrm{d}\beta_*^I}{\mathrm{d}\phi_*^I} \delta^I{}_{J_1} \delta_{J_1 J_2} = \frac{\kappa^2}{(\beta_*^I)^3} \frac{\mathrm{d}\beta_*^I}{\mathrm{d}N_*} \delta^I{}_{J_1} \delta_{J_1 J_2}, \tag{4.16}$$

and so on. Hence, the trajectory is constructed out of D independent trajectories, and there is no mixing among the different field components.

It is known that δN can be analytically solved also in different examples. Under the assumption of slow-roll, it has been shown that the number of e-foldings N can be computed analytically when the potential $V(\phi^I)$ is a separable product of potentials where each one depends on a single field ϕ^I [62], or when it is a separable sum [63]. This has been used to compute the non-Gaussianity in multi-field models of inflation, in the case of slow-roll evolution in two-field [63] and in multi-field models [64, 65]. The approach of [62, 63] has been extended beyond slow-roll, exploiting the H-J formalism in the case of a sum separable Hubble parameter [66–68]. Since by eq. (2.11) the Hubble parameter $H(\phi^I)$ is nothing but the superpotential $W(\phi^I)$ (up to a factor 2), this treatment can be straightforwardly applied to the case of a sum separable superpotential. The approach of [62] was applied also to the separable product superpotential (Hubble) case by Saffin in ref. [69]. As discussed in appendix C, where they overlap, our results agree with this reference. Note that as we discussed above, for the separable product superpotential, when the background is an attractor we can compute δN without introducing integration constants along the trajectory, as in ref. [62].

4.2 Primordial spectra

In what follows, we shall assume that the perturbations in the different fields $\delta\phi_*^I$ are uncorrelated at the time of horizon crossing:

$$\mathcal{P}_{\phi_*}^{IJ}(k) = \delta^{IJ} \mathcal{P}_{\phi_*^J}(k). \tag{4.17}$$

In the multi-field case, eq. (4.17) will hold provided that the linearized equations of motion for perturbations decouple from each other. For instance, when $W^{(I)}$ are exponentials, it is easy to show that we can go to a basis in field space where perturbations are decoupled from each other. The reason is that, as we shall see, we can always change variables so that the corresponding potential $V(\phi^I)$ depends only on one of the fields, while the other ones are massless. In this example, the slow roll parameter ε_1 need not be small. Note that for wavelengths well within the horizon, the field dependence in the potential is unimportant and perturbations of the different fields are effectively decoupled from each other. This suggests that in a more general setting the fields will be uncorrelated near the time of horizon crossing provided that $\varepsilon_2 \ll 1$.

Using eq. (4.17) in the power spectrum of $\zeta^{(I)}$, eq. (4.3), we have

$$\mathcal{P}_\zeta^{(IJ)}(k) = \delta^{KL} N_K^{(I)} N_L^{(J)} \mathcal{P}_{\phi_*^L}(k)\,. \tag{4.18}$$

Using eq. (4.15), we obtain

$$\mathcal{P}_\zeta^{(IJ)}(k) = \delta^{IJ}\mathcal{P}_\zeta^{(I)}(k)\,, \qquad \mathcal{P}_\zeta^{(I)}(k) \equiv \frac{\kappa^2}{(\beta_*^I)^2}\mathcal{P}_{\phi_*^I}(k)\,, \tag{4.19}$$

For canonical scalar fields in $d = 3$ dimensions, the field spectrum $\mathcal{P}_{\phi_*^J}$ is given by

$$\mathcal{P}_{\phi_*^I}(k) = \mathcal{P}_{\phi_*}(k) = \frac{H_*^2}{2k^3}\,, \tag{4.20}$$

and is the same for all I.

The result (4.19) is also valid for the case of a single field with arbitrary speed of sound. In this single field case

$$\mathcal{P}_{\phi_*}(k) = \frac{H_*^2}{2k^3 c_{s*} P_{X*}}\,, \qquad c_s^2 \equiv \frac{P_X}{P_X + 2XP_{XX}}\,, \tag{4.21}$$

where c_s is the speed of propagation of fluctuations and a $*$ denotes sound-horizon crossing, $k = a_* H_*/c_{s*}$.

Defining the cross spectra for the curvature and entropy perturbations by

$$\langle \zeta(\boldsymbol{k})\zeta(\boldsymbol{k}')\rangle \equiv (2\pi)^d \delta(\boldsymbol{k}+\boldsymbol{k}')\mathcal{P}_\zeta(k)\,, \tag{4.22}$$
$$\langle \zeta(\boldsymbol{k})S^{IJ}(\boldsymbol{k}')\rangle \equiv (2\pi)^d \delta(\boldsymbol{k}+\boldsymbol{k}')\mathcal{P}_{\zeta S^{IJ}}(k)\,, \tag{4.23}$$
$$\langle S^{IJ}(\boldsymbol{k})S^{KL}(\boldsymbol{k}')\rangle \equiv (2\pi)^d \delta(\boldsymbol{k}+\boldsymbol{k}')\mathcal{P}_{S^{IJ}S^{KL}}(k)\,. \tag{4.24}$$

and using eqs. (3.32), (3.34) and (4.19), we obtain

$$\mathcal{P}_\zeta(k) = \sum_{I=1}^{D}\left(\frac{\beta_{Ie}\beta_e^I}{\beta_e^2}\right)^2 \frac{\kappa^2}{(\beta_*^I)^2}\mathcal{P}_{\phi_*}\,, \tag{4.25}$$

$$\mathcal{P}_{\zeta S^{IJ}}(k) = d\left[\frac{\beta_{Ie}\beta_e^I}{\beta_e^2}\frac{\kappa^2}{(\beta_*^I)^2} - \frac{\beta_{Je}\beta_e^J}{\beta_e^2}\frac{\kappa^2}{(\beta_*^J)^2}\right]\mathcal{P}_{\phi_*}\,, \tag{4.26}$$

$$\mathcal{P}_{S^{IJ}S^{KL}}(k) = d^2 \sum_{M=1}^{D}\frac{\kappa^2}{(\beta_*^M)^2}(\delta_{IM}-\delta_{JM})(\delta_{KM}-\delta_{LM})\mathcal{P}_{\phi_*}\,, \tag{4.27}$$

where an index e denotes a quantity evaluated at $t = t_e$. Thus, the only non-vanishing components of $\mathcal{P}_{S^{IJ}S^{KL}}(k)$ are

$$\mathcal{P}_{S^{IJ}S^{IJ}}(k) = -\mathcal{P}_{S^{IJ}S^{JI}}(k) = d^2 \left[\frac{\kappa^2}{(\beta_*^I)^2} + \frac{\kappa^2}{(\beta_*^J)^2} \right] \mathcal{P}_{\phi_*} . \tag{4.28}$$

Since the power spectrum of ζ^I does not vary after the Hubble crossing, the auto-correlation of S^{IJ} does not vary either.

From eq. (3.34), the power spectrum of ζ is given by the sum of the conserved power spectra of ζ^I as

$$\mathcal{P}_\zeta(k) = \sum_{J=1}^{D} R_J \, \mathcal{P}_\zeta^{(J)}(k), \qquad R_J \equiv \left(\frac{\beta_J \beta^J}{\beta^2} \right)^2 . \tag{4.29}$$

Notice that when the trajectory converges at $t = t_e$, say in the direction of $I = 1$, satisfying

$$\beta_e^1 \gg \beta_e^I \sqrt{\frac{\beta_*^1}{\beta_*^I}} \tag{4.30}$$

for $I \neq 1$, the amplitude of ζ is determined solely by the one for $I = 1$ at $t = t_*$ as in the single field case as

$$\mathcal{P}_\zeta(k) \simeq \mathcal{P}_\zeta^{(1)}(k) . \tag{4.31}$$

Then, the influence of the components $I \neq 1$ does not explicitly show up in the power spectrum of ζ, while the change of these components can be still traced through their contributions to H_*. This property was pointed out by Garcia-Bellido and Wands in the different separable example, where the slow-roll approximation is employed [62].

Similarly, the bispectrum generated by the super-horizon evolution in eq. (4.7) can be rewritten using eq. (4.17),

$$\mathcal{B}_{\zeta,\mathrm{super}}^{(IJK)}(k_1, k_2, k_3) = N_{LM}^{(I)} N_L^{(J)} N_M^{(K)} \, \mathcal{P}_{\phi_*}(k_1) \mathcal{P}_{\phi_*}(k_2) + (2 \text{ perms}) . \tag{4.32}$$

Inserting eqs. (4.15) and (4.16) into eq. (4.32), we obtain

$$\mathcal{B}_{\zeta,\mathrm{super}}^{(IJK)}(k_1, k_2, k_3) = \delta^{IJ} \delta^{IK} \frac{\mathrm{d} \ln \beta_*^I}{\mathrm{d} N_*} \mathcal{P}_\zeta^{(I)}(k_1) \mathcal{P}_\zeta^{(I)}(k_2) + (2 \text{ perms}) . \tag{4.33}$$

The bispectrum for the entropy perturbations is trivially obtained from eq. (4.33), since S^{IJ} is linear in $\zeta^{(I)}$. In principle, the bispectrum of ζ can also be found from eq. (4.33) by using the nonlinear relation between ζ and $\zeta^{(I)}$.

5 δN and holographic inflation

In refs. [24–26], the primordial spectra were computed holographically by means of the dual quantum field theory which lives on the three-dimensional boundary. The computation from holography may provide an alternative way to address the primordial perturbations generated during inflation. In this section, we reexamine the primordial spectra derived from the δN formalism, comparing them to the prediction from holography.

5.1 Inflation from holography

In this subsection, we briefly overview the way to compute the primordial perturbations from the dual boundary theory, following refs. [25–27, 32]. During inflation, spacetime is quasi-de Sitter, i.e., the de Sitter symmetry SO(1, 4) is slightly broken by the time evolving inflaton field. In order to provide the boundary QFT which is dual to the inflationary spacetime, we need to slightly break the conformal symmetry in $\mathbb{R}^3$, which is also SO(1, 4). In particular, to address a model with D scalar fields, we consider a boundary QFT whose action is given by

$$S_{\rm QFT}[\chi] = S_{\rm CFT}[\chi] + \sum_{I=1}^{D} \int {\rm d}\Omega_d g^I \mathcal{O}_I(\boldsymbol{x})\,, \tag{5.1}$$

where ${\rm d}\Omega_d$ is the d-dimensional invariant volume and χ is the boundary field. The second term describes the deviation from the conformal field theory. Here, $\mathcal{O}_I(\boldsymbol{x})$ is a composite operator of χ and g^I are the coupling constants. Solving the renormalization group flow, we can compute the beta function,

$$\beta_g^I \equiv \frac{{\rm d}g^I}{{\rm d}\ln\mu}\,, \tag{5.2}$$

as a functional of g^I, where μ is the renormalization scale. In general, since the different components I can couple with each other, solving eq. (5.2) analytically is hard. However, in the case where the beta function is separable as $\beta_g^I = \beta_g^I(g^I)$, i.e., in the case where there is no correlation between the different components of $\mathcal{O}^I$, we can analytically solve eq. (5.2) as in a QFT with a single deformation operator. In such a case, we can also compute the auto-correlation functions of $\mathcal{O}(\boldsymbol{x})$ as a function of μ, analytically.

Assuming that the wave functions of the curvature and entropy perturbations are related to the generating functional of the dual quantum fields as

$$\psi_{\rm bulk}\left[\zeta,\, s^{I'}\right] = A\, Z_{\rm QFT}\left[\zeta,\, s^{I'}\right]\,, \tag{5.3}$$

where A is a normalization constant, we can compute the correlators of ζ and $s^{I'}$. Here, to include only independent degrees of freedom, we introduced the entropy perturbations $s^{I'}$ as $s^{I'} \equiv S^{1I'}$ with $I' = 2, \cdots, D$. In the boundary QFT, the primordial perturbations ζ and $s^{I'}$ should be treated as external fields. Once the wave function $\psi_{\rm bulk}$ is specified, we can compute all the correlators for ζ and $s^{I'}$. For instance, the n-point function for ζ is given by

$$\langle\zeta(\boldsymbol{x}_1)\zeta(\boldsymbol{x}_2)\cdots\zeta(\boldsymbol{x}_n)\rangle = \int D\zeta \prod_{I'=2}^{D} Ds^{I'}\, |\psi_{\rm bulk}|^2\, \zeta(\boldsymbol{x}_1)\zeta(\boldsymbol{x}_2)\cdots\zeta(\boldsymbol{x}_n)\,. \tag{5.4}$$

The cosmological evolution in the bulk is described by the D scalar fields as a function of time and of the spatial coordinates. To describe the bulk evolution by means of the dual boundary QFT, these bulk quantities should be related to the quantities in the boundary QFT. One may expect that the time evolution in the bulk will be described by the RG flow in the boundary and then the time evolution of ϕ^I will be determined by the RG flow of g^I. When we specify the relation between the time in the bulk and the renormalization scale in the boundary, $t = t(\mu)$, and also the relation between ϕ^I and g^I,

$$g^I = g^I\left(\phi^J\right)\,, \tag{5.5}$$

with the use of eq. (5.3), the correlators of ζ and $s^{I'}$ can be described by g^I and the correlators of $\mathcal{O}$ [25, 26]. Then, the correlators of the primordial perturbations can be holographically computed by solving the RG flow in the boundary.

In refs. [24, 70–72] it was argued that in the de Sitter limit the renormalization scale μ should be proportional to the scale factor a,

$$\mu \propto a\,. \tag{5.6}$$

However, the relations suggested in these references differ from each other when the solutions deviate from de Sitter spacetime. In ref. [25], considering the RG flow with two fixed points (a fixed point (FP) is a point where the beta function vanishes), which corresponds to the time evolution in cosmology from one de Sitter to another de Sitter, it was shown that with the choice of eq. (5.6), the power spectrum of the curvature perturbation ζ in single field models is conserved at large scales so that the holographic computation gives a result consistent with the standard cosmological perturbation theory. Meanwhile, a more subtle issue is left unresolved for the conservation of the bispectrum [25].

5.2 Comparison of the bulk and boundary computations

In this subsection, we compare the result from the boundary computation, which is obtained by solving the RG flow, to the one from the bulk computation which is obtained in the δN formalism. When the RG flow with D deformation operators is separable, i.e., it is given by the D copies of the RG flow with single deformation operator, using the conformal perturbation theory, we can derive the beta function β_g^I as [26]

$$\beta_g^I = \frac{\mathrm{d}g^I(\mu)}{\mathrm{d}\ln\mu} = \left(\Delta^I - d\right) g^I(\mu) + \frac{\pi^{d/2}}{\Gamma(d/2)}\frac{C_I}{c}\{g^I(\mu)\}^2 + \mathcal{O}\!\left(g^3\right)\,, \tag{5.7}$$

where Δ^I is the scaling dimension of $\mathcal{O}^I$, c is the central charge, and C_I is the structure constant. Solving eq. (5.7), we can compute $g^I(\mu)$. Once $g^I = g^I(\phi^J)$ and $t = t(\mu)$ are determined, $g^I(\mu)$ gives the time evolution of the scalar fields ϕ^I in the bulk.

Assuming the relation $g^I = \kappa\phi^I$ and eq. (5.6) leads to $\beta_g^I = \beta^I$, where β^I is defined in eq. (2.17). In this case, one can compute the primordial power spectra of ζ^I from the boundary QFT whose RG flow is separable. Under these assumptions, one finds [26]

$$\mathcal{P}_\zeta^{(IJ)}(k) = \frac{\delta^{IJ}}{\left[\beta_g^I(k)\right]^2}\mathcal{P}_{\phi^I}(k)\,, \tag{5.8}$$

where $\mathcal{P}_{\phi^I}(k)$ is the power spectrum of $\delta\phi^I$ in the flat gauge, computed from the two-point function of the boundary operator $\mathcal{O}^I$. Since $\beta_g^I = \beta^I$, this result agrees with eq. (4.19).

Moreover, in holography the power spectra of ζ and S^{IJ} are related to those of ζ^Is by eqs. (4.25)–(4.27) with $\beta_g^I = \beta^I$. Hence, the power spectra of ζ and S^{IJ} computed from the boundary QFT agree with those computed from cosmological perturbation theory in the bulk. Notice that, in the case of separable trajectories, the conserved power spectrum of ζ^I can be computed in the same way as in single field inflation. Therefore, the fact that both calculations agree on the conservation of $\mathcal{P}_\zeta^{IJ}(k)$ directly follows from the fact that they agree in the single field case [24, 25].

Finally, we note that instead of using the simple relation $g^I = \kappa\phi^I$, one may identify the couplings g^I in the boundary with the scalar fields ϕ^I in the bulk by more non-trivial relations $g^I = g^I(\phi^J)$. In this case, the relation between the spectra of ζ^I and those of the boundary operators may become more complicated, due to the Jacobian $||\partial g^I/\partial\phi^J||$. Yet, changing the identification $g^I = g^I(\phi^J)$ can be simply understood as a field redefinition.

6 Case studies

In this section, we compute the primordial spectra for D canonical scalar fields, using the formula derived in section 4.2. For our purposes, we consider a separable superpotential as a product of exponential superpotentials, i.e.,

$$W\left(\phi^I\right) = W_0 \exp\left[-\sum_{I=1}^{D} f^I\left(\phi^I\right)\right] , \tag{6.1}$$

where W_0 is constant. Since the superpotential $W(\phi^I)$ directly gives the Hubble parameter as $H = 2W$, the summation of f^I over all $I = 1, \cdots, D$ should increase in time so that H decreases in time.

It is instructive to consider the case where this superpotential describes D scalar fields with canonical Lagrangian,

$$P\left(X, \phi^I\right) = X - V\left(\phi^I\right) , \qquad X \equiv -\frac{1}{2}\sum_{I=1}^{D}\partial_\mu\phi^I\partial^\mu\phi^I . \tag{6.2}$$

In this case eqs. (2.7) and (2.12) become

$$H^2 = \frac{2\kappa^2}{d(d-1)}\left[X + V\left(\phi^I\right)\right] , \tag{6.3}$$

and

$$\dot\phi^I = -\frac{2(d-1)}{\kappa^2}\frac{\partial W\left(\phi^I\right)}{\partial\phi^I} , \qquad H = 2W\left(\phi^I\right) . \tag{6.4}$$

Using these equations we can compute the potential $V(\phi^I)$, which is given by

$$V\left(\phi^I\right) = V_0\left[1 - \frac{d-1}{d\kappa^2}\sum_{I=1}^{D}\left(\frac{\mathrm{d}f^I\left(\phi^I\right)}{\mathrm{d}\phi^I}\right)^2\right]\exp\left[-2\sum_{I=1}^{D} f^I\left(\phi^I\right)\right] , \qquad V_0 \equiv \frac{2d(d-1)W_0^2}{\kappa^2} . \tag{6.5}$$

Note that the potential is not, in general, separable. The beta function $\beta^I(\phi^I)$ is given by

$$\beta^I\left(\phi^I\right) = \kappa\frac{\mathrm{d}\phi^I}{\mathrm{d}N} = \frac{d-1}{\kappa}\frac{\mathrm{d}f^I}{\mathrm{d}\phi^I} . \tag{6.6}$$

In terms of $f^I(\phi^I)$, the slow-roll parameter ε_I is given by

$$\varepsilon_1^I = \frac{d-1}{\kappa^2}\left(\frac{\mathrm{d}f^I}{\mathrm{d}\phi^I}\right)^2 . \tag{6.7}$$

6.1 Constant β^I: power-law inflation

First, we consider the case where $f^I(\phi^I)$ is linear in ϕ^I,

$$f^I\left(\phi^I\right) = \frac{1}{d-1} p_I \kappa \phi^I \,, \tag{6.8}$$

where p_I is a dimensionless constant parameter. Then, the potential $V(\phi^I)$ becomes a separable product of exponential potentials for each ϕ^I,[9]

$$V\left(\phi^I\right) = V_0 \left(1 - \frac{1}{d(d-1)} \sum_{I=1}^{D} p_I^2\right) \exp\left(-\frac{2}{d-1} \sum_{I=1}^{D} \kappa p_I \phi^I\right) . \tag{6.9}$$

In the single field case, this is known as power-law inflation [75]. Using eq. (6.8) in eqs. (6.6) and (6.7) one finds that β^I and the slow-roll parameters ε_I become constant,

$$\beta^I = p_I \,, \qquad \varepsilon_1^I = \frac{p_I^2}{d-1} \,. \tag{6.10}$$

In this case, the coupling constant of the dual boundary theory g^I with $\beta^I = \beta_g^I$ blows up both at the IR and UV limits except for the trivial case with $p_I = 0$.

Solving eq. (6.6), we can compute the evolution of ϕ^I and H as

$$\kappa \phi^I(N) = \kappa \phi_*^I + p_I (N - N_*) \,, \tag{6.11}$$

$$H(N) \propto e^{-\varepsilon_1 N} \,, \tag{6.12}$$

where N_* is an integration constant and we remind the reader that $\varepsilon_1 \equiv \sum_I \varepsilon_1^I$. As expected, integrating eq. (6.12) in time we obtain the power law evolution

$$a(t) \propto t^{1/\varepsilon_1} \,. \tag{6.13}$$

Inflation requires $\varepsilon_1 \ll 1$.

Using eqs. (4.25)–(4.27) and assuming that all fields have the same power spectrum $\mathcal{P}_{\phi_*}(k)$, the power spectra of ζ and S^{IJ} are given by

$$P_{\zeta\zeta}(k) = \frac{\kappa^2}{(d-1)\varepsilon} \mathcal{P}_{\phi_*}(k) \,, \tag{6.14}$$

$$P_{\zeta S^{IJ}}(k) = 0 \,, \tag{6.15}$$

$$P_{S^{IJ} S^{IJ}}(k) = \frac{(d\kappa)^2}{d-1} \left(\frac{1}{\varepsilon_1^I} + \frac{1}{\varepsilon_1^J}\right) \mathcal{P}_{\phi_*}(k) \,, \tag{6.16}$$

where we used $\beta^2 = (d-1)\varepsilon_1$. The power spectrum of ζ is given by the same expression as the one for the single field case and the amplitude is frozen after $t = t_*$. For scale invariant field fluctuations in three dimensions $\mathcal{P}_{\phi_*}(k) = H_*^2/(2k^3)$ and the spectral index is given by

$$n_s - 1 = -\frac{4\varepsilon_1}{1 - 2\varepsilon_1} \,. \tag{6.17}$$

In this case, the bispectrum $B_{\zeta,\mathrm{super}}^{(I_1 I_2 I_3)}$ vanishes.

[9] The case where the potential is given by a separable sum of exponential potentials is known as assisted inflation, see for instance [73, 74].

6.2 Linear $\beta^I(\phi^I)$

Next, we consider the superpotential where $f^I(\phi^I)$ also includes the quadratic term,

$$f^I\left(\phi^I\right) = \frac{1}{d-1}\left[p_I \kappa\phi^I + q_I\left(\kappa\phi^I\right)^2\right] , \tag{6.18}$$

where p_I and q_I are constant parameters. In this case, the beta function $\beta^I(\phi^I)$ is given by the linear function as

$$\beta^I\left(\phi^I\right) = p_I + 2q_I\kappa\phi^I , \tag{6.19}$$

and $V(\phi^I)$ is given by

$$V(\phi^I) = V_0\left[1 - \frac{1}{d(d-1)}\sum_{I=1}^{D}\left(p_I + 2q_I\kappa\phi^I\right)^2\right]\exp\left\{-\frac{2}{d-1}\sum_{I=1}^{D}\left[p_I\kappa\phi^I + q_I\left(\kappa\phi^I\right)^2\right]\right\} . \tag{6.20}$$

For $q_I \neq 0$, the potential $V(\phi^I)$ is not a separable product and it is not easy to analytically solve the Klein-Gordon equations, which are not separable. However, since the superpotential is a separable product, eq. (6.6) can be easily solved, which gives

$$\kappa\phi^I(N) = \kappa\phi^I_* + \left(\kappa\phi^I_* + \frac{p_I}{2q_I}\right)\left(e^{2q_I(N-N_*)} - 1\right) . \tag{6.21}$$

As expected, in the limit $q_I \to 0$ we recover eq. (6.11). Using eq. (6.21), we obtain the beta function β^I,

$$\beta^I(N) = \beta^I_*\, e^{2q_I(N-N_*)} . \tag{6.22}$$

At late times, the beta function blows up for $q_I > 0$ and approaches 0 for $q_I < 0$. In the perspective of holography, where the late time in cosmology corresponds to the UV limit in the boundary QFT, the boundary theory dual to the latter case has the FP in the UV limit $\mu \to \infty$. During slow-roll inflation, β^I should be kept much smaller than 1, requiring

$$\beta^I_*\, e^{2q_I(N-N_*)} \ll 1 . \tag{6.23}$$

Using eq. (6.21) in eq. (6.7) we find

$$\varepsilon^I_1(N) = \varepsilon^I_{1*}\, e^{4q_I(N-N_*)} , \qquad \varepsilon^I_{1*} \equiv \frac{1}{d-1}\left(p_I + 2q_I\kappa\phi^I_*\right)^2 , \tag{6.24}$$

using which we have $f_I = [(d-1)\varepsilon^I_1 - p_I^2]/[4(d-1)q_I]$. Using this in eqs. (6.1) and (6.4), we can give the following expression for the Hubble parameter

$$H(N) = 2W_0 \exp\left[\sum_I \frac{p_I^2 - (d-1)\varepsilon^I_1(N)}{4(d-1)q_I}\right] . \tag{6.25}$$

For arbitrary values of q_I with $q_I \neq 0$, the Hubble parameter H decreases in time, which is simply because $\varepsilon_1 > 0$.

Since β^I varies in time for $q_I \neq 0$, the power spectrum of ζ varies also after $t = t_*$. Given that the scalar fields take values ϕ^I_e at $t = t_e$ (more precisely, for each separable trajectory of ϕ^I, the final time t_e is specified by a value of each field, ϕ^I_e), the power spectrum for ζ is given by the summation of the conserved power spectra for ζ^I as in eq. (4.29) with the ratio R_I given by

$$R_I = \frac{\left(p_I + 2q_I\kappa\phi_e^I\right)^4}{\left[\sum_{J=1}^{D}\left(p_J + 2q_J\kappa\phi_e^J\right)^2\right]^2}. \tag{6.26}$$

Similarly, using eqs. (4.26) and (4.27), we can compute the cross-correlation between ζ and S^{IJ} and the auto-correlation of S^{IJ}. In this case, since the beta function varies in time, $B^{(I_1I_2I_3)}_{\zeta,\mathrm{super}}$, given in eq. (4.33), takes a non-vanishing value.

6.3 Quadratic $\beta^I(\phi^I)$

Next, we consider $f^I(\phi^I)$ which includes a cubic term as

$$f^I(\phi^I) = \frac{1}{d-1}\left[p_I\kappa\phi^I + q_I\left(\kappa\phi^I\right)^2 + r_I\left(\kappa\phi^I\right)^3\right], \tag{6.27}$$

where p_I, q_I, and r_I are constant parameters. Now, the beta function $\beta^I(\phi^I)$ and the potential $V(\phi^I)$ are given by

$$\beta^I\left(\phi^I\right) = p_I + 2q_I\kappa\phi^I + 3r_I\left(\kappa\phi^I\right)^2, \tag{6.28}$$

$$\begin{aligned} V\left(\phi^I\right) = V_0&\left\{1 - \frac{1}{d(d-1)}\sum_{I=1}^{D}\left[p_I + 2q_I\kappa\phi^I + 3r_I\left(\kappa\phi^I\right)^2\right]^2\right\} \\ &\times \exp\left\{-\frac{2}{d-1}\sum_{I=1}^{D}\left[p_I\kappa\phi^I + q_I\left(\kappa\phi^I\right)^2 + r_I\left(\kappa\phi^I\right)^3\right]\right\}. \end{aligned} \tag{6.29}$$

For later use, we introduce

$$D_I \equiv q_I^2 - 3r_Ip_I\,. \tag{6.30}$$

For $D_I > 0$, the beta function β^I vanishes at two different values of ϕ^I, i.e., the dual boundary theory has two FPs. For $D_I = 0$, β^I vanishes at one value of ϕ^I, *i.e.*, the boundary theory has one FP. For $D_I < 0$, the beta function β^I does not vanish at any values of ϕ^I, i.e., the boundary theory has no FPs. Using the beta function, given in eq. (6.28), we obtain the power spectrum of ζ as in eq. (4.29) with

$$R_I = \frac{\left[p_I + 2q_I\kappa\phi_e^I + 3r_I\left(\kappa\phi_e^I\right)^2\right]^4}{\left\{\sum_{J=1}^{D}\left[p_J + 2q_J\kappa\phi_e^J + 3r_J\left(\kappa\phi_e^J\right)^2\right]^2\right\}^2}. \tag{6.31}$$

In the following, we study the background evolution of these three cases in turn.

6.3.1 $D_I < 0$: RG flow with no fixed point

First, we consider the case where $\beta^I(\phi^I)$ does not vanish. Solving eq. (6.6), we obtain

$$\kappa\phi^I(N) = -\frac{q_I}{3r_I} + \frac{\sqrt{-D_I}}{3r_I}\tan\theta^I(N) \tag{6.32}$$

with

$$\theta^I(N) \equiv \sqrt{-D_I}(N - N_*) + \tan^{-1}\left(\frac{E_I}{\sqrt{-D_I}}\right), \qquad E_I \equiv q_I + 3r_I\kappa\phi_*^I\,. \tag{6.33}$$

Inserting this solution into eq. (6.28), we can compute the time evolution of the beta function β^I as

$$\beta^I(N) = -\frac{D_I}{3r_I}\left[1 + \tan^2\theta^I(N)\right] . \tag{6.34}$$

The beta function starts to grow rapidly, when $\tan\theta^I(N)$ becomes $\mathcal{O}(1)$. Now, the Hubble parameter, given by

$$H(N) = H_0 \exp\left[-\sum_{I=1}^{D} \frac{(-D_I)^{3/2}}{27(d-1)r_I^2} \tan\theta^I(N)\left(\tan^2\theta^I(N) - 3\right)\right] \tag{6.35}$$

with

$$H_0 \equiv 2W_0 \exp\left[\sum_{I=1}^{D} \frac{q_I}{27(d-1)r_I^2}\left(9r_I p_I - 2q_I^2\right)\right] , \tag{6.36}$$

decreases monotonically in time.

In this case, after the slow-roll time evolution, inflation ends when $\theta_I(N) \simeq \pi/4$ for at least one of the Is and, afterwards, the Hubble parameter starts to decrease more rapidly. Therefore, this case can provide a graceful exit to inflation. Meanwhile, in the boundary side, the dual QFT does not have any FPs and the RG flow is dominated by irrelevant deformations in UV. It may be interesting to study such boundary QFT.

6.3.2 $D_I = 0$: RG flow with one fixed point

Next, we consider the case where the beta function β^I vanishes only at $\kappa\phi^I = -q_I/3r_I$. In this case, the time evolution of $\phi^I(N)$ and $\beta^I(N)$ are given by

$$\kappa\phi^I(N) = -\frac{q_I}{3r_I} - \frac{1}{3r_I\left(N - N_* - E_I^{-1}\right)} , \tag{6.37}$$

and

$$\beta^I(N) = \frac{1}{3r_I}\frac{1}{\left(N - N_* - E_I^{-1}\right)^2} . \tag{6.38}$$

At late times with $(N - N_* - E_I^{-1}) \gg 1$, ϕ^I approaches the constant value $-q_I/3r_I$, where the beta function β^I vanishes. In this case, the boundary theory has one FP in the UV. When all components satisfy $D_I = 0$, the Hubble parameter becomes constant at late times and the universe becomes the de Sitter spacetime. This solution does not provide a realistic model of inflation because there is no graceful exit.

6.3.3 $D_I > 0$: RG flow with two fixed points

Finally, we consider the case where the beta function β^I vanishes at two different values:

$$\kappa\phi^I_\pm = \frac{-q_I \pm \sqrt{D_I}}{3r_I} . \tag{6.39}$$

In this case, solving eq. (6.28), we obtain

$$\phi^I(N) = \frac{\phi^I_+ + \phi^I_- e^{2\theta^I(N)}}{1 + e^{2\theta^I(N)}} , \tag{6.40}$$

and

$$\beta^I(N) = -\frac{4D_I}{3r_I}\frac{e^{2\theta^I(N)}}{\left[1+e^{2\theta^I(N)}\right]^2} \tag{6.41}$$

with

$$\theta^I(N) \equiv \sqrt{D_I}(N-N_*) - \tanh^{-1}\left(\frac{E_I}{\sqrt{D_I}}\right), \qquad E_I \equiv q_I + 3r_I\kappa\phi_*^I . \tag{6.42}$$

Notice that both in the early time and late time limits, where $\theta \to \pm\infty$, the beta function $\beta^I(N)$ vanishes, approaching the de Sitter spacetime. In this solution, ϕ^I takes the constant values ϕ_-^I and ϕ_+^I in the limits $N \to -\infty$ and $N \to \infty$, respectively. If all components of the scalar fields satisfy $D_I > 0$, the solution describes the transition from one de Sitter to another de Sitter. In this case, the dual boundary theory has 2 FPs both in the IR and UV and its RG flow is driven only by relevant deformations. Such boundary theory was studied by means of the conformal perturbation theory in refs. [24–26]

If all components satisfy $D_I \geq 0$, the universe becomes the de Sitter spacetime at late times. Only if there exists at least one component $\bar{I}$ with $D_{\bar{I}} < 0$, inflation can terminate as discussed in section 6.3.1. Notice that the components with $D_I \geq 0$, whose β^I decrease in time, will satisfy $|\beta_{Ie}| \ll |\beta_{\bar{I}e}|$ at sufficiently late times. Then, R_Is become negligibly small for $I \neq \bar{I}$ and hence they do not explicitly contribute to the spectra of ζ, while they can still contribute implicitly through the Hubble parameter and the slow-roll parameters at the Hubble crossing time.

7 Conclusion

In this paper, we reviewed the superpotential formalism for multi-field inflation, and extended it to include the case of non-minimal kinetic terms. The superpotential is useful in characterizing the attractor behaviour of inflationary trajectories, as well as to assess the validity of the separate universe approximation. Furthermore, the logarithm of the superpotential plays an interesting role in the dual description of inflation, as the c-function for the RG flow in the boundary theory, whose gradient is related to the beta functions.

Using the δN formalism, we obtain simple expressions for the power spectra for adiabatic and entropy perturbations in the case when the superpotential is given as a separable product. In that case, the trajectory for each field is convergent even when the whole trajectory is not, and the power spectra can easily be found by solving the corresponding separable background trajectories.

The bulk solution for the separable product superpotential corresponds to a boundary QFT with a separable RG flow, where the deformation operators are mutually uncorrelated. In such case, we showed that the power spectra of the adiabatic and entropy perturbations computed from the δN formalism agree with the ones computed by solving the RG flow of the dual boundary theory.

The separable case we addressed in this paper is described by D copies of the single field case. The power spectra of the adiabatic and entropy perturbations can be expressed in terms of such single field power spectra by linear relations. Because of that, the agreement of the bulk and boundary computations for the curvature perturbation in the single field model directly implies the agreement for the adiabatic and entropy perturbations. It would be very interesting to check the agreement in more non-trivial multi field models.

Finally, with a view to phenomenological applications, we have considered some case studies of RG flows with a polynomial c-function, with terms up to quadratic order in the

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fields. These contain a range of possible behaviours from the infrared to the UV, which may hopefully illustrate the results which should be expected in more realistic scenarios.

Acknowledgments

Y. U. would like to thank C. Byrnes, Y. Takamizu and D. Wands for their valuable comments. J. G. and Y. U. are partially supported by MEC FPA2010-20807-C02-02 and AGAUR 2009-SGR-168. Y. U. would like to thank University of Padua, IPhT at CEA/Saclay, the Centro de Ciencias de Benasque Pedro Pascual, CERN, and Institute for Advanced Study for their hospitalities during the period of this work. Y. U. is supported by JSPS Grant-in-Aid for Research Activity Start-up under Contract No. 26887018 and the National Science Foundation under Grant No. NSF PHY11-25915. F.V. acknowledges partial financial support from *Programme National de Cosmologie and Galaxies* (PNCG) of CNRS/INSU, France.

A Second order equations of motion from the superpotential

Here, we show that the usual second order equations of motion (2.5) and (2.7) follow from the first order equation of motion (2.12) with (2.11), for any superpotential W which satisfies the H-J equation (2.13).

First, we square (2.12) to obtain

$$P_{IJ}P_{KL}X^{JL} = \frac{2(d-1)^2}{\kappa^4}W_I W_K. \tag{A.1}$$

Here, we are using the notation

$$W_I \equiv \frac{\partial W}{\partial \phi^I}. \tag{A.2}$$

In what follows, we assume that P_{IJ} is an invertible matrix and we shall denote its inverse as P^{IJ}. We use these matrices to raise and lower the field indices. In particular

$$X_{IK} \equiv P_{IJ}P_{KL}X^{JL} = \frac{2(d-1)^2}{\kappa^2}W_I W_K, \tag{A.3}$$

and we define

$$X \equiv P_{IJ}X^{IJ} = P^{IJ}X_{IJ}. \tag{A.4}$$

The H-J equation (2.13) can then be written as

$$\frac{2d(d-1)}{\kappa^2}W^2 = 2X - P, \tag{A.5}$$

where it is understood that any occurrence of $\dot{\phi}^I$ is replaced by its expression in terms of ϕ^J and $W_I(\phi^J)$ through eqs. (2.10) and (2.12).

Taking the total derivative of (A.5) with respect to ϕ^I, and using (2.12), we have

$$-dHP_{IJ}\dot{\phi}^J = 2\frac{\mathrm{d}X}{\mathrm{d}\phi^I} - \frac{\partial P}{\partial \phi^I} - P_{KL}\frac{\mathrm{d}X^{KL}}{\mathrm{d}\phi^I}, \tag{A.6}$$

where the ϕ^I dependence of $P(\phi^I, X^{IJ})$ has been separated into its explicit dependence and its dependence through field velocities contained in X^{IJ}. Using

$$P_{KL}\frac{\mathrm{d}X^{KL}}{\mathrm{d}\phi^I} = \frac{\mathrm{d}X}{\mathrm{d}\phi^I} - X^{KL}\frac{\mathrm{d}P_{KL}}{\mathrm{d}\phi^I} = \frac{\mathrm{d}X}{\mathrm{d}\phi^I} + X_{KL}\frac{\mathrm{d}P^{KL}}{\mathrm{d}\phi^I}, \tag{A.7}$$

and

$$\frac{\mathrm{d}X}{\mathrm{d}\phi^I} = P^{KL}\frac{\mathrm{d}X_{KL}}{\mathrm{d}\phi^I} + \frac{\mathrm{d}P^{KL}}{\mathrm{d}\phi^I}X_{KL}, \tag{A.8}$$

we have

$$\frac{\partial P}{\partial \phi^I} = dHP_{IJ}\dot{\phi}^J + P^{KL}\frac{\mathrm{d}X_{KL}}{\mathrm{d}\phi^I}. \tag{A.9}$$

We can evaluate the last term as follows:

$$P^{KL}\frac{\mathrm{d}X_{KL}}{\mathrm{d}\phi^I} = P^{KL}\frac{4(d-1)^2}{\kappa^4}\frac{\partial^2 W}{\partial\phi^K\partial\phi^I}W_L = \frac{-2(d-1)}{\kappa^2}\frac{\mathrm{d}W_I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(P_{IJ}\dot{\phi}^J\right). \tag{A.10}$$

With this, eq. (A.9) coincides with the second order equation of motion for ϕ^I, eq. (2.5), while the Friedmann equation (2.7) is also satisfied because W solves (A.5).

B Dilution of $\partial W/\partial c_K$ with cosmic expansion

In this appendix, we find the time dependence of the derivative of the complete solution of the H-J equation, $W(\phi^K, c_K)$, with respect to the integration constants. From the Friedmann equation (2.7), we have

$$\frac{4d(d-1)}{\kappa^2}W\frac{\partial W}{\partial c_J} = \rho_{KL}\frac{\partial X^{KL}}{\partial c_J}, \tag{B.1}$$

where we note that ρ only depends on c_K through the kinetic variables X^{KL}. Now, from (A.3), we have

$$P_{KI}P_{LJ}X^{KL} = \frac{2(d-1)^2}{\kappa^4}W_IW_J, \tag{B.2}$$

where, again, we are using the notation (A.2). Taking derivative of (B.2) with respect to c_K, and then contracting with the "inverse metric" P^{IJ}, we immediately find

$$\left(2P_{IJ,LM}X^{IJ} + P_{LM}\right)\frac{\partial X^{LM}}{\partial c_K} = \frac{4(d-1)^2}{\kappa^4}P^{IJ}W_I\frac{\partial W_J}{\partial c_K}. \tag{B.3}$$

Noting that $\rho = 2P_{IJ}X^{IJ} - P$, eq. (B.3) can be rewritten as

$$\rho_{LM}\frac{\partial X^{LM}}{\partial c_K} = -2\frac{(d-1)}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial W}{\partial c_K}, \tag{B.4}$$

where we used (2.12) to express the right hand side as a total time derivative. Here ρ_{LM} denotes the symmetrized derivative of ρ with respect to X^{LM}.

Substituting eq. (B.4) in eq. (B.1) we obtain

$$W\frac{\partial W}{\partial c_J} = -\frac{1}{2d}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial W}{\partial c_J}, \tag{B.5}$$

and using $H = 2W$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\partial W}{\partial c_J}\right) = -dH, \tag{B.6}$$

which leads to

$$\frac{\partial W}{\partial c_J} = A^J e^{-d\int H\mathrm{d}t} = A^J a^{-d}, \tag{B.7}$$

where A^J are integration constants. Hence, the dependence of W on the integration constants becomes smaller in time along the dynamical trajectories, diluting with volume.

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C Alternative way to compute δN

In this appendix, we derive the primordial spectra of ζ in an alternative way, following refs. [62–64]. When several scalar fields contribute to the background evolution, the background equations of motion can be solved analytically only for some special cases, e.g., the case where the background evolution can be described by separable equations [62–64]. In ref. [62], Garcia-Bellido and Wands computed the power spectrum of ζ for the two-field model whose potential is given by a separable product as $V(\phi_1, \phi_2) = V_1(\phi_1)V(\phi_2)$ under the slow-roll assumption. In ref. [63], it was shown that a similar analysis can be done also for a separable summation potential $V(\phi_1, \phi_2) = V_1(\phi_1)+V(\phi_2)$. This analysis was extended to the case with arbitrary number of the scalar fields [64] (see also ref. [76]). In these discussions, it's crucial that the Klein-Gordon equations for the scalar fields can be recast into the first order equation by employing the slow-roll approximation.

With the use of the superpotential $W(\phi^I)$, which is related to the Hubble parameter as $H(\phi^I) = 2W(\phi^I)$, the field equations for the scalar fields can be recast into the first order equations without employing the slow-roll approximation. In refs. [66, 67], under the assumption that the Hubble parameter is given by the separable summation as $H(\phi^I) = \sum_{I=1}^{D} H_I(\phi^I)$, Byrnes and Tasinato computed δN, following the method by Garcia-Bellido and Wands [62]. This analysis was extended to a non-canonical scalar field in ref. [68] and to the case with the separable summation superpotential (or the Hubble parameter) in ref. [69]. In appendix, we summarize the computation of δN in the method by Garcia-Bellido and Wands [62] for the separable product $W(\phi^I)$. The overlapped part with ref. [69] agrees.

When the superpotential $W(\phi^I)$ is given by the separable product as in eq. (4.9) and P_{IJ} becomes diagonal as in eq. (4.8), the beta function β^I is given by a functional of ϕ^I as

$$\kappa\frac{\mathrm{d}\phi^I}{\mathrm{d}N} = \beta^I(\phi^I)\,. \tag{C.1}$$

To solve the background trajectory, we introduce the integrals of motion C_i as

$$C_i \equiv \int \frac{\mathrm{d}\phi^i}{\beta^i(\phi^i)} - \int \frac{\mathrm{d}\phi^{i+1}}{\beta^{i+1}(\phi^{i+1})} \tag{C.2}$$

with $i = 1, \cdots, D-1$. Using eq. (C.1), we can verify that C_i actually stays constant along the trajectory. With the aid of the constant parameters C_i, the change of the e-folding along the trajectory can be expressed only by one of the fields, say ϕ^1, as

$$N\left(t_e,\, t_*,\, \{C_i\}_{i=1}^{\mathcal{N}-1}\right) = \int_{t_*}^{t_e} H\mathrm{d}t = \int_{\phi^1_*}^{\phi^1_e} \frac{\kappa\mathrm{d}\phi^1}{\beta^1(\phi^1)}\,. \tag{C.3}$$

On the second equality, we used $\mathrm{d}t = \mathrm{d}\phi^1/\dot{\phi}^1$ and eqs. (2.11) and (C.1). Unlike δN^I, which is determined only by $\delta\phi^I_*$ for the separable product case, the e-folding number (C.3) depends also on ϕ^I_* with $I \neq 1$, since ϕ^1_e depends on C_is, which are determined by using ϕ^I_* with $I = 1, \cdots, D$. Now, taking the variance with respect to ϕ^I_*, we obtain

$$\mathrm{d}N = -\frac{\kappa\mathrm{d}\phi^1_*}{\beta^1_*} + \frac{1}{\beta^1_e}\sum_{I=1}^{\mathcal{N}} \frac{\partial\phi^1_e}{\partial\phi^I_*}\kappa\mathrm{d}\phi^I_*\,. \tag{C.4}$$

In the following, we compute $\partial\phi_e^1/\partial\phi_*^I$. The integrals of motion C_i can be expressed in terms of ϕ_*^I and hence we obtain

$$\frac{\partial\phi_e^I}{\partial\phi_*^J} = \sum_{i=1}^{D-1} \frac{\partial\phi_e^I}{\partial C_i}\frac{\partial C_i}{\partial\phi_*^J}\,, \tag{C.5}$$

Using eq. (C.2), we obtain

$$\frac{\partial C_i}{\partial\phi_*^J} = \frac{1}{\beta_*^J}(\delta_{i\,J} - \delta_{i\,J-1})\,. \tag{C.6}$$

Choosing the uniform Hubble slicing, we specify the final time t_e as the time when the Hubble parameter takes a particular value as

$$H_e = 2W\left(\phi_e^I\right) = 2\prod_{I=1}^{D} W^{(I)}\left(\phi_e^I\right)\,. \tag{C.7}$$

Taking the derivative of H_e with respect to C_i and dividing it by H_e, we obtain

$$0 = \sum_{I=1}^{D} \beta_{Ie}\frac{\partial\phi_e^I}{\partial C_i}\,. \tag{C.8}$$

Next, introducing

$$\tilde{C}_I \equiv \sum_{i=1}^{I-1} C_i = \int \frac{\mathrm{d}\phi^1}{\beta^1\left(\phi^1\right)} - \int \frac{\mathrm{d}\phi^I}{\beta^I\left(\phi^I\right)} \tag{C.9}$$

with $I = 1, \cdots, D$, we compute $\partial\phi_e^I/\partial C_i$. Taking the derivative of $\tilde{C}_I$ with respect to C_i, we obtain

$$\frac{\partial\tilde{C}_I}{\partial C_i} = \frac{1}{\beta_e^1}\frac{\partial\phi_e^1}{\partial C_i} - \frac{1}{\beta_e^I}\frac{\partial\phi_e^I}{\partial C_i}\,, \tag{C.10}$$

where we noted that values of ϕ_e^I depend on C_i chosen for each trajectory. Equation (C.10) is recast into

$$\frac{\partial\phi_e^I}{\partial C_i} = \beta_e^I\left(\frac{1}{\beta_e^1}\frac{\partial\phi_e^1}{\partial C_i} - \Theta_{i\,I}\right)\,, \tag{C.11}$$

with

$$\Theta_{i\,I} \equiv \frac{\partial\tilde{C}_I}{\partial C_i} = \begin{cases} 1 \ (i \le I-1) \\ 0 \ (i > I-1) \end{cases}\,. \tag{C.12}$$

Using eqs. (C.8) and (C.11), we obtain

$$\frac{\partial\phi_e^I}{\partial C_i} = \beta_e^I\left[\frac{\sum_{J=i+1}^{D}\beta_{Ie}\beta_e^I}{\beta_e^2} - \Theta_{iI}\right] \tag{C.13}$$

with

$$\beta_e^2 \equiv \sum_{I=1}^{\mathcal{N}} \beta_{Ie}\beta_e^I\,. \tag{C.14}$$

Using eqs. (C.6) and (C.13), we obtain

$$\frac{\partial \phi_e^1}{\partial \phi_*^I} = -\frac{\beta_e^1}{\beta_*^I}\left[\frac{\beta_{Ie}\beta_e^I}{\beta_e^2} - \delta_{1I}\right], \tag{C.15}$$

Inserting eq. (C.15) into eq. (C.4), we arrive at the compact expression:

$$\mathrm{d}N = -\sum_{I=1}^{D} \frac{\beta_{Ie}\beta_e^I}{\beta_e^2}\frac{\kappa \mathrm{d}\phi_*^I}{\beta_*^I}. \tag{C.16}$$

Using eq. (C.16), we can obtain

$$N_I = \frac{\partial N}{\partial \phi_*^I} = -\frac{\beta_{Ie}\beta_e^I}{\beta_e^2}\frac{\kappa}{\beta_*^I}, \tag{C.17}$$

$$N_{IJ} = \frac{\partial^2 N}{\partial \phi_*^I \partial \phi_*^J} = \delta_{IJ}\frac{\beta_{Ie}\beta_e^I}{\beta_e^2}\frac{\kappa^2}{(\beta_*^I)^3}\frac{\mathrm{d}\beta_*^I}{\mathrm{d}N_*}, \tag{C.18}$$

and so on. As in the case with the separable summation $W(\phi^I)$, $N_{I_1\cdots I_n}$ can be immediately computed for a given $W_I(\phi^I)$ and take non-vanishing values only if $I_1 = \cdots = I_n$. The power spectrum of the curvature perturbation computed from eq. (C.17) agrees with eq. (4.25).

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