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DO 3n-5 EDGES FORCE A SUBDIVISION OF K_5?

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Do 3n-5 edges force a subdivision of K_5?

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Abstract

A conjecture of Dirac states that every simple graph with n vertices and $3n-5$ edges must contain a subdivision of K_5. We prove that a topologically minimal counterexample is 5-connected, and that no minor-minimal counterexample contains $K_4 - e$. Consequently, we prove Dirac's conjecture for all graphs that can be imbedded in a surface with Euler characteristic at least -2.

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1. Introduction

Let H be a simple undirected graph. An *elementary subdivision* of H is a graph obtained from H by removing some edge $e = xy$ and adding a new vertex z together with two new edges xz and zy. A *subdivision* of H is a graph obtained from H by a succession of elementary subdivisions. If a subdivision of H is isomorphic to a subgraph of G, we write $TH \subset G$, where TH represents an arbitrary subdivision of H. A vertex of $TK_p (p \geq 4)$ with degree at least three is called a *branch* vertex.

A conjecture due to Dirac [2], and reported by Thomassen [8], states that any simple graph with n vertices and $3n - 5$ edges contains a subdivision of K_5. By Kuratowski's Theorem, no planar graph contains a subdivision of K_5. Thus Dirac's conjecture, if true, would be sharp. Thomassen [7] proved that $4n - 10$ edges force a TK_5. In [3], Dirac showed that, if $\delta(G) \geq 3$, then G contains a subdivision of K_4. A similar result by Pelikán [6] and Thomassen [7] established that $\delta(G) \geq 4$ forces G to contain a subdivision of $K_5 - e$. More generally, Mader [5] proved that, if $\delta(G) \geq 3(2)^{p-2} - 2p \ \ (p > 3)$, then $TK_p \subset G$.

A simple graph G with n vertices is called a *counterexample* if $|E(G)| \geq 3n - 5$ and $TK_5 \not\subset G$. Let $\mathcal{D}$ be the set of all counterexamples. A *minor* of G is a subgraph obtained from G by a sequence of edge deletions, vertex deletions, and edge contractions. A graph is *minor-minimal* in $\mathcal{D}$ provided it is a counterexample but no minor is a counterexample. Similarly, a graph is *(topologically) minimal* in $\mathcal{D}$ provided it is a counterexample and contains no subdivision of a smaller counterexample. Observe that any minor-minimal counterexample is also a (topologically) minimal counterexample.

In section 3 we prove that any minimal counterexample is 5-connected. From this we deduce, in section 4, that no minor-minimal counterexample contains $K_4 - e$. Finally, in section 5, we prove Dirac's conjecture for all graphs that can be imbedded in a surface with Euler characteristic at least -2.

2. Menger's Theorem and Extensions

We make use of several fundamental results which we list here. The reader is referred to Bollobás [1] for further details.

A *vertex cut* of G is a subset of vertices whose removal disconnects G. A *k-separator* of G is a vertex cut of k vertices. The *connectivity* of G is the least k such that there exists a k-separator of G. If k is the connectivity of G, we write $\kappa(G) = k$ and say that G is *k-connected.*

Theorem 1 (Menger). *A non-trivial graph is k-connected if and only if every pair of vertices is connected by k disjoint paths.*

Let S be a set of vertices in the graph G and let x be a vertex not in S. An *x-S fan* is a set of $|S|$ paths from x to S, any two of which share only the vertex x.

Theorem 2 (Dirac). *A graph G is k-connected if and only if $|G| \geq k + 1$ and for any k-set $S \subset V(G)$ and vertex $x \in V(G) - S$ there is an $x - S$ fan.*

The following two theorems follow as corollaries of the previous one.

Theorem 3 (Dirac). *If G is k-connected and $k \geq 2$, then for any set of k vertices there is a cycle containing all of them.*

Suppose $X, Y \subset V(G)$. We say that X is *linked* to Y if there are $|X|$ vertex disjoint paths from X to Y. Notice that the paths linking X to Y cannot share any vertices including initial and terminal vertices.

Theorem 4 (Dirac). *Let $|G| \geq 2k$. G is k-connected if and only if whenever V_1 and V_2 are disjoint k-sets of vertices, then V_1 is linked to V_2.*

3. 5-connectivity

Let G be a (topologically) minimal counterexample as defined in the introduction. In this section we show that G is 5-connected. We begin by examining the minimum degree. Observe that a minimal counterexample with n vertices has $3n - 5$ edges.

Lemma 1. *If G is minimal in $\mathcal{D}$, then $\delta(G) = 5$.*

Proof: The average degree is less than six, so the minimum degree is at most five. If the minimum degree is less than four, then we may delete a vertex of degree at most three from G, obtaining a smaller graph with $3(n-1) - 5$ edges and no subdivision of K_5, which contradicts minimality of G. Hence, it suffices to show that the minimum degree is not four.

Suppose, for a contradiction, that $\delta(G) = 4$. Let $v \in V(G)$ have $d_G(v) = 4$ with neighbors a, b, c, d. There must be a pair of these neighbors, say c and d, that are not adjacent, otherwise the five vertices $\{v, a, b, c, d\}$ form a K_5. Deleting the edges va and vb, then contracting v to edge cd yields a subgraph of G in $\mathcal{D}$, contradicting that G is a minimal counterexample. □

From Lemma 1, by counting edges and degrees, it is easy to deduce that a minimal counterexample must have at least ten vertices.

Suppose S is a set of vertices of G. $G[S]$ denotes the subgraph induced by S, and $E(S)$ are the edges of $G[S]$.

Lemma 2. *If G is minimal in $\mathcal{D}$, then $\kappa(G) \geq 3$.*

Proof: Suppose, for a contradiction, that G is 2-connected with a 2-separator $\{x, y\}$. Let C_1 be one component of $G - \{x, y\}$, and $C_2 = G - (\{x, y\} \cup C_1)$. Define $G_i = G[C_i \cup \{x, y\}]$ for $i = 1, 2$. Lemma 1 ensures that the number of vertices in each G_i $(i = 1, 2)$ is at least six. Because G_1 and G_2 are sufficiently large subgraphs of G, the minimality of G implies that they do not contain a subdivision of K_5; thus they each must have at most $3n_i - 6$ edges, where n_i represents the number of vertices in G_i. Observing $n_1 + n_2 = n + 2$, we find

$$3n - 5 = |E(G)| \leq |E(G_1)| + |E(G_2)| \leq (3n_1 - 6) + (3n_2 - 6) = 3n - 6$$

a contradiction. $\square$

Suppose G is a minimal in $\mathcal{D}$ with S a $\kappa(G)$-separator of G. Let C_1 be a component of $G - S$ and $C_2 = G - (S \cup C_1)$. Define $G_i = G[C_i \cup S]$, for $i = 1,2$. We say that S *divides* G into G_1 and G_2. Let n_i and e_i represent the number of vertices and edges of G_i, respectively. Observe that $n_1 + n_2 = n + \kappa(G)$ and, because G is a minimal counterexample, $e_i < 3n_i - 5$, for $i = 1,2$.

We strengthen the ideas of the previous lemma by augmenting each G_i with edges corresponding to paths in G. More precisely, consider a pair of non-adjacent vertices $x, y \in S$, and a path P connecting x to y in $G_2 - (S - \{x,y\})$. Now $H = G_1 + \{xy\}$ is a simple graph. Furthermore, if $TK_5 \subset H$, then $TK_5 \subset G$. Therefore, by the minimality of G, $|E(H)| < 3n_1 - 5$ which implies that $e_1 < 3n_1 - 6$. Thus we have used the path P to reduce the number of edges in G_1.

In general, suppose G is minimal $\mathcal{D}$ with S a $\kappa(G)$-separator that divides G into G_1 and G_2. Let P be a path in $G_i - (S - \{x,y\})$ connecting two vertices of $x, y \in S$ with $xy \notin E(G)$. We call P a ***substituting path*** for G_j (where $j = \{1,2\} - i$) and say P ***substitutes*** for xy (see figure 1). Define $\sigma(G_i)$ to be the maximum number of internally vertex-disjoint substituting paths for G_i that pairwise do not share the same initial and terminal vertex. Observe that, if some pair of vertices in $G[S]$ are not adjacent, then $\sigma(G_i) \geq 1$, for $i = 1,2$. We make implicit use of this observation throughout the rest of the paper. The following lemma is the essence of this section.

Lemma 3. *Suppose G is minimal in $\mathcal{D}$, and S is a $\kappa(G)$-separator dividing G into G_1 and G_2. Then*

$$7 + |E(S)| + \sigma(G_1) + \sigma(G_2) \leq 3|S| \tag{1}$$

Proof: For each $i = 1,2$, form the simple graph H_i from G_i by adding the $\sigma(G_i)$ edges corresponding to the substituting paths for G_i. By construction, $TK_5 \subset H_i$ implies $TK_5 \subset G$; hence $TK_5 \not\subset H_i$. Consequently, by the minimality of G, $|E(H_i)| < 3n_i - 5$ and $e_i < 3n_i - 5 - \sigma(G_i)$. Now,

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - |E(G_1) \cap E(G_2)| \\ &\leq 3(n_1 + n_2) - 12 - \sigma(G_1) - \sigma(G_2) - |E(S)| \end{aligned}$$

So the result follows from $n_1 + n_2 = n + |S|$ and $|E(G)| = 3n - 5$. □

To establish the 5-connectivity of a minimal counterexample, we shall use Lemma 3 repeatedly, forcing contradictions using equation (1).

Lemma 4. *If G is minimal in $\mathcal{D}$, then $\kappa(G) \geq 4$.*

Proof: As in Lemma 2, we argue by contradiction. Suppose that $S = \{x, y, z\}$ is a 3-separator, dividing G into G_1 and G_2. By Lemma 2, S is a $\kappa(G)$-separator of G.

If $|E(S)| = 3$, then equation (1) immediately yields a contradiction. We conclude that there is some pair of non-adjacent vertices in S, say x and y. Because S is a minimum separator, there is a substituting path for both G_1 and G_2, substituting for xy. That is, $\sigma(G_i) \geq 1$ for $i = 1, 2$, implying $E(S) = \emptyset$ by equation (1).

Because G is 3-connected, Theorem 3 implies there is a cycle containing x, y and z. The cycle segments P_{xy}, P_{yz} and P_{xz} can be considered as three vertex-disjoint paths. Indeed the three paths P_{xy}, P_{xz}, and P_{yz}, are three substituting paths substituting for xy, xz, and yz since $E(S) = \emptyset$. Thus, $\sigma(G_1) + \sigma(G_2) \geq 3$, and we again obtain a contradiction via equation (1). We conclude that $\kappa(G) > 3$. □

Observe that if G is a minimal counterexample, then G may not contain a K_4. To see this, consider a set $U \subset V(G)$ with $G[U]$ isomorphic to K_4. For any vertex $x \in V(G) - U$ there exists an $x - U$ fan by Lemma 4 and Theorem 2. This implies $TK_5 \subset G$. We use this observation to prove the following useful lemma. Let $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ denote the neighborhood of the vertex x in the graph G.

Lemma 5. *Suppose G is minimal in $\mathcal{D}$, and S is a 4-separator of G that divides G into G_1 and G_2. For $i = 1, 2$, S contains at most one vertex x such that $|N_{G_i}(x) - S| = 1$.*

Proof: By contradiction. Suppose $x, y \in S$ such that $N_{G_1}(x) - S = \{u\}$ and $N_{G_1}(y) - S = \{v\}$. Note that $u \neq v$, otherwise $\kappa(G) = 3$ contradicting Lemma 4. Because $G[S]$ is not isomorphic to K_4, $\sigma(G_i) \geq 1$, for $i = 1, 2$. Hence $e_i \leq 3n_i - 7$. Moreover, $H = G_1 - \{x, y\}$ has at most $3(n_1 - 2) - 7$

edges, by similar reasoning. Therefore we obtain the following contradiction:

$$\begin{aligned} 3n-5 = |E(G)| &\leq 2 + |E(H)| + |E(G_2)| \\ &\leq 2 + (3n_1 - 13) + (3n_2 - 7) \\ &\leq 3n - 6 \end{aligned}$$

since $n_1 + n_2 = n + 4$. □

Theorem 5. *If G is minimal in $\mathcal{D}$, then $\kappa(G) = 5$.*

Proof: As in the previous lemmas, we assume that G is 4-connected and obtain a contradiction. To this end, suppose $S = \{w, x, y, z\}$ is a 4-separator of G that divides G into G_1 and G_2. Because $G[S]$ is not isomorphic to K_4, $\sigma(G_i) \geq 1$, for $i = 1, 2$. From equation (1), we conclude that $|E(S)| \leq 3$.

Let P_j and E_j denote a path and independent set on j vertices, respectively; $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2. So, $G[S]$ is isomorphic to one of K_3, P_4, $K_{1,3}$, $P_2 \cup P_2$, $P_3 \cup E_1$, $P_2 \cup E_2$, or E_4. To prove that G is 5-connected, it remains to exclude these seven cases.

Case 1: K_3. Suppose $\{x, y, z\}$ form a triangle. There are four vertex-disjoint paths from any vertex $u \in G_1 - S$ to $v \in G_2 - S$ since $\kappa(G) \geq 4$. Consequently G contains a subdivision of K_5 with branch vertices $\{x, y, z, u, v\}$.

Case 2: P_4. Suppose $E(S) = \{wx, xy, yz\}$. By equation (1), it suffices to show $\sigma(G_1) + \sigma(G_2) \geq 3$. Let $v \in N_{G_1}(z) - S$. Because G is 4-connected, Theorem 2 guarantees a fan from w to $\{x, y, z, v\}$ consisting of four vertex-disjoint paths $P_{wx}, P_{wy}, P_{wz}, P_{wv}$. The paths P_{wy} and P_{wz} each lie completely in G_1 or G_2 since $\{w, x, y, z\}$ is a 4-separator. Similarly, P_{wv} must lie completley in G_1. If $P_{wz} \in G_2$, then P_{wz} is a substituting path for G_1 and $P_{wv} + vz$ is a substituting path for G_2; so together with P_{wy}, $\sigma(G_1) + \sigma(G_2) \geq 3$.

Suppose $P_{wz}, P_{wv} \in G_1$. To show $\sigma(G_1) + \sigma(G_2) \geq 3$, it suffices to find vertex-disjoint paths P_{xz} and P_{wz} in G_1 that avoid y. Consider a path P_{xz} connecting x to z in G_1 such that P_{xz} avoids the vertices w, y (if no such path exists then $\{w, x, y\}$ is a 3-separator, contradicting Lemma 4). If P_{xz} avoids either P_{wz} or P_{wv}, then we have found the desired paths. Otherwise, let u be the vertex closest to x where P_{xz} intersects one of these paths. Without loss of generality, we may assume that $u \in P_{wz}$. Let P_{uz} be the segment of P_{wz} from u to z. Then $P_{xu} + P_{uz}$ and $P_{wv} + vz$ are the two desired paths.

Case 3: $K_{1,3}$. Suppose $E(S) = \{wx, wy, wz\}$. By Lemma 4 and Theorem 3, there is a cycle in $G - w$ containing x, y, z. This cycle determines three substituting paths P_{xy}, P_{xz}, and P_{yz}. Hence, $\sigma(G_1) + \sigma(G_2) \geq 3$ and equation (1) yields a contradiction.

Case 4: $P_2 \cup P_2$. Suppose $E(S) = \{wx, yz\}$. To obtain a contradiction from equation (1), it suffices to show, for $i = 1, 2$, that $\sigma(G_i) \geq 2$. We show $\sigma(G_2) \geq 2$. The other case is symmetric.

Observe that, by Lemma 5, there is at most one vertex of S, say z, such that $|N_{G_1}(z) - S| = 1$. So there are two vertices $a, b \in N_{G_1}(w) - S$,

Now $G - \{w, x\}$ is 2-connected. Hence, by Theorem 4 there are two disjoint paths linking $\{a, b\}$ and $\{y, z\}$. Because $\{y, z\}$ is a 2-cut in $G - \{w, x\}$, these two paths must lie entirely in G_1. These paths substitute for edges wy and wz, and so $\sigma(G_2) \geq 2$.

Case 5: $P_3 \cup E_1$. Suppose $E(S) = \{wx, xy\}$. In this case, we show that $\sigma(G_1) + \sigma(G_2) \geq 4$ by showing that, for some $i \in \{1, 2\}$, $\sigma(G_i) \geq 3$. Equation (1) provides the contradiction.

Because $\delta(G) = 5$, there is some $j \in \{1, 2\}$ such that there exist three vertices $a, b, c \in N_{G_j}(z) - S$. By theorem 4, there exist vertex disjoint paths linking $\{a, b, c\}$ to $\{w, x, y\}$ in $G - z$. These three paths must all lie in G_j. Therefore they form three substituting paths P_{zw}, P_{zx}, and P_{zy} for G_i, where $i = \{1, 2\} - j$.

Case 6: $P_2 \cup E_2$. Suppose $E(S) = \{wx\}$. By Lemma 5 and $\delta(G) = 5$, we may assume, without loss of generality, that there are three vertices $a, b, c \in N_{G_1}(y) - S$. Arguing as in the previous case, theorem 4 implies the existence of three substituting paths for G_2, P_{yw}, P_{yx}, and P_{yz} by linking $\{a, b, c\}$ with $\{w, x, z\}$ in $G - y$. Hence, $\sigma(G_2) \geq 3$.

Furthermore, by Lemma 5 and $\delta(G) = 5$, either $|N_{G_2}(y) - S| \geq 2$ or $|N_{G_2}(z) - S| \geq 2$. In either case, linking the neighborhood vertices with $\{w, x\}$ in $G - \{y, z\}$ shows that $\sigma(G_1) \geq 2$. Thus, $\sigma(G_1) + \sigma(G_2) \geq 5$, and equation (1) yields a contradiction.

Case 7: E_4. In this case, it suffices to show that $\sigma(G_1) + \sigma(G_2) \geq 6$. Observe that, applying the method in the previous case, if there is a vertex of S, say w, such that $|N_{G_i}(w) - S| \geq 3$, then $\sigma(G_j) \geq 3$, where $j = \{1,2\} - i$. Thus, it is enough to consider the case that, for some $i \in \{1,2\}$, for all $v \in S$, $|N_{G_i}(v) - S| \leq 2$. Without loss of generality, suppose $i = 1$.

Applying the method of the previous case, it is easy to show $\sigma(G_2) \geq 2$. Hence, $e_2 \leq n_2 - 8$. Consider $H = G_1 - S$. If H has at least three vertices (i.e. $n_1 - 4 \geq 3$), then $|E(H)| \leq 3(n_1 - 4) - 6$, by the minimality of G (This is clearly true if $n_1 - 4 \geq 5$. The remaining cases, $n_1 - 4 \in \{3, 4\}$, follow because G is simple). Therefore,

$$\begin{aligned} 3n - 5 = |E(G)| &\leq |E(H)| + |E(G_2)| + 8 \\ &\leq 3(n_1 - 4) - 6 + 3n_2 - 8 + 8 \\ &= 3n - 6 \end{aligned}$$

This contradiction implies H has exactly two vertices (the minimum degree prohibits H having a single vertex).

So, H is consists of two adjacent vertices, u and v, each of which is adjacent to every vertex of S. Suppose $G - \{u, v\}$ is 3-connected. In this case, theorem 3 guarantees that $\{x, y, z\}$ lie on a cycle of $G - \{u, v\}$. Consequently, G contains a subdivision of K_5; the branch vertices are u, v, x, y, z. This is a contradiction.

Therefore, $G - \{u, v\}$ must be 2-connected, with a 2-separator S'. However, in this case, we may form a 4-separator $\{u, v\} \cup S'$ of G with at least one edge. This reduces to a previous case. □

4. Forbidden subgraphs

Recall that, in the previous section, K_4 was forbidden from any minimal graph in $\mathcal{D}$. Applying similar arguments and 3-connectivity, we now extend these results and summarize them in the following theorem. Let $G_1 + G_2$ denote the *join* of G_1 and G_2; it is the graph obtained from G_1 and G_2 by joining each vertex of G_1 to each vertex of G_2.

Theorem 6. *No minimal graph in $\mathcal{D}$ contains K_4, $K_{3,3}$, $K_2 + E_3$ or $K_{2,4}$.*

Proof: We prove only that $K_2 + E_3$ is forbidden; the other proofs are similar and are omitted. Suppose that G is minimal in $\mathcal{D}$, and $K_2 + E_3 \subset G$ such that x, y are the vertices of the K_2 portion of $K_2 + E_3$. By Theorem 3, there is a cycle in $G - \{x, y\}$ containing the three vertices of E_3, since $G - \{x, y\}$ is 3- connected. This implies $TK_5 \subset G$. $\square$

The aim of this section is to forbid $K_4 - e$ in any minor-minimal graph in $\mathcal{D}$. To prove this we require some preliminary definitions and technical lemmas. Graph L is defined as shown in figure 2. A *branch vertex* of a subdivision is a vertex of degree at least three; and, a *branch path* is a path between branch vertices. In any subdivision of L, the branch vertices of degree three are called *minor* branch vertices, and the branch vertices of degree four are called *major* branch vertices. The following lemma is presented by Thomassen in [Th74]:

Lemma 6 (Thomassen). *Let $G' = G/_{xy}$, the graph obtained by contracting edge xy in G.*
(a) If $TK_5 \subset G'$ such that $xy \in V(G')$ is not a branch vertex, then $TK_5 \subset G$.
(b) If $TK_5 \subset G'$ with vertex $xy \in V(G')$ a branch vertex, then either $TK_5 \subset G$ such that x or y is a branch vertex, or $TL \subset G$ such that x and y are minor branch vertices.

Lemma 7. *If G is minor-minimal in $\mathcal{D}$ then, for every $x, y \in V(G)$ with $xy \in E(G)$, there is a subdivision of L in G such that x and y are minor branch vertices.*

Proof: Let G be minor-minimal in $\mathcal{D}$, with $x, y \in V(G)$ such that $xy \in E(G)$. Since the graph $K_2 + E_3$ is forbidden from G, $G/_{xy}$ has at most three fewer edges than G. Hence $|E(G/_{xy})| \geq 3|V(G/_{xy})| - 5$, and $G/_{xy}$ contains a TK_5. By Lemma 6, G contains a subdivision of L such that x and y are minor branch vertices. $\square$

From Lemma 7, we may now obtain more detailed structural information about any minor-minimal graph in $\mathcal{D}$ with a triangle. We introduce a few definitions to refine our view of TL and describe this structure.

Label the minor branch vertices of TL, x and y, and the major branch vertices a,b,c, and d as in figure 3. The four branch paths between $\{x, y\}$ and $\{a, b, c, d\}$ are designated P_1, P_2, P_3, and P_4 and are called *P-paths*. P is the set of vertices in $V(G) - \{x, y\}$ that appear in a P-path. The six branch paths between the major branch vertices are labelled $R_1, \ldots, R_6$ and are called *R-paths*. R is the set of vertices in $V(G) - \{a, b, c, d\}$ that appear in an R-path. R_i and R_j are *adjacent* if they are incident to the same branch vertex, and *parallel* if they are not. For example, R_1 and R_2 are adjacent; R_1 and R_6 are parallel. $\{R_1, R_2, R_5, R_6\}$ are the *middle* R-paths, and $\{R_3, R_4\}$ the *outside* R-paths. If Q is a path with a single endpoint in $R - \{a, b, c, d\}$, we define $\Phi(Q)$ to be the R-path that contains the endpoint of Q in R. If S is a set of paths with endpoints in $R - \{a, b, c, d\}$, $\Phi(S)$ is defined to be the set of R-paths that contain the endpoints of S in R.

Lemma 8. *Suppose G is minor-minimal in $\mathcal{D}$ with a triangle $\{x, y, z\}$. Then, G contains a subdivision of L such that x and y are minor branch vertices. Furthermore, given R and P as defined above,*
(1) z is separated from P by R in $G - \{x, y\}$,
(2) If $z \notin R$, then there are three disjoint paths in $G - \{x, y\}$ from z to R such that all interior vertices avoid $V(TL)$, and all three endpoints are either

(a) all in the same R-path, or

(b) incident to three different R-paths, which are pairwise adjacent, though not all incident to the same major branch vertex.

Proof: Let G be minor-minimal in $\mathcal{D}$ with a triangle $\{x, y, z\}$. By Lemma 7, G contains a subdivision of L with minor branch vertices x and y.

If z is a vertex of a P-path, then $TK_5 \subset G$ with branch vertices a, b, c, d and, either x or y depending upon which P-path contains z. More generally, if there is a path from z to P using only vertices of $V(G) - V(TL)$, then $TK_5 \subset G$, as shown in figure 4. Thus, $z \notin P$, and no path from z to P avoids $V(TL)$; that is, the vertices in R separate z from P in $G - \{x, y\}$, and statement (1) has been established.

Suppose $z \notin R$ (if not, statement (2) is vacuous). Because G is 5-connected, there are three disjoint paths from z to $\{a, b, c\}$ in $G - \{x, y\}$. Each of these paths must contain a vertex in R, since R separates z from $\{a, b, c\}$ in $G - \{x, y\}$. Let Z_1, Z_2, and Z_3 be the three disjoint paths from z to R defined by these three paths. Call these paths Z-paths, and let Z be the set Z-paths.

Suppose two Z-paths paths have endpoints in parallel R-paths. If the parallel R-paths are both middle R-paths, there is a TK_5 in G with branch vertices $\{c, d, x, y, z\}$, as shown in figure 5. Otherwise the endpoints are in R_3 and R_4, and $\{a, c, x, y, z\}$ are branch vertices of a TK_5 (see figure 6).

Suppose the endpoints of the Z-paths lie in three different R-paths all incident to the same major branch vertex. Without loss of generality, we may assume $\Phi(Z) = \{R_1, R_2, R_3\}$; they are all incident to a. In this case, $\{b, c, d, y, z\}$ are branch vertices of a TK_5 as shown in figure 7.

Suppose $\Phi(Z)$ consists of two adjacent R-paths. Without loss of generality, we may assume they are incident to a. In this case, $\{y, z, b, c, d\}$ are the branch vertices of a TK_5 (figure 8).

For every $1 \leq i < j \leq 3$, $\Phi(Z_i)$ and $\Phi(Z_j)$ cannot be parallel, and hence must be equal or mutually adjacent. But if $\Phi(Z)$ consists of three R-paths all incident to a single branch vertex, then $\Phi(Z)$ must consist of a single R-path. This shows that the endpoints of the Z-paths are either,

(a) all in the same R-path, or

(b) incident to three different R-paths, which are pairwise adjacent, though not all incident to the same major branch vertex.

These are the configurations given in the statement of the lemma. □

We now can state the main result of this section:

Theorem 7. *No minor-minimal graph in $\mathcal{D}$ contains $K_4 - e$.*

Proof: We prove the theorem by contradiction. Suppose G is minor-minimal in $\mathcal{D}$ such that w, x, y, and z induce a $K_4 - e$. Let x and y be the vertices of degree three in the induced $K_4 - e$. By Lemma 7, there is a subdivision of L in G with x and y as minor branch vertices. Label this TL as in the previous lemma. Also define the P-paths and R-paths as in the previous lemma.

We divide the proof into three cases depending upon whether all, one, or none of z and w are in R. To prove the the theorem, it suffices to exclude these three cases.

Case 1: $w, z \in R$. We consider three subcases according to the placement of w and z in R: the same R-path, adjacent R-paths, or parallel R-paths.

Case 1.1: w and z are in the same R-path. If w and z are both in R_1, there is a TK_5 $\{c, w, x, y, z\}$, as shown in figure 9. Similar arguments apply for the other R-paths. (Figure 10 shows the case where w and z are in R_3.)

Case 1.2: w and z are in adjacent R-paths, say R_w and R_z. By symmetry, it suffices to consider the case that one of R_w, R_z is an outside R-path, and the case that they are both middle R-paths: $R_w = R_1$, $R_z = R_3$; and, $R_w = 1$, $R_z = R_2$. If $R_w = R_1$ and $R_z = R_3$, then $\{a, b, x, y, z\}$ are the branch vertices of a TK_5, as shown in figure 11. If $R_w = R_1$ and $R_z = R_2$, then $\{w, x, y, z, d\}$ are the branch vertices of a TK_5, as shown in figure 12.

Case 1.3: w and z are in parallel R-paths, say R_w and R_z. By symmetry, it suffices to consider when these R-paths are both middle or both outside R-paths: $R_w = R_1$, $R_z = R_6$; and, $R_w = R_3$, $R_z = R_4$. If $R_w = R_1$ and $R_z = R_6$, then $\{a, b, w, x, y\}$ are the branch vertices of a TK_5, as shown in figure 13. If $R_w = R_3$ and $R_z = R_4$, a subdivision of K_5 appears as in figure 14.

Case 2: $|R \cap \{z, w\}| = 1$. Without loss of generality, assume $w \in R$. By symmetry, there are only two subcases to consider: $w \in R_1$ or $w \in R_3$. Because $z \notin R$, Lemma 8 guarantees three disjoint paths from z to R. Call these three paths Z-paths. By Lemma 8, either $\Phi(Z)$ is a single R-path, or $\Phi(Z)$ consists of three pairwise adjacent R-paths, not all incident to the same major branch vertex. We may assume that $\Phi(Z)$ is not a single R-path because, in this case, one can form a new subdivision of L in G such that $z, w \in R$ and x, y are the minor branch vertices, by redirecting the R-path in $\Phi(Z)$ through z (this reduces to case 1). We also may assume no Z-path ends at w since, in such a case, G contains a subdivision of K_5 with branch vertices $\{w, x, y, z\}$ plus one vertex in $\{a, b, c, d\}$ depending upon the location of w and $\Phi(Z)$ in R (another Z-path is used to complete a path from z to the fifth branch vertex).

Case 2.1: $w \in R_1$. Because $\Phi(Z)$ consists of pairwise adjacent R-paths not all incident to one major branch vertex, some Z-path ends in an outside R-path. Therefore $\{a, c, x, y, w\}$ are the branch vertices of a TK_5, as in figure 15.

Case 2.2: $w \in R_3$. $\Phi(Z)$ consists of pairwise adjacent R-paths, not all incident to the same major branch vertex. By symmetry, we may assume, without loss of generality, that $\Phi(Z)$ contains R_2; that is, $\Phi(Z) = \{R_2, R_3, R_6\}$ or $\{R_2, R_1, R_4\}$. In either case, $\{w, x, y, z, a\}$ are the branch vertices of a TK_5, as shown in figure 16 (which shows the case where a Z-path ends in R_6).

Case 3: $R \cap \{w, z\} = \emptyset$. Because both w and z are neighbors to x and y, Lemma 8 guarantees three disjoint paths from z to R, and three disjoint paths from w to R. Let Z_1, Z_2 and Z_3 be the three disjoint paths from z to R (the Z-paths), and Z the set of Z-paths. Similarly, let W_1, W_2 and W_3 be the three disjoint paths from w to R (the W-paths), and W the set of W-paths. Observe that, by definition, only terminal vertices of Z-paths or W-paths are vertices of R.

By Lemma 8, either $\Phi(Z)$ is a single R-path, or $\Phi(Z)$ consists of three pairwise adjacent R-paths, not all incident to the same major branch vertex. We may assume that $\Phi(Z)$ is not a single R-path because, in this case, one can form a new subdivision of L in G such that $z \in R$ and x, y are the minor branch vertices, by redirecting the R-path in $\Phi(Z)$ through z (this reduces to case 2). The same argument shows that $\Phi(W)$ is not a single R-path.

Because $\Phi(Z)$ and $\Phi(W)$ each consist of three pairwise adjacent R-paths not all incident to the same branch vertex, we may assume, without loss of generality, that $\Phi(Z_1) = R_1$ and $\Phi(W_1) = R_3$. If Z_1 and W_1 do not intersect, then G contains a subdivision of K_5 with branch vertices $\{x, y, b, c, d\}$, as shown in figure 17. Hence, Z_1 and W_1 must intersect.

Reorder the W-paths so that W_1 is the first W-path that Z_1 intersects, and u is a vertex of their intersection closest to z. Our immediate goal is to construct, from the Z-paths and W-paths, three internally disjoint paths: one zw-path, one zR-path (Q_z), and one wR-path (Q_w). If Z_2 does not meet any W-path, then we let $Q_z = Z_2$, $Q_w = W_2$, and form the zw- path with the initial segments of Z_1 and W_1 that meet at u. Otherwise, Z_2 first intersects some W-path, say W_i, at some vertex v. If $W_i \neq W_1$, then let $Q_w = W_j (j = \{2,3\} - \{i\})$, Q_z the path formed by the initial segment of Z_2 from z to v and the final segment of W_i from v to R, and form the zw-path from the initial segments of Z_1 and W_1. If $W_i = W_1$, we may assume, without loss of generality, that u is closer to v along W_1. In this case, let Q_z be the path formed by the initial segment of Z_2 and the final segment of W_1, let $Q_w = W_2$, and form the zw-path from the initial segments of Z_1 and W_1.

The zw-path together with the edges in the $K_4 - e$ form a subdivision of K_4 in G. To show that G has a subdivision of K_5, it suffices to show that some vertex in $\{a,b,c,d\}$ can be the fifth branch vertex of a TK_5 involving $\{w,x,y,z\}$. The branch paths from the fifth branch vertex are constructed using Q_w, Q_z, P-paths, and R-paths.

Suppose $\Phi(Q_w) = \Phi(Q_z)$. If Q_w and Q_z end in the same vertex $q \in R$, then $\{q,w,x,y,z\}$ are the branch vertices of a TK_5. If Q_w and Q_z do not share a common endpoint, but $\Phi(Q_z) = \Phi(Q_w) = R_1$ say, then $\{a,w,x,y,z\}$ are the branch vertices of a TK_5 (figure 18). Other cases where $\Phi(Q_z) = \Phi(Q_w)$ are similar.

Suppose $\Phi(Q_w) \neq \Phi(Q_z)$. By symmetry, we may assume that $\Phi(Q_z)$ is incident to a, while $\Phi(Q_w)$ is not. It suffices to find four vertex disjoint paths: one path from each of w,x,y,z to a. P_2 connects x and a. A segment of $\Phi(Q_z)$ plus Q_z connects z and a. A path in $\{R_1, R_2\} - \Phi(Q_z)$ plus a path in $\{P_3, P_4\}$ connect y and a. The remaining R-paths and Q_w contain a path connecting w and a. Thus, $\{a,w,x,y,z\}$ are the branch vertices of a TK_5. □

5. Genus

We assume the reader is familiar with the notation and results found in [4]. Let S be a closed, connected 2-manifold. We denote the ***Euler characteristic*** of a cellular imbedding, $G \to S$ of a connected graph G into S by $\chi(G \to S)$; its value is $|V(G)| - |E(G)| + f$, where f is the number of faces of the imbedding. The Euler characteristic is an invariant of the surface S. Let $\chi(S)$ be the Euler characteristic of S (so $\chi(G \to S) = \chi(S)$ for any cellular imbedding of any G into S).

Theorem 8. *Suppose G is a simple graph on n vertices that is minor-minimal in $\mathcal{D}$, and $G \to S$ a cellular imbedding of G into S, a closed, connected 2-manifold. Then,*

$$\chi(S) \leq \lfloor 5/3 - n/4 \rfloor .$$

Proof: Let $\chi = \chi(S)$, α = number of triangles in G, and f_i = the number of i-sided faces in the imbedding $G \to S$. Now, $\chi = n - (3n - 5) + f$, since $|E(G)| = 3n - 5$. On the other hand,

$$3\alpha + 4(f - \alpha) \leq \sum_{i \geq 3} i f_i = 2(3n - 5).$$

Combining these two, we find

$$-4\chi \geq 2n - 10 - \alpha \tag{2}$$

so it suffices to show that $\alpha \leq (3n - 10)/3$.

Theorem 7 implies that every edge of G is in at most one triangle. Furthermore, every vertex of degree five in G is incident to an edge in no triangle, otherwise G has a $K_4 - e$. Because G has at least ten vertices of degree five, there are at least five edges of G that appear in no triangle. Thus, at most $3n - 10$ edges are in triangles, and $\alpha \leq (3n - 10)/3$. □

We say that Dirac's conjecture holds for a surface S if every simple graph G with n vertices, $3n - 5$ edges, and a cellular imbedding into S, contains a subdivision of K_5 (the conjecture holds vacuously for the sphere). In this section, we use Theorem 8 to prove that Dirac's conjecture holds for several surfaces. First we prove a technical lemma.

Lemma 9. *Suppose G is minor-minimal in $\mathcal{D}$, and $F = \{v \in V(G) : d_G(v) = 5\}$. Then, the girth of $G[F]$ is at least five.*

Proof: We prove that $G[F]$ does not have a triangle or four-cycle.

Suppose, to the contrary, that $x_1, x_2, x_3 \in F$ form a triangle of G. By Theorem 7, $N_G(x_i) \cap N_G(x_j) = \{x_k\}$ for $\{i,j,k\} = \{1,2,3\}$. Furthermore, for each $i = 1,2,3$, there exist a pair of vertices $y_i, z_i \in N_G(x_i)$ such that $y_i z_i \notin E(G)$. Consider $H = G + \{y_1 z_1, y_2 z_2, y_3 z_3\} - \{x_1, x_2, x_3\}$. H has $n - 3$ vertices and $3(n - 3) - 5$ edges. By the minimality of G, $TK_5 \subset H$ contradicting $TK_5 \not\subset G$. Thus, $G[F]$ has no triangle.

Suppose $x_1, x_2, x_3, x_4 \in F$ form a four-cycle. By Theorem 7, we may assume $N_G(x_i) \cap N_G(x_j) = \emptyset$, for $i - j$ odd. Furthermore, one can show that, for each $i = 1, \ldots, 4$, there exist a pair of vertices $y_i, z_i \in N_G(x_i)$ such that $y_i z_i \notin E(G)$ and $\{y_i, z_i\} \cap \{y_j, z_j\} = \emptyset$ for all $j \neq i$. Now consider

$H = G + \{y_i z_i\}_{i=1}^4 - \{x_i\}_{i=1}^4$. H has $n-4$ vertices and $3(n-4)-5$ edges, so by the minimality of G, $TK_5 \subset H$. This contradicts $TK_5 \not\subset G$. □

The conclusion of Lemma 9 may be extended in the case that G has large girth. In particular, if G has girth at least five, then $G[F]$ must be acyclic.

Corollary 1. *Suppose G is a simple graph with n vertices, $3n-5$ edges, and a cellular imbedding into a surface S with $\chi(S) \geq -2$. Then $TK_5 \subset G$.*

Proof: We show that no minor-minimal counterexample can be imbedded into a surface with Euler characteristic greater than -3. To this end, let G be a minor-minimal counterexample with an imbedding $G \to S$ into a surface S with $\chi(S) \geq -2$. By Theorem 8, $\chi(S) \leq 5/3 - n/4$, so $n \leq 14$. By remarks following Lemma 1, $n \geq 10$.

Observe that G must contain a triangle T; otherwise, by equation (2), $-4\chi(S) \geq 2n - 10 \geq 10$. By Lemma 9, T must contain a vertex of degree six. Counting the neighborhood of T reveals that $n \geq 13$ since Theorem 7 implies the neighborhoods of vertices in T are disjoint.

Case 1: $n = 13$. Suppose that G has a vertex with degree at least eight. An edge count reveals that the remaining vertices must then all have degree five. Because every triangle contains the high degree vertex and G has no $K_4 - e$, G has at most four triangles so, by equation (2), $-4\chi(S) \geq 2n - 10 - 4 \geq 12$.

Thus, the maximum degree of G is seven, which implies that G has three vertices of degree six and ten vertices of degree five. If a triangle of G contains two vertices of degree six, then $n \geq 14$ because the neighbors of the triangle are all distinct. So, every triangle in G contains exactly one degree six vertex. Because G has no $K_4 - e$, we conclude that G has at most seven triangles and $-4\chi(S) \geq 2n - 10 - 7 \geq 9$, which is a contradiction.

Case 2: $n = 14$. By the proof of Theorem 8, G has at most ten triangles. On the other hand, equation (2) implies that G has at least ten triangles. Consequently, G must have exactly ten triangles.

If G has a vertex v with degree at least eight, then an edge count reveals that G must have a vertex u of degree six. Now every triangle contains either u or v by Theorem 9. However v is in at most four triangles and u is in at most three triangles; that is, G has at most seven triangles, a contradiction.

So the maximum degree of G is seven. If there is a vertex of degree seven, then there are at most three vertices with degree more than five. Hence, G has at most nine triangles, a contradiction.

The remaining case is when G has exactly four degree six vertices and exactly ten degree five vertices. Let F be the set of degree five vertices, and $S = \{a, b, c, d\}$ the set of degree six vertices. Note that $|E(F)| = 13 + |E(S)|$. Also, $G[F]$ is connected since G is 5-connected and $G[F] = G - S$. In particular, $G[F]$ does not have isolated vertices.

If there is a vertex $v \in F$ with $d_{G[F]}(v) = 5$, then $G[F] - \{v\} - N_G(v)$ has four vertices and at least four edges, contradicting that the girth of $G[F]$ is at least five. Therefore, $\Delta(G[F]) \leq 4$.

Suppose there is a vertex $v \in F$ with $d_{G[F]}(v) = 4$. Let $N_G(v) \cap S = \{a\}$ and $N_G(v) \cap F = \{x_1, x_2, x_3, x_4\}$. If $d_{G[F]}(x_1) = 1$ say, then $x_j \notin N_G(a)$ $(2 \leq j \leq 4)$ since $K_4 - e \not\subset G$, so there must be a pair, say x_2, x_3 such that $|N_G(x_2) \cap N_G(x_3) \cap \{b, c, d\}| \geq 2$. However, $G[\{v, b, c, d, x_1, x_2, x_3\}]$ must then contain $K_{3,3}$ contradicting Theorem 6. On the other hand, if $d_{G[F]}(x_i) \geq 2$ for $i = 1, \ldots, 4$, then $G[\{v, b, c, d\} \cup N_G(v)]$ must contain $K_{3,3}$, by similar reasoning.

Therefore, $\Delta(G[F]) = 3$. Notice that this implies that $\delta(G[F]) = 2$. To see this, consider, for a contradiction, a vertex $v \in F$ with $d_{G[F]}(v) = 1$. Now $d_G(v) = 5$, so v must be adjacent to every vertex of S. A neighbor of v in $G[F]$ must have at least two neighbors in S (since $\Delta(G) = 3$). Therefore S, v, and the neighbor of v in $G[F]$ must induce $K_4 - e$, a contradiction.

Subcase A: $|E[S]| \geq 3$. In this case, $G[F]$ has at least 16 edges and so it must contain a vertex of degree four, contradicting $\Delta(G[F]) \leq 3$.

Subcase B: $|E(S)| = 2$. Consider two adjacent vertices $c, d \in S$. If c and d share no common neighbor, then the edge cd appears in no triangle; consequently each of c and d appear in at most two triangles. However, if c and d have a common neighbor $w \in F$, then $(N_G(c) \cup N_G(d)) \cap N_G(w) = \emptyset$ because G has no $K_4 - e$. Therefore, there exists a common neighbor of c and d, say $z \in F - w$, since

$|E(S)| = 2$ and $\delta(G[F]) = 3$. However, this implies that G contains a $K_4 - e$, namely $\{c, d, w, z\}$. Hence, c and d appear in at most two triangles. Because c and d were arbitrary adjacent vertices of S and $|E(S)| = 2$, there must be three vertices of S that appear in at most two triangles. That is, G has at most nine triangles, since each triangle of G must contain a vertex of S. This is a contradiction.

Subcase C: $|E(S)| = 1$. In this case, $|E[F]| = 14$. Because $\Delta(G[F]) = 3$ and $\delta(G[F]) = 2$, $G[F]$ must have exactly two vertices of degree two, say u and v. If $w \in N_G(u) \cap N_G(v) \cap F$, then $K_4 - e \subset G[\{u, v, w\} \cup S]$, a contradiction. Similarly, if u and v are adjacent, then $K_4 - e \subset G[\{u, v\} \cup S]$. So, we may assume $N_G(v) \cap N_G(v) \cap F = \emptyset$, and $uv \notin E(G)$.

Suppose, without loss of generality, $E(S) = \{cd\}$. If $\{c, d\} \subset N_G(v)$, then $K_4 - e \subset G[v \cup N_G(v) \cup S]$. Thus, we may assume $|N_G(v) \cap \{c, d\}| = 1$. The same argument applies to u. Thus, there are two cases to consider: $N_G(v) \cap S \neq N_G(u) \cap S$, and $N_G(v) \cap S = N_G(u) \cap S$. Let $H = G[\{u, v\} \cup N_G(u) \cup N_G(v) \cup S]$.

Suppose $N_G(v) \cap S \neq N_G(u) \cap S$. Without loss of generality, assume $c \in N_G(v)$ and $d \in N_G(u)$. Figure 19 shows the ten vertices of H, the edges forced into H by degree requirements and $K_4 - e \not\subset G$, and a new vertex $z \in N_G(a) \cap N_G(b) - H$. The vertex z must exist since a and b each have six neighbors in G while a has only four neighbors in H, b has only three neighbors in H, and there are only four vertices in $G - H$. Thus G contains a subdivision of K_5 as shown by the bold lines in the figure.

Similarly, suppose $N_G(v) \cap S = N_G(u) \cap S$. Figure 20 shows the ten vertices of H, the edges forced into H by degree requirements and $K_4 - e \not\subset G$, and a vertex $z \in N_G(b) \cap N_G(c) - H$ guaranteed by arguing as in the previous paragraph. Thus G contains a subdivision of K_5 as shown by the bold lines in the figure.

Subcase D: $E(S) = \emptyset$. In this case, $|E[F]| = 13$. Because $\Delta(G[F]) \leq 3$ and $\delta(G[F]) = 2$, $G[F]$ has a set T of four vertices of degree two.

Suppose there are two vertices $u, v \in T$, such that $N_G(u) \cap S = N_G(v) \cap S$; without loss of generality, $N_G(u) \cap S = \{a, b, c\} = N_G(v) \cap S$. If u and v are adjacent, then a, b, c, u, v form a

$K_2 + E_3$. Similarly, if u and v share a common neighbor $w \in F$, then w must have a neighbor among a, b, c so a $K_4 - e$ is formed. Thus $N_G(v) \cap F = \{x, y\}$ and $N_G(u) \cap F = \{p, q\}$ such that $p, q, x, y \in F - T$. Since $K_4 - e \not\subset G$, $\{p, q, x, y\} \subset N_G(d)$. We may assume that $x \in N_G(a)$ and $y \in N_G(b)$. Now there are three cases according to whether $S - (N_G(p) \cup N_G(q))$ is equal to a, b, or c. The three cases are shown in figures 21,22, and 23. The figures include a vertex $z \notin \{u, v\} \cup N_G(u) \cup N_G(v) \cup S$ adjacent to two vertices of S (the existence of z can be established by considering the neighborhoods of vertices adjacent to z in S). In each case a subdivision of K_5 is indicated by bold lines.

Thus, we may assume that no pair of vertices in T share the same three neighbors in S; that is, $G[S \cup T]$ is isomorphic to $K_{4,4}$ minus a one-factor. Because no pair of vertices of T are adjacent, some pair of vertices $u, v \in T$ share a common neighbor $z \in N_G(u) \cap N_G(v) \cap F$. Let $w = N_G(u) \cap F - \{z\}$. Without loss of generality, assume $N_G(u) \cap S = \{b, c, d\}$ and $N_G(v) \cap S = \{a, c, d\}$ (so $N_G(z) \cap S = \{a, b\}$ and $a \in N_G(w)$). However, one can now see that there is a subdivision of K_5 in $G[S \cup T \cup \{w, z\}]$ with branch vertices a, b, u, v, z. □

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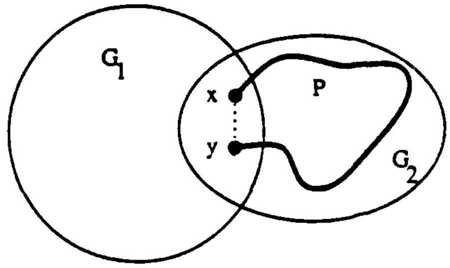

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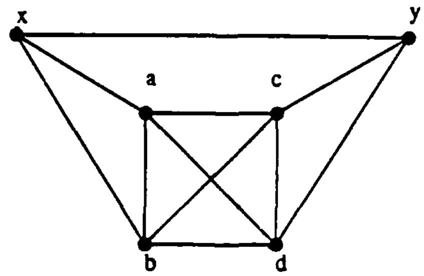

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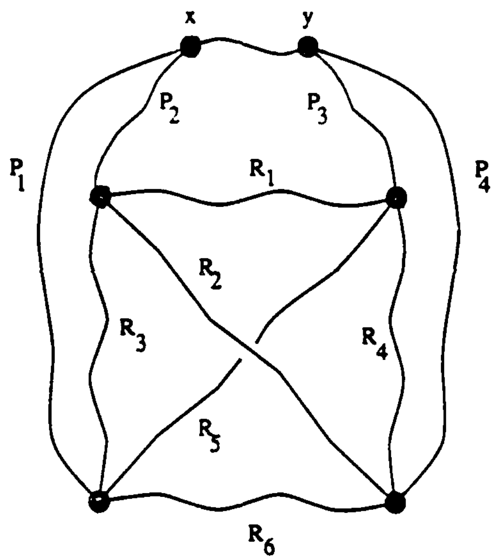

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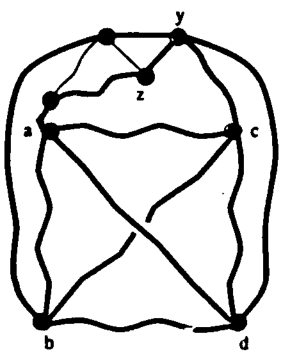

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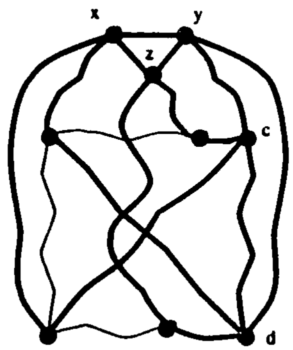

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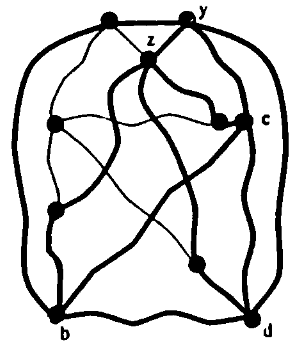


figure 7.

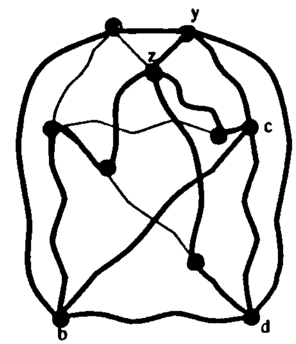
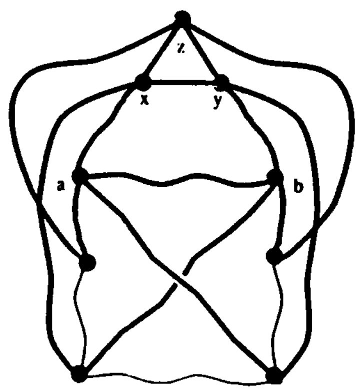


figure 8.



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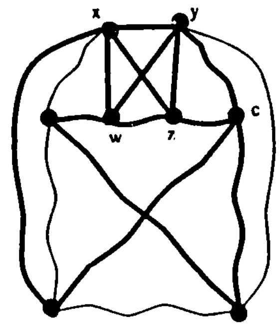

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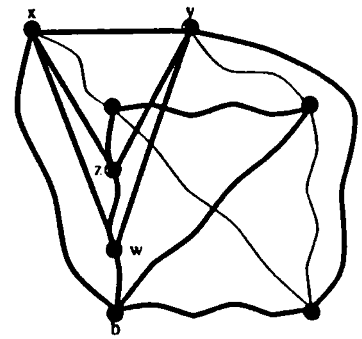

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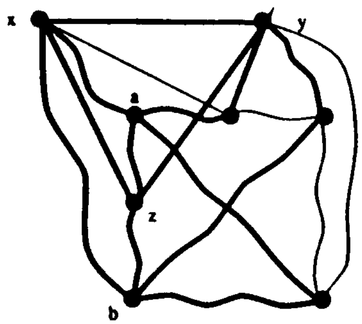

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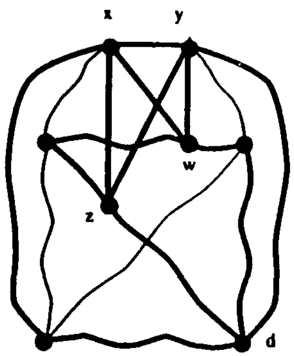

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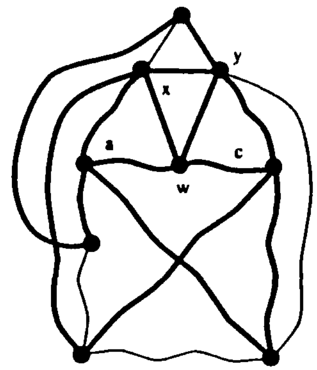

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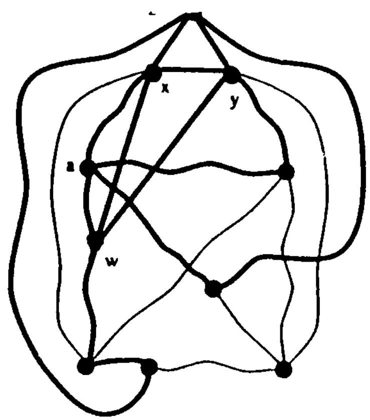

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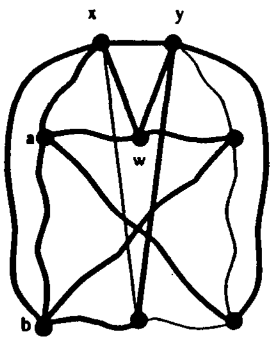

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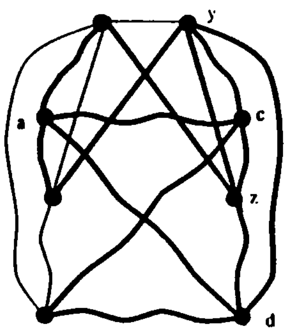

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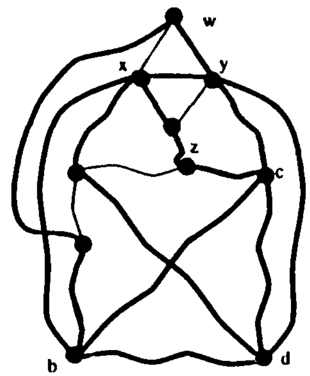

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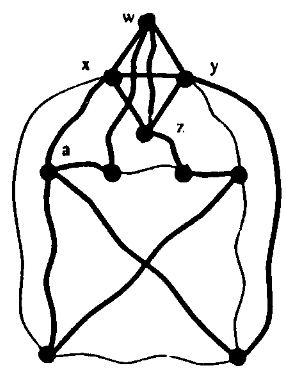

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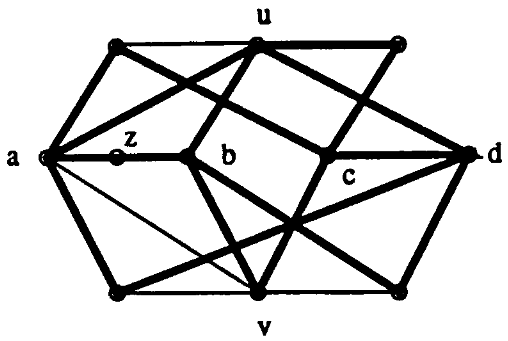

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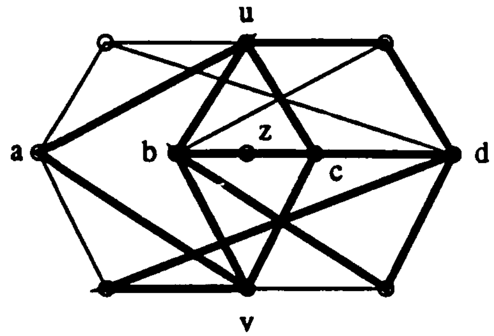

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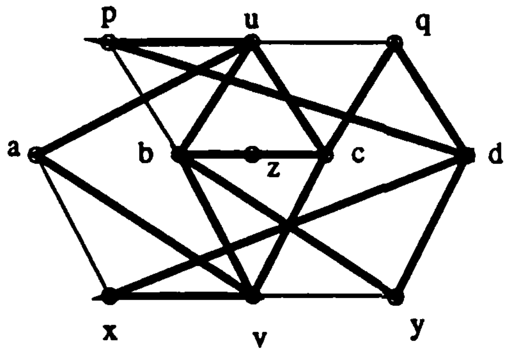

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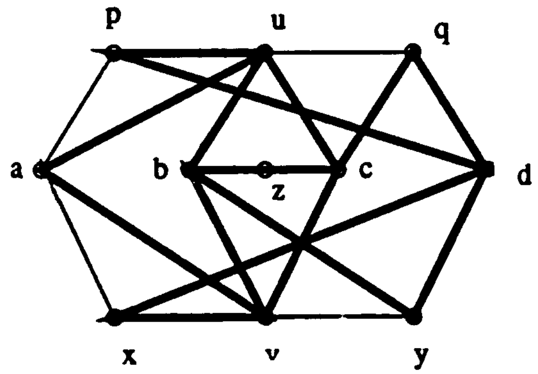

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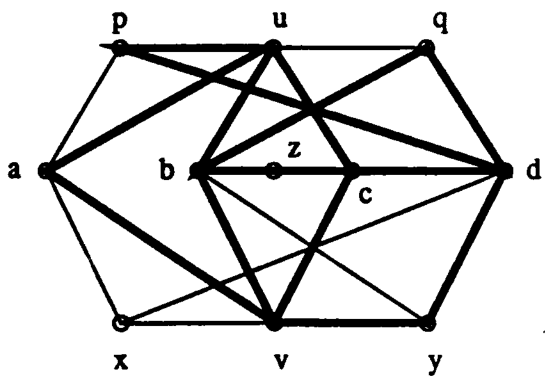

figure 23.