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Do AM Hercules white dwarfs have toroidal internal fields?

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Summary. Cropper (1988) has determined the orientations of the magnetic poles of eight AM Hercules white dwarfs. These data imply a quasi-static aligning torque, which I compare to previous theories. The fact of alignment, but not the position angles, is consistent with our earlier magnetostatic model. The angles may be explained if the dominant torque is gravitational and the white dwarfs are dynamically prolate. I construct an explicit but simplified internal model of such a star in which its figure is distorted from sphericity by a toroidal internal magnetic field, and discuss some implications.

1 Introduction

The AM Hercules stars are binaries in which an accreting magnetic degenerate (white) dwarf rotates synchronously with its orbital period (see Liebert & Stockman 1985 for a review). This remarkable phenomenon poses the question of how synchronism is produced and maintained.

Joss, Katz & Rappaport (1979; henceforth JKR) treated the non-degenerate star as a sphere of high electrical conductivity. Because such a sphere effectively excludes the degenerate dwarf's magnetic field (except for a thin electromagnetic skin depth at the surface) there is a magnetostatic energy which depends on the relative orientation of the degenerate dwarf's magnetic field and the line of centres of the two stars. Synchronism, once achieved, may be maintained by a substantial non-dissipative magnetostatic torque. This torque is capable of locking the system into exact synchronism even in the presence of a steady accretion torque, so long as the latter is not too large. In the non-synchronous state there is, in addition, a small dissipative torque resulting from resistive losses in the skin layer.

Campbell (1983) assumed that the effective electric resistivity of the non-degenerate star is large, so that any external field penetrates deeply. Although standard microscopic theory gives a low plasma resistivity, the expected presence of turbulent convection and the plausible occurrence of 'anomalous' plasma resistivity justify the use of his assumptions, at least as a working hypothesis. Campbell found a large dissipative synchronizing torque. Because this torque is effectively frictional (it depends on the time rate of change of the orientation of the degenerate dwarf in the corotating frame, rather than on its orientation) it cannot maintain exact synchronism in the presence of a steady accretional torque.

Lamb *et al.* (1983) argued for a large synchronising torque driven by dissipation within the magnetosphere between the two stars. Because their torque is driven by asynchronous rotation, it too is effectively frictional, and cannot maintain exact synchronism in the presence of a steady accretional torque.

Campbell (1985, 1986) proposed, as did JKR, that if the secondary star is intrinsically magnetized the coupling of the two frozen-in dipoles would give an additional magnetostatic interaction. This mechanism may lock the white dwarf's rotation into exact synchronism with its companion's rotation. Coupling to the orbital motion is dissipative.

The recent discovery by Stockman, Schmidt & Lamb (1988) that V1500 Cygni has a polarimetric period slightly shorter than its photometric period suggests that the synchronization process may be directly observable in this system, if they are correct in arguing that the photometric period is the orbital period (rather than the orbital sideband of the polarimetric period). The calculation of JKR represents a lower limit to the synchronization rate, obtained by assuming only classical microscopic resistivity. The calculation of Lamb *et al.* (1983) represents an upper limit to the electromagnetic dissipation rate, obtained by essentially assuming the dissipative torque to approximate the magnetostatic torque. The calculations of Campbell (1983) made intermediate assumptions concerning the dissipation, and found intermediate synchronization rates. Tidal dissipation may also contribute to synchronization, although tides, being dissipative, cannot maintain exact synchronism.

Cropper (1988) reported an extensive compilation of data on AM Her stars. These data answer questions raised during the early work on AM Her stars, and raise new ones. His most important results concern the orientation of the degenerate dwarf's dipole moment within the binary. This is assumed to be coincident with the accretion column, whose location is inferred from observations.

Describe the orientation of the dipole axis in the co-rotating frame by the angles (θ, ϕ), where θ is the angle between it and the orbital angular momentum, and ϕ is the angle between the dipole's projection onto the orbital plane and the line of centres, increasing in the direction of orbital motion. These angles may be thought of as the colatitude and longitude of the magnetic pole. The geometry is illustrated in fig. 1 of JKR; Cropper uses the same geometry with the notation (β, ψ) instead of (θ, ϕ).

Cropper determined these angles for eight AM Herculis stars. In every case ϕ is between $-20°$ and $70°$, with two values close to (probably consistent with) $0°$, five in the range $20°$ to $70°$, and one about $-15°$. He found this anisotropy to be of high statistical significance, and pointed out that it implies that synchronous rotation must be maintained to extreme accuracy – to within a fraction of a rotation in the entire recognisable lifetime of an AM Her star. This exact synchronism may be explained by a quasi-static synchronizing potential, such as those proposed by JKR and Campbell (1985, 1986). Dissipative synchronizing torques are inadequate because they produce a steady drift in ϕ when there is a steady accretion torque.

It is harder to explain the preferred range of ϕ. The model of JKR predicts that the minimum of the magnetostatic potential should occur at $\phi = 90°$ (see their equation 24). A steady accretion torque will displace the degenerate star from the potential minimum. If this torque has the expected sense it will increase ϕ. This prediction is contradicted by the data.

If both stars are intrinsically magnetized exact synchronism may be achieved at any value of ϕ. This hypothesis can explain the observed synchronism, but does not evidently explain the observed existence of a preferred range in ϕ.

It is therefore necessary to find a quasi-static synchronizing torque capable of explaining the observed values of ϕ. JKR pointed out that the Maxwell stress within a magnetized star will distort its figure, and that the resulting gravitational torque will generally be larger than the magnetostatic torque. Unfortunately, a dipole magnetic field will produce an oblate dynamical figure (Chandrasekhar & Fermi 1953). This is apparent from the required toroidal currents; like currents attract, so the Lorentz forces pull matter away from the magnetic axis (across which the direction of current changes) and towards the equator. The minimum gravitational energy of such an oblate star occurs when its companion lies in its equatorial plane. The gravitational torque produces the same unsatisfactory alignment at $\phi = 90°$ as the magnetostatic torque.

This problem may be resolved if the degenerate dwarf is dynamically prolate, with its longest axis (axis of smallest moment of inertia) along its magnetic axis. A star with a toroidal internal magnetic field will have such a prolate figure. I assume that the symmetry axis of the toroidal field coincides with the observed dipole axis. The toroidal field must be confined within the stellar interior because it is produced by poloidal currents; if this field and currents extended above the surface the resulting configuration would not be force-free, but in the near-vacuum above the surface there are no countervailing forces to balance a non-zero Lorentz force.

The purpose of this paper is to develop this model for alignment, in which the aligning torque is gravitational and results from the distortion of the degenerate dwarf's figure by an internal toroidal magnetic field. Such a model can explain Cropper's two chief results: the achievement of exact synchronism, and the observed range of values of ϕ. Magnetostatic models can explain synchronism, but only the present model has explained the observed range of ϕ. In Section 2 I present an oversimplified but explicit model of such a dynamically prolate star, and calculate the gravitational torques on it. In Section 3 I discuss the implications of this model.

2 Prolate stars

The equations of magnetohydrostatics (Ferraro & Plumpton 1966; Roberts 1967) are simple, but relatively few explicit solutions exist. Our problem is complicated by our desire to calculate explicitly the dynamical figure of the resulting configuration. This naturally leads us to consider models of incompressible fluid whose figure is given by their surface shape. Such a model (and, more generally, any barytropic model) can be in equilibrium only if the Lorentz force is curl-free, which imposes an additional awkward constraint. Some explicit solutions were given by Ferraro (1954) and Prendergast (1956). Wentzel (1961) found solutions in which the stars are dynamically prolate, as required here. Ostriker & Hartwick (1968) constructed numerical models of rotating magnetized degenerate stars, which are prolate if the effects of rotation are less than those of the toroidal magnetic field. Mestel (1954) has pointed out that in realistic stars an additional thermodynamic variable is present, whose adjustment may readily lead to an equilibrium configuration (though at the price of Eddington-Sweet-like circulation), but this does not help construct an explicit incompressible model. Fortunately, models of toroidal fields which satisfy the required conditions are much simpler than the models of poloidal or mixed toroidal–poloidal fields investigated by earlier authors.

Our fundamental equations are Ampère's law, neglecting the displacement current,

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}, \tag{1}$$

and the Lorentz force acting on an uncharged current:

$$\vec{F} = \frac{1}{c} \vec{J} \times \vec{B}. \tag{2}$$

For an explicit example take a current density within the degenerate star

$$\vec{J}=\frac{J_0 R_D}{\varphi}\hat{z}, \tag{3}$$

where R_D is the stellar radius, and I work in cylindrical coordinates (φ, z, ϕ_c). The return current flows along the stellar surface. The resulting magnetic field is

$$\vec{B}=\frac{4\pi}{c} J_0 R_D \hat{\phi}_c \equiv B_0 \hat{\phi}_c \tag{4}$$

within the star, and zero outside it. The force density is

$$\vec{F}=-\frac{4\pi}{c^2}\frac{J_0^2 R_D^2}{\varphi}\hat{\varphi}, \tag{5}$$

which squeezes matter towards the magnetic axis, making the star prolate. The Lorentz force field may be described as the result of a scalar potential

$$\begin{aligned}\phi_B &= \frac{4\pi}{c^2} J_0^2 R_D^2 \ln \varphi \\ &= \frac{4\pi}{c^2} J_0^2 R_D^2 \ln(r \sin \theta_b),\end{aligned} \tag{6}$$

where (r, θ_b, ϕ_b) defines a spherical coordinate system whose axis is the star's magnetic axis.

The assumed current and force distributions (3) and (5) are singular on the axis and are convectively unstable (Tayler 1973; see also Goossens & Tayler 1980 and Goossens, Biront & Tayler 1981). I assume this field distribution in the same spirit as the assumption of an incompressible fluid: more realistic assumptions would lead to quantitatively different but qualitatively similar results, and would require extensive numerical calculation. The magnetic field B_0 should therefore be taken as a parameter estimating the magnitude of the toroidal field, rather than as a quantitative measure of it.

An incompressible star of density ρ will be bounded by a surface of constant total potential $\phi_G+\phi_B$, where $\phi_G=-GM_D/r$ is the Newtonian gravitational potential of a spherical star of mass M_D. I ignore the effect on the figure of the star of the deviation from sphericity of the gravitational field. If the magnetic potential ϕ_B is small in comparison to ϕ_G the star is nearly spherical, and its deviation from sphericity is given by

$$\delta(\theta_b)=-\frac{1}{4\pi}\frac{B_0^2 R_D^2}{GM_D \rho}\ln \sin \theta_b+\delta_0, \tag{7}$$

where δ_0 is an angle-independent quantity which may be ignored. The weak singularities at $\theta_b=0, \pi$ are unphysical but of no significance.

I have so far neglected the surface current required to close the current loop. It is readily found to be $J_0 R_D \hat{\theta}_b$ at all points on the stellar surface. The magnetic field drops from its interior value of $B_0\hat{\phi}_c$ to zero through the surface. The cylindrical and spherical azimuthal vectors are equal: $\hat{\phi}_c=\hat{\phi}_b$. Then the net surface force density, integrated through the surface current sheet, is

$$\vec{F}_s = \frac{B_0^2}{8\pi}\hat{r}. \tag{8}$$

This is spherically symmetric and does not change the dynamical figure of the star. It is directed radially outward, and must be opposed by some inward force. The weight of an overlying atmosphere could provide the necessary force. This atmosphere must have a density ρ_A less than ρ because its outer surface will be a spherical equipotential surface; if its density were ρ the star (including atmosphere) would be dynamically spherical. I assume $\rho_A \ll \rho$ so that the effect of the atmosphere on the stellar dynamical figure may be neglected. If the surface current were distributed through a finite thickness the complication of the atmosphere would not have to be considered, but then our simple explicit solution for $\vec{B}$ and $\vec{J}$ would not hold everywhere.

The dominant aspherical part of the gravitational potential of a figure of revolution is given by (Kovalevsky 1966)

$$U_2 = \frac{G}{r^3}(C-A)\left(\frac{1}{2} - \frac{3}{2}\sin^2\theta_b\right), \tag{9}$$

where A, B and C are the principal components of the moment of inertia tensor, with $A=B$ and C the moment about the axis of rotational symmetry. For a prolate body $C<A, B$. By direct calculation equation (7) gives

$$C-A = -\frac{1}{6}\frac{B_0^2 R_D^6}{GM_D}. \tag{10}$$

The gravitational potential becomes

$$\begin{aligned} U_2 &= \frac{B_0^2 R_D^6}{4M_D r^3}\sin^2\theta_b \\ &= -\frac{B_0^2 R_D^6}{4M_D D^3}\sin^2\theta\cos^2\phi, \end{aligned} \tag{11}$$

where isotropic terms have been dropped, the potential has been evaluated at the location of the non-degenerate companion star at the orbital separation D, and the value of θ_b appropriate to that star has been expressed in terms of the orientation of the degenerate dwarf's dipole axis using $\cos\theta_b = \sin\theta\cos\phi$. The potential minimum occurs for $\theta = \pi/2$ and $\phi = 0, \pi$, corresponding to the magnetic dipole axis along the line of centres of the stars, as required.

The orientation-dependent part of the potential energy is given by

$$V(\theta,\phi) = -\frac{M_c B_0^2 R_D^6}{4M_D D^3}\sin^2\theta\cos^2\phi, \tag{12}$$

where M_c is the mass of the companion star. JKR demonstrated that θ is essentially constant for a given star, so only the dependence of V on ϕ is of interest. The order of magnitude of this expression is equivalent to an estimate of the gravitational torque in §V of JKR, if $B_0 R_D^3$ is identified with an effective magnetic dipole moment, even though the toroidal magnetic field has no moment.

3 Discussion

The data of Cropper show that only one (or perhaps two) of eight AM Hercules stars have orientations consistent with the values $\phi=0, \pi$ predicted by (12). One possible resolution of this problem is that the degenerate dwarfs have comparable distortions produced by poloidal and toroidal magnetic fields. The resulting stars would be triaxial if the symmetry axes of the poloidal and toroidal fields did not coincide. Depending on the details of their geometry, any equilibrium orientation is possible. This asymmetric triaxial model does not naturally explain the observed concentration of values of ϕ. Such a triaxial star (or a magnetic moment on the non-degenerate companion) may, however, be the best explanation of the one star (E1114+182) for which ϕ is observed to be negative.

A more promising hypothesis is that five (or perhaps six) of the stars are displaced from their equilibrium positions at $\phi=0$ by a steady accretion torque. The greatest possible magnitude of gravitational torque (along the axis of symmetry) is

$$|\tau_{\max}|=\frac{M_c B_0^2 R_D^6 \sin^2\theta}{4M_D D^3} \approx 2.5\times 10^{35}\left(\frac{10M_c}{M_D}\right)\left(\frac{B_0 R_D^3}{10^{35}\text{G cm}^3}\right)^2\left(\frac{10^{11}\text{ cm}}{D}\right)^3 \sin^2\theta \text{ dyne cm,} \tag{13}$$

and is found for $\phi=\pi/4$, consistent with the orientations of five of the stars. This numerical estimate is rough, because it depends on our oversimplified model of a magnetized incompressible star and on uncertain values of the parameters, but is of the same order of magnitude as the magnetostatic torques considered by JKR (for comparable poloidal and toroidal fields), and is probably consistent with the accretion torques. There is a natural selection effect tending to produce values of ϕ approaching $\pi/4$: if the accretion torque were any larger synchronism would be broken, while if it were significantly smaller (so that $\phi\to 0$) the accretional luminosity would be lower and the system would be less likely to have been discovered.

This model predicts a correlation between values of ϕ and the accretional luminosity L_{acc}, according to $L_{acc}\propto \partial V\partial\phi \propto \sin\phi\cos\phi$. The large variation in other parameters among binary systems means that the correlation should be rather loose.

The prolate star model requires that the toroidal magnetic field significantly exceeds the poloidal field. This may naturally be explained if the toroidal field is produced by shearing a pre-existing field during a differentially rotating phase of the star's evolution.

Free oscillations may be excited about an equilibrium value of ϕ. It is not possible to predict the amplitude of their excitation *a priori*. For the simple case of a negligible accretion torque the period is obtained from (12):

$$P=2\pi\left(\frac{2M_D D^3 C}{M_c B_0^2 R_D^6 \sin^2\theta}\right)^{1/2} \sim 10^3\left(\frac{M_D}{10M_c}\right)^{1/2}\left(\frac{10^{35}\text{ G cm}^3}{B_0 R_D^3}\right)\left(\frac{D}{10^{11}\text{ cm}}\right)^{3/2}\left(\frac{C}{10^{50}\text{ g cm}^2}\right)^{1/2} \text{d,} \tag{14}$$

comparable to the value for the period of oscillations found by JKR considering magnetostatic interactions.

My numerical estimates of the interaction energies of AM Hercules stars, taking the toroidal field $B_0 \sim 10^8$ G are comparable to those found by JKR for similar magnitudes of poloidal fields B_p. Observed poloidal fields are typically $\sim 3 \times 10^7$ G. The ratio of the gravitational to the magnetostatic interaction energies is

$$\frac{V_g}{V_m} \sim \frac{M_c}{M_D} \frac{D^3}{R_c^3} \frac{B_0^2}{B_p^2}. \tag{15}$$

If $B_0 = B_p$ this is in the range 1–10, and is essentially determined by Roche lobe geometry. It could be much larger if the internal toroidal field much exceeds the poloidal field.

Purely magnetostatic models suffer from the problem that, in the presence of an accretion torque, even a small amount of resistivity permits a slow creep (on the secondary's resistive diffusion time) in ϕ. The resulting interaction between the stars is complex because the magnetostatic interaction must now include the effects of the nearly static magnetic dipole moment induced in the secondary by the creep of the primary's magnetic field through it. In these models the velocity of magnetic field creep may be estimated as

$$v_c \sim \eta / R_c, \tag{16}$$

where R_c is the radius of the non-degenerate companion and η is the magnetic diffusivity, or

$$\dot{\phi} \sim \frac{\eta}{R_c D}. \tag{17}$$

The condition (Cropper 1988) that $\dot{\phi}$ is less than 1 rad in the synchronized lifetime t_l of an AM Her star gives

$$\eta < \frac{R_c D}{t_l}. \tag{18}$$

If a value is assumed for t_l, then the magnetic diffusivity η [and an effective conductivity $\sigma = c^2/(4\pi\eta)$] may be bounded, given the purely magnetostatic assumption.

If V1500 Cygni shows rapid synchronization, that would imply a small value of σ (or, alternatively, rapid tidal dissipation). Such a result would not be contradicted by a small value of η (large σ) apparently implied by (18), because if phase-locking in the synchronous state is maintained by gravitational torques, as proposed in this paper, there is no phase creep at all and expressions (16)–(18) are inapplicable.

A remarkable feature of these results is that it may be possible to infer the existence of a toroidal magnetic field, entirely buried within the star, from its effects on the star's figure and gravitational potential. In principle, this technique could be more widely applicable. For example, the solar surface shape constrains any magnetic (as well as rotational) distortion of its internal mass distribution.

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