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**DO MANAGERS WITH LIMITED LIABILITY TAKE MORE  
RISKY DECISIONS? AN INFORMATION ACQUISITION MODEL**

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# **Do Managers with Limited Liability Take More Risky Decisions? An Information Acquisition Model**

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## **Abstract**

Risk-neutral individuals take more risky decisions when they have limited liability. Risk-neutral managers may not when acting as agents under contract and taking costly actions to acquire information before taking decisions. Limited liability makes it optimal to increase the reward for outcomes relatively more likely to arise from desirable than from undesirable actions. The resulting decisions may be less, rather than more, risky. Making a decision after acquiring information provides an additional reason to those in the classic principal-agent literature for using contracts with pay increasing in the return. Further results on the form of contracts are also derived.

*Keywords:* Managers, risky decisions, limited liability, principal-agent contracts, asymmetric information

*JEL classifications:* D82, D86, J33, M52

# 1 Introduction

It is widely held that the 2008 financial crisis was exacerbated, if not caused, by executives making more risky decisions than shareholders would have thought appropriate. Alan Greenspan commented: “I made a mistake in presuming that the self-interests of organizations, specifically banks and others, were such that they were best capable of protecting their own shareholders” (Testimony to Congress, quoted in *The Guardian*, 24 October 2008). The cause is widely thought to be the use of bonuses for performance combined with limited liability. Limited liability introduces a convexity into the utility function because all outcomes below the level at which limited liability binds result in the same utility, so a risk-neutral person with limited liability makes decisions like a risk lover. But that is not a complete argument. Executives work as agents under a contract and, in drawing up that contract, a firm’s owner (principal) should take the limited liability into account and design the contract to counteract it. So does limited liability, combined with performance bonuses, really result in more risky decisions when an optimal contract is used?

In answer to that question, this paper formalizes the following intuition. A risk-neutral executive or manager can always be induced to make the decision preferred by a risk-neutral principal with the same information by a contract that makes compensation an increasing linear function of the return to the principal. The reason is that replacing the principal’s linear objective function by a different, but order-preserving, linear objective function does not affect the maximizing decision — it just corresponds to an affine transformation of a von Neumann-Morgenstern utility function. So, if all the information for making the decision were readily available, there would be no problem: the principal could simply use a linear contract with a very small slope and with an intercept that satisfies the limited liability constraint and is sufficient to induce the agent to participate. It is not so simple if the agent has to be induced to incur effort to acquire information on which to base the decision. The principal could, of course, still avoid distorting decisions by using a linear contract with a slope steep enough to induce effort. But with an intercept sufficiently large to satisfy limited liability, that might prove a very expensive way for the principal to induce effort. The cost of inducing effort could be reduced by reducing the reward for outcomes that are less likely to occur if the agent has not actually put in the effort to acquire the appropriate information. If those outcomes are the less risky ones, the agent will optimally end up taking more risky decisions than the principal would given the same information. But those outcomes might alternatively be the riskier ones, in which case the agent will optimally end up making less risky decisions.

Straightforward as this intuition is, deriving it formally in a general setting is not so straightforward. As Grossman and Hart (1983) noted, principal-agent models are notoriously difficult for deriving precise results in general formulations. But it is nev-

ertheless informative to formalize that intuition in a simple setting. That is what this paper does.

The general framework is the following. A manager is employed to make a decision and is rewarded on the resulting return. Before making the decision, the manager can take a (costly) action that reveals privately-observed information about the returns from the possible decisions. The riskiness of the optimal decision can then be compared with and without limited liability. The costly action is an essential part of this. Without that, as noted above, there is no reason to distort decisions for incentive reasons. The framework is a direct extension of the classic principal-agent model studied by Mirrlees (1999), Holmström (1979) and Grossman and Hart (1983). If the information generated by the manager's action were verifiable and payment could be based directly on that, the framework would reduce to that classic model.

A number of papers in the literature have analysed optimal contracts for risk-neutral agents with limited liability in this general framework, for example, Diamond (1998), Biais and Casamatta (1999), Palomino and Prat (2003), Feess and Walzl (2004), and Gromb and Martimort (2007).<sup>1</sup> But of these, only Palomino and Prat (2003), in an application to delegated portfolio choice, address the question of the impact of optimal contract choice on the riskiness of the agent's decisions. They conclude that portfolio managers may make either more, or less, risky decisions when they have limited liability but their result relies on the "first-order approach" being valid, which they cannot guarantee. Moreover, they limit the distributions of returns for each decision to those characterised by two parameters, the mean and riskiness of a portfolio. Biais and Casamatta (1999), Feess and Walzl (2004), and Gromb and Martimort (2007) are concerned with other issues. But in their models, the agent can in any case choose between only two decisions, which limits the scope for answering the question because marginal adjustments in riskiness are in general not possible. Biais and Casamatta (1999) and Feess and Walzl (2004), moreover, consider no more than three levels of return to the principal on which payment to the agent can be based, so the same incentives can be achieved by different forms of contract.

The most general of these formulations is in Diamond (1998). In that paper, there are only a finite number of possible levels of gross return to the principal, but a continuum of decisions that vary the probabilities of these returns. For this structure, Diamond (1998) establishes the "near-linearity" result that, when a linear contract is the only form of contract that induces the agent to make efficient decisions, an optimal contract converges to a linear contract as the ratio of the principal's gross return to the agent's cost of taking appropriate action increases. The intuition is that a linear

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<sup>1</sup>Malcomson (2009) considers a risk-averse agent and the literature for that case. Innes (1990) analyses the classic principal-agent problem with no additional decision but with the restrictions that the rewards to both agent and principal are monotone in the outcome. Such monotonicity restrictions arise naturally in some cases in which the agent's action generates private information that is used subsequently in making a decision, though Innes (1990) does not derive them in that way.

contract always induces the agent to make efficient decisions. When it is the only form that does so, it is natural that the optimal contract converges to linear when the cost of inducing action is small relative to the principal's concern with making efficient decisions. Diamond (1998) does not, however, consider the effect of limited liability on the riskiness of the agent's decisions.

The present paper derives results on this issue that do not depend on an unsubstantiated "first-order approach". It does so, moreover, with a continuum of possible decisions and a continuum of possible outcomes. Specifically, the manager-agent makes a decision that can be either successful, with known positive return, or unsuccessful with zero return. An example is the price to bid to supply a good or service; a successful bid yields a return equal to the price less the cost of supply, an unsuccessful bid a return of zero. The manager's action reveals information about the probability that different decisions will be successful. There is a continuum of possible decisions, each with a different return if successful, and thus a continuum of possible returns. As a result, the manager can always make marginally more or less risky decisions. To anticipate the conclusion, when the manager takes costly action to acquire information before making a decision, limited liability may result in either a more risky or a less risky decision than the risk-neutral principal would choose given the same information. The analysis gives precise conditions under which each occurs. When limited liability results in *less* risky decisions, managers are induced to make decisions in a way that makes it look as if they are risk-averse — limited liability not only reduces the profits of firms but also biases their decisions in a risk-averse direction.

The model also provides insights into when the "near-linearity" result of Diamond (1998) fails to hold. An example is when it is crucial for the principal to ensure that the decision results in a successful outcome (because, say, the future of the company depends on winning a contract). The principal then wants the manager to make the decision with the highest return that the manager knows will be successful given the information available. In that case, *any* contract with payment non-decreasing in the return will induce the manager not to make a decision with return lower than is optimal. That results in sufficient flexibility for a linear contract not to be the only form that induces the agent to make efficient decisions. This case is sufficiently straightforward to solve explicitly for an optimal contract. A linear contract turns out not to be optimal because other contracts ensure efficient decisions at lower cost to the principal. Moreover, an optimal contract is not a debt contract, as derived by Innes (1990) when rewards to both principal and agent are assumed to be monotone. Nor is it a combination of debt, equity and share options that Biais and Casamatta (1999) found optimal with just two possible decisions and three possible levels of return to the principal.

This paper is organised as follows. Section 2 sets out the model. Section 3 develops the example of bidding to supply a good or service. Section 4 specifies formally

the principal's optimal contracting problem. Section 5 analyses the agent's decisions under an optimal contract. Section 6 considers the form an optimal contract takes. Section 7 concludes. Proofs of propositions are in an appendix.

## 2 The model

A risk-neutral principal (owner) employs an agent (manager) to make a decision. Each possible decision has one of two outcomes, failure with return zero, or success with known non-negative return  $b \in [0, \bar{b}]$ , with  $\bar{b}$  finite. Of the possible decisions, there is just one that, if successful, yields return  $b$ . Decisions can, therefore, be labelled by their return if successful. By taking action  $a \in \{\underline{a}, \bar{a}\}$  before making the decision, the agent acquires, with probability density  $f(s; a)$ , a signal  $s \in [\underline{s}, \bar{s}]$  about the probability  $\pi(b; s)$  that the decision  $b$  will be successful. The action has a direct utility cost to the agent, the decision does not. In this respect the environment, like those in the papers cited in the Introduction, differs from the multi-task agency environment of Holmstrom and Milgrom (1991). Without loss of generality, signals can be ordered so that the likelihood ratio  $LR(s) \equiv f(s; \underline{a}) / f(s; \bar{a})$  is non-increasing in  $s$ . A higher signal is then relatively more likely to have arisen from  $\bar{a}$  than from  $\underline{a}$ . By itself, this has no economic implications. Only when combined with a decision rule  $b(s)$  that specifies the decision as a function of  $s$  does it have implications for the likelihood ratio of a decision arising from action  $\bar{a}$  rather than  $\underline{a}$ . The principal observes only the return, not the decision and the action taken by the agent.

A number of different economic interpretations can be placed on this structure. The decision  $b$  may be the amount to bid to supply a good or service to a purchaser whose reservation value is unknown. If the bid is successful, the return is  $b$  (measured net of the cost of supply). If it is unsuccessful, the return is zero. The signal is then information about the purchaser's reservation value. This interpretation is developed further in an example below. Alternatively the decision may be which of a set of mutually exclusive projects to undertake. The returns  $b$  (in the case of success) and zero (in the case of failure) are then measured gross of the investment in the project.

The agent is risk-neutral but has limited liability that rules out payments from the principal of less than  $\underline{P}$ . As conventional in principal-agent models, the agent's utility function is additively separable in income and action, so the agent's utility from being paid  $P$  and taking action  $a$  can be written  $P - v(a)$ , where  $v(a)$  is the disutility of taking action  $a$  and  $v(\bar{a}) > v(\underline{a})$ . The agent's reservation utility for accepting a contract with the principal is  $\underline{U}$ .

It is natural to assume  $\pi(b; s)$  is non-increasing in  $b$  for given  $s$  — a lower bid, for example, is at least as likely to be accepted by the purchaser as a higher one. The return from the decision  $b = 0$  is the same whether or not it is successful, so effectively  $\pi(0; s) = 1$ . It is convenient to assume that  $\pi(b; s)$  is strictly decreasing in  $b$  where



feasible (that is,  $\pi_b(b; s) \equiv \partial\pi(b; s) / \partial b < 0$  for all  $(b, s)$  such that  $\pi(b; s) > 0$ ), twice differentiable with respect to  $b$  and  $s$  except possibly for  $\pi(b; s) = 0$  and  $\pi(b; s) = 1$  and, to avoid trivial decisions, has the property that for each  $s \in [\underline{s}, \bar{s}]$  there exists some  $b \in (0, \bar{b}]$  for which  $\pi(b; s) > 0$ . A useful benchmark is an *efficient decision rule*  $b^*(\cdot)$  the principal would use if receiving the signal  $s$  directly. Such a rule satisfies

$$b^*(s) \in \arg \max_{b \in [0, \bar{b}]} b\pi(b; s), \text{ for all } s \in [\underline{s}, \bar{s}]. \quad (1)$$

The sequence of events is as follows. As in the classic principal-agent model, the principal makes a “take it or leave it” offer of a contract to the agent. If accepted, the agent chooses an action  $a$ , receives a signal  $s$ , and makes a decision  $b$ , in that order. The return from the decision is then realised and the agent is paid according to the contract. The essential difference from the classic principal-agent model is that the signal received by the agent is not itself verifiable. If it were, the incentive issues here would reduce to those of the classic model.

The assumption that the principal does not observe the decision itself, only the return from it, has the implication that payment to the agent cannot be conditioned explicitly on the decision. That follows the motivating literature discussed in the Introduction. It may, of course, not be appropriate for some applications in which the decision itself is observable whatever the outcome. But some care is needed in determining when the decision is actually observable in a way that is useful for payment. In the bidding interpretation, for example, the return corresponds to an accounting transaction. An unsuccessful bid does not and one can imagine collusion between the purchaser and the agent about the level of a losing bid becoming a significant issue if payment to the agent depended on it. Moreover, in the present context, knowing the decision corresponds to knowing the distribution of returns given any signal. A characteristic of the 2008 financial crisis is that some senior bank executives appeared to have no idea what distribution of returns their subordinates were choosing when, for example, buying such financial instruments as collateralized debt obligations (CDOs). Many such instruments are highly complex, so the correspondence between observable choices and probability distributions may not be at all apparent.

The assumption that there are only two possible outcomes from each decision, one of which is the same for all decisions, is essentially for analytical tractability. The role of the assumption is to give a direct link between the decision and the return so that changing the payment for return  $b$  affects only the payoff from making decision  $b$ , not that from making any other decision. But that also helps greatly in sharpening the intuition underlying the results. It is, moreover, entirely natural in the context of bidding, where the agent either wins, with the return corresponding to the bid, or loses with a return of zero. That makes bidding a natural application to illustrate the results.

### 3 Bidding example

This section develops the bidding interpretation. In this, the principal wishes the agent to tender a bid to supply a good or service to a buyer whose reservation value is unknown. Let  $\theta \in [0, \bar{\theta}]$ , with  $\bar{\theta} > 0$ , be the buyer's privately known valuation of the good or service, with density function  $h(\theta)$  capturing the principal's and agent's common beliefs about  $\theta$ . By taking action  $a \in \{\underline{a}, \bar{a}\}$ , with  $v(\underline{a}) < v(\bar{a})$ , the agent acquires with probability  $\rho(a)$ , with  $\rho(\bar{a}) > \rho(\underline{a})$ , information about the buyer's reservation value and thus about the optimal price to bid. That information corresponds to the signal  $s \in (\underline{s}, \bar{s}]$ , with density function denoted  $\zeta(s; \theta)$  conditional on information being acquired. It might, for example, be that  $s$  is the price the buyer paid for a complementary good. With probability  $1 - \rho(a)$ , the agent acquires no information, corresponding to  $\underline{s}$ . The price to bid corresponds to the decision  $b$ .

The buyer accepts the bid  $b$  if  $\theta \geq b$ . The probability of acceptance given action  $a$  and signal  $s \in (\underline{s}, \bar{s}]$ , which corresponds to  $\pi(b, s)$  in the general model, is

$$\pi(b, s) \equiv \Pr(\theta \geq b \mid s) = \frac{\int_b^{\bar{\theta}} \zeta(s; \theta) h(\theta) d\theta}{\int_0^{\bar{\theta}} \zeta(s; \theta) h(\theta) d\theta}. \quad (2)$$

It might be that  $s$  is the price the buyer paid for a complementary good, in which case one might expect the probability of a given bid being accepted to increase with  $s$  ( $\partial \pi(b, s) / \partial s > 0$ ) and the rate at which that probability declines with the bid to decrease ( $\partial^2 \pi(b, s) / \partial b \partial s > 0$ ). For simplicity, let the cost of supply be zero so that  $b$  is the return to the principal from a successful bid. The return from an unsuccessful bid is 0. In this example,

$$\begin{aligned} f(s; a) &= \rho(a) \int_0^{\bar{\theta}} \zeta(s; \theta) h(\theta) d\theta, \text{ for } s \in (\underline{s}, \bar{s}], a \in \{\underline{a}, \bar{a}\}, \\ f(\underline{s}; a) &= 1 - \rho(a), \text{ for } a \in \{\underline{a}, \bar{a}\}, \end{aligned} \quad (3)$$

so  $LR(s)$  is non-increasing in  $s$ , as assumed.

Important for the results that follow is the effect of changing  $s$  on

$$\frac{\pi_b(b, s)}{\pi(b, s)} = - \frac{\zeta(s; b) h(b)}{\int_b^{\bar{\theta}} \zeta(s; \theta) h(\theta) d\theta}, \text{ for } b < \bar{\theta}, s \in (\underline{s}, \bar{s}]. \quad (4)$$

To illustrate the possibilities, suppose first that  $1/s$  is an upper bound on the true value of  $\theta$ , that is,  $1/s \geq \theta$  or  $s \leq 1/\theta$ . Then a higher  $s$  provides more information and  $\zeta(s; \theta) = 0$  for  $s > 1/\theta$ . In this case, a bid  $b > 1/s$  is not optimal because it will be rejected with probability 1. Suppose, moreover, that  $s$  has a uniform distribution on

$(1/\bar{\theta}, 1/\theta]$  so that

$$\xi(s; \theta) = \begin{cases} \frac{1}{(1/\theta) - (1/\bar{\theta})}, & s \in (1/\bar{\theta}, 1/\theta]; \\ 0, & \text{otherwise.} \end{cases}$$

Use of this in (4) gives that  $\pi_b(b, s) / \pi(b, s)$  is strictly decreasing in  $s$  for  $b < 1/s$ . As an alternative, suppose that  $s$  is a lower bound on the true value of  $\theta$ , that is,  $s \leq \theta$ . Then a higher  $s$  provides more information and  $\xi(s; \theta) = 0$  for  $s > \theta$ . In this case, a bid  $b < s$  is not optimal because there is always a slightly higher bid that will be accepted with probability 1. The uniform distribution is slightly awkward in this case but suppose  $s$  has a triangular distribution on  $(\underline{\theta}, \theta]$  with its mode at its lower support so that

$$\xi(s; \theta) = \begin{cases} \frac{2(\theta-s)}{(\theta-\underline{\theta})^2}, & s \in (\underline{\theta}, \theta]; \\ 0, & \text{otherwise.} \end{cases}$$

Use of this in (4) gives that  $\pi_b(b, s) / \pi(b, s)$  is strictly increasing in  $s$  for  $b > s$ . Thus each effect is plausible under some circumstances.

## 4 Contracts between principal and agent

A contract between principal and agent specifies the payment  $P(\cdot)$  to the agent as a function of the return to the principal. (The contract could specify a lottery conditional on  $b$  but, since both parties are risk neutral, that would have no advantage over a deterministic payment equal to the expected value.) The agent's limited liability requires that  $P(b) \geq \underline{P}$  for all  $b \in [0, \bar{b}]$ . For decision  $b$ , the return is  $b$  if the outcome is successful and 0 if it is not. Thus, an agent facing contract  $P(\cdot)$  who receives signal  $s$  and makes decision  $b$  has expected monetary compensation

$$u(b, s, P(\cdot)) \equiv P(b) \pi(b; s) + P(0) [1 - \pi(b; s)], \text{ for all } s \in [\underline{s}, \bar{s}]. \quad (5)$$

The agent's expected utility before the signal  $s$  is known from adopting the decision rule  $b(\cdot)$  given action  $a$  is then

$$U(a, b(\cdot), P(\cdot)) \equiv \int_{\underline{s}}^{\bar{s}} u(b(s), s, P(\cdot)) f(s; a) ds - v(a). \quad (6)$$

The principal's expected payoff from decision  $b$  given signal  $s$  and contract  $P(\cdot)$  is

$$r(b, s, P(\cdot)) \equiv [b - P(b)] \pi(b; s) - P(0) [1 - \pi(b; s)], \text{ for all } s \in [\underline{s}, \bar{s}], \quad (7)$$

that from the agent choosing action  $a$  and decision rule  $b(\cdot)$

$$R(a, b(\cdot), P(\cdot)) \equiv \int_{\underline{s}}^{\bar{s}} r(b(s), s, P(\cdot)) f(s; a) ds. \quad (8)$$

An optimal contract maximises the principal's payoff subject to feasibility, individual rationality and incentive compatibility for the agent. The optimal contract problem can be written in the way standard with principal-agent problems as

$$\max_{a, b(\cdot), P(\cdot)} R(a, b(\cdot), P(\cdot)) \quad \text{subject to} \quad (9)$$

$$U(a, b(\cdot), P(\cdot)) \geq \underline{U}; \quad (10)$$

$$a \in \arg \max_{a' \in \{a, \bar{a}\}} U(a', b(\cdot), P(\cdot)); \quad (11)$$

$$b(s) \in \arg \max_{b \in [0, \bar{b}]} u(b, s, P(\cdot)) \text{ for all } s \in [\underline{s}, \bar{s}]; \quad (12)$$

$$P(b) \geq \underline{P}, \text{ for all } b \in [0, \bar{b}]. \quad (13)$$

Constraint (10) is the individual rationality condition that the agent has expected utility from the decision rule  $b(\cdot)$  when choosing action  $a$  under contract  $P(\cdot)$  no lower than from declining the contract. Constraint (11) ensures that the agent receives at least as much expected utility from choosing action  $a$  as from choosing any other action. Constraint (12) ensures that the decision rule  $b(\cdot)$  maximises the agent's payoff for each  $s$ . (Writing the constraints this way implicitly assumes that an agent who is indifferent between two values of  $a$  or  $b$  chooses that preferred by the principal. Mixed strategies could be handled at the cost of complicating the presentation.) Finally, constraint (13) ensures the contract respects the agent's limited liability.

**Proposition 1** *The following properties apply to the principal's problem (9)–(13).*

1. *There is no loss to the principal from using a contract with payment non-decreasing in the return.*
2. *An efficient decision rule  $b^*(\cdot)$  can be implemented by a contract  $P(\cdot)$  of the form*

$$P(b) = \underline{u} + kb, \text{ for all } b \in [0, \bar{b}], \quad \text{for any constant } k \geq 0, \quad (14)$$

*for constant  $\underline{u}$  such that, given  $k$ , the contract satisfies both the agent's individual rationality constraint (10) for some  $a \in \{a, \bar{a}\}$  and the limited liability constraint (13).*

The first property in Proposition 1 arises because the agent will never make decision  $b(s)$  if there is some decision  $b < b(s)$  for which the successful outcome has payment  $P(b) > P(b(s))$  given that  $b$  has probability of success no lower than  $b(s)$ . The possibility of the agent making a decision conditional on the outcome of the action thus provides a reason for contracts to be monotone in practice in addition to

those discussed in the literature, see Hart and Holmström (1987). In particular, it does not depend on a Monotone Likelihood Ratio Property (MLRP) assumption with genuine economic implications — the MLRP assumption on  $LR(s)$  here amounts purely to a labelling. The second property holds because, using the linear contract (14) in (5) ensures (12) has the same set of maximizers as (1). For  $k = 0$ , the agent is indifferent to the choice of  $b$  but, in that case, has been assumed to do whatever the principal would prefer. With  $k = 0$ , of course, the agent has no incentive to take the costly action  $\bar{a}$ . But if it is optimal for the principal to induce the agent to take action  $\underline{a}$ , a contract of the form in (14) with  $k = 0$  and  $\underline{u}$  satisfying (10) and (13), one of them with equality, is optimal. The discussion that follows, therefore, considers the case in which it is optimal for the principal to induce the agent to choose action  $\bar{a}$ .

## 5 Optimal decision rules

If limited liability were not binding, it would be optimal for the principal to use a contract of the form in (14) with  $k = 1$ , which corresponds to the principal selling the project to the agent at a fixed price and the agent receiving all the return from it. The agent would then select  $b^*(s)$  given  $s$ , so  $b^*(s)$  is an appropriate benchmark for an agent without limited liability. This section explores the decision rule it is optimal for the principal to induce the agent to adopt when limited liability is binding. The assumption used for this is the following.

**Assumption 1**  $\pi(b; s)$  satisfies the following:

1.  $\pi(b; s) \in (0, 1)$  for all  $b \in (0, \bar{b})$ ,  $\pi(\bar{b}; s) = 0$ , for all  $s \in [\underline{s}, \bar{s}]$ ;
2.  $\pi_b(b; s) / \pi(b; s)$  is continuous non-increasing in  $b$  for all  $b \in [0, \bar{b})$  and  $s \in [\underline{s}, \bar{s}]$ ;
3.  $\pi_b(b; s) / \pi(b; s)$  is either strictly increasing or strictly decreasing in  $s$  for all  $b \in (0, \bar{b})$  and  $s \in [\underline{s}, \bar{s}]$ .

This assumption guarantees  $\pi_b(b; s) < 0$  for all  $s$  and  $b < \bar{b}$ . The assumption in Part 1 that decision  $\bar{b}$  has zero probability of success is not restrictive — in practice that will be the case for  $\bar{b}$  set sufficiently high. More restrictive is the implication that the probability of success becomes zero at the same  $b$  for all  $s$ . That can be avoided but at the cost of complicating the proofs. The assumption in Part 2 is common to many standard distributions. The bidding example of Section 3 illustrates that either case in Part 3 can hold. The economic implications are discussed further later.

The following proposition establishes that there is a unique efficient decision rule under Assumption 1 and specifies some of its characteristics.

**Proposition 2** *Under Assumption 1:*

1.  $b^*(s) \in (0, \bar{b})$  for all  $s \in [\underline{s}, \bar{s}]$ ;
2.  $b^*(s)$  is the unique solution to the first-order condition

$$b^*(s) \pi_b(b^*(s); s) + \pi(b^*(s); s) = 0, \text{ for all } s \in [\underline{s}, \bar{s}]; \quad (15)$$

3.  $b^*(s)$  is strictly increasing (decreasing) if  $\pi_b(b; s) / \pi(b; s)$  is strictly increasing (decreasing) in  $s$  for all  $b \in (0, \bar{b})$  and  $s \in [\underline{s}, \bar{s}]$ .

Part 3 of this result can be illustrated with the bidding example from Section 3. The case there with  $\pi_b(b; s) / \pi(b; s)$  strictly increasing in  $s$  has  $s$  as a lower bound on the true reservation price of the purchaser. In that case, higher  $s$  makes a higher bid more desirable, so  $b^*(s)$  is increasing in  $s$ . The case with  $\pi_b(b; s) / \pi(b; s)$  strictly decreasing in  $s$  has  $1/s$  as an upper bound on the true reservation price of the purchaser. Then a higher  $s$  makes a lower bid more desirable, so  $b^*(s)$  is decreasing in  $s$ . In both cases, having a higher  $s$  is information valuable to the principal, so action to increase the probability of higher  $s$  is also valuable.

The next proposition gives results on implementing decision rules that are helpful for characterising an optimal contract. By Proposition 1, the principal need consider only contracts that are non-decreasing on  $[0, \bar{b}]$  and hence differentiable almost everywhere. To simplify the analysis, attention is restricted to contracts that are differentiable everywhere.

**Proposition 3** *Suppose Assumption 1 holds.*

1. A decision rule  $b(\cdot)$  is implementable by the contract  $P(\cdot)$  only if, for all  $s \in [\underline{s}, \bar{s}]$ ,  $b(s) \in (0, \bar{b})$  and  $P(b(s)) > P(b)$  for all  $b < b(s)$ .
2. Necessary and sufficient conditions for a contract  $P(\cdot)$  accepted by the agent to implement  $b(\cdot)$  with  $b(s) \in (0, \bar{b})$  for all  $s \in [\underline{s}, \bar{s}]$  are that  $b(s)$ :
  - (a) satisfies the first-order condition

$$P'(b(s)) \pi(b(s); s) + [P(b(s)) - P(0)] \pi_b(b(s); s) = 0, \text{ for all } s \in [\underline{s}, \bar{s}], \quad (16)$$

and (b) is non-decreasing (non-increasing) if  $\pi_b(b; s) / \pi(b; s)$  is strictly increasing (decreasing) in  $s$  for all  $b \in (0, \bar{b})$  and  $s \in [\underline{s}, \bar{s}]$ .<sup>2</sup>

3. A necessary condition for the agent to make an efficient decision  $b^*(s)$  for all  $s \in [\underline{s}, \bar{s}]$  is that  $P(\cdot)$  satisfies

$$P(b) = P(0) + kb, \text{ for all } b \in [b^*(\underline{s}), b^*(\bar{s})], \text{ with } k > 0. \quad (17)$$

---

<sup>2</sup>This result also holds if “non-decreasing” and “non-increasing” are replaced by “strictly increasing” and “strictly decreasing” respectively because  $b'(s) = 0$  is in fact inconsistent with the first-order condition (16) holding for all  $s$ . The result is stated as in Proposition 3 because it is more convenient later to work with a weak, rather than a strict, inequality.

Part 1 of Proposition 3 establishes that only decision rules with decisions interior to  $(0, \bar{b})$  can be implemented and they require contracts with payment increasing with  $b$ . Part 2 establishes conditions under which the first-order condition for the agent's decision is sufficient, as well as necessary, for an optimum conditional on  $s$  so that one can adopt a "first-order approach" — the requirement that the agent's decision must be optimal can be characterised by the first-order condition for optimality. The underlying idea is the same as in the application of the "first-order approach" in Rogerson (1985) and Jewitt (1988) but there the application is to the agent's choice of action, not the agent's decision based on a signal that results from that action. The proof makes use of the property that the agent's first-order condition must hold for *all*  $s$ . This imposes restrictions on how the agent's expected utility varies with  $s$ , which in turn imposes restrictions on  $b(\cdot)$  that turn out to be sufficient as well as necessary. Specifically,  $b(\cdot)$  must be non-decreasing or non-increasing according to whether  $\pi_b(b; s) / \pi(b; s)$  is increasing or decreasing in  $s$ , a property of the exogenously given probability distribution. Thus, for a probability distribution with the appropriate property, the only constraint other than the first-order condition that needs to be applied to ensure incentive compatibility is one on the direction that  $b(s)$  changes with  $s$ . This is a standard, and straightforward, constraint to impose. Part 3 of Proposition 3 establishes that a contract linear in the return  $b$  is not only sufficient to induce the agent to adopt  $b^*(\cdot)$  but also necessary when Assumption 1 holds.

Proposition 3 specifies properties of the decision rules it is *feasible* for the principal to induce the agent to use and of the contracts required to do so. The next proposition concerns the properties of the decision rules it is *optimal* for the principal to induce. With a risk-neutral agent, the only reason to distort the agent's decision away from what is efficient is to reduce the effect of limited liability. In general, it is not obvious what direction that distortion takes. The next result shows that it can be towards either more risky decisions or less risky decisions.

**Proposition 4** *Suppose Assumption 1 holds.<sup>3</sup> Then, if limited liability is binding under an optimal contract, that contract implements:*

1.  $b(s) > b^*(s)$  for all  $s \in (\underline{s}, \bar{s})$ ,  $b(\underline{s}) = b^*(\underline{s})$  and  $b(\bar{s}) \geq b^*(\bar{s})$  if  $\pi_b(b; s) / \pi(b; s)$  is strictly increasing in  $s$  for all  $b \in (0, \bar{b})$  and  $s \in [\underline{s}, \bar{s}]$ ;
2.  $b(s) < b^*(s)$  for all  $s \in (\underline{s}, \bar{s})$ ,  $b(\underline{s}) = b^*(\underline{s})$  and  $b(\bar{s}) \leq b^*(\bar{s})$  if  $\pi_b(b; s) / \pi(b; s)$  is strictly decreasing in  $s$  for all  $b \in (0, \bar{b})$  and  $s \in [\underline{s}, \bar{s}]$ .

Proposition 4 is illustrated in Figure 1. From Proposition 3, only a contract linear in  $b$  with positive slope induces the agent both to take action  $\bar{a}$  and to select decision  $b^*(s)$  for all  $s$ . Let  $\hat{P}(b)$  in the figure illustrate the linear contract with the *least steep*

<sup>3</sup>An appropriate constraint qualification for the optimal control problem of choosing  $b(\cdot)$  is also assumed to be satisfied, see Lemma 2 in the appendix.

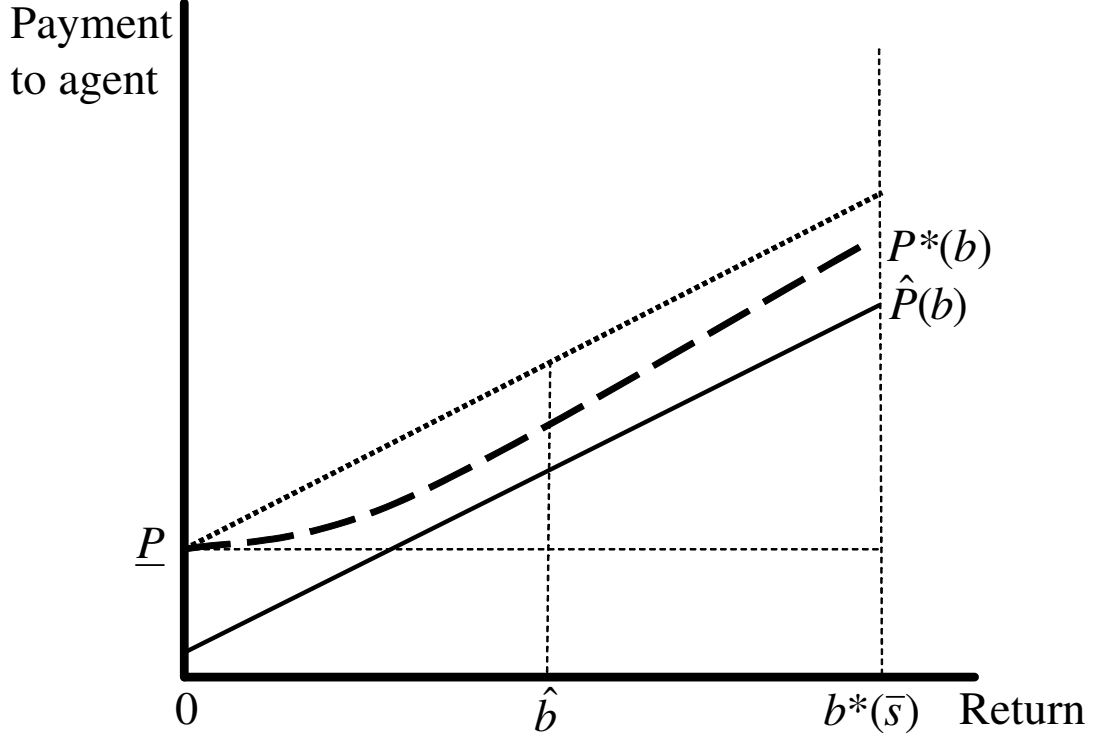


Figure 1: Illustration of Proposition 4

slope that, in the absence of limited liability, would satisfy the incentive compatibility constraint for the agent to choose action  $\bar{a}$  and also, with equality, the individual rationality constraint. Binding limited liability implies  $\underline{P} > \hat{P}(0)$ . One possible response of the principal would be to continue to implement  $b^*(s)$  for all  $s$  by using a linear contract with the same slope but with  $P(0)$  increased to  $\underline{P}$ . That corresponds to the dotted line in Figure 1. The incentive compatibility condition corresponding to (11) for the agent to choose action  $\bar{a}$  is

$$\begin{aligned}
 P(0) + \int_{\underline{s}}^{\bar{s}} [P(b(s)) - P(0)] \pi(b(s); s) f(s; \bar{a}) ds - v(\bar{a}) \\
 \geq P(0) + \int_{\underline{s}}^{\bar{s}} [P(b(s)) - P(0)] \pi(b(s); s) f(s; \underline{a}) ds - v(\underline{a}), \quad (18)
 \end{aligned}$$

so adding the same constant to  $P(b)$  for all  $b$  leaves it still satisfied. But the expected payment to the agent is given by the left-hand side of (18), so adding a positive constant involves paying the agent more than required to satisfy the individual rationality constraint. However, with the definition  $LR(s) \equiv f(s; \underline{a}) / f(s; \bar{a})$ , (18) can, when limited liability binds so that  $P(0) = \underline{P}$ , be written

$$\int_{\underline{s}}^{\bar{s}} [P(b(s)) - \underline{P}] \pi(b(s); s) f(s; \bar{a}) [1 - LR(s)] ds \geq v(\bar{a}) - v(\underline{a}). \quad (19)$$



Thus the expected payment to the agent can be reduced while still satisfying (19) by reducing payments that are above  $\underline{P}$  for returns  $b(s)$  for  $s$  such that the likelihood ratio  $LR(s)$  is greater than 1. Since  $s$  is ordered so that  $LR(s)$  is non-increasing, that corresponds to reducing payments for returns  $b(s)$  for low  $s$ .

For Part 1 of Proposition 4,  $\pi_b(b(s); s) / \pi(b(s); s)$  is strictly increasing in  $s$ . By Proposition 2,  $b^*(s)$  is then strictly increasing, so reducing payments for returns  $b(s)$  for low  $s$  corresponds to reducing payments for low returns. Because that increases the left-hand side of (19), it enables payments to be reduced for  $b(s)$  for  $s$  such that  $LR(s) < 1$  too but never such as to make the contract concave because then a less steep linear contract would also have satisfied the constraint. Suppose the optimal contract corresponds to  $P^*(b)$  in Figure 1. The effect on the decision for given  $s$  can be seen from the agent's first-order condition (16) which, with binding limited liability (and so  $P(0) = \underline{P}$ ), can be written

$$-\frac{\pi_b(b(s); s)}{\pi(b(s); s)} = \frac{P'(b(s))}{P(b(s)) - \underline{P}}. \quad (20)$$

Consider the return  $\hat{b}$  in Figure 1 at which the dashed line has a steeper slope than the dotted line and let  $s'$  and  $s''$  denote the values of  $s$  for which  $\hat{b}$  is chosen under the contracts represented by the dashed and the dotted lines respectively. At  $\hat{b}$ , the denominator on the right hand side of (20) is smaller for the contract represented by the dashed line and the numerator is larger. Thus, from (20),  $s'$  and  $s''$  must satisfy

$$-\frac{\pi_b(\hat{b}; s')}{\pi(\hat{b}; s')} > -\frac{\pi_b(\hat{b}; s'')}{\pi(\hat{b}; s'')}. \quad (21)$$

When  $\pi_b(b(s); s) / \pi(b(s); s)$  is strictly increasing in  $s$ , that implies  $s' < s''$ . Moreover, with  $b(s)$  and  $b^*(s)$  both increasing and  $\hat{b} = b(s') = b^*(s'')$ , it follows that  $b(s'') > b^*(s'')$ . That conclusion applies for  $\pi_b(b(s); s) / \pi(b(s); s)$  strictly increasing in  $s$  as in Part 1 of Proposition 4. For  $\pi_b(b(s); s) / \pi(b(s); s)$  strictly decreasing in  $s$  as in Part 2,  $b(s'') < b^*(s'')$ . The proof of Proposition 4 establishes that these conclusions apply for all  $s \in (\underline{s}, \bar{s})$ .

In both cases, the response to limited liability is to reduce the payment for returns  $b(s)$  for those  $s$  for which  $LR(s)$  is greater than 1. The difference between them is the effect that has on the decision rule. For  $b^*(s)$  increasing, higher  $b$  is more attractive to the principal for higher  $s$  and, by definition,  $LR(s)$  is lower for higher  $s$ . So reducing payments for returns with higher  $LR(s)$  reduces payments for lower  $b$ , making higher, more risky, decisions more attractive to the agent and thus  $b(s) > b^*(s)$ . The opposite applies for  $b^*(s)$  decreasing. The conclusion that the direction of the distortion from efficiency can go either way mirrors that in Palomino and Prat (2003) in their application to delegated portfolio choice. Their result, however, relies on the "first-order approach" being valid, which they cannot guarantee. For the model used here, it has

been established that the first-order approach is indeed valid.

The essential point here is the following. Start from a linear contract that respects limited liability and provides sufficient incentive for action  $\bar{a}$ . The principal can reduce the total expected payment to the agent and still achieve the same incentive for action by reducing payments for returns to decisions that are made for signals that are more likely to arise from  $\underline{a}$  than from  $\bar{a}$ . Given the ordering of signals, these are lower signals. If less risky decisions (lower  $b$ ) are relatively more attractive to the principal for low signals, this will induce the agent to make more risky decisions than under a linear contract. If, on the other hand, more risky decisions (higher  $b$ ) are relatively more attractive to the principal for low signals, it will induce the agent to make less risky decisions than under a linear contract. Since the linear contract induces efficient decisions, small changes in decisions involve little efficiency loss, so some deviation of this sort is always worthwhile in order to reduce the total expected payment to the agent. This general insight seems robust. The precise conditions in Assumption 1 ensure that (1) a linear contract is the only starting point for efficient decisions and (2) efficient decisions are always interior so that small deviations from them always involve only small efficiency losses. The assumption that there are only two possible outcomes for each decision, one of which is the same for all decisions, makes it straightforward to identify how the riskiness of decisions changes with the signal.

Under the conditions of Part 2 of Proposition 4, the agent is induced, for all values of the information signal apart from those at the two extremes of its support, to make a less risky decision, with a lower return and a higher probability of success, than the principal would choose given the same information. Thus the principal chooses a contract that more than compensates for the incentive an agent otherwise has to make a more risky decision if limited liability is imposed on a contract that would be optimal in the absence of limited liability. In that case, managers with limited liability are induced to take decisions in a way that makes it look as if they are risk-averse — limited liability not only reduces the profits of firms but also biases their decisions in a less risky direction.

## 6 Optimal contract form

Proposition 3 implies that, when Assumption 1 holds, only a contract with payment linear in the return induces the agent to select  $b^*(s)$  for all  $s$ , a condition under which Diamond (1998) establishes the “near-linearity” result described in the Introduction. But Diamond (1998) recognises that this is not always the case. This section uses an example of the model of Section 2 to give some insight into when it is not.

The model in Diamond (1998) is similar to that used here in having the agent make a decision that affects the probabilities of the possible outcomes. It differs in having only a finite number of possible outcomes, with the possible outcomes resulting from

a decision not restricted to just two. (It also differs in that it is not possible for the agent to influence the probabilities of the outcomes without first taking a costly action but that difference is not important here.) The “near-linearity” result can be stated as follows. Let  $c = v(\bar{a}) - v(\underline{a})$  and express the returns from decision  $b$ , if successful, as  $\lambda b$  for  $\lambda > 0$ , with the probability of success being unaffected by the value of  $\lambda$ . Then, when the only contracts that induce the agent to select an efficient decision rule are linear, an optimal contract converges to a linear contract as  $c/\lambda$  converges to zero. The essential intuition is that, as  $\lambda$  gets large relative to  $c$ , the gain to the principal from decisions being efficient dwarfs the additional cost of inducing the agent to take action  $\bar{a}$  with a linear contract than with some other contract.

To illustrate why the same conclusion does not necessarily apply when contracts that are not linear induce efficient decisions, consider a principal who wants to ensure that the agent’s decision has a successful outcome with a return no less than is necessary to achieve this. That is plausible in some circumstances. Winning a particular contract may be crucial to the survival of a firm, the situation of Rolls-Royce with its RB211 jet engines in the 1970s. Of course, that is possible only when some decision has a certainly successful outcome given the information available. The following assumption formalizes these properties.

**Assumption 2** *For each  $s \in [\underline{s}, \bar{s}]$ , there exists  $b \in (\underline{b}, \bar{b}]$  such that  $\pi(b; s) = 1$ . The principal chooses a contract that ensures the agent uses a decision rule  $b(\cdot)$  satisfying  $b(s) = \max \left\{ b \in [\underline{b}, \bar{b}] \mid \pi(b; s) = 1 \right\}$  for all  $s \in [\underline{s}, \bar{s}]$ . For sufficiently small  $\varepsilon > 0$ ,  $\pi(b(s) + \varepsilon; s) \in (0, 1)$  for all  $s \in [\underline{s}, \bar{s}]$ .*

The reason for the final part of Assumption 2 is that, if  $\pi(b(s) + \varepsilon; s) = 0$  for all  $s \in [\underline{s}, \bar{s}]$ , the agent’s signal would fully reveal which decisions would have successful outcomes whatever the agent’s action, so there would be no reason for wanting the agent to choose  $\bar{a}$  rather than  $\underline{a}$ . For an example of a distribution satisfying Assumption 2, let  $b'$  denote the decision with the highest return that will actually be successful and suppose  $f(s; a)$  has support  $[ab', b']$  for  $a \in \{\underline{a}, \bar{a}\}$  with  $0 < \underline{a} < \bar{a} < 1$ . Then the signal  $s$  is a random draw from a distribution with both lower and upper supports no greater than  $b'$  and the decision  $b = s$  will thus certainly have a successful outcome. Moreover, higher  $a$  results in a signal  $s$  drawn from a distribution concentrated closer to  $b'$  and, as  $a$  approaches one,  $s$  corresponds to  $b'$ . This is illustrated for the case of  $f(s; a)$  uniform in Figure 2. Under straightforward conditions,  $b = s$  is also an efficient decision. Let  $\pi_{b_+}(b; s)$  denote the right-hand derivative at  $\pi(b; s) = 1$ . Then if

$$s\pi_{b_+}(s; s) + 1 < 0,$$

or equivalently  $\pi_{b_+}(b; s) < -1/s$ , the decision  $b = s$  is a local optimum for the maximand in (1). Under straightforward conditions, it is also a global optimum. Then

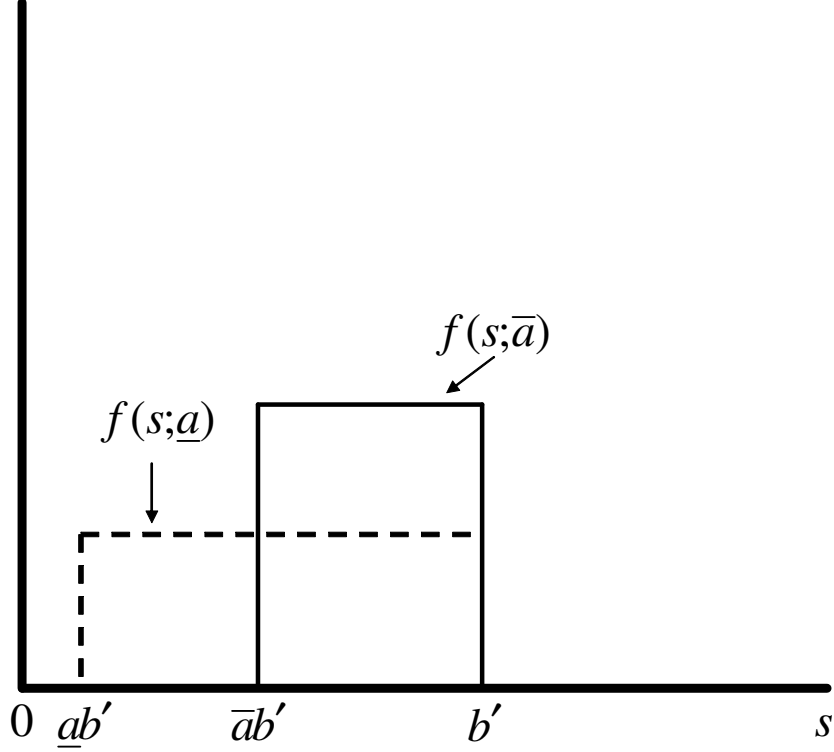


Figure 2: Example of  $f(s; a)$  for case of certain success

$b^*(s) = s$ , so an efficient decision rule ensures that the decision always has a successful outcome.

By Proposition 1, contracts can be restricted to those that are non-decreasing. Then limited liability binds for  $P(0)$  and, if satisfied by  $P(0)$ , is satisfied for all  $b > 0$ . Thus, the condition corresponding to (12) for the agent to choose  $b(s)$  given signal  $s$  is

$$b(s) \in \arg \max_{b \in [0, \bar{b}]} P(b) \pi(b; s) + \underline{P} [1 - \pi(b; s)], \text{ for all } s \in [\underline{s}, \bar{s}]. \quad (22)$$

Under Assumption 2, with  $b(\cdot)$  such that  $\pi(b(s); s) = 1$  for all  $s$  and hence  $\pi(b; s) = 1$  for all  $b < b(s)$ , this condition is automatically satisfied for all  $b \leq b(s)$  if  $P(\cdot)$  is non-decreasing. The following result is concerned with  $b > b(s)$ .

**Proposition 5** *Suppose Assumption 2 holds. A necessary condition for contract  $P(\cdot)$  to implement  $b(\cdot)$  is that*

$$P'(b(s)) \leq -\pi_{b+}(b(s); s) [P(b(s)) - \underline{P}], \text{ for all } s \in [\underline{s}, \bar{s}]. \quad (23)$$

*For  $P(\cdot)$  non-decreasing, this condition is also sufficient if  $b(s)$  and  $\pi_b(b; s) / \pi(b; s)$  are either both non-decreasing in  $s$  or both non-increasing in  $s$  for all  $b \in [0, \bar{b}]$  and  $s \in [\underline{s}, \bar{s}]$ .*

The implication of (23) is that, to ensure the agent makes a decision with an outcome that is always successful,  $P(\cdot)$  must not only not decrease but also not increase

too fast. If it does, the agent's payoff is increased by choosing a slightly higher  $b$ , with a higher reward if successful but with a probability of success strictly less than one.

Suppose the constraint (23) is not binding. Then limited liability reduces the principal's payoff only because it is not possible to satisfy the incentive compatibility condition (11) for the agent to choose action  $\bar{a}$  while satisfying the agent's individual rationality condition (10) with equality. The incentive compatibility condition (11) in this case can be written in a form similar to (19) that, when binding, becomes

$$\int_{\underline{s}}^{\bar{s}} P(b(s)) f(s; \bar{a}) [1 - LR(s)] ds = v(\bar{a}) - v(\underline{a}). \quad (24)$$

As discussed in connection with Proposition 4, the expected payment to the agent can be reduced while keeping the incentive compatibility condition (24) satisfied by reallocating rewards from  $b(s)$  for  $s$  such that  $LR(s)$  is high to  $b(s)$  for  $s$  such that  $LR(s)$  is low. Since  $s$  is ordered so that  $LR(\cdot)$  is non-increasing, that corresponds to increasing the payments for low  $b$  if  $b(\cdot)$  is decreasing and for high  $b$  if  $b(\cdot)$  is increasing. In the former case, the principal would wish to focus all payments above  $\underline{P}$  on the lowest  $b$  to be chosen for any  $s$ , but that would clearly not satisfy  $P(\cdot)$  non-decreasing. In the latter case, the principal would wish to focus payments above  $\underline{P}$  on the highest  $b$  to be chosen for any  $s$ , but that would necessarily violate the constraint (23) that  $P(\cdot)$  not increase too fast. The following result applies to the case  $b(\cdot)$  everywhere strictly increasing so the inverse function  $b^{-1}(\cdot)$  exists.

**Proposition 6** *Suppose Assumption 2 holds. Then the following form of contract is optimal if  $b(\cdot)$  is strictly increasing and  $\pi_b(b; s) / \pi(b; s)$  is non-decreasing in  $s$  for all  $b \in [0, \bar{b}]$  and  $s \in [\underline{s}, \bar{s}]$ :*

$$P(b) = \begin{cases} \underline{P}, & \text{for } b \in [0, b(\underline{s})], \\ \underline{P} + [P(b(\underline{s})) - \underline{P}] \exp \left[ - \int_{b(\underline{s})}^b \pi_{b_+}(b; b^{-1}(b)) db \right], & \text{for } b \in [b(\underline{s}), b(\bar{s})], \\ P(b(\bar{s})), & \text{for } b \in [b(\bar{s}), \bar{b}], \end{cases} \quad (25)$$

with  $P(b(\underline{s}))$  ensuring that the incentive compatibility condition (24) is satisfied.

Proposition 6 gives the formula for an optimal contract under the conditions specified. Everything in that formula is data for the model apart from  $P(b(\underline{s}))$ , which is set so that the incentive compatibility condition (24) is satisfied when the formula in (25) is substituted for  $P(b)$ . The formula is linear in the exponential term and so will not, in general, result in a contract linear in  $b$ . If, for example,  $\pi_{b_+}(b; b^{-1}(b))$  were constant,  $P(b)$  would be exponentially increasing in  $b$ . Note that the middle line in (25) approaches a constant as  $\pi_{b_+}(b; b^{-1}(b))$  approaches zero (because then the exponential term approaches one), in which case the contract approaches one with provision for just two levels of payment,  $\underline{P}$  and  $P(b(\underline{s}))$ , the former of which is never actually paid

because it applies only to returns that do not occur. That is like a fixed salary  $P(b(\underline{s}))$ , with  $\underline{P}$  corresponding to the threat of being fired for a return so low that it cannot occur if the agent uses the decision rule  $b(\cdot)$ . But that limit can never actually be reached; for  $\pi_{b^+}(b; b^{-1}(b)) = 0$ , it would not be optimal for the principal to implement  $b(\cdot)$  because it would always be profitable to accept a small risk of an unsuccessful outcome in order to relax the limited liability constraint.

Proposition 6 illustrates that the “near-linearity” result of Diamond (1998) may not hold when a non-linear contract can induce the agent to select an efficient decision rule. That proposition applies to any  $b(\cdot)$  that satisfies Assumption 2 and, hence, an efficient decision rule  $b^*(\cdot)$  when that does so. Then, although a linear contract would implement  $b^*(\cdot)$ , the non-differentiability of the probability  $\pi(b, s)$  for  $b = b^*(s)$  gives sufficient flexibility for a linear contract not to be the only contract that induces the agent to choose  $b^*(\cdot)$ . Moreover, the non-linear contract in (25) does that at lower cost to the principal, no matter what the magnitude of the agent’s disutility of action  $\bar{a}$  (conditional, of course, on it being worthwhile to induce the agent to take that action).

It is also instructive to compare the result in Proposition 6 with results in two papers in which optimal contract forms are derived explicitly for a risk-neutral agent with limited liability, Innes (1990), and Biais and Casamatta (1999). Innes (1990, Proposition 1) considers a classic principal-agent model (corresponding to the signal  $s$  being verifiable and, hence,  $b(\cdot)$  being contractible) with the additional assumption that both the principal’s and the agent’s rewards are monotonic in the return. Under assumptions corresponding to  $b(\cdot)$  increasing, as in Proposition 6, he shows that a debt contract is optimal. With limited liability, a debt contract has  $P(b) = \underline{P}$  (and hence  $P'(b) = 0$ ) for  $b < \hat{b}$  and  $P'(b) = 1$  for  $b > \hat{b}$  for some  $\hat{b}$ . The contract in (25) clearly does not correspond to that. The reason for the difference lies in the nature of the constraint on how fast  $P(\cdot)$  can increase. A contract monotonic for the principal implies that the reward to the agent cannot increase faster than the realised return, that is,  $P'(\cdot) \leq 1$ . Thus in Innes (1990) the constraint  $P'(\cdot) \leq 1$  replaces the constraint (23). As discussed above, with  $LR(\cdot)$  non-increasing, payment to the agent is reduced while maintaining incentives for action by transferring rewards from lower values of  $b$  to higher values, which results in the constraint  $P'(\cdot) \leq 1$  becoming binding. As long as limited liability is not so tight as to make that constraint bind for all  $b$ , a debt contract, with a flat section for low  $b$ , is optimal. Thus, although the contracts differ, the underlying rationale for a debt contract in Innes (1990) is similar to that for the contract in Proposition 6. The difference is that the constraint (23) is not an assumption but is derived from the need to make the decision rule incentive compatible for the agent.

In Biais and Casamatta (1999), there is only a single decision that is ever worthwhile to the principal, no matter what signal the agent receives. Thus there is no issue of trading off a different decision for a lower expected payment to the agent. They show that an optimal contract can be implemented in terms of three instruments, debt,

equity and share options. Thus their result, like that in Proposition 6, serves to emphasize that a debt contract is not in general optimal when a risk-neutral agent makes a decision as well as choosing how hard to work at acquiring information. But with only three possible outcomes in their model, it is not surprising that just three instruments can implement an optimal contract. It remains to be seen what standard instruments (if any) can do so when, as in the model studied here, the optimal decision for the principal depends on the signal privately received and there is a continuum of possible outcomes.

## 7 Concluding remarks

This paper has been concerned with whether, when taking costly action to acquire information before making decisions, a risk-neutral manager with limited liability makes more risky or less risky decisions than one without limited liability when operating under an optimal contract with a principal. The answer depends on the nature of the environment. The manager's limited liability restricts the incentives for action that can be attained at given expected cost to the principal. As a result, the principal wants to place greater reward on those outcomes that have higher likelihood of arising from the action the principal wishes the manager to take. In the notation used here, that corresponds to rewarding the agent more for signals  $s$  for which the likelihood ratio  $LR(s) (\equiv f(s; \underline{a}) / f(s; \bar{a}))$  is low. The resulting decisions will be more risky if those signals are ones for which the principal wants the agent to take more risky decisions and less risky otherwise.

A contract in which the reward to the manager is non-decreasing and linear in the return to the manager's decision will always induce the manager to adopt an efficient decision rule. Diamond (1998) establishes the "near-linearity" result that, if such a linear contract is the only form that does this, an optimal contract converges to a linear contract as the ratio of the principal's gross return to the agent's cost of taking appropriate action increases. The present paper gives an insight into when that result does not apply. In the model used here, the non-decreasing property of the contract still holds — there is always an optimal contract that is non-decreasing. That result does not depend on the likelihood ratio  $LR(\cdot)$  being monotone. It arises because a decreasing contract gives the manager an incentive to make decisions that effectively "throw away" returns. The fact that the manager makes a decision, as well as taking an action, thus provides a reason for contracts to be monotone in practice additional to those discussed in the earlier literature. But in an example in which efficient decisions occur where the probability of success has corners (and so is non-differentiable), the optimal contract is not linear and does not converge to a linear contract no matter what the agent's cost of action, so the "near-linearity" result of Diamond (1998) does not apply. The non-differentiability gives sufficient flexibility for a non-linear contract

to induce the agent to adopt an efficient decision rule and does so at a lower expected cost to the principal.

## Appendix Proofs

**Proof of Proposition 1.** Part 1. Suppose  $P(\cdot)$  implements decision rule  $b(\cdot)$  with  $P(b') > P(b(s))$  for some  $b' < b(s)$  for some  $s$ . By (5), the agent's payoff from choosing  $b(s)$  conditional on  $s$  is

$$u(b(s), s, P(\cdot)) = P(b(s)) \pi(b(s); s) + P(0) [1 - \pi(b(s); s)].$$

If  $\pi(b(s); s) = 0$ , the return to the principal is the same as if the agent had chosen  $b = 0$ , so there is no loss to the principal from implementing  $b = 0$  instead of  $b(s)$ . If  $\pi(b(s); s) > 0$ , it must be that  $P(b(s)) \geq P(0)$  because otherwise the agent would achieve a higher payoff from choosing  $b = 0$ . The agent's payoff from choosing  $b'$  instead of  $b(s)$  is

$$u(b', s, P(\cdot)) = P(b') \pi(b'; s) + P(0) [1 - \pi(b'; s)].$$

But this is greater than  $u(b(s), s, P(\cdot))$  for  $P(b') > P(b(s))$  because  $\pi(b; s)$  is non-increasing in  $b$ , contradicting that  $P(\cdot)$  implements  $b(\cdot)$ . So it must be that  $P(b(s)) \geq P(b)$  for all  $b < b(s)$  for all  $s$ .

Now consider  $b$  that is not to be implemented for any  $s$ . If there exists  $b(s) < b$  for some  $s$ , let  $\hat{b}(b) = \max b(s)$  over  $s$  such that  $b(s) < b$ . If not, let  $\hat{b}(b) = 0$ . The preceding argument does not rule out  $P(b) < P(\hat{b}(b))$  even though  $b > \hat{b}(b)$ . But, as long as  $P(b) \leq P(\hat{b}(b))$ , the agent cannot gain by choosing  $b$  in preference to  $\hat{b}(b) < b$  because  $\pi(\hat{b}(b); s) \geq \pi(b; s)$ . Since  $b$  is never chosen if  $b(s)$  is implemented,  $P(b)$  does not affect the agent's incentive for action or the principal's payoff. Thus the principal does not lose by setting  $P(b) = P(\hat{b}(b))$ . But then the contract is non-decreasing.

Part 2. Use of  $P(\cdot)$  from (14) in (5) and the resulting expression for  $u(b, s, P(\cdot))$  in (12) gives

$$b(s) \in \arg \max_{b \in [0, \bar{b}]} [\underline{u} + kb\pi(b; s)], \text{ for all } s \in [s, \bar{s}],$$

which for  $k \geq 0$  defines the same set of maximizers as (1). ■

**Proof of Proposition 2.** The maximand in (1) is zero for  $b = 0$ . It is also zero for  $b = \bar{b}$  because, by Assumption 1,  $\pi(\bar{b}; s) = 0$ . By assumption, for each  $s$  there exists some  $b > 0$  for which  $\pi(b; s) > 0$ , so the agent's payoff from selecting some  $b \in (0, \bar{b})$  given any  $s$  is strictly positive, establishing Part 1.



In view of Part 1, the first-order condition for the problem in (1) holds with equality and hence takes the form in (15). Since  $\pi_b(b; s) < 0$  for  $\pi(b; s) \in (0, 1)$  and hence, by Assumption 1, for all  $b \in (0, \bar{b})$ , that condition can be written

$$b^*(s) = -\frac{\pi(b^*(s); s)}{\pi_b(b^*(s); s)}, \quad \forall s. \quad (26)$$

Also by Assumption 1,  $\pi_b(b; s) / \pi(b; s)$  is continuous non-increasing in  $b$  and hence so is  $-\pi(b; s) / \pi_b(b; s)$ . Moreover,  $-\pi(b; s) / \pi_b(b; s)$  is strictly positive for  $b = 0$  and strictly less than  $\bar{b}$  for some  $b$  because  $\pi(b; s) \rightarrow 0$  as  $b \rightarrow \bar{b}$  and  $\pi_b(b; s) < 0$  for  $\pi(b; s) > 0$  for all  $s$ . Thus (26) has a unique solution. To see that this solution is a maximum, note first that  $\pi_b(b; s) / \pi(b; s)$  non-increasing in  $b$  for all  $s$  implies

$$\frac{\partial}{\partial b} \left( \frac{\pi_b(b; s)}{\pi(b; s)} \right) = \frac{\pi(b; s) \pi_{bb}(b; s) - \pi_b(b; s)^2}{\pi(b; s)^2} \leq 0, \quad \forall b,$$

or

$$\pi_{bb}(b; s) \leq \frac{\pi_b(b; s)^2}{\pi(b; s)}, \quad \forall b. \quad (27)$$

The second derivative with respect to  $b$  of the maximand in (1) evaluated at  $b^*(s)$  is

$$\begin{aligned} & b^*(s) \pi_{bb}(b^*(s); s) + 2\pi_b(b^*(s); s) \\ &= -\frac{\pi(b^*(s); s)}{\pi_b(b^*(s); s)} \pi_{bb}(b^*(s); s) + 2\pi_b(b^*(s); s) \\ &\leq \pi_b(b^*(s); s) < 0, \end{aligned}$$

the equality following from the first-order condition (26) and the weak inequality from (27), given  $\pi_b(b^*(s); s) < 0$ . Thus the second-order sufficient condition for a maximum is satisfied at the unique solution to (15), establishing Part 2.

Part 3 follows directly from (26) and Assumption 1. ■

**Proof of Proposition 3.** Part 1. In view of (5), (12) can be written

$$b(s) \in \arg \max_{b \in [0, \bar{b}]} P(b) \pi(b; s) + P(0) [1 - \pi(b; s)], \quad \text{for all } s \in [\underline{s}, \bar{s}].$$

To induce the agent to take action  $a = \bar{a}$ , it must be that  $P(\hat{b}) > P(0)$  for some  $\hat{b}$  for which  $\pi(\hat{b}; s) > 0$  for some  $s$ . It follows from Assumption 1 that  $\hat{b} < \bar{b}$  and, hence,  $\pi(\hat{b}; s) > 0$  for all  $s$ . Consider  $b(s)$  that, for some  $s$ , has  $P(b(s)) = P(0)$  or  $b(s) = \bar{b}$ . Then the agent's expected utility from choosing  $b(s)$  given  $s$  is  $P(b(s)) = P(0)$ , in the former case because  $b(s) = 0$ , in the latter because  $\pi(\bar{b}; s) = 0$ . But the agent could

have chosen  $\hat{b}$  when observing  $s$  and this would have yielded expected utility

$$P(0) + \left[ P(\hat{b}) - P(0) \right] \pi(\hat{b}; s),$$

which is greater than  $P(0)$  given  $\pi(\hat{b}; s) > 0$ . Thus  $b(s)$  is not implementable. Finally, suppose  $P(b(s)) \leq P(\hat{b})$  for some  $\hat{b} < b(s)$ . Since  $P(b(s)) > P(0)$ ,  $P(\hat{b}) > P(0)$  too. Then the agent's payoff is strictly increased by choosing  $\hat{b}$  rather than  $b(s)$  because  $\pi(b; s)$  is strictly decreasing in  $b$  for each  $s$  for  $b < \bar{b}$ . Thus it must be that  $P(b(s)) > P(b)$  for all  $b < b(s)$ .

Part 2. Since  $\pi(b; s)$  is differentiable in  $b$  for  $\pi(b; s) \in (0, 1)$ , necessity of the first-order condition (16) follows from  $b(s)$  interior to  $[0, \bar{b}]$  and differentiability of  $P(b)$ . Let  $z(b) = P(b) - P(0)$  and

$$W(b(s), s) = z'(b(s)) + z(b(s)) \frac{\pi_b(b(s); s)}{\pi(b(s); s)}.$$

The first-order condition (16) can then be written  $\pi(b(s); s) W(b(s), s) = 0$  for all  $s$ . By Part 1,  $b(\cdot)$  is implementable only if  $b(s) \in (0, \bar{b})$  for all  $s$  and, by Assumption 1,  $\pi(b(s); s) \in (0, 1)$  for all such  $b(s)$ . So to implement  $b(\cdot)$  requires  $W(b(s), s) = 0$  for all  $s$ . It is sufficient for (16) to have at most one solution for each  $s$ , and for this solution to be a maximum, that

$$\pi_b(b(s); s) W(b(s), s) + \pi(b(s); s) W_b(b(s), s) < 0 \text{ for all } s.$$

With  $W(b(s), s) = 0$  and  $\pi(b(s); s) > 0$ , that will certainly hold if  $W_b(b(s), s) < 0$  for each  $s$ . Because (16) must hold for all  $s$ , its total derivative with respect to  $s$  must equal zero. With  $W(b(s), s) = 0$  and  $\pi(b(s); s) > 0$  for all  $s$ , that implies

$$\frac{dW(b(s), s)}{ds} = W_b(b(s), s) b'(s) + W_s(b(s), s) = 0. \quad (28)$$

From the definition of  $W(b(s), s)$ ,

$$W_s(b(s), s) = z(b(s)) \frac{\partial}{\partial s} \left( \frac{\pi_b(b(s); s)}{\pi(b(s); s)} \right),$$

which, given  $P(b(s)) > P(0)$  and hence  $z(b(s)) > 0$ , is either strictly positive or strictly negative under the conditions in Part 2(b). Thus (28) cannot be satisfied if  $b'(s) = 0$ . It can therefore be written

$$\begin{aligned} W_b(b(s), s) &= -W_s(b(s), s) / b'(s) \\ &= -\frac{z(b(s))}{b'(s)} \frac{\partial}{\partial s} \left( \frac{\pi_b(b(s); s)}{\pi(b(s); s)} \right). \end{aligned}$$

Thus  $W_b(b(s), s) > 0$  if  $b'(s)$  and  $\frac{\partial}{\partial s}(\pi_b(b(s); s) / \pi(b(s); s))$  have opposite signs, so any solution to the first-order condition cannot be a maximum, establishing necessity of Part 2(b). Moreover,  $W_b(b(s), s) < 0$  under the conditions in Part 2(b), which establishes sufficiency of those and the first-order conditions.

Part 3. From Proposition 2,  $b^*(s) \in (0, \bar{b})$  and satisfies (26). Moreover, the first-order condition for the agent (16) must hold for all  $s$ . It is then clear from comparison with (26) that the agent chooses  $b^*(s)$  for all  $s \in [\underline{s}, \bar{s}]$  only if

$$\frac{P'(b)}{P(b) - P(0)} = \frac{1}{b}, \text{ for all } b \in [b^*(\underline{s}), b^*(\bar{s})]. \quad (29)$$

Define  $z(b) = P(b) - P(0)$  for given  $P(0)$ . Then (29) can be written

$$\frac{z'(b)}{z(b)} = \frac{1}{b}, \text{ for all } b \in [b^*(\underline{s}), b^*(\bar{s})],$$

which, by integration, has solution

$$z(b) = kb, \text{ for all } b \in [b^*(\underline{s}), b^*(\bar{s})], \quad (30)$$

for some constant of integration  $k$ . Part 3 follows because the solution for  $P(\cdot)$  implied by (30) is that given by (17) and, by the argument in Part 1,  $k > 0$ . ■

A series of preliminary lemmas is useful for proving Proposition 4.

**Lemma 1** *Define*

$$h(b, s; a) \equiv \pi(b; s) f(s; a), \quad b \in [\underline{b}, \bar{b}], s \in [\underline{s}, \bar{s}], a \in \{a, \bar{a}\}, \quad (31)$$

$$P_0 \equiv P(0), \quad (32)$$

$$\zeta(s) \equiv P(b(s)), \quad s \in [\underline{s}, \bar{s}], \quad (33)$$

and

$$Z(a, b(\cdot), \zeta(\cdot), P_0) \equiv \int_{\underline{s}}^{\bar{s}} \left\{ \zeta(s) h(b(s), s; a) + P_0 [h(0, s; a) - h(b(s), s; a)] \right\} ds - v(a). \quad (34)$$

Then the principal's problem (9)–(13) can be written

$$\max_{b(\cdot), \zeta(\cdot), \chi(\cdot), P_0} \int_{\underline{s}}^{\bar{s}} \left\{ [b(s) - \zeta(s)] h(b(s), s; \bar{a}) - P_0 [h(0, s; \bar{a}) - h(b(s), s; \bar{a})] \right\} ds \quad (35)$$

subject to the dynamic constraints

$$\zeta'(s) = -\chi(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})}, \text{ for all } s \in [\underline{s}, \bar{s}]; \quad (36)$$

$$b'(s) = \chi(s), \text{ for all } s \in [\underline{s}, \bar{s}]; \quad (37)$$

and the inequality constraints

$$Z(\bar{a}, b(\cdot), \zeta(\cdot), P_0) - \underline{U} \geq 0; \quad (38)$$

$$Z(\bar{a}, b(\cdot), \zeta(\cdot), P_0) - Z(\underline{a}, b(\cdot), \zeta(\cdot), P_0) \geq 0; \quad (39)$$

$$P_0 - \underline{P} \geq 0; \quad (40)$$

$$\text{for } \frac{h_b(b, s; \bar{a})}{h(b, s; \bar{a})} \text{ strictly increasing in } s: \quad \chi(s) \geq 0, \text{ for all } s \in [\underline{s}, \bar{s}]; \quad (41)$$

$$\text{for } \frac{h_b(b, s; \bar{a})}{h(b, s; \bar{a})} \text{ strictly decreasing in } s: \quad \chi(s) \leq 0, \text{ for all } s \in [\underline{s}, \bar{s}]; \quad (42)$$

with free boundaries at  $s = \underline{s}$  and  $s = \bar{s}$ .

**Proof.** Since  $\pi(0; s) = 1$  for all  $s$ , the definition (31) implies  $h(0, s; a) = f(s; a)$  and

$$\frac{h_b(b, s; a)}{h(b, s; a)} = \frac{\pi_b(b; s)}{\pi(b; s)}, \text{ for all } (b, s, a). \quad (43)$$

Then, with the further definitions (32)–(33), the principal's objective function in (9) can, in view of (7) and (8), be written as (35). With, in addition, the definition (34), the constraints (10) and (11) can be written as (38) and (39).

Next consider the constraint (12). Under Assumption 1, by Part 2 of Proposition 3, necessary and sufficient conditions to induce the agent to select  $b(s)$  for all  $s$  are that the first-order condition (16) is satisfied and that

$$b'(s) \geq 0 \text{ for all } s \in [\underline{s}, \bar{s}] \text{ if } \pi_b(b; s) / \pi(b; s) \text{ strictly increasing in } s, \quad (44)$$

$$b'(s) \leq 0 \text{ for all } s \in [\underline{s}, \bar{s}] \text{ if } \pi_b(b; s) / \pi(b; s) \text{ strictly decreasing in } s. \quad (45)$$

The constraint (12) can, therefore, be replaced by these. They can, in turn, be re-written in terms of  $h(b, s; a)$ ,  $\zeta(s)$  and  $\chi(s)$  as in (36), (37), (41) and (42) by the following argument. From (33),

$$\zeta'(s) = P'(b(s)) b'(s). \quad (46)$$

With this notation and (37), the first-order condition (16) can be written

$$\zeta'(s) = -\chi(s) [\zeta(s) - P_0] \frac{\pi_b(b(s); s)}{\pi(b(s); s)},$$

which, in view of (43), can be written as (36). In view of (37) and (43), the conditions

(44) and (45) can be written as (41) and (42).

The remaining constraint in the principal's problem is (13). By Part 1 of Proposition 3, the first-order condition ensures  $b$  will be selected for  $s$  only if  $P(b) > P(0) \equiv P_0$ . Now consider  $b$  that is not to be selected for any  $s$ . If there exists  $b(s) < b$  for some  $s$ , let  $\hat{b}(b) = \max b(s)$  over  $s$  such that  $b(s) < b$ . If not, let  $\hat{b}(b) = 0$ . Then it is sufficient to set  $P(b) = P(\hat{b}(b)) \geq P(0) \equiv P_0$ . Thus (40) is sufficient to ensure that (13) is satisfied. ■

**Lemma 2** *Define the Hamiltonian*

$$\begin{aligned} H(\bar{a}, b(s), \zeta(s), P_0, s) &= [b(s) - \zeta(s) + P_0] h(b(s), s; \bar{a}) - P_0 h(0, s; \bar{a}) \\ &\quad - \psi_1(s) \chi(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} + \psi_2(s) \chi(s), \text{ for all } s \in [\underline{s}, \bar{s}], \end{aligned} \quad (47)$$

where  $\psi_1(s)$  and  $\psi_2(s)$  are multipliers attached to the constraints (36) and (37) respectively. Then, provided an appropriate constraint qualification is satisfied (see Léonard and Long (1992)), Maximum Principle conditions for the principal's problem in Lemma 1, in which  $\lambda, \mu, \phi$  and  $v(s)$  are multipliers attached in this order to the inequality constraints (38)–(41), are for all  $s \in [\underline{s}, \bar{s}]$ :

$$\begin{aligned} \psi_1'(s) &= -\frac{\partial H(\bar{a}, \cdot)}{\partial \zeta(s)} - \lambda \frac{\partial Z(\bar{a}, \cdot)}{\partial \zeta(s)} - \mu \left[ \frac{\partial Z(\bar{a}, \cdot)}{\partial \zeta(s)} - \frac{\partial Z(\underline{a}, \cdot)}{\partial \zeta(s)} \right] \\ &= h(b(s), s; \bar{a}) - \lambda h(b(s), s; \bar{a}) - \mu [h(b(s), s; \bar{a}) - h(b(s), s; \underline{a})] \\ &\quad + \psi_1(s) \chi(s) \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})}; \end{aligned} \quad (48)$$

$$\begin{aligned} \psi_2'(s) &= -\frac{\partial H(\bar{a}, \cdot)}{\partial b(s)} - \lambda \frac{\partial Z(\bar{a}, \cdot)}{\partial b(s)} - \mu \left[ \frac{\partial Z(\bar{a}, \cdot)}{\partial b(s)} - \frac{\partial Z(\underline{a}, \cdot)}{\partial b(s)} \right] \\ &= -h(b(s), s; \bar{a}) - [b(s) - \zeta(s) + P_0] h_b(b(s), s; \bar{a}) \\ &\quad - \lambda [\zeta(s) - P_0] h_b(b(s), s; \bar{a}) \\ &\quad - \mu [\zeta(s) - P_0] [h_b(b(s), s; \bar{a}) - h_b(b(s), s; \underline{a})] \\ &\quad + \psi_1(s) \chi(s) [\zeta(s) - P_0] \frac{h(b(s), s; \bar{a}) h_{bb}(b(s), s; \bar{a}) - h_b(b(s), s; \bar{a})^2}{h(b(s), s; \bar{a})^2}; \end{aligned} \quad (49)$$

$$\begin{aligned} 0 &= \frac{\partial H(\bar{a}, \cdot)}{\partial \chi(s)} + \lambda \frac{\partial Z(\bar{a}, \cdot)}{\partial \chi(s)} + \mu \left[ \frac{\partial Z(\bar{a}, \cdot)}{\partial \chi(s)} - \frac{\partial Z(\underline{a}, \cdot)}{\partial \chi(s)} \right] \\ &= -\psi_1(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} + \psi_2(s) + v(s); \end{aligned} \quad (50)$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial P_0} \int_{\underline{s}}^{\bar{s}} H(\bar{a}, \cdot) ds + \lambda \frac{\partial Z(\bar{a}, \cdot)}{\partial P_0} + \mu \left[ \frac{\partial Z(\bar{a}, \cdot)}{\partial P_0} - \frac{\partial Z(\underline{a}, \cdot)}{\partial P_0} \right] + \phi \\
&= \int_{\underline{s}}^{\bar{s}} \left\{ -[h(0, s; \bar{a}) - h(b(s), s; \bar{a})] + \lambda [h(0, s; \bar{a}) - h(b(s), s; \bar{a})] \right. \\
&\quad \left. + \mu \{ [h(0, s; \bar{a}) - h(b(s), s; \bar{a})] - [h(0, s; \underline{a}) - h(b(s), s; \underline{a})] \} \right. \\
&\quad \left. + \psi_1(s) \chi(s) \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right\} ds + \phi; \tag{51}
\end{aligned}$$

plus the original equality constraints

$$\zeta'(s) = -\chi(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})}; \tag{52}$$

$$b'(s) = \chi(s); \tag{53}$$

the complementary slackness conditions

$$Z(\bar{a}, \cdot) - \underline{U} \geq 0; \quad \lambda \geq 0; \quad \lambda [Z(\bar{a}, \cdot) - \underline{U}] = 0; \tag{54}$$

$$Z(\bar{a}, \cdot) - Z(\underline{a}, \cdot) \geq 0; \quad \mu \geq 0; \quad \mu [Z(\bar{a}, \cdot) - Z(\underline{a}, \cdot)] = 0; \tag{55}$$

$$P_0 - \underline{P} \geq 0; \quad \phi \geq 0; \quad \phi [P_0 - \underline{P}] = 0; \tag{56}$$

for  $\frac{h_b(b, s; \bar{a})}{h(b, s; \bar{a})}$  strictly increasing in  $s$  for all  $s \in [\underline{s}, \bar{s}]$ :

$$\chi(s) \geq 0; \quad \nu(s) \geq 0; \quad \nu(s) \chi(s) = 0; \tag{57}$$

for  $\frac{h_b(b, s; \bar{a})}{h(b, s; \bar{a})}$  strictly decreasing in  $s$  for all  $s \in [\underline{s}, \bar{s}]$ :

$$\chi(s) \leq 0; \quad \nu(s) \leq 0; \quad \nu(s) \chi(s) = 0; \tag{58}$$

and the boundary conditions

$$\psi_1(\underline{s}) = \psi_2(\underline{s}) = 0; \tag{59}$$

$$\psi_1(\bar{s}) = \psi_2(\bar{s}) = 0. \tag{60}$$

**Proof.** Direct consequence of the Maximum Principle. ■

**Lemma 3** The Maximum Principle conditions (48), (49), (50), (52), and (53) imply

$$\left[ 1 - \frac{\psi_1'(s)}{h(b(s), s; \bar{a})} \right] = \lambda + \mu \left( 1 - \frac{f(s; \underline{a})}{f(s; \bar{a})} \right) - \frac{\psi_1(s) \chi(s) h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a}) h(b(s), s; \bar{a})}; \tag{61}$$

$$\begin{aligned}
&- \pi_b(b(s); s) f(s; \bar{a}) \left[ \frac{\pi(b(s); s)}{\pi_b(b(s); s)} + b(s) \right] + \nu'(s) \\
&= \psi_1(s) [\zeta(s) - P_0] \frac{\partial}{\partial s} \left( \frac{\pi_b(b(s); s)}{\pi(b(s); s)} \right), \text{ for all } s \in [\underline{s}, \bar{s}]. \tag{62}
\end{aligned}$$

**Proof.** Condition (61) follows directly from (48). Conditions (49) can, in view of the definition of  $h(\cdot)$  in (31), be rewritten

$$\begin{aligned} & \frac{h(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} + \frac{\psi'_2(s)}{h_b(b(s), s; \bar{a})} \\ &= -[b(s) - \zeta(s) + P_0] - \left[ \lambda + \mu \left( 1 - \frac{f(s; \underline{a})}{f(s; \bar{a})} \right) \right] [\zeta(s) - P_0] \\ & \quad + \frac{\psi_1(s) \chi(s)}{h(b(s), s; \bar{a})} [\zeta(s) - P_0] \left[ \frac{h_{bb}(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} - \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right]. \end{aligned} \quad (63)$$

Add  $[\zeta(s) - P_0]$  times (61) to (63) noting that, from Proposition 3,  $\zeta(s) \equiv P(b(s)) > P_0$  for all  $s$  and from (53) that  $\chi(s) = b'(s)$ , to get

$$\begin{aligned} & [\zeta(s) - P_0] \left[ 1 - \frac{\psi'_1(s)}{h(b(s), s; \bar{a})} \right] + \frac{h(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} + \frac{\psi'_2(s)}{h_b(b(s), s; \bar{a})} \\ &= \frac{\psi_1(s) b'(s)}{h(b(s), s; \bar{a})} [\zeta(s) - P_0] \left\{ \frac{h_{bb}(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} - \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right. \\ & \quad \left. - \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right\} - [b(s) - \zeta(s) + P_0] \\ &= \psi_1(s) [\zeta(s) - P_0] \frac{b'(s)}{h(b(s), s; \bar{a})} \left\{ \frac{h_{bb}(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} \right. \\ & \quad \left. - 2 \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right\} - [b(s) - \zeta(s) + P_0]. \end{aligned} \quad (64)$$

From (50), it follows that

$$\psi_2(s) = \psi_1(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - v(s) \quad (65)$$

so, differentiating with respect to  $s$ ,

$$\begin{aligned} \psi'_2(s) &= \psi'_1(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - v'(s) \\ & \quad + \psi_1(s) \left\{ [\zeta(s) - P_0] \left[ \left( \frac{h_{bb}(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - \frac{h_b(b(s), s; \bar{a})^2}{h(b(s), s; \bar{a})^2} \right) b'(s) \right. \right. \\ & \quad \left. \left. + \frac{\partial}{\partial s} \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right) \right] + \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \zeta'(s) \right\}. \end{aligned}$$

Substitution for  $\zeta'(s)$  from (52) and use of (53) gives

$$\begin{aligned}
\psi_2'(s) &= \psi_1'(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - v'(s) \\
&\quad + \psi_1(s) [\zeta(s) - P_0] \left\{ \left[ \left( \frac{h_{bb}(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - \frac{h_b(b(s), s; \bar{a})^2}{h(b(s), s; \bar{a})^2} \right) \right. \right. \\
&\quad \left. \left. - \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right)^2 \right] b'(s) + \frac{\partial}{\partial s} \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right) \right\} \\
&= \psi_1'(s) [\zeta(s) - P_0] \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - v'(s) \\
&\quad + \psi_1(s) [\zeta(s) - P_0] \left\{ \left[ \frac{h_{bb}(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right. \right. \\
&\quad \left. \left. - 2 \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right)^2 \right] b'(s) + \frac{\partial}{\partial s} \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right) \right\}.
\end{aligned}$$

Use of this in (64) gives

$$\begin{aligned}
[\zeta(s) - P_0] &\left[ 1 - \frac{\psi_1'(s)}{h(b(s), s; \bar{a})} \right] + \frac{h(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} \\
&+ \frac{\psi_1'(s)}{h(b(s), s; \bar{a})} [\zeta(s) - P_0] - \frac{v'(s)}{h_b(b(s), s; \bar{a})} \\
&= \psi_1(s) [\zeta(s) - P_0] \left\{ \frac{b'(s)}{h(b(s), s; \bar{a})} \left[ \frac{h_{bb}(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} - 2 \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right] \right. \\
&\quad \left. - \frac{1}{h_b(b(s), s; \bar{a})} \left[ \left( \frac{h_{bb}(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - 2 \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right)^2 \right) b'(s) \right. \right. \\
&\quad \left. \left. + \frac{\partial}{\partial s} \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right) \right] \right\} - [b(s) - \zeta(s) + P_0]
\end{aligned}$$

or, re-arranging and cancelling terms,

$$\begin{aligned}
&\frac{h(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} - \frac{v'(s)}{h_b(b(s), s; \bar{a})} \\
&= -b(s) + \frac{\psi_1(s) b'(s)}{h(b(s), s; \bar{a})} [\zeta(s) - P_0] \left\{ \frac{h_{bb}(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} \right. \\
&\quad \left. - 2 \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} - \frac{h_{bb}(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} + 2 \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right\} \\
&\quad - \frac{\psi_1(s)}{h_b(b(s), s; \bar{a})} [\zeta(s) - P_0] \frac{\partial}{\partial s} \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right),
\end{aligned}$$



from which all the terms in  $\psi_1(s) b'(s)$  cancel to give

$$\begin{aligned} \frac{h(b(s), s; \bar{a})}{h_b(b(s), s; \bar{a})} - \frac{v'(s)}{h_b(b(s), s; \bar{a})} \\ = -b(s) - \frac{\psi_1(s)}{h_b(b(s), s; \bar{a})} [\zeta(s) - P_0] \frac{\partial}{\partial s} \left( \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right). \end{aligned}$$

In view of (43), this can be re-arranged to give (62). ■

**Lemma 4** *Suppose Assumption 1 holds. Then, provided the constraint  $b'(s) \geq 0$  ( $b'(s) \leq 0$ ) is not binding in the neighbourhoods of  $\underline{s}, \bar{s}$ , an optimal contract implements  $b(s)$  with the property  $b(s) = b^*(s)$  for  $s = \underline{s}, \bar{s}$ .*

**Proof.** Any solution for  $b(s)$  must satisfy the the Maximum Principle conditions (48)-(60), so Lemma 3 applies. It follows from (59) that  $\psi_1(\underline{s}) = 0$ , so the right-hand side of (62) equals zero for  $s = \underline{s}$ . It follows from (57) that, provided the constraint on  $b'(\underline{s}) [\equiv \chi(\underline{s})]$  is not binding in the neighbourhood of  $\underline{s}$ ,  $v(s) = 0$  in that neighbourhood and thus  $v'(\underline{s}) = 0$  also. A corresponding argument applies for  $s = \bar{s}$ . Since  $\pi_b(b(s); s) = 0$  is ruled out by Assumption 1 and Proposition 3, it must be that, for both  $\underline{s}$  and  $\bar{s}$ , the term in square brackets on the left-hand side of (62) equals zero. It then follows from (26) that  $b(s) = b^*(s)$  for  $s = \underline{s}, \bar{s}$ . ■

**Proof of Proposition 4.** Note that  $\int_{\underline{s}}^{\bar{s}} h(0, s; a) ds = 1$  for all  $a$  because  $h(0, s; a) = f(s; a)$  by definition. Use of this in (51) allows that condition to be re-arranged as

$$\begin{aligned} \left[ 1 - \int_{\underline{s}}^{\bar{s}} h(b(s), s; \bar{a}) ds \right] &= \lambda \left[ 1 - \int_{\underline{s}}^{\bar{s}} h(b(s), s; \bar{a}) ds \right] \\ &\quad - \mu \left[ \int_{\underline{s}}^{\bar{s}} h(b(s), s; \bar{a}) ds - \int_{\underline{s}}^{\bar{s}} h(b(s), s; \underline{a}) ds \right] \\ &\quad + \int_{\underline{s}}^{\bar{s}} \psi_1(s) \chi(s) \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} ds + \phi. \end{aligned} \quad (66)$$

Integration of (48) over  $s \in [\underline{s}, \bar{s}]$  gives

$$\begin{aligned} \int_{\underline{s}}^{\bar{s}} \psi_1'(s) ds &= \int_{\underline{s}}^{\bar{s}} h(b(s), s; \bar{a}) ds - \lambda \int_{\underline{s}}^{\bar{s}} h(b(s), s; \bar{a}) ds \\ &\quad - \mu \int_{\underline{s}}^{\bar{s}} [h(b(s), s; \bar{a}) - h(b(s), s; \underline{a})] ds \\ &\quad + \int_{\underline{s}}^{\bar{s}} \left[ \psi_1(s) \chi(s) \frac{h_b(b(s), s; \bar{a})}{h(b(s), s; \bar{a})} \right] ds. \end{aligned} \quad (67)$$

Now  $\int_{\underline{s}}^{\bar{s}} \psi_1'(s) ds = \psi_1(\bar{s}) - \psi_1(\underline{s})$  and, from (59) and (60),  $\psi_1(\bar{s}) = \psi_1(\underline{s}) = 0$ . Thus, the left-hand side of (67) equals zero. Use of that in (66) gives

$$\lambda = 1 - \phi. \quad (68)$$

Consider first values of  $s$  for which  $\psi_1(s) = 0$ . By Lemma 2, any solution for  $b(s)$  must satisfy the Maximum Principle conditions (48)-(60), so Lemma 3 applies. By Lemma 3, (61) must hold. Use of (68) in (61) for any such  $s$  allows (61) to be written

$$-\frac{\psi_1'(s)}{h(b(s), s; \bar{a})} = -\phi + \mu \left[ 1 - \frac{f(s; \underline{a})}{f(s; \bar{a})} \right]. \quad (69)$$

Since  $s$  is ordered such that  $f(s; \underline{a}) / f(s; \bar{a})$  is non-increasing in  $s$ , there exists  $\hat{s}$  such that

$$\frac{f(s; \underline{a})}{f(s; \bar{a})} \geq 1 \text{ for } s \leq \hat{s}; \quad \frac{f(s; \underline{a})}{f(s; \bar{a})} < 1 \text{ for } s > \hat{s}.$$

Note that  $\phi, \mu > 0$  when the lower bound on  $P_0$  is binding. Thus the right-hand side of (69) is negative for  $s = \underline{s}$  and increasing in  $s$ . So, either the right-hand side of (69) is negative for all  $s$  or there exists an  $\bar{s} > \hat{s}$  such that it is negative for  $s \leq \bar{s}$  and positive for  $s > \bar{s}$ . It follows either that, for any  $s \leq \bar{s}$  for which  $\psi_1(s) = 0$ ,  $\psi_1'(s) > 0$  and, for any  $s > \bar{s}$  for which  $\psi_1(s) = 0$ ,  $\psi_1'(s) < 0$ , or that  $\psi_1'(s) > 0$  for all  $s$  for which  $\psi_1(s) = 0$ . The second of these cannot, however, be the case because we know from (59) and (60) that  $\psi_1(\underline{s}) = \psi_1(\bar{s}) = 0$ . Thus, as  $s$  increases from  $\underline{s}$ ,  $\psi_1(s)$  becomes positive and cannot change sign because, to do so,  $\psi_1'(s)$  would have to become negative and, since  $\psi_1'(s)$  would remain negative for all higher  $s$ ,  $\psi_1(s)$  would not be able to satisfy  $\psi_1(\bar{s}) = 0$ . Thus  $\psi_1(s) > 0$  for all  $s \in (\underline{s}, \bar{s})$ .

From Lemma 3, (62) must hold. When  $\psi_1(s) > 0$  and  $\zeta(s) > P_0$  for all  $s \in (\underline{s}, \bar{s})$ , the right-hand side of (62) is positive if  $\pi_b(b; s) / \pi(b; s)$  is everywhere strictly increasing in  $s$  and negative if  $\pi_b(b; s) / \pi(b; s)$  is everywhere strictly decreasing in  $s$ . The term multiplying the square bracket on the left-hand side is positive under Assumption 1. Thus, provided the constraint on  $b'(s) [\equiv \chi(s)]$  is not binding for any  $s$ , so that  $v(s) = 0$  for all  $s$  from (57) and thus also  $v'(s) = 0$  for all  $s$ , it must be that

$$\begin{aligned} \frac{\pi(b(s); s)}{\pi_b(b(s); s)} + b(s) &> 0 \text{ for } s \in (\underline{s}, \bar{s}) \text{ if } \frac{\partial}{\partial s} \left( \frac{\pi_b(b; s)}{\pi(b; s)} \right) > 0 \text{ for all } s; \\ \frac{\pi(b(s); s)}{\pi_b(b(s); s)} + b(s) &< 0 \text{ for } s \in (\underline{s}, \bar{s}) \text{ if } \frac{\partial}{\partial s} \left( \frac{\pi_b(b; s)}{\pi(b; s)} \right) < 0 \text{ for all } s. \end{aligned}$$

But  $\pi_b(b; s) / \pi(b; s)$  is non-increasing in  $b$ , so the left-hand sides of both these are increasing in  $b(s)$ . It follows from (26) that  $b(s) > b^*(s)$  for  $s \in (\underline{s}, \bar{s})$  if  $\pi_b(b; s) / \pi(b; s)$  is everywhere strictly increasing in  $s$  and  $b(s) < b^*(s)$  for  $s \in (\underline{s}, \bar{s})$  if  $\pi_b(b; s) / \pi(b; s)$  is everywhere strictly decreasing in  $s$ .

That establishes the result for all  $s \in (\underline{s}, \bar{s})$  provided the constraint on  $b'(s)$  does not bind for any  $s$  (and hence  $v(s) = v'(s) = 0$  for all  $s$ ). To show that the result is unaffected by having that constraint bind for some  $s$ , start with the case  $\pi_b(b; s) / \pi(b; s)$  everywhere strictly increasing in  $s$  in which that constraint takes the form  $b'(s) \geq 0$ . Consider first  $s = \bar{s}$ . It follows from Lemma 4 that, if the constraint did not bind,

$b(\bar{s}) = b^*(\bar{s})$ . Thus the constraint can bind only if  $b(\bar{s} - \varepsilon) > b^*(\bar{s})$  as  $\varepsilon \rightarrow 0$  from above and the effect of the constraint cannot be to reduce  $b(\bar{s})$ . Thus certainly  $b(\bar{s}) \geq b^*(\bar{s})$ . Moreover, by Part 3 of Proposition 2,  $b^*(s)$  is strictly increasing so, if  $b(\bar{s} - \varepsilon) = b(\bar{s}) \geq b^*(\bar{s})$ , it is certainly the case that  $b(\bar{s} - \varepsilon) > b^*(\bar{s} - \varepsilon)$ . In addition, for any  $s$  such that  $b(s) > b^*(s)$ , a similar argument implies  $b(s - \varepsilon) > b^*(s - \varepsilon)$  as  $\varepsilon \rightarrow 0$  from above even when the constraint binds. Thus  $b(s) > b^*(s)$  for all  $s \in (\underline{s}, \bar{s})$  whether or not the constraint binds. Finally,  $b(\underline{s} + \varepsilon) > b^*(\underline{s} + \varepsilon)$  as  $\varepsilon \rightarrow 0$  from above implies  $b(\underline{s} + \varepsilon) > b^*(\underline{s})$ , so having  $b(\underline{s}) = b^*(\underline{s})$ , as implied by Lemma 4 if the constraint does not bind, cannot result in the constraint binding. Thus  $b(\underline{s}) = b^*(\underline{s})$ . That completes the proof of Part 1 of the proposition.

Now consider  $\pi_b(b; s) / \pi(b; s)$  everywhere strictly decreasing in  $s$  for which the constraint on  $b'(s)$  takes the form  $b'(s) \leq 0$ . Again from Lemma 4, if the constraint does not bind at  $s = \bar{s}$ , then  $b(\bar{s}) = b^*(\bar{s})$ . Thus the constraint can bind at  $s = \bar{s}$  only if  $b(\bar{s} - \varepsilon) < b^*(\bar{s})$  as  $\varepsilon \rightarrow 0$  from above and the effect of the constraint cannot be to increase  $b(\bar{s})$ . Thus certainly  $b(\bar{s}) \leq b^*(\bar{s})$ . Moreover, by Part 3 of Proposition 2,  $b^*(s)$  is strictly decreasing so, if  $b(\bar{s} - \varepsilon) = b(\bar{s}) \leq b^*(\bar{s})$ , it is certainly the case that  $b(\bar{s} - \varepsilon) < b^*(\bar{s} - \varepsilon)$ . In addition, for any  $s$  such that  $b(s) < b^*(s)$ , a similar argument implies  $b(s - \varepsilon) < b^*(s - \varepsilon)$  as  $\varepsilon \rightarrow 0$  from above even when the constraint binds. Thus  $b(s) < b^*(s)$  for all  $s \in (\underline{s}, \bar{s})$  whether or not the constraint binds. Finally,  $b(\underline{s} + \varepsilon) < b^*(\underline{s} + \varepsilon)$  as  $\varepsilon \rightarrow 0$  from above implies  $b(\underline{s} + \varepsilon) < b^*(\underline{s})$ , so having  $b(\underline{s}) = b^*(\underline{s})$ , as implied by Lemma 4 if the constraint does not bind, cannot result in the constraint binding. Thus  $b(\underline{s}) = b^*(\underline{s})$ . That completes the proof of Part 2. ■

The following lemma is useful for proving Proposition 5.

**Lemma 5** Suppose  $b(s)$  satisfies, for  $s, s', s'' \in [\underline{s}, \bar{s}]$ ,

$$\underline{P} + [P(b(s')) - \underline{P}] \pi(b(s'); s') \geq \underline{P} + [P(b(s'')) - \underline{P}] \pi(b(s''); s'); \quad (70)$$

$$\underline{P} + [P(b(s'')) - \underline{P}] \pi(b(s''); s'') \geq \underline{P} + [P(b(s)) - \underline{P}] \pi(b(s); s''). \quad (71)$$

Then, for  $\pi(b(s''); s''), \pi(b(s); s'') > 0$  and either  $s \leq s'' \leq s'$  or  $s' \leq s'' \leq s$ ,

$$\underline{P} + [P(b(s')) - \underline{P}] \pi(b(s'); s') \geq \underline{P} + [P(b(s)) - \underline{P}] \pi(b(s); s') \quad (72)$$

if  $b(s)$  and  $\pi_b(b; s) / \pi(b; s)$  are either both non-decreasing in  $s$  or both non-increasing in  $s$  for all  $s \in [\underline{s}, \bar{s}]$  and all  $b \in [0, \bar{b}]$ .

**Proof.** The two inequalities hypothesised in the lemma respectively imply

$$\begin{aligned} [P(b(s')) - \underline{P}] \pi(b(s'); s') &\geq [P(b(s'')) - \underline{P}] \pi(b(s''); s'), \\ [P(b(s'')) - \underline{P}] &\geq [P(b(s)) - \underline{P}] \frac{\pi(b(s); s'')}{\pi(b(s''); s'')}. \end{aligned}$$

Together these imply

$$[P(b(s')) - \underline{P}] \pi(b(s'); s') \geq [P(b(s)) - \underline{P}] \frac{\pi(b(s''); s')}{\pi(b(s''); s'')} \pi(b(s); s'').$$

Since  $P(b(s)) \geq \underline{P}$ , the lemma thus certainly holds if

$$\frac{\pi(b(s''); s')}{\pi(b(s''); s'')} \pi(b(s); s'') \geq \pi(b(s); s'), \text{ for } s \leq s'' \leq s' \text{ and } s' \leq s'' \leq s,$$

or, equivalently,

$$\frac{\pi(b(s''); s')}{\pi(b(s''); s'')} \geq \frac{\pi(b(s); s')}{\pi(b(s); s'')}, \text{ for } s \leq s'' \leq s' \text{ and } s' \leq s'' \leq s. \quad (73)$$

This clearly holds with equality for  $s = s''$ . But

$$\begin{aligned} & \frac{d}{ds} \left( \frac{\pi(b(s); s')}{\pi(b(s); s'')} \right) \\ &= \frac{b'(s)}{\pi(b(s); s'')^2} [\pi(b(s); s'') \pi_b(b(s); s') - \pi(b(s); s') \pi_b(b(s); s'')] \\ &= b'(s) \frac{\pi(b(s); s')}{\pi(b(s); s'')} \left[ \frac{\pi_b(b(s); s')}{\pi(b(s); s')} - \frac{\pi_b(b(s); s'')}{\pi(b(s); s'')} \right], \end{aligned}$$

which is non-positive for  $s' \leq s''$  and non-negative for  $s'' \leq s'$  if  $\pi_b(b; s) / \pi(b; s)$  and  $b(s)$  are either both non-decreasing in  $s$  or both non-increasing in  $s$  for all  $s \in [\underline{s}, \bar{s}]$  and all  $b \in [0, \bar{b}]$ . This implies (73). ■

**Proof of Proposition 5.** Under Assumption 2, the decision rule  $b(\cdot)$  to be implemented always satisfies  $\pi(b(s); s) = 1$ . Thus a necessary condition for  $b(s)$  to satisfy (22) over  $b \geq b(s)$  for signal  $s$  is that

$$P'(b(s)) + [P(b(s)) - \underline{P}] \pi_{b_+}(b(s); s) \leq 0, \quad \text{for all } s \in [\underline{s}, \bar{s}]. \quad (74)$$

This necessary condition can be re-written as (23). An implication of Lemma 5 is the following. Consider  $b$  not to be selected for any  $s$ . If there exists  $b(s) < b$  for some  $s$ , let  $\hat{b}(b) = \max b(s)$  over  $s$  such that  $b(s) < b$ . If not, let  $\hat{b}(b) = 0$ . Let  $P(\cdot)$  be set such that  $P(b) = P(\hat{b}(b))$ , as a result of which no such  $b$  is chosen for any  $s$ . Then if, under the conditions given,  $P(\cdot)$  is such that the agent prefers  $b(s')$  to  $b(s'')$  given signal  $s'$  and  $b(s'')$  to  $b(s)$  given signal  $s''$  for any signal  $s$  the opposite side of  $s''$  from  $s'$ , the agent also prefers  $b(s')$  to  $b(s)$  given signal  $s'$ . Letting  $s''$  approach  $s'$  establishes that, under the conditions of Lemma 5, a local optimum is also a global optimum. As a result, (74) (and hence, (23)) is not only necessary but also sufficient to ensure that no  $b > b(s)$  is preferred to  $b(s)$  when  $b(s)$  and  $\pi_b(b; s) / \pi(b; s)$  are either both non-

decreasing in  $s$  or both non-increasing in  $s$  for all  $s \in [\underline{s}, \bar{s}]$  and all  $b \in [0, \bar{b}]$ . From the argument in the text, we already know that no  $b < b(s)$  is preferred to  $b(s)$  if  $P(\cdot)$  is non-decreasing. ■

**Proof of Proposition 6.** By Proposition 5, (23) must hold. For  $b(s)$  increasing, the constraint (23) necessarily binds for all  $b$  by the argument in the text. Define  $z(b) = P(b) - \underline{P}$ . Then, with  $s = b^{-1}(b)$ , (23) can be written

$$z'(b) \leq -z(b) \pi_{b_+}(b; b^{-1}(b)), \text{ for all } b \in [b(\underline{s}), b(\bar{s})],$$

which, when binding for all  $b$ , can be re-written for  $z(b) > 0$

$$\frac{z'(b)}{z(b)} = -\pi_{b_+}(b; b^{-1}(b)), \text{ for all } b \in [b(\underline{s}), b(\bar{s})].$$

This has solution

$$z(b) = K \exp \left[ - \int_{b(\underline{s})}^b \pi_{b_+}(b; b^{-1}(b)) db \right], \text{ for all } b \in [b(\underline{s}), b(\bar{s})],$$

with  $K = z(b(\underline{s}))$ . Translating that solution back into the original notation establishes the result. ■

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