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# Documenting Riemann's Impact on the Theory of Complex Functions

*Along with Euler, Gauss, and Hilbert, Bernhard Riemann is one of the most illustrious mathematicians of all time. His prominence in the field of complex analysis may be appreciated simply by noting that in the* Mathematisches Wörterbuch by Naas-Schmid [1961, Vol. 2, 510–524], the entries related to Riemann's function-theoretical work (Riemann mapping theorem, Riemann differential equation, Riemann surface, Riemann-Roch theorem, Riemann theta-function, Riemann sphere, Riemann zeta-function, etc.) take up almost as much space as those related to all the work of Euler or Gauss in total. Riemann's contribution to complex analysis rests, on the one hand, on his publications, especially his inaugural dissertation on the foundations of complex analysis (1851) and his articles on the theory of hypergeometric and Abelian functions (1856–57), but his several lecture courses in this field are also very important.

These were given in the years from 1855 to 1862, and regularly began with an introductory general part. Riemann then turned either to the theory of elliptic and Abelian functions or to hypergeometric series and related transcendental functions. Much of the more advanced parts of Riemann's courses were published in his Collected Papers [Riemann 1990, 599–692] and in a book by H. Stahl [Riemann 1899], but not his introductory lectures on general complex analysis. Thus the latter gradually fell into oblivion, despite their intrinsic interest, and despite their decisive influence on later developments through the closely related writings of Durège, Hankel, Koenigsberger, Neumann, Prym, Roch, and Thomae. It therefore seemed appropriate to publish them in a critical edition [Neuenschwander 1996], in order to make them accessible to a broader circle of readers. For this edition, I also prepared an extensive bibliography of the history of the impact and influence of Riemann's function theory.

This newly assembled bibliography is mainly intended to close—at least in the field of complex analysis—the considerable gap existing between the bibliographies of Purkert and Neuenschwander, which were appended to the reprint of *Bernhard Riemann's Gesammelte Mathematische Werke* [Riemann 1990]. The period from 1892 to 1944, not systematically covered there, was scrutinized, using the indexes of names in *Bibliotheca Mathematica* (1887–1914), and the *Revue semestrielle des Publications mathématiques* (1893–1934/35), and examining the sections on "Geschichte und Philosophie" and "Funktionentheorie" (or "Analysis") in the abstracting journal *Jahrbuch über die Fortschritte der Mathematik* (1868/71–1942/44). Additionally, I consulted the holdings of older books in the libraries of the Institutes of Mathematics at the Universities of Göttingen and Zürich. Generally, I included only publications which contained *explicit* reference to Riemann's work (e.g., quotations with exact indication of location), but even this criterion left more than 1,000 out of those 8,000 titles provisionally selected on the basis of the reviews and the systematic library inspection, the originals were then looked up, to make sure that they met the conditions for inclusion

Naturally, the new bibliography is in no way comprehensive The literature which refers to Riemann's pioneering work is almost boundless Purkert, for example, examining about ten journals from the first 25 years after the death of Bernhard Riemann, already found more than 500 publications referring to his work According to database surveys, the continuation and extension of Purkert's research up to the present would have to take into account about 30,000 publications, all of which obviously cannot be examined individually within a reasonable time Nevertheless, I hope that the new bibliography will become a useful tool for further investigations

In the following I will try to illustrate some of its possible applications, by surveying the impact of Riemann's function-theoretical work in the four most important European countries Germany, France, Italy, and Great Britain Special attention will be given to the developments in Great Britain because they have not yet been analysed in detail For more specific information, the reader may turn to the bibliography itself

## Germany: Early and Sustained Reception of Riemann's Methods

For a preliminary impression of the situation in Germany, let us first look at the references to Riemann's work in August Leopold Crelle's influential *Journal fur die reine und angewandte Mathematik* It is noteworthy that the first of these references goes back to Helmholtz [*Crelle* **55** (1858), 25–55], who, as we know, later discussed Riemann's hypotheses concerning the foundations of geometry Helmholtz was followed in chronological order by Lipschitz, Clebsch, Christoffel, Schwarz, Brill, Fuchs, Gordan, Luroth, and Weber, all of whom, even though they were not Riemann's immediate pupils, did a great deal to disseminate his ideas, as did his own students Roch, Thomae, and Prym [*Crelle* **61** (1863)–**70** (1869)] Clebsch and Brill—like Klein and Noether—published their later work primarily in the *Mathematische Annalen*, which started to appear in 1869, thus, consulting Crelle's *Journal* alone provides only incomplete results for them According to our bibliography, other important early promoters of Riemann's ideas in the German-speaking world were Cantor, Dedekind, Du Bois-Reymond, Durège, Hankel, Koenigsberger, Neumann, Schlafli, and, in his later years, Schottky [1]

## Italy: Enthusiastic Appreciation of Riemann's Work

An impressive picture of the extent to which Riemann's work was appreciated in Italy is given by a similar study of the references to his writings in the *Annali di Matematica pura ed applicata*, a journal which played an outstanding role in the dissemination of Riemann's thoughts As early as 1859, Enrico Betti, who was to become Riemann's friend, translated his dissertation [*Annali* **2** (1859), 288–304, 337–356], to which he soon returned in an extensive article on the theory of elliptic functions [*Annali* **3** (1860), 65–159, 298–310, **4** (1861), 26–45, 57–70, 297–336] In the same volumes, we also find a report by Betti on Riemann's treatise concerning the propagation of planar air waves [*Annali* **3** (1860), 232–241], and a report by Angelo Genocchi on Riemann's investigation of the number of primes less than a given bound [*Annali* **3** (1860), 52–59] In a later volume of the same journal [*Annali* (2) **3** (1869–70), 309–326], we find a French translation of Riemann's hypotheses on the foundations of geometry by the French mathematician Jules Houel, who typically did not publish his translation in a French journal, but in the *Annali* We should also mention Eugenio Beltrami and Felice Casorati, the latter having already presented Riemann's theories in 1868 in a book [Casorati 1868] and in special lectures given in Milan [Armenante & Jung 1869, Casorati & Cremona 1869] In view of this excellent introduction and trail-blazing, it is not surprising that Riemann's theories were widely known in Italy, and that they were quoted in more than thirty articles in the *Annali* alone up to 1890 [2] As to Riemann's own stays in Italy, and other followers of Riemann there, see the articles of Bottazzini, Dieudonné, Loria, Neuenschwander, Schering, Tricomi, Volterra, and Weil cited in the bibliography

## France: Hesitant Reception

In France, the situation was radically different Skimming through the pages of the *Journal de Mathématiques pures et appliquées* edited by Joseph Liouville, one finds almost no references to Riemann's papers before 1878, and in other French publications up to 1880 they also seem to be relatively sparse Furthermore, a certain critical reserve as to the usefulness of Riemann's methods quite often comes through Briot and Bouquet, for instance, write in the foreword to the second edition of their *Théorie des fonctions elliptiques* [Briot & Bouquet 1875, I f ]

*In Cauchy's theory, the path of the imaginary* [*complex*] *variable is characterized by the movement of a point on a plane To represent those functions which assume several values for the same value of the variable, Riemann regarded the plane as formed of several sheets, superimposed and welded together, in order to allow the variable to pass from one sheet to another, while traversing a connecting* [*branch*] *line The concept of a many-sheeted surface presents some difficulties, despite the beautiful results which Riemann achieved by this method, it did not seem to us to offer any advantage for our own objective Cauchy's idea is very suitable for representing multivalued functions, it is sufficient to join to the value of the variable the corresponding value of the function, and*

[1] For bibliographical details on particular articles by these authors citing Riemann see [Neuenschwander 1996 131–232]

[2] The authors of other articles in the *Annali* containing references to Riemann are Ascoli Beltrami Casorati Cesaro Christoffel Dini Lipschitz Pascal Schlafli Schwarz Tonelli Volterra etc For details see [Neuenschwander 1996]

*when the variable has described a closed curve and the value of the function has thereby changed, to indicate this change by an index*

Only Houel, mentioned above, tried quite early to promote Riemann's ideas, and, together with Gaston Darboux, he several times deplored their not being better known in France [Gispert 1985, 386–390 *et passim*]. A review of Houel's pioneering book in this field [Houel 1867–1874, parts 1 and 2 (1867/68)] concludes with the following, quite instructive passage [*Nouvelles Annales de Mathematiques* (2) **8** (1869), 136–143]:

*May the reception of this work encourage the author to keep his promise to give us, in a* third part, *an exposition of Riemann's theories, which up to now have been almost unknown in our country, and which our neighbours cultivate with so much ardour and success*[3]

In spite of Houel's efforts, the situation in France seems to have changed fundamentally only after 1880, when a new generation of French mathematicians took over, the most prominent among them were Henri Poincare (1854–1912), Paul Appell (1855–1930), Emile Picard (1856–1941), and Édouard Goursat (1858–1936). With these authors the reception of the works of Riemann's followers (mainly Clebsch and Fuchs) now began to lead to a more thorough reading of Riemann's own writings—as numerous quotations in their publications prove. As a manifestation of the increasing appreciation of Riemann's methods among French mathematicians at that time, we may take the fact that, in 1882, Georges Simart wrote a dissertation of 123 pages, now almost completely forgotten, in which he tried to acquaint French mathematicians with Riemann's function-theoretical treatises. In Simart's introduction, we find the following statement, which confirms my own assessment [Simart 1882, 1]:

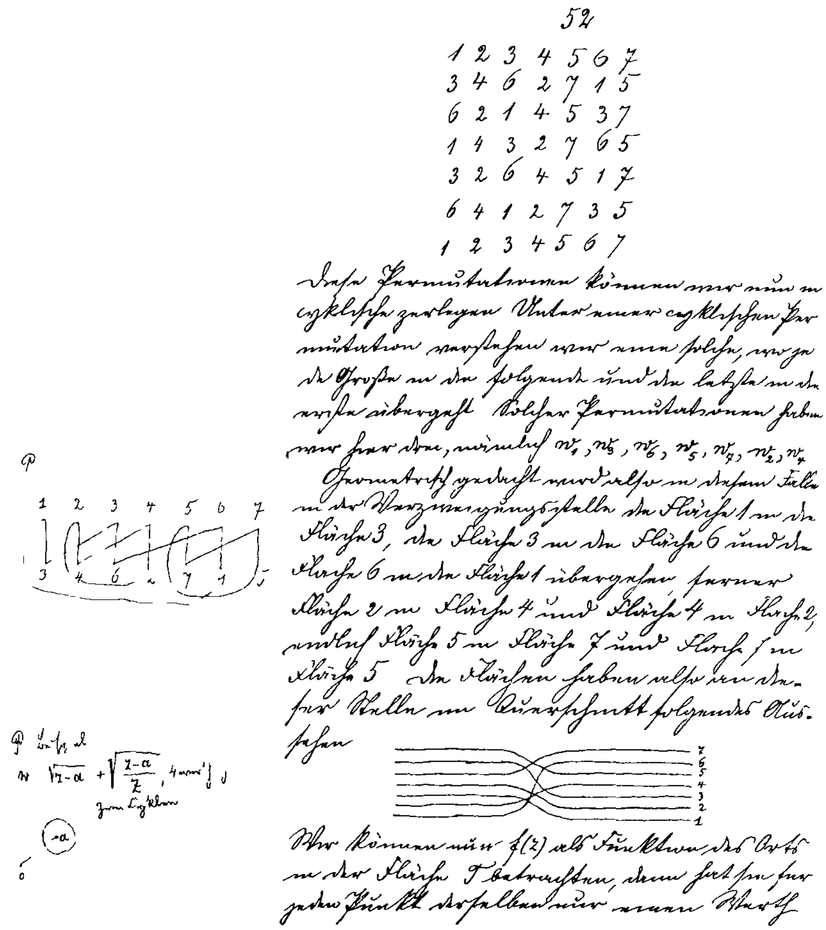

**Figure 1** Discussion and representation of a branch point of an algebraic function on a Riemann surface, taken from Riemann's lectures in the summer semester of 1861. The reproduced page comes from a set of lecture notes made by Eduard Schultze with annotations by Hermann Amandus Schwarz; it is kept in the Schwarz Nachlass at the Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften in Berlin. There exists a similar representation in the (probably) original notes by Schultze which are now kept in the Rare Book and Manuscript Library at Columbia University in New York. Both manuscripts were discovered by the author, in 1979 and 1986, respectively. For further information and a transcription of the text, see Neuenschwander 1988 and 1996, pp. 75 f. and 84, MS *S* and $S_s$.

*We know the magnificent results which Riemann arrived at in his two Memoirs concerning the general theory of analytic functions and the theory of Abelian functions, but the methods which he used—and explained perhaps too succinctly—are not well known in France [. . .] At the same time, however, Riemann's method continues to have many adherents in Germany; it serves as a basis for many important publications by famous geometers, such as Koenigsberger, Carl Newmann [sic], Klein, Dedekind, Weber, Prym, Fuchs, etc. The reading of these Memoirs requires a knowledge of Riemannian surfaces [and thus, finally, of Riemann's own writings], the use of which has become standard at certain German universities*

## Great Britain: A Forgotten Interest

In contrast to what is known about the reception of Riemann's theories in the three countries already surveyed, their reception in Great Britain is little studied. Let me therefore provide a more detailed account.

In Great Britain Riemann's theories seem to have become known and more widely disseminated only after his death. One of the first British mathematicians who cited Riemann frequently was Arthur Cayley, who, already in 1865/66, mentioned the investigations of Clebsch and

[3]Other scattered early citations of Riemann's writings are to be found in Bertrand, Darboux, Elliot, Emmanuel, Hermite, Jonquieres, Jordan, Marie and Tannery. For further information, see [Neuenschwander 1996].

Riemann on Abelian Integrals in his papers on the transformation and higher singularities of a plane curve. Cayley also discussed Riemann's work in later years in great detail and with appreciation. This can be seen from his note on Riemann's posthumously published early student paper on generalized integration and differentiation (1880), or from his presidential address to the British Association for the Advancement of Science (1883). Other early citations of Riemann's papers in Great Britain are by W. Thomson (1867 ff.) and J. C. Maxwell (1869 ff.). They occur in connection with Helmholtz's papers on vortex motion and with Listing's studies on topology.

A few years later, W. K. Clifford, H. J. S. Smith, and J. W. L. Glaisher also thoroughly analysed Riemann's papers and, on several occasions, emphasized their great importance. Clifford, for example, as early as 1873, translated Riemann's famous paper on the hypotheses which lie at the base of geometry into English for the journal *Nature*, and in 1877 he published his influential memoir on the canonical form and dissection of a Riemann surface. In November 1876 Smith, in his presidential address *On the Present State and Prospects of some Branches of Pure Mathematics* to the London Mathematical Society, reported in detail on Riemann's achievements, taking into account as well the newly published fragments in Riemann's Collected Mathematical Works. Smith's elaborate address seems to mark an initial highpoint in the appreciation of Riemann's works in Great Britain. Because it has not yet been studied in this context, I will treat it here in more detail.

At the beginning of his address Smith outlined recent progress in number theory and in particular the investigations on the number of primes less than a given bound. He wrote [Smith 1876–77, 16–18]:

*As to our knowledge of the series of the prime numbers themselves, the advance since the time of Euler has been great, if we think of the difficulty of the problem, but very small if we compare what has been done with what still remains to do. We may mention, in the first place, the undemonstrated, and indeed conjectural, theorems of Gauss and Legendre as to the asymptotic value of the number of primes inferior to a given limit $x$ [...] The memoir of Bernhard Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse," contains (so far as I am aware) the only investigation of the asymptotic frequency of the primes which can be regarded as rigorous [...] No less important than the investigation of Riemann, but approaching the problem of the asymptotic law of the series of primes from a different side, is the celebrated memoir, "Sur les Nombres Premiers," by M. Tchébychef [...] The method of M. Tchébychef, profound and inimitable as it is, is in point of fact of a very elementary character, and in this respect contrasts strongly with that of Riemann, which depends throughout on very abstruse theorems of the Integral Calculus.*

Smith then went on to speak of some branches of analysis which appeared to him to promise much for the immediate future. Here he focussed on the advancement of the "Integral Calculus." He mentioned, among other works, Riemann's memoir on the hypergeometric series and the unfinished memoir on linear differential equations with algebraic coefficients. In doing so he stressed the "great beauty and originality" of Riemann's reasoning and the "fertility of the conceptions of Cauchy and of Riemann." In his closing remarks Smith described the work of English mathematicians in the field during the last ten years, naming among others Glaisher, Cayley, and Clifford. He was convinced that nothing so hindered the progress of mathematical science in England as the want of advanced treatises on mathematical subjects, and

*that there are at least three treatises which we greatly need—one on Definite Integrals, one on the Theory of Functions in the sense in which that phrase is understood by the school of Cauchy and of Riemann, and one (though he should be a bold man who would undertake the task) on the Hyperelliptic and Abelian Integrals.*

Smith called on his colleagues to close this gap, and to some extent they did.[4]

As early as about 1871 the British Association for the Advancement of Science set up a special Committee on Mathematical Tables to which belonged, besides Smith, A. Cayley, G. G. Stokes, W. Thomson, and J. W. L. Glaisher. The purposes for which the Committee was appointed were (1) to form as complete a catalogue as possible of existing mathematical tables, and (2) to reprint or calculate tables necessary for the progress of the mathematical sciences. The Committee decided to begin with the first task, and in 1873, presented by the care of Glaisher a huge catalogue of 175 pages on mathematical tables which was printed in

[4] In a letter addressed to I. Todhunter, Smith gives a similar, more direct and succinct estimate of the present state of Mathematics in Great Britain, France, and Germany. It proves once again how highly he regarded the work of Riemann and Weierstrass: "But I so heartily agree with much, or rather with most of your book [*Conflict of Studies* 1873] that I should not have troubled you with this letter, if it were not that I cannot wholly subscribe to your estimate of the present state of Mathematics. All that we have, one may say, comes to us from Cambridge, for Dublin has not of late quite kept up the promise she once gave. Further, I do not think that we have anything to blush for in a comparison with France; but France is at the lowest ebb, is conscious that she is so, and is making great efforts to recover her lost place in Science.

Again, in Mixed Mathematics, I do not know whom we need fear. Adams, Stokes, Maxwell, Tait, Thomson will do to put against any list, even though it may contain Helmholtz and Clausius.

But in Pure Mathematics I must say that I think we are beaten out of sight by Germany, and I have always felt that the *Quarterly Journal* is a miserable spectacle as compared with Crelle ['s *Journal*] or even Clebsch and Neumann [*Mathematische Annalen*]. Cayley and Sylvester have had the lion's share of the modern Algebra (but even in Algebra the whole of the modern theory of equations, substitutions, etc., is French and German). But what has England done in Pure Geometry, in the Theory of Numbers, in the Integral Calculus? What a trifle the symbolic methods, which have been developed in England, are compared with such work as that of Riemann and Weierstrass!" [H. J. S. Smith, *Collected Mathematical Papers*, Introduction, Vol. 1, lxxxv–lxxxvi]

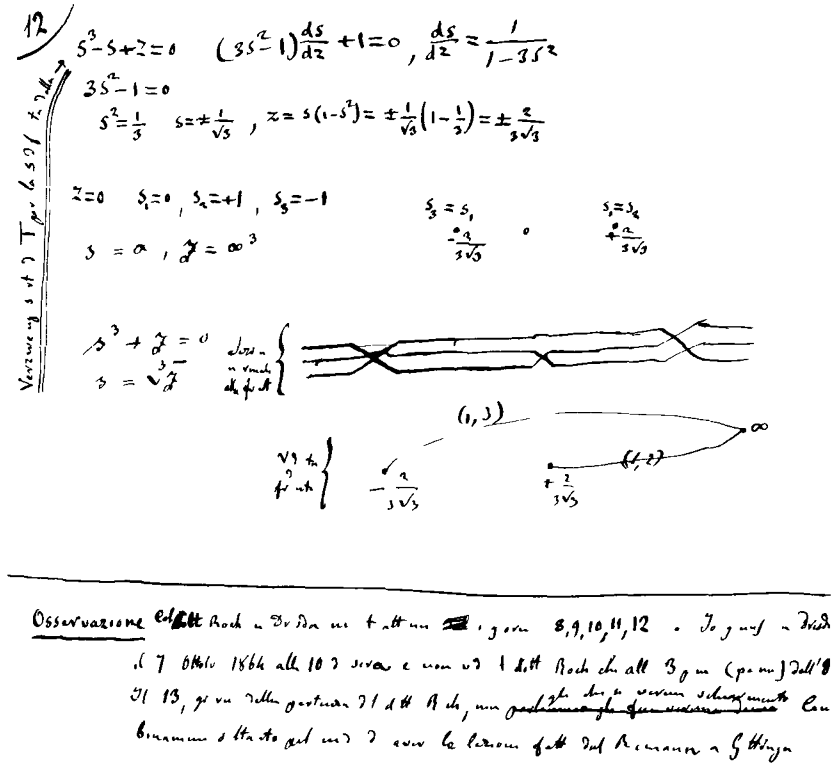

**Figure 2** **Discussion and representation of the branch points of the Riemann surface defined by the equation $s^3 - s + z = 0$ from Casorati's notes of conversations with Gustav Roch in Dresden in October 1864 The notes comprise 12 numbered pages At the end they give some information on his encounter with Roch (October 8–13) and Casorati's wishes to have a copy of Riemann's lectures The last page reproduced above from Casorati's notes of conversations with Roch contains one of the first schematic representations of a Riemann surface in cross-section ("Sezione normale alle fronte") and viewed from above ("Veduta di fronte") outside Riemann's own manuscripts Such representations later became very widespread, as can be seen from similar drawings in Neumann 1865, Houel 1867–1874 Clifford 1877, Bobek 1884, Amstein 1889, etc This way of representing a Riemann surface goes back in fact to Riemann himself, as can now be inferred from page 52 from Schultze's lecture notes, which is published here for the first time (Casorati's conversation notes were discovered, studied, and in part edited by the author during research in Pavia from Spring 1976 onwards ) For further information on Casorati's notes and the Casorati Nachlass, see Neuenschwander 1978, especially pp 4, 19, 73 ff**

the annual *Report of the British Association* In 1875 a continuation of this report was presented, compiled by Cayley It included, under the heading "Art 1 Divisors and Prime Numbers," new additional references to the table of the frequency of primes in Gauss's *Collected Works* and to the related approximate formulae by Gauss and Legendre Both reports contained *no* reference to Riemann's famous paper on prime numbers It is only mentioned in later *Association Reports* on Mathematical Tables (1877 ff ), and in a series of papers on the enumeration of the primes and on factor tables which Glaisher presented to the Cambridge Philosophical Society, the first one being read on December 4, 1876 In these papers Glaisher thanked Smith for his help on the relevant literature, and in a note referred explicitly back to Smith's above-mentioned address to the London Mathematical Society in November 1876 for specific references

Further details on the very intense later scientific interchange between Smith and Glaisher and their common admiration for Riemann can be gathered from Glaisher's Introduction to his edition of Smith's *Collected Mathematical Papers* and from Smith's papers themselves [5] From 1877 onwards J W L Glaisher's father James was engaged in the project of the construction of factor tables of numbers from 3,000,000 up to 6,000,000, and he once again documented thereby the great superiority of Riemann's formula for the numbers of primes as compared with those of Legendre and Tchebycheff James Glaisher's tables were cited in their turn by various mathematicians in Scandinavian countries (Oppermann 1882 f , Gram 1884 ff , Lorenz 1891), where Riemann's function-theoretical work had already excited interest before, as is shown by the publications of Bonsdorff, Mittag-Leffler, and some others Moreover, in 1884 W W Johnson published a detailed account of James Glaisher's factor tables and the distribution of primes in the American journal *Annals of Mathematics*, which introduced Riemann's investigations and the work of the British mathematicians in the New World

Finally, attention should be paid to Andrew Russell Forsyth, who went to live in the town of Cambridge and studied nothing but mathematics in the same year, 1876, that Smith and Glaisher started to draw attention to Riemann's work In his obituary notice, E T Whittaker described Forsyth's *Theory of Functions* (1893) as having had a greater influence on British mathematics than any work since Newton's *Principia* According to Whittaker [1942, 218], perhaps the most original feature of this work was the melding of the three methods associated with the names of Cauchy, Riemann, and Weierstrass, which in the continental books were regarded as separate branches of mathematics Furthermore, one can read in the Royal Society Biographical Memoir of Henry Frederick Baker by W V D Hodge that Forsyth "rendered Cambridge mathematics great service by his efforts to bring about closer co-operation between mathematicians in this country and those on the continent of Europe, and he made it easier for his

[5] In addition to Smith s London address discussed above see also his paper *On the integration of discontinuous functions* (1875) with a detailed analysis of Riemann s memoir on the representation of a function by a trigonometric series his paper *On some discontinuous series considered by Riemann* (1881) and his *Memoir on the Theta and Omega Functions* which was written to accompany the Tables of the Theta Functions calculated by J W L Glaisher

successors, including Baker, to get the full benefit of the work of the great German masters of the late nineteenth century."

As can be seen from Whittaker's obituary notice and Forsyth's own recollections of his undergraduate days [Forsyth 1935], there is a definite link between Forsyth's later work in complex analysis and his early studies in Cambridge, where he came into repeated contact with Glaisher, Cayley, and the works of Smith, who taught at Oxford. Cayley and Glaisher were also to be very instrumental in Forsyth's career in pure mathematics and helped him to overcome that "Cambridge atmosphere" in which "all were reared to graduation on applied mathematics."[6] According to Whittaker, it was Glaisher who suggested to Forsyth that he write his first book, the *Treatise on differential equations* (1885). His first two principal memoirs on theta functions (1881/1883) and on Abel's theorem and Abelian functions (1882/1884) were presented by Cayley, on the other hand, to the Royal Society. In the first memoir Forsyth gives a list of 22 principal papers in the field, including among others Riemann's *Theorie der Abelschen Functionen* as well as twelve other papers by the German mathematicians Jacobi, Richelot, Rosenhain, Gopel, Weierstrass, Koenigsberger, Kummer, Borchardt, and Weber. The second memoir on Abel's theorem and Abelian functions contained a large section on Weierstrass's approach, which clearly shows that around 1882 Forsyth was already fully aware of the achievements of the German mathematicians in this field.

## Conclusion

These bibliographical investigations make it evident that Riemann's ideas were more positively accepted in Great Britain, apparently even earlier, than in France. It seems they were among the more important stimuli which, through the efforts of Cayley, Clifford, Smith, Glaisher, etc., later led to the flourishing of English pure mathematics and function theory under Hobson, Forsyth, Mathews, Baker, Barnes, Hardy, Littlewood, Titchmarsh, and many others. In England and Italy Riemann's theories entered open territories, where they found mathematical communities eager to catch up in pure mathematics. In England this interest served to broaden to the "Cambridge atmosphere," which was nearly totally oriented towards natural philosophy and applied mathematics, in Italy it fitted in with a desire to built up a modern mathematical education for the newly founded nation (cf. Neuenschwander 1986). In France, on the other hand, Riemann's ideas first faced a cold reception from the well-established tradition of Cauchy and Briot & Bouquet.

Allowing for a certain politically motivated enthusiasm, Weierstrass was therefore probably right when, in a letter to Casorati dated 25 March 1867 (written in the early period before the publication of Riemann's Collected Mathematical Works in 1876), he especially emphasized Italy's role in the dissemination of Riemann's ideas. In that letter (see [Neuenschwander 1978, 72])[7] he writes:

*The happy rise of science in your fatherland will nowhere else be followed with more vivid interest than here in North Germany, and you may be certain that the* State *of Italy has nowhere else so many sincere and disinterested friends. With pleasure we are therefore ready to continue the alliance between you and us—which in the political field has had such good results—also in science, so that also in that area the barriers may more and more be overcome, with which unhappy politics has for so long separated two generally congenial nations. The paper which you sent to me proves to me once again that our scientific endeavours are better understood and appreciated in Italy than in France and England, particularly in the latter country, where an overwhelming formalism threatens to stifle any feeling for deeper investigation. How significant it is that our Riemann, whose loss we cannot sufficiently deplore, is studied and honoured outside Germany, only in Italy, in France he is certainly acknowledged externally but little understood, and in England [at least before 1867], he has remained almost unknown.*

***Note added in proof.*** In a recent issue of this journal (vol. 19, no. 4, Fall 1997) there appeared an article by Jeremy Gray which gives the reader a very welcome summary in English of Riemann's introductory lectures on general complex analysis. Gray's paper is largely based on my preprint *Riemanns Vorlesungen zur Funktionentheorie Allgemeiner Teil*, Darmstadt 1987 and on Roch 1863/65 (cf. Neuenschwander 1996, p. 15) but does not mention the substantially expanded version which appeared in Spring 1996 and which has been reviewed in MR 97d:01041 and Zbl. 844.01020. The published version also contains, besides what was in the original preprint, the above-mentioned extensive bibliography of papers and treatises that were influenced by Riemann's work, as well as a list of all known lecture notes of his courses on complex analysis. From this list [Neuenschwander 1996, 81 ff.], one can infer that Cod. Ms. Riemann 37 comprises notes of three different courses (not just one, as suggested by Gray), and its first part is thereby quite easily datable to the Winter semester 1855/56. On page 111 of Cod. Ms. Riemann 37 one

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[6]According to Forsyth's own recollections, as early as his student years he took an exceptional interest in pure mathematics and went for one term in his third year to Cayley's lectures, where at the beginning the very word plunged him "into complete bewilderment." From other passages of Forsyth's recollections one may see that already at that time he made his first timid ventures outside the range of Cambridge textbooks and ploughed through, among others, a large part of Durege's *Elliptische Functionen*. Further details may be gathered from the following personal confession by Forsyth: "Something of differential equations beyond mere examples, the elements of Jacobian elliptic functions, and the mathematics of Gauss's method of least squares, I had learnt from Glaisher's lectures; and by working at the matter of the one course by Cayley which had been attended, I began to understand that pure mathematics was more than a collection of random tools mainly fashioned for use in the Cambridge treatment of natural philosophy. Otherwise very nearly the whole of such knowledge of pure mathematics as is mine began to be acquired only after my Tripos degree. In that Cambridge atmosphere we all were reared to graduation on applied mathematics." [Forsyth 1935, 172]

[7]Notice that his characterization of the situation in England is strikingly similar to that of Smith (see note 4) or Forsyth (see note 6).

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reads "Fortsetzung im Sommersemester 1856 [Continuation in the Summer semester of 1856]," and on page 193 of the same manuscript "Theorie der Functionen complexer Großen mit besonderer Anwendung auf die Gauß'sche Reihe $F(\alpha, \beta, \gamma, x)$ und verwandte Transcendenten Wintersemester 1856/57" Furthermore, it should be noted that Riemann's lectures on general complex analysis developed gradually from 1855 to 1861 [Neuenschwander 1996, 11 f ], and that in 1861 he also treated Weierstrass's factorization theorem for entire functions with prescribed zeros, introducing to this end logarithms as convergence-producing terms [Neuenschwander 1996, 62–65], which is quite remarkable, although not mentioned by Gray 1997

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**BIBLIOGRAPHY**

Armenante, A Jung, G [Relazione sulle Lezioni complementali date nell Istituto tecnico superiore a Milano nello scorso anno (1868–69) dagli egregi Professori Brioschi Cremona e Casorati] *Giornale di Matematiche* **7** (1869), 224–234

Bottazzini, U Riemanns Einfluß auf E Betti und F Casorati *Archive for History of Exact Sciences* **18** (1977/78), 27–37

Bottazzini U *Il calcolo sublime storia dell'analisi matematica da Euler a Weierstrass* Boringhieri, Torino 1981 Revised and enlarged English version Springer, New York 1986

Bottazzini U Enrico Betti e la formazione della seuola matematica pisana In O Montaldo, L Grugnetti (eds ) *La storia delle matematiche in Italia* Atti del Convegno, Cagliari, 29–30 settembre e 1 ottobre 1982 Universita di Cagliari, Cagliari 1983 229–276

Bottazzini, U Riemann in Italia In *Conferenza Internazionale nel 125° Anniversario della morte di Georg Friedrich Bernhard Riemann 20 Luglio 1991 Verbania Atti del convegno*, 31–40

Briot, C , Bouquet, J -C *Theorie des fonctions elliptiques* Deuxieme edition Gauthier-Villars Paris 1875

Briot C *Theorie des fonctions abeliennes* Gauthier-Villars Paris 1879

Casorati F *Teorica delle funzioni di variabili complesse* Fusi, Pavia, 1868

Casorati, F , Cremona L Intorno al numero dei moduli delle equazioni o delle curve algebriche di un dato genere *Reale Istituto Lombardo di Scienze e Lettere Rendiconti* (2) **2** (1869), 620–625 = F Casorati *Opere* vol 1, 313–315 L Cremona, *Opere matematiche*, vol 3, 128–132

Dieudonne J The beginnings of Italian algebraic geometry In *Symposia mathematica*, vol 27, Academic Press, London/New York 1986, 245–263

Forsyth, A R Old Tripos days at Cambridge *The Mathematical Gazette* **19** (1935), 162–179

Gillispie, C C (ed ) *Dictionary of scientific biography*, 16 vols Scribner, New York 1970–1980

Gispert, H Sur la production mathematique française en 1870 (Etude du tome premier du Bulletin des Sciences mathematiques) *Archives Internationales d'Histoire des Sciences* **35** (1985), 380–399

Gispert, H La France mathematique La societe mathematique de France (1870–1914) Suivi de cinq etudes par R Bkouche, C Gilain C Houzel, J -P Kahane, M Zerner *Cahiers d'Histoire et de Philosophie des Sciences* (2), No 34 Blanchard, Paris, 1991

Houel, J *Theorie elementaire des quantites complexes* Gauthier-Villars Paris, 1867–1874

Klein F Riemann und seine Bedeutung fur die Entwickelung der modernen Mathematik *Jahresbericht der Deutschen Mathematiker-Vereinigung* **4** (1894–95) [Berlin 1897], 71–87 = *Gesammelte mathematische Abhandlungen*, Bd 3, 482–497 English translation *Bulletin of the American Mathematical Society* **1** (1895), 165–180

Klein, F *Vorlesungen uber die Entwicklung der Mathematik im 19 Jahrhundert* Teile 1 und 2, Grundlehren der mathematischen Wissenschaften, Bde 24 und 25 Springer Berlin, 1926–1927 Reprint Berlin/Heidelberg/New York 1979 English translation Brookline, Mass 1979

Loria, G (ed ) Lettere al Tardy di Genocchi Betti e Schlafli *Atti della Reale Accademia dei Lincei* Serie quinta Rendiconti, Classe di scienze fisiche, matematiche e naturali **24 1** (1915), 516–531

Loria, G *Storia delle matematiche* 3 vol STEN Torino 1929–1933 Seconda edizione riveduta e aggiornata Hoepli, Milano, 1950

Naas, J , Schmid, H L (Hrsg ) *Mathematisches Worterbuch mit Einbeziehung der theoretischen Physik*, 2 Bde Akademie-Verlag/ Teubner, Berlin/Stuttgart, 1961

Neuenschwander, E Der Nachlaß von Casorati (1835–1890) in Pavia *Archive for History of Exact Sciences* **19** (1978), 1–89

Neuenschwander E Uber die Wechselwirkungen zwischen der franzosischen Schule, Riemann und Weierstraß Eine Ubersicht mit zwei Quellenstudien *Archive for History of Exact Sciences* **24** (1981), 221–255 English version in *Bulletin of the American Mathematical Society*, New Ser **5** (1981) 87–105

Neuenschwander, E Der Aufschwung der italienischen Mathematik zur Zeit der politischen Einigung Italiens und seine Auswirkungen auf Deutschland In *Symposia Mathematica* **27** (1986), 213–237

Neuenschwander, E A brief report on a number of recently discovered sets of notes on Riemann's lectures and on the transmission of the Riemann Nachlass *Historia Mathematica* **15** (1988), 101–113 Reprinted in Riemann 1990, 855–867

Neuenschwander E Secondary literature on B Riemann In Riemann 1990, 896–910

Neuenschwander, E *Riemanns Einfuhrung in die Funktionentheorie Eine quellenkritische Edition seiner Vorlesungen mit einer Bibliographie zur Wirkungsgeschichte der Riemannschen Funktionentheorie* Abhandlungen der Akademie der Wissenschaften in Gottingen, Mathematisch-Physikalische Klasse, Dritte Folge Nr 44 Vandenhoeck & Ruprecht, Gottingen 1996

Purkert, W Arbeiten bis 1891 In Riemann 1990, 869–895

Riemann, B *Elliptische Functionen* Vorlesungen von Bernhard Riemann Mit Zusatzen herausgegeben von Hermann Stahl Teubner, Leipzig 1899

Riemann, B *Gesammelte mathematische Werke wissenschaftlicher Nachlass und Nachtrage* Nach der Ausgabe von H Weber und R Dedekind neu herausgegeben von R Narasimhan Springer/Teubner, Berlin, etc /Leipzig, 1990

Schering, E Zum Gedachtnis an B Riemann In *Gesammelte mathematische Werke von Ernst Schering* Bd 2 Berlin 1909 367–383 434–447 Partially reprinted in Riemann 1990, 827–844

Simart G *Commentaire sur deux Memoires de Riemann relatifs a la theorie generale des fonctions et au principe de Dirichlet* Theses presentees a la Faculte des Sciences de Paris Gauthier-Villars, Paris 1882

Smith, H J S On the present state and prospects of some branches of pure mathematics *Proceedings of the London Mathematical Society* **8** (1876–77), 6–29 = *Collected Mathematical Papers*, vol 2, 166–190

Tricomi F G Bernhard Riemann e l'Italia *Universita e Politecnico di Torino, Rendiconti del Seminario Matematico* **25** (1965–66), 59–72

Volterra, V Betti, Brioschi, Casorati, trois analystes italiens et trois manieres d'envisager les questions d'analyse In *Compte rendu du deuxieme Congres International des Mathematiciens tenu a Paris du 6 au 12 août 1900* Gauthier-Villars, Paris, 1902, 43–57 = *Opere matematiche*, vol 3, 1–11

Weil A Riemann, Betti and the birth of topology *Archive for History of Exact Sciences* **20** (1979), 91–96 **21** (1979/80), 387

Whittaker, E T Andrew Russell Forsyth In *Obituary Notices of Fellows of the Royal Society 1942–1944*, vol 4 no 11 (1942) 209–227

Wirtinger W Riemanns Vorlesungen uber die hypergeometrische Reihe und ihre Bedeutung In *Verhandlungen des dritten Internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, Teubner, Leipzig, 1905 121–139 Reprinted in Riemann 1990 719–738