Dodgson's Determinant-Evaluation Rule Proved by TWO-TIMING MEN and WOMEN

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Bijections are where it's at —Herb Wilf

Dedicated to Master Bijectionist Herb Wilf, on finishing 13/24 of his life

I will give a bijective proof of the Reverend Charles Lutwidge **Dodgson's Rule**([D]):

$$\det\left[(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}\right]\cdot\det\left[(a_{i,j})_{\substack{2\le i\le n-1\\2\le j\le n-1}}\right]=$$

$$\det\left[(a_{i,j})_{\substack{1\le i\le n-1\\1\le j\le n-1}}\right]\cdot\det\left[(a_{i,j})_{\substack{2\le i\le n\\2\le j\le n}}\right]-\det\left[(a_{i,j})_{\substack{1\le i\le n-1\\2\le j\le n}}\right]\cdot\det\left[(a_{i,j})_{\substack{2\le i\le n\\1\le j\le n-1}}\right]\quad . \qquad (Alice)$$

Consider n men, $1,2,\ldots,n$, and n women $1',2'\ldots,n'$, each of whom is married to exactly one member of the opposite sex. For each of the $n!$ possible (perfect) matchings π, let

$$weight(\pi):=sign(\pi)\prod_{i=1}^{n}a_{i,\pi(i)}\quad ,$$

where $sign(\pi)$ is the sign of the corresponding permutation, and for $i=1,\ldots,n$, Mr. i is married to Ms. $\pi(i)'$.

Except for Mr. 1, Mr. n, Ms. $1'$ and Ms. n' all the persons have affairs. Assume that each of the men in $\{2,\ldots,n-1\}$ has exactly one mistress amongst $\{2',\ldots,(n-1)'\}$ and each of the women in $\{2',\ldots,(n-1)'\}$ has exactly one lover amongst $\{2,\ldots,n-1\}$[2]. For each of the $(n-2)!$ possible (perfect) matchings σ, let

$$weight(\sigma):=sign(\sigma)\prod_{i=2}^{n-1}a_{i,\sigma(i)}\quad ,$$

where $sign(\sigma)$ is the sign of the corresponding permutation, and for $i=2,\ldots,n-1$, Mr. i is the lover of Ms. $\sigma(i)'$.

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2 Somewhat unrealistically, a man's wife may also be his mistress, and equivalently, a woman's husband may also be her lover.

Let $A(n)$ be the set of all pairs $[\pi, \sigma]$ as above, and let $weight([\pi, \sigma]) := weight(\pi)weight(\sigma)$. The left side of $(Alice)$ is the sum of all the weights of the elements of $A(n)$.

Let $B(n)$ be the set of pairs $[\pi, \sigma]$, where now n and n' are unmarried but have affairs, i.e. π is a matching of $\{1, \ldots, n-1\}$ to $\{1', \ldots, (n-1)'\}$, and σ is a matching of $\{2, \ldots, n\}$ to $\{2', \ldots, n'\}$, and define the weight similarly.

Let $C(n)$ be the set of pairs $[\pi, \sigma]$, where now n and $1'$ are unmarried and 1 and n' don't have affairs. i.e. π is a matching of $\{1, \ldots, n-1\}$ to $\{2', \ldots, n'\}$, and σ is a matching of $\{2, \ldots, n\}$ to $\{1', \ldots, (n-1)'\}$, and *now* define $weight([\pi, \sigma]) := -weight(\pi)weight(\sigma)$.

The right side of $(Alice)$ is the sum of all the weights of the elements of $B(n) \cup C(n)$.

Define a mapping

$$T : A(n) \to B(n) \cup C(n) \quad ,$$

as follows. Given $[\pi, \sigma] \in A(n)$, define an alternating sequence of men and women: $m_1 := n, w_1, m_2, w_2, \ldots, m_r, w_r = 1'$ or n', such that $w_i :=$wife of(m_i), and $m_{i+1} :=$lover of(w_i). This sequence terminates, for some r, at either $w_r = 1'$, or $w_r = n'$, since then m_{r+1} is undefined, as $1'$ and n' are lovers-less women. To perform T, change the relationships $(m_1, w_1), (m_2, w_2), \ldots, (m_r, w_r)$ from marriages to affairs (i.e. Mr. m_i and Ms. w_i get divorced and become lovers, $i = 1, \ldots, r$), and change the relationships $(m_2, w_1), (m_3, w_2), \ldots, (m_r, w_{r-1})$ from affairs to marriages. If $w_r = 1'$ then $T([\pi, \sigma]) \in C(n)$, while if $w_r = n'$ then $T([\pi, \sigma]) \in B(n)$.

The mapping T is weight-preserving. Except for the sign, this is obvious, since all the relationships have been preserved, only the nature of some of them changed. I leave it as a pleasant exercise to verify that also the sign is preserved.

It is obvious that $T : A(n) \to B(n) \cup C(n)$ is one-to-one. If it were onto, we would be done. Since it is not, we need one more paragraph.

Call a member of $B(n) \cup C(n)$ *bad* if it is not in $T(A(n))$. I claim that the sum of all the weights of the bad members of $B(n) \cup C(n)$ is zero. This follows from the fact that there is a natural bijection S, easily constructed by the readers, between the bad members of $C(n)$ and those of $B(n)$, such that $weight(S([\pi, \sigma])) = -weight([\pi, \sigma])$. Hence the weights of the bad members of $B(n)$ and $C(n)$ cancel each other in pairs, contributing a total of zero to the right side of $(Alice)$. □

A small Maple package, *alice*, containing programs implementing the mapping T, its inverse, and the mapping S from the bad members of $C(n)$ to those of $B(n)$, is available from my Home Page `http://www.math.temple.edu/~zeilberg`.

Reference

[D] C.L. Dodgson, *Condensation of Determinants*, Proceedings of the Royal Society of London **15**(1866), 150-155.