

Does ambiguity aversion reinforce risk aversion? Applications to portfolio choices and asset prices

Christian Gollier¹

Toulouse School of Economics (LERN and EIF)

March 14, 2008

¹The first version of this paper was written while the author was visiting Beijing University. The author thanks Alain Chateauneuf, Jean-Marc Tallon, Thomas Mariotti, Jean Tirole, François Salanié and seminar participants at Paris 1, Georgia State, Helsinki and Toulouse for helpful comments.

Abstract

It is often suggested that ambiguity aversion makes individuals more precautionary, thereby offering a potential explanation for the equity premium puzzle. We show that this is not true in general. We consider a model in which risk-and-ambiguity-averse agents can invest in an unambiguously safe asset and in an ambiguous risky asset. We exhibit some sufficient conditions to guarantee that, *ceteris paribus*, an increase in ambiguity aversion reduces the demand for the ambiguous risky asset, and raises the equity premium. For example, this is the case when the set of plausible distributions of returns can be ranked according to the monotone likelihood ratio order. We also show how ambiguity aversion distorts the price kernel.

Keywords: smooth ambiguity aversion, monotone likelihood ratio, equity premium, portfolio choice, price kernel, central dominance.

1 Introduction

In many circumstances, it is difficult to assess the precise probability distribution to describe the uncertainty faced by a decision maker (DM). These situations can be described by a set of plausible priors that contains more than one distribution. Two schools of thought have proposed an axiomatized solution to this choice problem. Relying on the Sure Thing Principle, Savage (1954) claims that it is intuitive to expect individuals to form subjective beliefs on these priors that they will use to measure their expected utility. In short, agents should be probabilistically sophisticated. This theory of Subjective Expected Utility (SEU) has been subject to criticisms because of its poor performance to explain observed behaviors in uncertain contexts, as illustrated by the Ellsberg paradox. Gilboa and Schmeidler (1989) proposed an alternative decision criterion that performs better in these contexts. Under their model of ambiguity aversion, for each possible choice *ex ante*, the DM computes the expected utility conditional to each possible prior, and takes the minimum to evaluate the welfare generated by that choice. They then select the choice that maximizes their welfare. Agents who behave according to this maxmin model exhibit a form of choice-sensitive pessimism. A smooth version of this maxmin expected utility model with multiple priors has recently been proposed by Klibanoff, Marinacci and Mukerji (2005).

In this paper, as in Chen and Epstein (2002) and in Mukerji, Sheppard and Tallon (2005), we explore the consequences of ambiguity aversion on the optimal portfolio allocation and on equilibrium asset prices. We examine a two-asset model with one safe and unambiguous asset, and another risky and ambiguous asset. Hansen, Sargent and Tallerini (1999), Chen and Epstein (2002) and Klibanoff, Marinacci and Mukerji (2005) have suggested that this multiple-priors utility model is helpful to solve the equity premium

puzzle. This puzzle is based on the observation that the equity premium in most developed countries during the last century has been around 4 and 6 percents per year, whereas its theoretical level obtained by using the standard probabilistically sophisticated modeling does not exceed half a percent per year. In other words, the subjective expected utility model requires an unrealistically large degree of risk aversion to explain why households invested so much in the less risky assets in spite of the large observed equity premium. Chen and Epstein (2002) suggested that "*part of the [equity] premium is due to the greater ambiguity associated with the return to equity, which reduces the required degree of risk aversion*" to explain the puzzle. They considered a model in which the growth process of the economy is surrounded by ambiguity. They obtained an equity premium composed of two positive terms, one coming from risk aversion, and the other originating from the representative agent's ambiguity aversion. Klibanoff, Marinacci and Mukerji (2005) considered a simple numerical example to show that "*ambiguity aversion acts as an extra risk aversion*".¹ Thus, the general idea is that ambiguity aversion reinforces risk aversion to make people more reluctant to undergo ambiguous risky acts.²

¹They also consider an illustration in which the safe asset is ambiguous, in which case ambiguity aversion makes the risky asset more attractive. In this paper, we assume that the safe asset is unambiguous.

²The same idea can be found in the debate on the precautionary principle (O'Riordan and Cameron (1995), CEC (2000), and Gollier (2001a)). This principle, which appears in various international texts as in the Conference of Rio on Environment and Development or the Maastricht Treaty. It states that "*lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation*". This principle has widely been interpreted as a recommendation for reducing the collective risk exposure in the presence of ambiguous probabilities. In short, ambiguity aversion should make us behave in a more risk-averse way. It has frequently been used in Europe against genetically modified organisms, or for an increase in the effort to reduce emissions of greenhouse gases. Gollier, Jullien and Treich (2000) examine these questions by considering an environmental decision problem that is equivalent to the portfolio choice model considered in this paper.

The starting point of this paper is that it is generally not true that ambiguity aversion reinforces the effect of risk aversion to induce a reduction in the demand for the ambiguous risky asset. For cleverly chosen - but still not spurious - multiple-priors for the return of the risky asset, we show that the introduction of ambiguity aversion *increases* the investor's demand for the risky asset. The intuition for why such counterexamples may exist can be explained as follows. Ambiguity aversion implies that the DM selects the relevant prior for the optimal choice in a more pessimistic way than what the probabilistically sophisticated DM would do. For example, if the DM has two possible priors, one being riskier than the other in the sense of Rothschild and Stiglitz (1970), the ambiguity-averse maxmin DM will use the riskier distribution for the excess return, whereas the subjective-expected-utility-maximizer DM will use a less risky distribution. How does this differentiated choice of priors affect the demand for the risky asset? This question was first raised by Rothschild and Stiglitz (1971), who showed that an increase in the riskiness of the risky asset does not necessarily reduce the demand for it by all risk-averse agents. Consequently, it is not generally true that the ambiguity-averse DM will have a smaller demand for the risky asset than the ambiguity-neutral one.

The main objective of the paper is to characterize conditions under which ambiguity aversion reduces the optimal exposure to uncertainty. This can be done by restricting either the set of utility functions and/or the set of possible priors. If we assume that the set of priors can be ranked according to the first-degree or second-degree stochastic dominance orders (FSD/SSD), we exhibit some simple sufficient conditions on the utility function to obtain the comparative static property of the introduction of ambiguity aversion. This result is derived from the observation that, when determining whether to purchase more or less of the risky asset than the SEU agent, the ambiguity-

averse agent will use a an implicit distribution for the risky asset that is FSD or SSD dominated by the distribution used by the probabilistically sophisticated agent. This observation allows us to reinterpret the effect of ambiguity aversion as the effect of a FSD/SSD shift in beliefs in the standard expected utility model. We then rely on some sufficient conditions on the utility function existing in the EU literature that guarantee that these differences in implicit beliefs yield a reduction in the demand for the risky asset. Rothschild and Stiglitz (1971), Fishburn and Porter (1976) and Hadar and Seo (1990) provided these sufficient conditions. By restricting the set of possible beliefs, we are able to relax these conditions on the utility functions. This is in line with the EU literature on the effect of a change in beliefs on the demand for equity (Eeckhoudt and Hansen (1980), Meyer and Ormiston (1985), Black and Bulkley (1989), Dionne, Eeckhoudt and Gollier (1993), Athey (2002)). Gollier (1995) characterized the stochastic order that guarantees that the change in the distribution of the risky asset return yields a reduction of the optimal risk exposure by all risk-averse investors. A change in riskiness that satisfies this comparative static property is said to be centrally dominated (CD). As explained earlier, second-degree stochastic dominance (SSD) does not imply CD. Gollier (1995) and Chateauneuf and Lakhnati (2005) also showed that CD does not implies SSD. Gollier (1997) exhibited the necessary and sufficient condition for a change in the distribution of the growth of the economy to raise the equity premium. Unaware of this literature, Cecchetti, Lam and Mark (2000) and Abel (2002) more recently reexamined these questions.

In the first part of the paper, we examine the effect of ambiguity aversion on the demand for the risky asset by an agent who takes prices as given. The decision model is presented in Section 2. Section 3 provides a counterexample in which an increase in ambiguity aversion raises the demand for the

ambiguous risky asset. In Section 4, we review the literature on the effect of a change in beliefs on the demand for the risky asset in the standard EU model. In Section 5, we show that an increase in ambiguity aversion has an effect on the demand for the risky asset that is qualitatively equivalent to a change in beliefs in the EU model. We show that if the set of plausible distributions can be ranked according to the FSD (SSD) order, then an increase in ambiguity aversion deteriorates the implicit beliefs in the sense of the FSD (SSD) order. We use these observations in Section 6 to obtain sufficient condition for the comparative static property of more ambiguity aversion. In the second part of the paper (Sections 7 and 8), we examine a Lucas economy with a representative agent facing ambiguous probabilities. We show how the ambiguity aversion of the representative agent affects the equity premium, the price kernel of the economy, and individual asset prices.

2 The multiple-priors utility model applied to the portfolio problem

Our model is static with two assets. The first asset is safe and unambiguous with a rate of return that is normalized to zero. The risky asset has a return x whose distribution is ambiguous in the sense that it is sensitive to some parameter θ whose true value is unknown. The investor is initially endowed with wealth w_0 . If he invests α in the risky asset, his final wealth will be $w_0 + \alpha x$ conditional to a realized return x of the risky asset.

The ambiguity of the risky asset is characterized by a set $\Pi = \{F_1, \dots, F_n\}$ of subjectively plausible cumulative probability distributions for \tilde{x} . Let \tilde{x}_θ denote the random variable distributed as F_θ . Based on his subjective information, the investor associates a probability distribution (q_1, \dots, q_n) over Π , with $\sum_{\theta=1}^n q_\theta = 1$, where $q_\theta \geq 0$ is the subjective probability that F_θ be the true

probability distribution of excess returns. We hereafter denote $\tilde{\theta}$ for the random variable $(1, q_1; 2, q_2; \dots; n, q_n)$. Following Klibanoff, Marinacci and Mukerji (2005), we assume that the preferences of the investor exhibit smooth ambiguity aversion. For each plausible probability distribution F_θ , the investor computes the expected utility $U(\alpha, \theta) = Eu(w_0 + \alpha \tilde{x}_\theta) = \int u(w_0 + \alpha x) dF_\theta(x)$ conditional to F_θ being the true distribution. We assume that u is increasing and concave, so that $U(\cdot, \theta)$ is concave in the investment α in the risky asset, for all θ . Ex ante, for a given portfolio allocation α , the welfare of the agent is measured by $V(\alpha)$ with

$$\phi(V(\alpha)) = \sum_{\theta=1}^n q_\theta \phi(U(\alpha, \theta)) = \sum_{\theta=1}^n q_\theta \phi(Eu(w_0 + \alpha \tilde{x}_\theta)),$$

The shape of ϕ describes the investor's attitude towards ambiguity (or parameter uncertainty). $V(\alpha)$ can be interpreted as the certainty equivalent of the uncertain conditional expected utility $U(\alpha, \tilde{\theta})$. A linear ϕ means that the investor is neutral to ambiguity. In such a case, the DM is indifferent to any mean-preserving spread of $U(\alpha, \tilde{\theta})$, and $V(\alpha)$ can be represented by a subjective expected utility functional $V^{SEU}(\alpha) = Eu(w_0 + \alpha \tilde{x})$, where \tilde{x} is the random variable that is distributed as $(\tilde{x}_1, q_1; \dots; \tilde{x}_n, q_n)$. On the contrary, a concave ϕ is synonymous of ambiguity aversion in the sense that the DM dislikes any mean-preserving spread of the conditional expected utility $U(\alpha, \tilde{\theta})$.

An interesting particular case arises when the absolute ambiguity aversion $\eta(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -\eta^{-1} \exp(-\eta U)$. As proved by Klibanoff, Marinacci and Mukerji (2005), the ex ante welfare $V(\alpha)$ tends to maxmin expected utility functional $V^{MEU}(\alpha) = \min_\theta Eu(w_0 + \alpha \tilde{x}_\theta)$ when the degree of absolute ambiguity aversion ϕ tends to infinity. Thus, the Gilboa and Schmeidler (1989)'s maxmin criteria is a special case of this model.

The optimal portfolio allocation α^* maximizes the ex ante welfare of the investor $V(\alpha)$. Because ϕ is increasing, α^* is the solution of the following program:

$$\alpha^* \in \arg \max_{\alpha} \sum_{\theta=1}^n q_{\theta} \phi(Eu(w_0 + \alpha \tilde{x}_{\theta})). \quad (1)$$

If ϕ and u are strictly concave, the objective function is concave in α and the solution to program (1), when it exists, is unique.

To illustrate, consider the following special case. Suppose that the plausible distributions \tilde{x}_{θ} are all normally distributed with the same variance σ^2 , and with $E\tilde{x}_{\theta} = \theta$.³ Suppose also that the investor's preferences exhibit constant absolute risk aversion $-u''(z)/u'(z) = A$, i.e., $u(z) = -A^{-1} \exp -Az$. As is well-known, this implies that the Arrow-Pratt approximation is exact.⁴ This implies in turn that

$$U(\alpha, \theta) = -A^{-1} \exp -A(w_0 + \alpha\theta - 0.5A\alpha^2\sigma^2) \in \mathbb{R}_-. \quad (2)$$

Moreover, suppose that ϕ exhibits constant relative ambiguity aversion on the relevant domain of U , which is \mathbb{R}_- . This means that $\phi(U) = -(-U)^{1+\gamma}/(1+\gamma)$. This function is increasing in \mathbb{R}_- and is concave in this domain if γ is positive. Under this specification, we have that

$$V(\alpha) = -A^{-1} (\exp -A(w_0 - 0.5A\alpha^2\sigma^2)) \left(E \exp -A(1+\gamma)\alpha\tilde{\theta} \right)^{1/(1+\gamma)}. \quad (3)$$

In order to get an analytical solution to this problem, suppose that the parameter uncertainty on the equity premium θ is also normal, in the sense that $\tilde{\theta}$ is normally distributed with mean μ_0 and variance σ_0^2 . This latter parameter is a measure of the degree of ambiguity faced by the investor.

³It is easy to extend this to the case of an ambiguous variance of returns.

⁴For a simple proof, see for example Gollier (2001b, page 57).

Using again the property that the Arrow-Pratt approximation is exact for exponential functions and normal distributions, we obtain that

$$E \exp -A(1 + \gamma)\alpha\tilde{\theta} = \exp -A(1 + \gamma)\alpha(\mu_0 - 0.5A(1 + \gamma)\alpha\sigma_0^2). \quad (4)$$

Combining these two formulas yields

$$V(\alpha) = -A^{-1} \exp -A(w_0 + \alpha\mu_0 - 0.5A\alpha^2(\sigma^2 + (1 + \gamma)\sigma_0^2)). \quad (5)$$

The optimal demand for the risky asset is thus equal to

$$\alpha^* = \frac{\mu_0}{A(\sigma^2 + (1 + \gamma)\sigma_0^2)}. \quad (6)$$

We see that, when the risky asset is ambiguous ($\sigma_0^2 > 0$), the demand for the risky asset is decreasing in the relative degree γ of ambiguity aversion of the investor. In this example, risk aversion and ambiguity aversion go into the same direction. Compared to the ambiguity neutral agent ($\gamma = 0$), ambiguity aversion $\gamma > 0$ has an effect on the demand for the ambiguous risky asset that is equivalent to increasing absolute risk aversion by $100\gamma\sigma_0^2/(\sigma^2 + \sigma_0^2)$ %.

3 An example in which more ambiguity-averse agents raise their demand for the ambiguous asset

The above example together with the findings by Chen and Epstein (2002), Klibanoff, Marinacci and Mukerji (2005) and Mukerji, Sheppard and Tallon (2005) suggest that ambiguity aversion reinforces risk aversion in situations where the more risky actions are also more ambiguous. In this section, we show that this intuitive idea is not true in general.

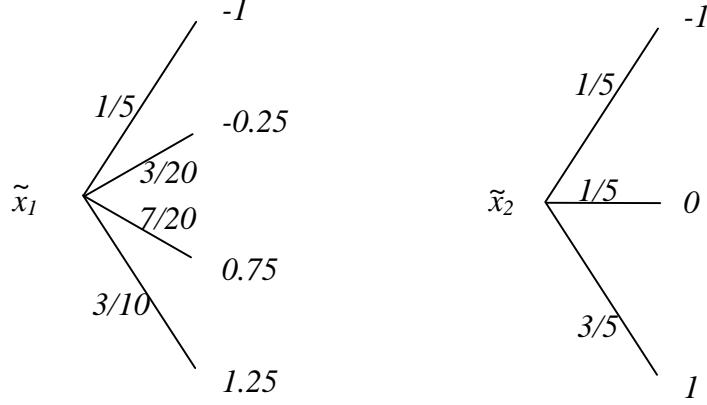


Figure 1: The conditional distributions of the excess return of the risky asset in the counterexample.

In our counterexample, there are only $n = 2$ plausible distribution functions F_1 and F_2 for the excess return of the risky asset. The corresponding conditional distributions are depicted in Figure 1. It is easy to check that \tilde{x}_1 is riskier than \tilde{x}_2 in the sense of Rothschild and Stiglitz (1970). Indeed, \tilde{x}_1 is obtained from \tilde{x}_2 by adding the zero-mean noises $(-0.25, 3/4; 0.75, 1/4)$ and $(-0.25, 1/2; 0.25, 1/2)$ conditional to respectively $x_2 = 0$ and $x_2 = 1$. In addition, given all available information, investors believe that the less risky distribution F_1 has a probability $q_1 = 5\%$ to be the true distribution.

The investor's attitude toward risk is represented by the following concave utility function:

$$u(z) = \begin{cases} z & \text{if } z \leq 3 \\ 3 + 0.3(z - 3) & \text{if } z > 3 \end{cases} \quad (7)$$

Moreover, the investor's attitude towards ambiguity is characterized by con-

stant absolute risk tolerance η :

$$\phi(U) = -\frac{\exp -\eta U}{\eta}. \quad (8)$$

Finally, the investor's final wealth equals $w_0 = 2$. We have drawn in Figure 2 the ex ante welfare V of the investor as a function of the investment α in the risky asset for various non-negative degrees η of ambiguity aversion. As expected, V is concave in α , and an increase in ambiguity aversion yields a downward shift in welfare, for all $\alpha \neq 0$. In Figure 3, we reported the demand α^* for the risky asset as a function of η , the constant ambiguity aversion of the investor. Because of the kink at $\alpha = 1$ of the V curves for small values of η , the introduction of a small degree of ambiguity aversion has no effect on the demand for the risky asset, which is equal to $\alpha^* = 1$. However, above a threshold η_{\min} approximately equal to 20, the introduction of ambiguity aversion *increases* α^* above the optimal investment of the ambiguity-neutral investor, and *the demand for the risky and ambiguous asset is increasing in the degree of ambiguity aversion*. When η tends to infinity – a case that correspond to the Gilboa and Schmeidler's maxmin model, the optimal investment in the risky asset tends to $\alpha^* = 4/3$. The ambiguity-averse maxmin investor has a demand for the risky asset that is 33% larger than the demand of the ambiguity-neutral one!

Let us more specifically compare the demands of respectively the maxmin investor ($\eta = +\infty$) and the ambiguity-neutral one ($\eta = 0$). Because of his risk aversion, the maxmin agent behaves as if the riskier distribution F_1 would be certain, in which case the optimal investment is $\alpha^* = 4/3$. On the contrary, the ambiguity-neutral investor behaves as if the distribution of excess returns would be $F^{SEU} = (F_1, 5\%; F_2, 95\%)$, which is less risky than F_1 in the sense of Rothschild and Stiglitz. The reader can check that, conditional to beliefs F^{SEU} , the optimal investment in the risky asset equals $\alpha^* = 1$. In spite of

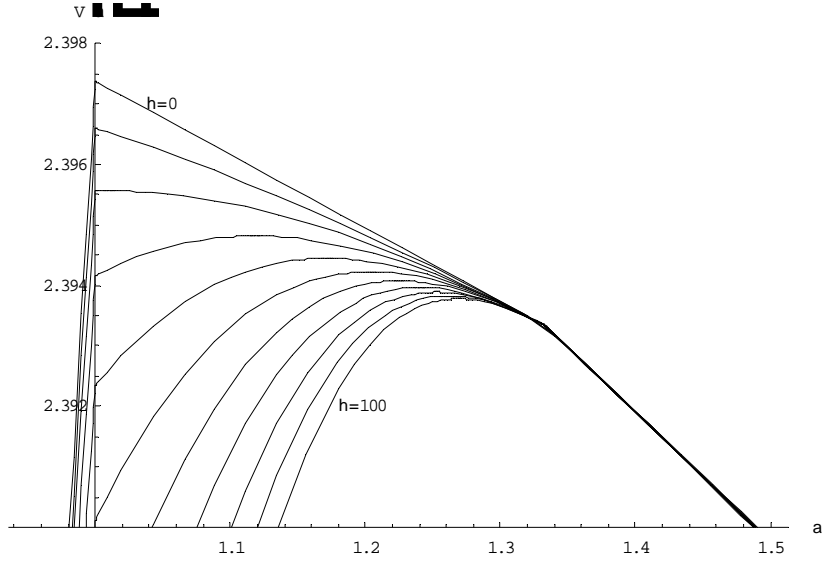


Figure 2: $V(\alpha)$ for $\eta = 0, 10, 20, \dots, 100$.

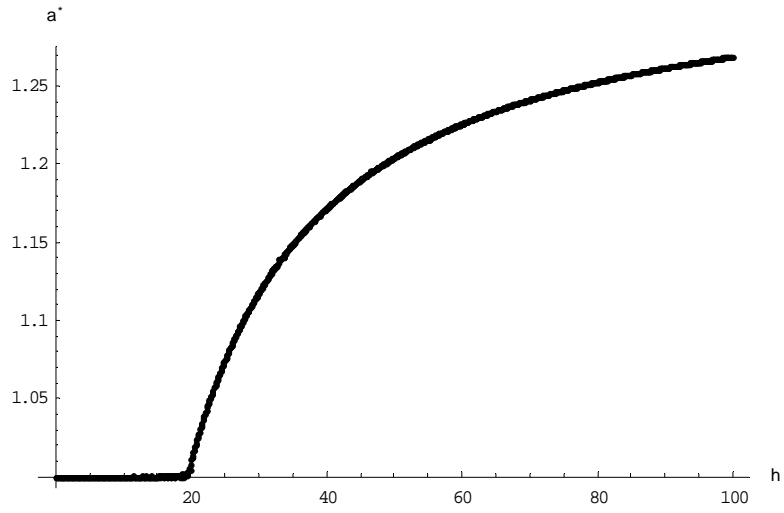


Figure 3: The demand for the risky asset as a function of the degree η of absolute ambiguity aversion.

subjectively perceiving a safer risky asset, the ambiguity-neutral DM invests less in it. This comment enlightens the link between the effect of ambiguity aversion on the demand for the risky asset, and the effect of an increase in the riskiness of the risky asset on the demand for it in the classical SEU model. Because it is not true in general that this increase in risk reduces the demand for the risky asset in that SEU model, it is not true in general that ambiguity aversion reduces this demand, both in the maxmin MEU model and its smooth version.

4 A short overview of the literature on the effect of a change in risk on the demand for the risky asset

In this section, we consider the ambiguity-free version of the portfolio problem presented in section 2. If the investor believes that the distribution of excess returns is \tilde{x}_a , the optimal investment α_a^* in the risky asset is such that

$$E\tilde{x}_a u'(w_0 + \alpha_a^* \tilde{x}_a) = 0. \quad (9)$$

Without loss of generality, let us assume that $E\tilde{x}_a$ is positive, which implies that α_a^* is positive.⁵ Consider a change in beliefs from \tilde{x}_a to another random variable \tilde{x}_b . Because $Eu(w_0 + \alpha \tilde{x}_b)$ is concave in α , the optimal investment α_b^* in the risky asset is less than α_a^* if and only if

$$E\tilde{x}_b u'(w_0 + \alpha_a^* \tilde{x}_b) \leq 0. \quad (10)$$

Let us first explain why SSD is neither necessary nor sufficient to guarantee that (9) implies (10), i.e., that the demand is reduced. Normalizing α_a^*

⁵If $E\tilde{x}_a$ is negative, replace \tilde{x} by $-\tilde{x}$ and α by $-\alpha$.

to unity, condition (10) can be rewritten as

$$E\tilde{x}_b u'(w_0 + \tilde{x}_b) \leq E\tilde{x}_a u'(w_0 + \tilde{x}_a), \quad (11)$$

As observed by Hadar and Seo (1990), condition (11) would be satisfied when \tilde{x}_b is SSD-dominated by \tilde{x}_a if function $h(x) = xu'(w_0 + x)$ is increasing and concave in x . Assuming that u is three times continuously differentiable, observe that

$$h'(x) = u'(w_0 + x) [1 - R(w_0 + x) + w_0 A(w_0 + x)],$$

where $A(z) = -u''(z)/u'(z)$ and $R(z) = zA(z)$ are respectively the absolute and relative risk aversion of the investor. Observe that a sufficient condition for h to be nondecreasing is that relative risk aversion be less than unity. We can also check that

$$h''(x) = -u''(w_0 + x) [(P^r(w_0 + x) - 2) - w_0 P(w_0 + x)]$$

where $P(z) = -u'''(z)/u''(z)$ and $P^r(z) = zP(z)$ are respectively absolute and relative prudence (Kimball (1990)). This condition implies that h is concave if relative prudence is less than 2, and prudence is positive. These findings can be summarized in the following proposition.

Proposition 1 *Suppose that the vNM utility function u is three times continuously differentiable. Then, a change in beliefs on the excess return from \tilde{x}_a to \tilde{x}_b implies inequality (11), i.e., it reduces the demand in the risky asset, if condition 1 or 2 is satisfied:*

1. \tilde{x}_b is riskier than \tilde{x}_a in the sense of Rothschild and Stiglitz, and relative prudence is positive and less than 2;
2. \tilde{x}_b is dominated by \tilde{x}_a in the sense of First-degree Stochastic Dominance (FSD), and relative risk aversion is less than unity.

None of these sufficient conditions is really satisfactory. Starting with condition 2, it is often claimed that relative risk aversion is larger than unity. Arguments have been provided based on introspection (Drèze (1981), Kandel and Stambaugh (1991), Gollier (2001)) or on the equity premium puzzle that can be solved in the canonical model only with a degree of relative risk aversion exceeding 40. A similar problem arises for condition 1, which requires that relative prudence be positive and less than 2. The positiveness of relative prudence relies on the observation that agents are prudent, i.e., that they raise their saving rate when their future incomes become riskier. The problem with condition 1 comes from the requirement that relative prudence be less than 2. Because $P^r(z) = 1 + R(z)$ when relative risk aversion is constant – a classical assumption in macroeconomics and finance – $P^r \leq 2$ is equivalent to $R \leq 1$ for that specification of the utility function. Thus, condition $P^r \leq 2$ seems as unrealistic as condition $R \leq 1$.

Rather than limiting the set of utility functions yielding an unambiguous effect of a FSD (SSD) change in beliefs, an alternative approach consists in searching for the set of changes in beliefs that yield an unambiguous reduction in the demand for the risky asset by all risk-averse investors. To do this, let us first introduce the following concepts, which rely on the location-weighted-probability functions T_a and T_b that are defined as follows:

$$T_a(x) = \int_{x_-}^x t dF_a(t) \quad \text{and} \quad T_b(x) = \int_{x_-}^x t dF_b(t), \quad (12)$$

where F_a and F_b are the cumulative distribution functions of \tilde{x}_a and \tilde{x}_b whose supports are assumed to be bounded in $[x_-, x_+]$.

Definition 1 *Consider two random variables $(\tilde{x}_a, \tilde{x}_b)$ with support in $[x_-, x_+]$ and $E\tilde{x}_a > 0$. We say that random variable \tilde{x}_b is dominated by \tilde{x}_a in the sense of Linear Stochastic Dominance of factor m , i.e., $\tilde{x}_b \preceq_{LSD(m)} \tilde{x}_a$, if*

$T_b(x) \leq mT_a(x)$ for all x in $[x_-, x_+]$. We say that \tilde{x}_b is centrally dominated (CD) by \tilde{x}_a , i.e., $\tilde{x}_b \preceq_{CD} \tilde{x}_a$, if there exists a nonnegative scalar m such that $\tilde{x}_b \preceq_{LSD(m)} \tilde{x}_a$.

Proposition 2 *Suppose that $E\tilde{x}_a$ is positive. All risk-averse investors reduce their demand for the risky asset due to a change in beliefs from \tilde{x}_a to \tilde{x}_b if and only if \tilde{x}_b is centrally dominated by \tilde{x}_a .*

Proof: The proof is essentially in Gollier (1995). Observe that we restricted the definition of CD to the set of LSD with a *nonnegative* factor m . When assuming that $E\tilde{x}_a > 0$, this is without loss of generality. Indeed, under this condition, if \tilde{x}_b is dominated by \tilde{x}_a in the sense of LSD with a negative m , it must be that it is also dominated in the sense of LSD of factor 0. This claim is proved as follows: Suppose that $\tilde{x}_b \preceq_{LSD(m)} \tilde{x}_a$ with $m < 0$. This implies in particular that $T_b(x_+) \leq mT_a(x_+)$, or equivalently that $E\tilde{x}_b \leq mE\tilde{x}_a$, which is nonpositive. Because T_b is first decreasing and then increasing with $T_b(x_-) = 0$, the fact that $T_b(x_+) \leq 0$ implies that $T_b(x) \leq 0$ for all $x \in [x_-, x_+]$. This implies in turn that \tilde{x}_b is dominated by \tilde{x}_a in the sense of $LSD(0)$. Thus, there is no restriction to limit the search of factor m to the set of nonnegative scalars. ■

SSD-dominance is not sufficient for CD-dominance. The numerical illustration presented in the previous section illustrates this fact. Indeed, \tilde{x}_1 is SSD-dominated by \tilde{x}_2 , but \tilde{x}_1 is not centrally riskier than \tilde{x}_2 . This observation is at the origin of our counterexample.

Here is a partial list of stochastic orders that have been shown to belong to CD:

- Strong Increase in Risk (Meyer and Ormiston (1985)): The excess return \tilde{x}_b is a strong increase in risk with respect to \tilde{x}_a if they have the

same mean and if any probability mass taken out of some of the realizations of \tilde{x}_a is transferred out of the support of this random variable.⁶

- Simple Increase in Risk (Dionne and Gollier (1992)): Random variable \tilde{x}_b is a simple increase in risk with respect to \tilde{x}_a if they have the same mean and $x(F_a(x) - F_b(x))$ is nonnegative for all x .⁷
- Monotone Likelihood Ratio order (MLR) (Milgrom (1981), Landsberger and Meilijson (1990) and Ormiston and Schlee (1993)): We say that \tilde{x}_b is dominated by \tilde{x}_a in the sense of MLR if there exists a scalar c in $[x_-, x_+]$ and a nonincreasing function r such that $F_a(x) = 0$ for all $x < c$ and $F_b(x) = F_b(c) + \int_c^x r(t)dF_a(t)$ for all $x \geq c$. Notice that MLR is a subset of FSD.
- Monotone Probability Ratio order (MPR) (Eeckhoudt and Gollier (1995), Athey (2002)): When the two random variables have the same support, we say that \tilde{x}_b is dominated by \tilde{x}_a in the sense of MPR if the cumulative probability ratio $F_b(x)/F_a(x)$ is nonincreasing. It can be shown that MPR is more general than MLR, but is still a subset of FSD: $MLR \Rightarrow MPR \Rightarrow FSD$.

5 Effect of an increase of ambiguity aversion

The beliefs of investors is represented by the set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ of the excess return of the risky asset, together with the a priori distribution (q_1, \dots, q_n) on these marginals. We compare two agents with the same beliefs and the same concave utility function u , with $u' \geq 0$ and $u'' \leq 0$, but with

⁶See also Eeckhoudt and Hansen (1980), Black and Bulkley (1989), Dionne, Eeckhoudt and Gollier (1993).

⁷Chateauneuf and Lakhani (2005) propose a generalized concept that combines the features of strong and simple increases in risk.

different attitudes toward ambiguity represented by concave functions ϕ_1 and ϕ_2 . The demand for the risky asset by agent ϕ_1 is expressed by α_1^* which must satisfy the following first-order condition:

$$\phi_1'(V_1(\alpha_1^*))V_1'(\alpha_1^*) = \sum_{\theta=1}^n q_\theta \phi_1'(U(\alpha_1^*, \theta)) E\tilde{x}_\theta u'(w_0 + \alpha_1^* \tilde{x}_\theta) = 0. \quad (13)$$

Because $U(0, \theta) = u(w_0)$ and $U_\alpha(0, \theta) = u'(w_0)E\tilde{x}_\theta$, we see that $V_1'(0) = \sum_{\theta} q_\theta E\tilde{x}_\theta$. By the concavity of V_1 , this implies that α_1^* must have the same sign than the unconditional expectation of the excess return, $\sum_{\theta} q_\theta E\tilde{x}_\theta$. Notice that we can rewrite the above condition as

$$E\tilde{y}_1 u'(w_0 + \alpha_1^* \tilde{y}_1) = 0, \quad (14)$$

where \tilde{y}_1 is a compound random variable which equals \tilde{x}_θ with probability \hat{q}_θ^1 , $\theta = 1, \dots, n$, such that

$$\hat{q}_\theta^1 = \frac{q_\theta \phi_1'(U(\alpha_1^*, \theta))}{\sum_{t=1}^n q_t \phi_1'(U(\alpha_1^*, t))}. \quad (15)$$

Thus, the ambiguity-averse agent ϕ_1 behaves as a SEU agent who would distort his beliefs from (q_1, \dots, q_n) to the "implicit probability distribution" $(\hat{q}_1^1, \dots, \hat{q}_n^1)$. Notice that the distortion functional described by equation (15) is endogenous, as it depends upon the portfolio allocation α_1^* selected by the agent.

Following Klibanoff, Marinacci and Mukerji (2005), we assume that the agent with function ϕ_2 is more ambiguity-averse than agent ϕ_1 in the sense that there exists an increasing and concave transformation function k such that $\phi_2(U) = k(\phi_1(U))$ for all U in the relevant domain. We would like to characterize conditions under which the more ambiguity-averse agent ϕ_2 has a smaller demand for the risky asset than agent ϕ_1 : $\alpha_2^* \leq \alpha_1^*$. By the

concavity of V_2 , this would be the case if and only if

$$\sum_{\theta=1}^n q_{\theta} \phi'_2(U(\alpha_1^*, \theta)) E \tilde{x}_{\theta} u'(w_0 + \alpha_1^* \tilde{x}_{\theta}) \leq 0. \quad (16)$$

As for agent ϕ_1 , this condition can be rewritten as

$$E \tilde{y}_2 u'(w_0 + \alpha_1^* \tilde{y}_2) \leq 0, \quad (17)$$

where \tilde{y}_2 is a compound random variable which equals \tilde{x}_{θ} with probability \hat{q}_{θ}^2 , $\theta = 1, \dots, n$, such that

$$\hat{q}_{\theta}^2 = \frac{q_{\theta} \phi'_2(U(\alpha_1^*, \theta))}{\sum_{t=1}^n q_t \phi'_2(U(\alpha_1^*, t))}. \quad (18)$$

To sum up, the change in preferences from (u, ϕ_1) to (u, ϕ_2) reduces the demand for the risky asset if (14) implies (17). By analogy to what has been done in the previous section, we obtain the following lemma. It builds a bridge between our comparative static analysis in this paper and the literature on the comparative statics of a change in risk in the SEU model.

Lemma 1 *The change in preferences from (u, ϕ_1) to (u, ϕ_2) reduces the demand for the risky asset if the SEU agent with utility function u reduces his demand for the risky asset when his beliefs about the excess return shift from $\tilde{y}_1 \sim (\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1)$ to $\tilde{y}_2 \sim (\tilde{x}_1, \hat{q}_1^2; \dots; \tilde{x}_n, \hat{q}_n^2)$.*

It is important to notice that agent ϕ_2 does actually not use beliefs \hat{q}^2 to determine his optimal portfolio allocation, but knowing \hat{q}^2 is enough to determine whether his demand for the risky asset is smaller than α_1^* . It is now useful to examine how agents ϕ_1 and ϕ_2 differentially distort their implicit probability distribution to determine whether α_2^* is smaller than α_1^* . A first answer to this question is provided by the following result.

Lemma 2 Rank the set of marginals so that $U(\alpha_1^*, 1) \leq U(\alpha_1^*, 2) \leq \dots \leq U(\alpha_1^*, n)$. The following two conditions are equivalent:

1. Agent ϕ_2 is more ambiguity-averse than agent ϕ_1 , i.e., there exists a concave function k such that $\phi_2(U) = k(\phi_1(U))$ for all U ;
2. Beliefs \hat{q}^2 is dominated by \hat{q}^1 in the sense of the monotone likelihood ratio order, for any set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ satisfying the above-mentioned ranking.

Proof: Because ϕ_1 and ϕ_2 are increasing in U , there exists an increasing function k such that $\phi_2(U) = k(\phi_1(U))$, or $\phi_2'(U) = k'(\phi_1(U))\phi_1'(U)$ for all U . Using definition (15) and (18), we obtain that

$$\frac{\hat{q}_\theta^2}{\hat{q}_\theta^1} = k'(\phi_1(U(\alpha_1^*, \theta))) \frac{\sum_{t=1}^n q_t \phi_1'(U(\alpha_1^*, t))}{\sum_{t=1}^n q_t \phi_2'(U(\alpha_1^*, t))} \quad (19)$$

for all $\theta = 1, \dots, n$. The Lemma is a direct consequence of (19), in the sense that the likelihood ratio $\hat{q}_\theta^2/\hat{q}_\theta^1$ is decreasing in θ if k' is decreasing in ϕ_1 . ■

An increase in ambiguity aversion is characterized by the MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if agent ϕ_1 prefers marginal \tilde{x}_θ than marginal $\tilde{x}_{\theta'}$, then, compared to agent ϕ_1 , the more ambiguity-averse agent ϕ_2 increases the implicit prior probability $\hat{q}_{\theta'}^2$ relatively more than the implicit prior probability \hat{q}_θ^2 . Lemma 2 provides a justification to say that more ambiguity aversion is equivalent to more pessimism, i.e., to a MLR deterioration of beliefs. To be more precise, more ambiguity aversion and more pessimism share the same comparative static direction. This result is central to prove our next proposition, in which we consider three dominance orders: first-degree stochastic dominance ($D_1 = \text{FSD}$), second-degree stochastic dominance ($D_2 = \text{SSD}$), and Rothschild and Stiglitz's increase in risk ($D_3 = \text{IR}$).

Proposition 3 *Suppose that the expected excess return of the risky asset is positive. Suppose that the set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ can be ranked according the stochastic order D_i , $i = 1, 2$, or 3 . It implies that an increase in ambiguity aversion deteriorates the implicit probability distribution of the return of the risky asset in the sense of the stochastic order D_i . In short, if $\exists k$ concave: $\phi_2 = k(\phi_1)$ and $\tilde{x}_1 \preceq_{D_i} \tilde{x}_2 \preceq_{D_i} \dots \preceq_{D_i} \tilde{x}_n$, then*

$$\tilde{y}_2 \sim (\tilde{x}_1, \hat{q}_1^2; \dots; \tilde{x}_n, \hat{q}_n^2) \preceq_{D_i} (\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1) \sim \tilde{y}_1.$$

Proof: Suppose that $\tilde{x}_1 \preceq_{D_i} \tilde{x}_2 \preceq_{D_i} \dots \preceq_{D_i} \tilde{x}_n$. It implies that $U(\alpha_1^*, 1) \leq U(\alpha_1^*, 2) \leq \dots \leq U(\alpha_1^*, n)$. We have to prove that $(\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1)$ is preferred to $(\tilde{x}_1, \hat{q}_1^2; \dots; \tilde{x}_n, \hat{q}_n^2)$ by all utility functions v in C_i , that is

$$\sum_{\theta=1}^n \hat{q}_\theta^2 Ev(\tilde{x}_\theta) \leq \sum_{\theta=1}^n \hat{q}_\theta^1 Ev(\tilde{x}_\theta),$$

where C_1 is the set of increasing functions, C_2 is the set of increasing and concave functions, and C_3 is the set of concave functions. By assumption, $Ev(\tilde{x}_\theta)$ is increasing in θ . The above inequality is obtained by combining this property with the fact that \hat{q}^2 is dominated by \hat{q}^1 in the sense of MLR (Lemma 2), a special case of FSD. ■

This is an important result, which states that, to determine whether he should reduce his demand for the risky asset compared to agent ϕ_1 , agent ϕ_2 uses an implicit probability distribution \tilde{y}_2 for the excess return that is either FSD, SSD or IR dominated by the distribution \tilde{y}_1 used by agent ϕ_1 to determine his own optimal investment. To illustrate, if the marginals can be ranked according to the IR stochastic order, as in our counterexample, then the more ambiguity-averse agents will use a riskier implicit distribution for the excess return.

6 When does an increase in ambiguity aversion reduce the demand for the risky asset?

The following proposition combines the results presented in Propositions 1 and 3.

Proposition 4 *Suppose that the vNM utility function u is three times continuously differentiable, and that the unconditional expectation of excess returns is positive. Any increase in ambiguity aversion reduces the demand for the risky asset if one of the following two conditions is satisfied:*

1. *The set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ can be ranked according to the Rothschild and Stiglitz's riskiness order, and relative prudence is positive and less than 2;*
2. *The set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ can be ranked according first-degree stochastic dominance, and relative risk aversion is less than unity.*

More generally, if the set of marginals can be ranked according to the SSD order, an increase in ambiguity aversion reduces the demand for the risky asset if relative risk aversion is less than unity, and relative prudence is positive and less than two. As said before these conditions are not very convincing. Therefore, an alternative strategy would consist in restricting the set of priors rather than the investor's attitude towards risk.

Proposition 5 *Suppose that the unconditional expectation of excess returns is positive. Any increase in ambiguity aversion reduces the demand for the risky asset if the set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ can be ranked according to both Second-degree Stochastic Dominance and Central Dominance, that is, if $\tilde{x}_\theta \preceq_{SSD} \tilde{x}_{\theta+1}$ and $\tilde{x}_\theta \preceq_{CD} \tilde{x}_{\theta+1}$ for all $\theta = 1, \dots, n-1$.*

Proof: See the Appendix. ■

We can exploit this result by relying on the literature on the comparative statics of a change in risk. To illustrate, because we know that MLR yields both first-degree stochastic dominance and central riskiness, we directly obtain the following corollary.

Corollary 1 *Suppose that the set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ can be ranked according to the monotone likelihood ratio order, and that the unconditional expectation of excess return is positive. Then, any increase in ambiguity aversion reduces the demand for the risky asset.*

In this case, we conclude that ambiguity aversion and risk aversion goes into the same direction. A more general corollary holds where the MLR order is replaced by the more general MPR order. We also obtain the same result if marginals can be ranked according to the strong riskiness order, or to the simple riskiness order, or to any mixture of these stochastic orders, as stated in Proposition 5.

7 The equity premium

We can use these results to determine the effect of ambiguity aversion on the equity premium. Consider two Lucas tree economies, $i = 1, 2$, with a risk-averse and ambiguity-averse representative agent whose preferences are characterized by increasing and concave functions (u, ϕ_i) . Each agent is endowed with a tree producing an uncertainty quantity of fruits at the end of the period. The distribution of fruits is subject to some parameter uncertainty. Parameter θ can take value $1, \dots, n$ with probabilities (q_1, \dots, q_n) . Let \tilde{c}_θ denotes the random variable of fruits conditional on θ . Ex ante, there is a market for trees. The unit (future) price of trees is denoted P . The

problem of the representative agent is to determine which share α of her tree should be sold ex ante:

$$\max_{\alpha} \phi_i^{-1} \left[\sum_{\theta=1}^n q_{\theta} \phi_i(Eu(\tilde{c}_{\theta} + \alpha(P - \tilde{c}_{\theta}))) \right]. \quad (20)$$

The first-order condition is written as

$$\sum_{\theta=1}^n q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta} + \alpha_i^*(P - \tilde{c}_{\theta}))) E[(P - \tilde{c}_{\theta})u'(\tilde{c}_{\theta} + \alpha_i^*(P - \tilde{c}_{\theta}))] = 0. \quad (21)$$

The market-clearing condition requires that $\alpha_i^* = 0$, which yields the following pricing formula for the equilibrium price P_i of trees in the economy:

$$\sum_{\theta=1}^n q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta})) E[(\tilde{c}_{\theta} - P_i)u'(\tilde{c}_{\theta})] = 0,$$

or, equivalently,

$$P_i = \frac{\sum_{\theta=1}^n q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta})) E[\tilde{c}_{\theta} u'(\tilde{c}_{\theta})]}{\sum_{\theta=1}^n q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta})) E u'(\tilde{c}_{\theta})}. \quad (22)$$

It implies that the equity premium equals

$$\psi_i = \frac{[\sum_{\theta=1}^n q_{\theta} E \tilde{c}_{\theta}] [\sum_{\theta=1}^n q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta})) E u'(\tilde{c}_{\theta})]}{\sum_{\theta=1}^n q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta})) E [\tilde{c}_{\theta} u'(\tilde{c}_{\theta})]} - 1. \quad (23)$$

Suppose that the representative agent in economy $i = 2$ is more ambiguity-averse than in economy $i = 1$ in the sense of Klibanoff, Marinacci and Mukerji (2005). Does it imply a reduction in the equilibrium price of trees ($P_2 \leq P_1$) and, therefore an increase in the equity premium ($\psi_2 \geq \psi_1$)? From our earlier results in this paper, it is without surprise that the answer to this question is ambiguous, since an increase in ambiguity aversion does not necessarily imply a reduction in the demand for the ambiguous trees. Here is a counterexample. Suppose that there are only two plausible distributions $\tilde{c}_1 = 2 + \tilde{x}_1$

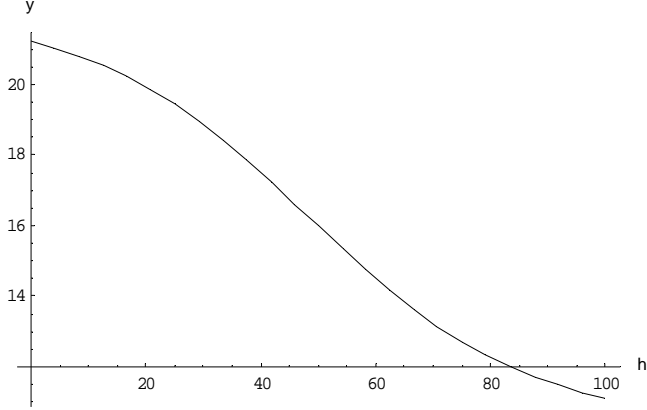


Figure 4: The equity premium ψ (in percent) has a function of the degree η of ambiguity aversion.

and $\tilde{c}_2 = 2 + \tilde{x}_2$ for the production of fruits per tree, where \tilde{x}_1 and \tilde{x}_2 are described in Figure 1. The beliefs of the representative agents are such that $q_1 = 1 - q_2 = 5\%$. The utility function u is the piecewise specification (7), whereas function ϕ_i exhibits constant absolute ambiguity aversion η_i . In Figure 4, we depicted the equity premium as a function of the degree η of absolute ambiguity aversion. The equity premium is smaller in more ambiguity-averse economies. The Savagian ambiguity-neutral economy has an equity premium equaling $\psi = 22.1\%$, whereas the Gilboa-Schmeidler maxmin economy has an equity premium equaling $\psi = 10.4\%$.

Because the left-hand side of the pricing formula (22) is decreasing in the price P_i , we obtain that P_2 is smaller than P_1 if and only the following inequality holds:

$$\sum_{\theta=1}^n q_{\theta} \phi_2'(Eu(\tilde{c}_{\theta})) E[(\tilde{c}_{\theta} - P_1) u'(\tilde{c}_{\theta})] \leq 0, \quad (24)$$

where P_1 is defined by (22) for $i = 1$. Technically, this condition is equivalent

to condition (16) with $\tilde{c}_\theta = w_0 + \alpha_1^* \tilde{x}_\theta$, $\alpha_1^* = 1$ and $P_1 = w_0$. We conclude this section with the following proposition, which is a direct consequence of this observation together with the results presented in the previous section.

Proposition 6 *An increase in ambiguity aversion raises the equity premium if one of the following conditions is satisfied:*

1. *The set of marginals $(\tilde{c}_1, \dots, \tilde{c}_n)$ can be ranked according to the Rothschild and Stiglitz's riskiness order, and relative prudence is positive and less than 2;*
2. *The set of marginals $(\tilde{c}_1, \dots, \tilde{c}_n)$ can be ranked according first-degree stochastic dominance, and relative risk aversion is less than unity;*
3. *The set of marginals $(\tilde{c}_1 - P_1, \dots, \tilde{c}_n - P_1)$ can be ranked according to both Second-degree Stochastic Dominance and Central Dominance, where P_1 is the initial price of equity.*

Notice that, contrary to SSD, central dominance is location-sensitive, in the sense that $\tilde{c}_1 \preceq_{CD} \tilde{c}_2$ does not necessarily imply that $\tilde{c}_1 - P_1 \preceq_{CD} \tilde{c}_2 - P_2$.⁸ However, several stochastic orders belonging to CD that we considered earlier in this paper are insensitive to the location of P_1 . This is the case in particular for the MLR order, for the MPR order and for strong increases in risk. This implies for example that the equity premium is increasing in the degree of ambiguity aversion of the representative agent if the set of marginals $(\tilde{c}_1, \dots, \tilde{c}_n)$ can be ranked according to the MLR order.

⁸Gollier (1997) defined the notion of Portfolio Dominance (PD) so that $\tilde{x}_1 \preceq_{PD} \tilde{x}_2$ if and only if $\tilde{x}_1 - P \preceq_{CR} \tilde{x}_2 - P$ for all P . PD is stronger than SSD.

8 Asset prices in an Arrow-Debreu economy

In this section, we extend the focus of our analysis to the effect of ambiguity aversion to the price of any Arrow-Debreu asset. In addition to allowing consumers to sell shares of their tree, we also allow them to trade claims of fruits contingent on the harvest. Suppose that there is a finite set of possible states of nature $s = 1, \dots, S$. The representative agent is endowed with c_s fruits in state s , $s = 1, \dots, S$. Assuming complete markets, the ambiguity-averse and risk-averse agent whose preferences are given by the pair (u, ϕ_i) solves the following problem:

$$\max_{(x_1, \dots, x_S)} \phi_i^{-1} \left[\sum_{\theta=1}^n q_{\theta} \phi_i \left(\sum_{s=1}^S p_{s|\theta} u(x_s) \right) \right], \text{ s.t. } \sum_{s=1}^S \Pi_s (x_s - c_s) = 0, \quad (25)$$

where $p_{s|\theta}$ is the probability of state s conditional to θ , x_s is the demand for the Arrow-Debreu security associated to state s , and Π_s is the price of that asset. The first-order conditions for this program are written as

$$u'(x_s) \left[\sum_{\theta=1}^n q_{\theta} \phi_i' \left(\sum_{s'=1}^S p_{s'|\theta} u(x_{s'}) \right) p_{s|\theta} \right] = \lambda \Pi_s, \quad (26)$$

for all s , where λ is the Lagrange multiplier associated to the budget constraint, which is hereafter normalized to unity. The market-clearing conditions impose that $x_s = c_s$ for all s , which implies that the following equilibrium state prices in economy i :

$$\Pi_s^i = \hat{p}_s^i u'(c_s), \quad (27)$$

for all s , where the implicit state probability \hat{p}_s^i is defined as follows:⁹

$$\hat{p}_s^i = \sum_{\theta=1}^n \hat{q}_{\theta}^i p_{s|\theta} \text{ with } \hat{q}_{\theta}^i = \frac{q_{\theta} \phi_i' (Eu(\tilde{c}_{\theta}))}{\sum_{t=1}^n q_t \phi_i' (Eu(\tilde{c}_t))}, \quad (28)$$

⁹Mukerji, Sheppard and Tallon (2005) were the first to make this point.

where \tilde{c}_θ is distributed as $(c_1, p_{1|\theta}; \dots, c_S, p_{S|\theta})$. The aversion to ambiguity of the representative agent affects the equilibrium state prices in a way that is equivalent to a distortion of beliefs in the EU model. This distortion takes the form of a transformation of the subjective prior distribution from (q_1, \dots, q_n) to $(\hat{q}_1^i, \dots, \hat{q}_n^i)$.¹⁰ Lemma 2 implies that \hat{q}^2 is dominated by \hat{q}^1 in the sense of MLR when $Eu(\tilde{c}_1) \leq Eu(\tilde{c}_2) \leq \dots \leq Eu(\tilde{c}_n)$ and ϕ_2 is more ambiguity-averse than agent ϕ_1 . The next proposition is a consequence of this observation.

Proposition 7 *Suppose that the set of marginals $(\tilde{c}_1, \dots, \tilde{c}_n)$ can be ranked according the stochastic order D_i , ($D_i = FSD, SSD$ or IR) . It implies that an increase in ambiguity aversion deteriorates the implicit probability distribution of consumption in the sense of the stochastic order D_i . In short, if $\exists k$ concave: $\phi_2 = k(\phi_1)$ and $\tilde{x}_1 \preceq_{D_i} \tilde{x}_2 \preceq_{D_i} \dots \preceq_{D_i} \tilde{x}_n$, then $(\tilde{c}_1, \hat{q}_1^2; \dots; \tilde{c}_n, \hat{q}_n^2) \preceq_{D_i} (\tilde{c}_1, \hat{q}_1^1; \dots; \tilde{c}_n, \hat{q}_n^1)$.*

Proof: The proof of this proposition is parallel to the proof of Proposition 3. ■

Thus, if the marginals can be ranked by FSD, an increase in ambiguity aversion yields a change in beliefs about the state probability distribution that is FSD-deteriorating (SSD-deteriorating). Similarly, an increase in ambiguity aversion makes the implicit probability distribution riskier if the marginals can be ranked by the riskiness order of Rothschild and Stiglitz.

The price kernel $(\pi_1^i, \dots, \pi_S^i)$ is defined as the vector of equilibrium state prices Π_s^i per unit of the state probability $\sum_\theta q_\theta p_{s|\theta}$:

$$\pi_s^i = \frac{\Pi_s^i}{\sum_{\theta=1}^n q_\theta p_{s|\theta}} = \frac{\hat{p}_s^i}{\sum_{\theta=1}^n q_\theta p_{s|\theta}} u'(c_s). \quad (29)$$

¹⁰It is noteworthy that, unlike the earlier sections, \hat{q}^2 is the implicit distorted belief used for pricing in the EU model with agent 2. This is due to the fact that the equilibrium conditions impose the same optimal portfolios for economies $i = 1$ and 2.

In an economy without ambiguity aversion, it is well-known that the state price per unit of probability is just a function of the aggregate consumption in that state. Indeed, in such an economy, \hat{p}_s^i equals $\sum_{\theta=1}^n q_{\theta} p_{s|\theta}$, and equation (29) can be rewritten as $\pi_s^i = u'(c_s)$. It is a decreasing function of c_s , a property that directly implies the positive equity premium.

In an economy with ambiguity aversion, this needs not to be the case anymore. In general, the state price per unit of probability will be a function of the aggregate consumption in that state, *and* of the degree of ambiguity of that state probability. To keep the analysis simple, suppose that $c_s \neq c_{s'}$ for all (s, s') , so that we can substitute index $s = 1, \dots, n$ by another index $s = c_1, \dots, c_S$. We see from (29) that ambiguity aversion adds a second multiplicative term $\hat{p}_s^i / \sum_{\theta=1}^n q_{\theta} p_{s|\theta}$ to $u'(c_s)$ in the pricing kernel. This second term is affected by an increase in ambiguity aversion that is described by Proposition 7. Suppose for example that the marginals can be ranked by the FSD order. Then, an increase in ambiguity aversion tends to transfer the implicit probability mass \hat{p} from the good states to the bad ones. This implies a clockwise shift in the price kernel, as illustrated in Figure 5a. If the marginals can be ranked according to their riskiness, an increase in ambiguity aversion tends to transfer the implicit probability mass to the extreme states. This implies convexifying the price kernel, as depicted in Figure 5b.

Notice that the price kernel needs not be decreasing in c in a model with ambiguity aversion, contrary to the classical result holding for the EU model. In particular, when the marginals can be ranked according to the Rothschild-Stiglitz riskiness order, it may be possible that the price kernel be increasing in c for large values of the GDP per capita c . This is due to the transfer of the implicit probability mass towards these states due to ambiguity aversion. This is in line with a recent observation by Rosenberg and Engle (2002) who computed the price kernel in the U.S.A. by using

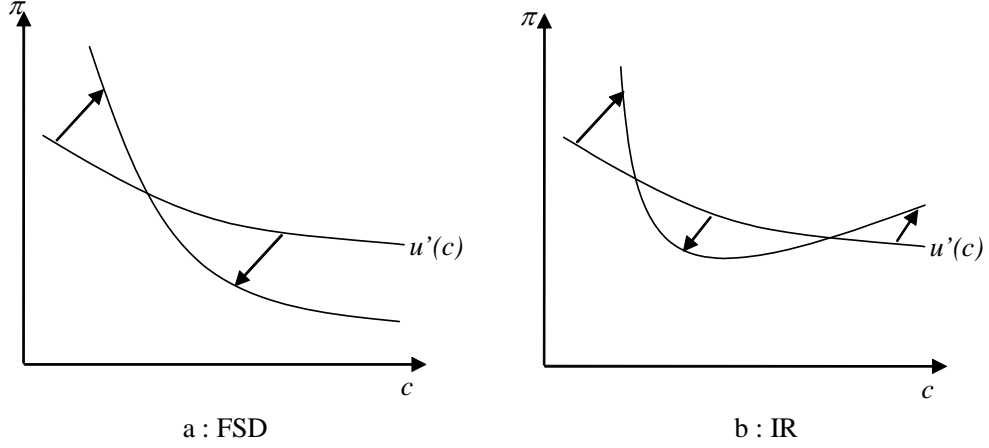


Figure 5: The effect of an increase in ambiguity aversion on the price kernel, when the marginals can be ranked by the FSD order (a), or by the Rothschild-Stiglitz riskiness order (b).

data on option prices. For low cs , they obtained an empirical price kernel described by curve R-E in Figure 6 that is much more convex than could be explained by the standard EU/CRRA model. Moreover, the price kernel is locally increasing for intermediate values of c . For larger cs , π is again decreasing and below what would be obtained by the EU/CRRA model. Therefore, Rosenberg and Engle's empirical price kernel can be explained by the ambiguity aversion of the representative agent and by a set of marginals that can be ranked according to the SSD order. Indeed, by Proposition 7, the ambiguity aversion implies a SSD-deteriorating shift in the implicit state probabilities, which would combine an increase in risk (shift from the classical curve A-N to curve R-R), and a FSD-deteriorating shift in the range of large c (shift from curve R-R to curve R-E).

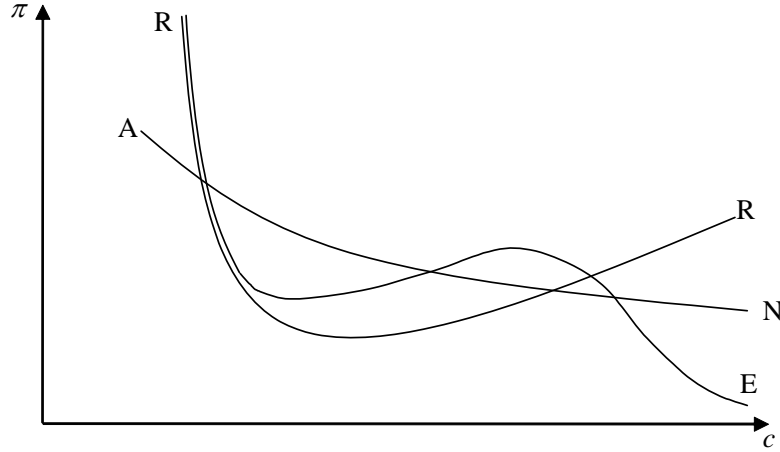


Figure 6: The Rosenberg and Engle (2002) empirical price kernel (R-E curve) can be obtained from the classical decreasing one (A-N curve) by a sequence of an increase in risk and a FSD-deteriorating shift.

9 Conclusion

In this paper, we explored the determinants of the demand for risky assets and of asset prices when investors are ambiguity-averse. We have shown that, contrary to the intuition, ambiguity aversion may yield an increase in the demand for the risky and ambiguous asset, and a reduction in the demand for the safe and unambiguous one. In the same fashion, it is not true in general that ambiguity aversion raises the equity premium in the economy. We have first shown that the qualitative effect of an increase in ambiguity aversion in these settings is equivalent to that of a shift in the beliefs of the investor in the standard EU model. If the set of plausible marginals can be ranked according to the FSD order, this shift is FSD deteriorating, whereas it is risk-increasing if these marginals can be ranked according to their riskiness.

The problem originates from the observation already made by Rothschild and Stiglitz (1971) and Fishburn and Porter (1976) that a FSD/SSD deteriorating shift in the distribution of the return of the risky asset has an ambiguous effect on the demand for that asset in the EU framework. We relied on the literature that emerged from this negative result to provide some sufficient conditions for any increase in ambiguity aversion to yield a reduction in the demand for the risky asset and an increase in the equity premium. We also examined the effect of ambiguity aversion on the price kernel.

References

- Athey, S.C., (2002), Monotone Comparative Statics Under Uncertainty, *Quarterly Journal of Economics*, 117(1) : 187-223.
- Abel, A.B., (2002), An exploration of the effects of optimism and doubt on asset returns, *Journal of Economic Dynamics and Control*, 26, 1075-1092.
- Black, J.M. and G. Bulkeley, (1989), A Ratio Criterion for Signing the Effect of an Increase in Uncertainty, *International Economic Review*, 30, 119-130.
- CEC (Commission of the European Communities), (2000), Communication from the commission on the Precautionary Principle, see at www.europa.eu.int.
- Chateauneuf, A. and G. Lakhnati, (2005), Increases in risk and demand for risky asset, mimeo, University Paris 1.
- Chen, Z., and L. Epstein, (2002), Ambiguity, risk, and asset returns in continuous time, *Econometrica*, 70, 1403-1443.
- Cecchetti, S.G., P.-S. Lam and N.C. Mark, (2000), Asset pricing with distorted beliefs: Are equity returns too good to be true?, *American Economic Review*, 90, 787-805.
- Dionne, G., L. Eeckhoudt and C. Gollier, (1993), Increases in Risk and Linear Payoffs, *International Economic Review*, 34, 309-319.
- Dionne, G., and C. Gollier, (1992), Comparative Statics Under Multiple Sources of Risk with Applications to Insurance Demand, *The Geneva Papers on Risk and Insurance Theory*, 17.

- Drèze, J. H. (1981). “Inferring Risk Tolerance from Deductibles in Insurance Contracts”, *The Geneva Papers* 6, 48-52.
- Eeckhoudt, L., and C. Gollier, (1995), Demand for Risky Assets and the Monotone Probability Ratio Order, *Journal of Risk and Uncertainty*, 11, 113-122.
- Eeckhoudt, L. and P. Hansen (1980), Minimum and maximum prices, uncertainty and the theory of the competitive firm, *American Economic Review*, Vol 70, 1064-1068.
- Fishburn, P., and B. Porter, (1976), Optimal portfolios with one safe and one risky asset: Effects of changes in rate of return and risk, *Management Science*, 22, 1064-73.
- Gilboa, I. and D. Schmeidler, (1989), Maximin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 141, 153.
- Gollier, C., (1995), The Comparative Statics of Changes in Risk Revisited, *Journal of Economic Theory*, 66, 522-535.
- Gollier, C., (1997), A Note on Portfolio Dominance, *Review of Economic Studies*, 64, 147-150.
- Gollier, C., (2001a), Should we beware of the Precautionary Principle?, *Economic Policy*, 16, 301-328.
- Gollier, C., (2001b), *The economics of risk and time*, MIT Press, Cambridge.
- Gollier, C., B. Jullien and N. Treich, (2000), Scientific Progress and Irreversibility: An Economic Interpretation of the Precautionary Principle, *Journal of Public Economics*, 2000, 75, 229-53.

- Hadar, J. and T.K. Seo, (1990), The Effects of Shifts in a Return Distribution on Optimal Portfolios, *International Economic Review*, 31, 721-736.
- Hansen, L.P., T.J. Sargent, and T.D. Tallarini, (1999), Robust permanent income and pricing, *Review of Economic Studies*, 66.
- Kandel, S., and R.F. Stambaugh, (1991), Assets returns and intertemporal preferences, *Journal of Monetary Economics*, 27, 39-71.
- Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53-73.
- Klibanoff, P., M. Marinacci and S. Mukerji, (2005), A smooth model of decision making under ambiguity, *Econometrica* 73(6), 1849-1892.
- Landsberger, M, and I. Meilijson, (1990), Demand for risky financial assets: A portfolio analysis, *Journal of Economic Theory*, 50, 204-13.
- Meyer, J. and M.B. Ormiston, (1985), Strong Increase in Risk and Their Comparative Statics, *International Economic Review*, 26, 425-437.
- Mukerji, S., K. Sheppard and J.-M. Tallon, (2005), Smooth ambiguity aversion and equity premium, paper presented at the RUD conference, Heidelberg.
- O’Riordan, T. and J. Cameron, (1995) , Interpreting the Precautionary Principle, Earthscan Publications, London.

- Ormiston, M. and E. Schlee, (1993), Comparative Statics Under Uncertainty for a Class of Economic Agents, *Journal of Economic Theory*, 61, 412-422.
- Rosenberg, J.V., and R.F. Engle, (2002), Empirical pricing kernel, *Journal of Financial Economics*, 64, 341-372.
- Rothschild, M. and J. Stiglitz, (1970), Increasing Risk: I. A Definition, *Journal of Economic Theory*, 2, 225-243.
- Rothschild, M. and Stiglitz, J.E., (1971), Increasing Risk II: Its Economic Consequences, *Journal of Economic Theory*, 3, pp. 225-243.
- Savage, L.J., (1954), *The foundations of statistics*, New York: Wiley. Revised and Enlarged Edition, New York: Dover (1972).

Appendix: Proof of Proposition 5

The following lemma is useful to prove Proposition 5. Let K denote interval $[\min_{\theta} \alpha_{\theta}^*, \max_{\theta} \alpha_{\theta}^*]$, where α_{θ}^* is the maximand of $Eu(w_0 + \alpha \tilde{x}_{\theta})$.

Lemma 3 *Consider a specific set of marginals $(\tilde{x}_1, \dots, \tilde{x}_n)$ and a concave utility function u . They characterize function U defined by $U(\alpha, \theta) = Eu(w_0 + \alpha \tilde{x}_{\theta})$. Consider a specific scalar α_1^* in K . The following two conditions are equivalent:*

1. *Any agent ϕ_2 that is more ambiguity-averse than agent ϕ_1 with demand α_1^* for the risky asset will have a demand for the risky asset that is smaller than α_1^* ;*
2. *There exists $\bar{\theta} \in \{1, \dots, n\}$ such that*

$$U(\alpha_1^*, \theta)U_{\alpha}(\alpha_1^*, \theta) \geq U(\alpha_1^*, \bar{\theta})U_{\alpha}(\alpha_1^*, \theta) \quad (30)$$

for all $\theta \in \{1, \dots, n\}$.

Proof: We first prove that condition 2 implies condition 1. Consider an agent $\phi_2 = k(\phi_1)$ that is more ambiguity-averse than agent ϕ_1 , so that the transformation function k is concave. The condition thus implies that

$$k'(\phi_1(U(\alpha_1^*, \theta)))U_{\alpha}(\alpha_1^*, \theta) \leq k'(\phi_1(U(\alpha_1^*, \bar{\theta})))U_{\alpha}(\alpha_1^*, \theta)$$

for all θ . Multiplying both side of this inequality by $q_{\theta}\phi_1(U(\alpha_1^*, \theta)) \geq 0$ and summing up over all θ yields

$$\sum_{\theta=1}^n q_{\theta}\phi_2'(U(\alpha_1^*, \theta))U_{\alpha}(\alpha_1^*, \theta) \leq k'(\phi_1(U(\alpha_1^*, \bar{\theta}))) \sum_{\theta=1}^n q_{\theta}\phi_1'(U(\alpha_1^*, \theta))U_{\alpha}(\alpha_1^*, \theta) = 0.$$

The last equality comes from the assumption that agent ϕ_1 selects portfolio α_1^* . Thus, condition (16) is satisfied, thereby implying that α_2^* is less than α_1^* .

We then prove that condition 1 implies condition 2. Without loss of generality, rank the θ s such that $U(\alpha_1^*, \theta)$ is increasing in θ . By contradiction, suppose that there exists a $\theta_0 < n$ such that $U_\alpha(\alpha_1^*, \theta_0) \geq 0$ and $U_\alpha(\alpha_1^*, \theta_0 + 1) \leq 0$. Select a prior distribution (q_1, \dots, q_n) so that $q_\theta = 0$ for all θ except for θ_0 and $\theta_0 + 1$. Select $q_{\theta_0} = q \in [0, 1]$ so that

$$q\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) + (1 - q)\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1) = 0, \quad (31)$$

so that agent ϕ_1 selects portfolio α_1^* . Consider any concave transformation function k . It implies that

$$\begin{aligned} \sum_{\theta=1}^n q_\theta \phi_1'(U(\alpha_1^*, \theta))U_\alpha(\alpha_1^*, \theta) &= qk'(\phi_1(U(\alpha_1^*, \theta_0)))\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) \\ &\quad + (1 - q)k'(\phi_1(U(\alpha_1^*, \theta_0 + 1)))\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1). \end{aligned}$$

Because $U_\alpha(\alpha_1^*, \theta_0 + 1) \leq 0$ and $k'(\phi_1(U(\alpha_1^*, \theta_0 + 1))) \leq k'(\phi_1(U(\alpha_1^*, \theta_0)))$, this is larger than

$$k'(\phi_1(U(\alpha_1^*, \theta_0))) [q\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) + (1 - q)\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1)] = 0.$$

It implies that condition (16) is violated, implying in turn that α_2^* is larger than α_1^* , a contradiction. ■

In Figure 7, we draw an example for which condition 2 in Lemma 3 is satisfied. If we rank the θ in such a way that $U(\alpha_1^*, \theta)$ is monotone in θ , this condition is essentially a single-crossing property of function $U_\alpha(\alpha_1^*, \theta)$. To illustrate, suppose that $u(z) = -A^{-1} \exp(-Az)$ and $\tilde{x}_\theta \sim N(\theta, \sigma^2)$, which implies that $U(\alpha, \theta)$ is increasing in θ and is given by equation (2). It implies that $U_\alpha(\alpha, \theta)$ has the same sign as $\theta - \alpha A \sigma^2$. It implies in turn that condition

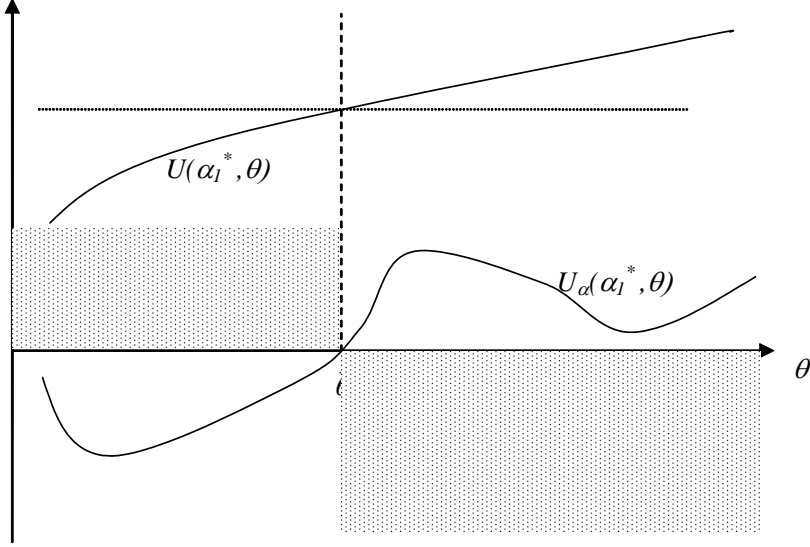


Figure 7: An example for which condition 2 in the lemma is satisfied.

2 in Lemma 3 is satisfied with $\bar{\theta} = \alpha A \sigma^2$. Our Lemma implies that ambiguity aversion reduces the demand for the risky asset in the CARA/Normal case. This was shown in Section 2 in the special case of power ϕ functions.

We need to prove a second lemma in order to prepare for the proof of Proposition 5.

Lemma 4 *Suppose that \tilde{x}_b is centrally dominated by \tilde{x}_a . Then, $E\tilde{x}_b u'(w_0 + \alpha \tilde{x}_b) \leq 0$ for any $\alpha \geq 0$ such that $E\tilde{x}_a u'(w_0 + \alpha \tilde{x}_a) \leq 0$.*

Proof: By assumption, there exists a positive scalar m such that $T_b(x) \leq mT_a(x)$. Integrating by part, we have that

$$\begin{aligned} E\tilde{x}_b u'(w_0 + \alpha \tilde{x}_b) &= \int_{x_-}^{x_+} u'(w_0 + \alpha x) x dF_b(x) \\ &= u'(w_0 + \alpha x_+) T_b(x_+) - \alpha \int_{x_-}^{x_+} u''(w_0 + \alpha x) T_b(x) dx. \end{aligned}$$

This implies that

$$\begin{aligned} E\tilde{x}_b u'(w_0 + \alpha\tilde{x}_b) &\leq m \left[u'(w_0 + \alpha x_+) T_a(x_+) - \alpha \int_{x_-}^{x_+} u''(w_0 + \alpha x) T_a(x) dx \right] \\ &= m E\tilde{x}_a u'(w_0 + \alpha\tilde{x}_a). \end{aligned}$$

By assumption, this is nonpositive. ■

We can now prove Proposition 5.

Proof of Proposition 5: Condition $\tilde{x}_\theta \preceq_{SSD} \tilde{x}_{\theta+1}$ implies that $U(\alpha, \theta + 1) \geq U(\alpha, \theta)$, whereas, by Lemma 4, condition $\tilde{x}_\theta \preceq_{CD} \tilde{x}_{\theta+1}$ implies that $U_\alpha(\alpha, \theta) \leq 0$ whenever $U_\alpha(\alpha, \theta + 1) \leq 0$. This latter result implies that there exists a $\bar{\theta}$ such that $(\theta - \bar{\theta})U_\alpha(\alpha, \theta) \leq 0$ for all θ . This immediately yields condition 2 in Lemma 3, which is sufficient for our comparative static property. ■