Does there exist the Lebesgue measure in the infinite-dimensional space?

A.M. Vershik

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To V. I. Arnold with profound respect.

Abstract

We consider the sigma-finite measures in the space of vector-valued distributions on the manifold X with characteristic functional

$$\Psi(f) = \exp\{-\theta \int_X \ln ||f(x)|| dx\},\ \theta > 0.$$

The collection of such measures constitutes a one-parameter semigroup relative to θ. In the case of scalar distributions and $\theta = 1$, this measure may be called the *infinite-dimensional Lebesgue measure. We prove that the weak limit of Haar measures on the Cartan subgroup of the group $SL(n, \mathbb{R})$ when n tends to infinity is that infinite dimensional Lebesgue measure.* This measure is invariant under the linear action of some infinite-dimensional Abelian group that can be viewed as an analog of an infinite-dimensional Cartan subgroup; this fact can be a justification of the name *Lebesgue* as a valid name for the measure in question. Application to the representation theory of the current groups was one of the reason to define this measure. The measure also is closely related to the Poisson–Dirichlet measures well known in combinatorics and probability theory.

The only known example of the analogous asymptotical behavior of the uniform measure on the homogeneous manifold is *classical Maxwell-Poincaré lemma which asserts that the weak limit of uniform measures on the Euclidean sphere of appropriate radius as dimension tends to infinity is the standard infinite-dimensional Gaussian measure.* Our situation is similar but all the measures are no more finite but sigma-finite. The result raises an important question about the existence of other types of the interesting asymptotic behavior of invariant measures on the homogeneous spaces of Lie groups.

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[*] Steklov Mathematical Institute at St. Petersburg. E-mail: `vershik@pdmi.ras.ru`. Partly supported by the grants RFBR 05-01-0089 and CRDF RUM1-2622-ST-04.

Dedication. On the 70th anniversary of Arkady Raikin, one actor addressed him with approximately these words:

"Some of us attend some performances of some of their friends-actors from time to time; however, ALL of us, without exception, have seen ALL your programmes".

Similarly, I want to say the mathematical analog of this:

"Some of us (mathematicians) sometimes read some papers written by some of their colleagues, but ALL of us, without exception, read ALL Arnold's papers!"

Contents

1 Introduction

1.1 On asymptotic approach to measure and integration in infinite-dimensional spaces

In his remarkable but less known, compared with other works, paper “Approximative properties of matrices of high finite order” ([34]), J. von Neumann wrote that experts in functional analysis neglect problems concerning spaces of high finite dimension in favor of the study of actually infinite-dimensional spaces. Possibly, in the last third of the 20th century the situation has slightly changed, but one still cannot say that we understand analysis in the spaces of dimension, say, 10^{24} better than in the infinite-dimensional Hilbert space (where “almost everything is clear”!)[1] Specifically, this is true in what concerns problems in measure theory and integration in infinite-dimensional spaces. Never-ceasing attempts to justify the notion of Feynman integral, which is so important to physicists, and to embed it into one or another general scheme of integration over a measure do not evoke interest or approval of physicists and apprehension of mathematicians. It is easy to understand this lack of enthusiasm: physical modelling is always or almost always based upon asymptotic constructions (in dimension, number of particles, some constants, etc.). On the contrary, mathematicians usually try to interpret these constructions as actually infinite (infinite-dimensional). This is productive and necessary within some limits but inevitably results in certain difficulty of interpretation when one tries to absolutize the limiting constructions. Certainly, it is impossible to say that asymptotic approach can be a substitute of actually infinite constructions, and there is no need in such substitution. It is important to understand what effects, in the infinite-

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[1] A subheading of Chapter 5 in the E. Borel’s book [2] reads: “Functions in high number of variables: areas and volumes in the geometry of 10^{24}-dimensional spaces”.

dimensional case, really survive, or grow out of asymptotic finite-dimensional properties, and how to obtain them. We will investigate an example of asymptotic behavior of measures on classical homogeneous spaces which leads to a remarkable limiting measure (to be precise, a one-parameter family of measures), the Lebesgue measure in the infinite-dimensional space. This measure (in different, actually infinite terms) was earlier discovered in connection with representation theory of current groups [16]. The role it plays in combinatorics and representation theory is probably not smaller than that of the Gaussian measure. Its properties and deep connections with, for instance, Poisson–Dirichlet measures are probably covered in this work for the first time (cf. [19]) and need further investigation.

1.2 About this paper

We begin Subsection 2 with the classical and well-known calculation, the so-called "Poincaré's Lemma" that substantiates the Maxwell (Gaussian) distribution of velocities in statistical physics. This example shows how the infinite-dimensional Gaussian measure ("white noise") arises as a limiting distribution of the radius-vector of a point on the Euclidean sphere as the dimension and the radius of the sphere tend to infinity coherently. The aim of the present paper is to demonstrate that, as in the above-mentioned example, which will systematically play the role of a reference example for the main theme of the paper, there exists another series of homogeneous spaces of the Cartan subgroup in $SL(n,\mathbb{R})$, on which the invariant measures, as in the case of Maxwell-Poincaré's Lemma, weakly converge to quite another, now sigma-finite, measure which reminds the infinite-dimensional Lebesgue measure. The symmetry group of the measure in question is as large as in the case of the Gaussian measure, but quite different one. This measure is related to the remarkable Poisson–Dirichlet measures of combinatorial origin. There is reason to compare the Wiener and our Lebesgue measures: they can be viewed as the measures corresponding to the extreme values in the segment $\alpha \in [0,2]$ whose inner points parameterize Lévy measures of stable laws; to be more precise, our measure is the derivative of these measures over α at the point 0. Apart from interest per se, the measures in question are used (and first appeared) in representation theory of current groups. However, our main aim is the description of these measures in the geometric and asymptotic aspects. In Sect. 2, we give proofs of the Maxwell-Poincaré Lemma which we use for comparison in many situations. This comparison is useful and allows us to outline further natural generalizations. We comment this lemma from various points of view.

In Sect. 3, we consider the orbits of the Cartan subgroup $SDiag(n,\mathbb{R})$ in the group $SL(n,\mathbb{R})$. It is convenient to start with the study of the positive part $SDiag_+(n\mathbb{R})$ of the Cartan subgroup and of its orbit, postponing the general case to Sect. 4. Further, we embed the orbits into the cone K^+ of positive step functions on the segment and define the weak convergence of the invariant measures on these orbits as the convergence of their Laplace transforms. The limit of the Laplace transformations of the properly normalized measures on $SDiag_+(n\mathbb{R})$ is the functional

$$\Phi_\theta(f) = \exp\Big\{ -\theta \int_X \ln f(x)dx \Big\}, \quad \theta > 0,$$

and we use several methods of finding it. This functional is defined on the set of functions whose logarithm has finite integral; it is invariant under all changes of variables keeping the measure invariant and under multiplication by functions whose logarithm has zero integral.

The main object of Sect. 4 is an explicit definition of the sigma-additive sigma-finite measure $\mathcal{L}_\theta^+$. First we define the weak distribution Ξ_θ (Subsect. 4.1) on the cone K whose Laplace transform is Φ_θ. Then we introduce the cone D_+ of discrete positive measures of finite mass defined on X, which is in duality with the cone K^+. Thus the weak distribution Ξ_θ may be viewed as a premeasure on D_+. We emphasize once more that our object is not finite but infinite weak distributions and measures. So the usual tools like projections, etc., cannot be applied here. The final step of the construction is the proof of the existence of a true sigma-additive measure that is a continuation of our weak distribution. This is done in a constructive way using an infinite ("poissonized", or conic) version of the Poisson–Dirichlet measures $PD(\theta)$, $\theta > 0$, which became popular in the last years. These measures are defined on the simplex of monotone positive series with sum one; we describe them in Appendix 1. We need their sigma-finite versions, PDC, the "conic Poisson–Dirichlet measures", which are defined on the cone of monotone convergent positive series. These measures are direct products of the Poisson–Dirichlet measure $D(\theta)$ and the measure on the half-line L_θ defined by the density $t^{\theta-1}/\Gamma(\theta)$. (The measure on the half-line is a "distribution" of the sum of the series.)

In Subsection 4.2, which plays a central role in our exposition, we define the principal object, the multiplicative measures $\mathcal{L}_\theta$, as *an image of the product of the Bernoulli measures* m^∞ *and the conic Poisson–Dirichlet measures* $PDK(\theta)$ described above.

These measures are eventually the weak limits of the measures defined on the sequence of $SDiag_+(n,\mathbb{R})$-orbits. The measure corresponding to $\theta = 1$ is called the multiplicative Lebesgue measure on the cone D_+.

Thus, our scheme of the introducing the multiplicative measures, the Lebesgue measure in particular, is the following.

We define the measures on the orbits of the Cartan subgroup, then find the limit of their Laplace transform; the latter is the Laplace transform of some weak distribution and we define the measure, which is a continuation of this distribution, by taking an explicit image of the conic Poisson–Dirichlet measures multiplied by the Bernoulli product measure.

In this apparently long way the concluding step does not depend on the preceding ones. This allows one to introduce the measures we are looking for directly, independently of the preceding steps. However, this economy of efforts conceals the asymptotic and geometric sense of the measure constructed. The reader who does not care of this sense can pass to the Subsection 4.2 immediately after the introduction.

We summarize the properties of these measures.

1. They are the weak limits of the measures on orbits of the positive Cartan subgroups;
2. Their Laplace transform is

$$\Phi_\theta(f) = \exp\Big\{ -\theta \int \ln f(x)dx \Big\}, \quad \theta > 0;$$

3. They are the images of the product of the Poisson–Dirichlet measures on the simplex of positive convergent series summing to one by the Bernoulli measure and the Lebesgue measure on the half-line.

On the other hand, these measures behave like the laws of Lévy processes, but with infinite probability: our measures are absolutely continuous and even equivalent to the laws of Lévy gamma processes on subordinators. Exactly in this way they were defined in [19], and eventually in this way they were discovered in [14, 15]. In Subsection 4.3, we connect these measures to Lévy gamma processes, subordinated or complete. In [16, 19], an opposite way to define the measures is adopted. They are defined via a gamma process by the introduction of densities. This method is less analytic and transparent, especially in the infinite-dimensional case.

In Subsection 4.4, we give an additive version of the description of these measures and show that they present the first example of a sigma-finite measure invariant under shifts by vectors of an infinite-dimensional Banach space.

Further, in Subsection 5.1, the main definition and all the other definitions are repeated in the case of signed measures; the cone is replaced by the vector space D, positive series by absolutely convergent ones, etc. This transition is easy, and the most important properties are already visible in the "positive" version. These extended and most important measures have the following properties.

1. They are the weak limits of measures on the orbits of the complete Cartan subgroup.
2. Their Laplace transform is

$$\Phi_\theta(f) = \exp\Big\{ -\theta \int \ln |f(x)|dx \Big\}, \quad \theta > 0$$

   (the logarithm is replaced by the logarithm of the modulus).

3. They are the images of the products of extended Poisson–Dirichlet measures on the octahedron composed by all decreasing (in modulus) absolutely convergent series, a Bernoulli measure, and the Lebesgue measure on the line.

4. Finally (the most important): *these measures are invariant relative to the group of multiplicators by the functions f with zero integral of* $\log|f|$*, they are projectively invariant relative the multiplication by the functions f with finite integral of* $\log|f|$*, and (Subsection 5.2) they are invariant under the changes of variables that leave the measure invariant.*[2]

In Subsection 5.3, we remind the connection of these measures with representation theory of current groups. Finally, in Subsection 5.4 we define a generalization of the Lebesgue and the Poisson–Dirichlet measures to the vector case, which is necessary for the representations of current groups with coefficients in the group $SO(n,1)$.

The first appendix contains the most important information about the Poisson–Dirichlet measures and their applications in probability, algebra, and number theory. In the second appendix we discuss the conditions that are imposed on the group of admissible shifts by the properties of invariance and quasi-invariance of the measures under this group, and we explain what is new in the additive approach to infinite-dimensional Lebesgue measures introduced here.

2 A brief historic digression: white noise according to Maxwell–Poincaré–Borel, and commentaries

2.1 Maxwell-Poincaré's Lemma

A remarkable example of asymptotic approach to infinite-dimensional objects is presented by the following way to introduce the Maxwell-Boltzmann distribution in mathematical physics. Consider the small canonical ensemble of the velocities of a system of identical particles with energy

$$H(v_1, \dots v_n) = \frac{1}{2}\sum_k ||v_k||^2.$$

Since we do not care about the dimension, the velocities may be treated as scalars ($d = 1$). A natural measure carried by the small ensemble is the normalized Lebesgue measure on the corresponding Euclidean sphere (because the measure must be orthogonally invariant). On the other hand, consider the canonical ensemble of velocities with Gibbs measure, i.e., the

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[2] I.e., they are invariant relative to the normalizer of the infinite-dimensional torus (= the group of multiplication operators).

measure with density $\exp\{-H(v_1,\dots v_n)\}$, $c > 0$, on it. When normalized, it becomes the standard Gaussian measure. Then we increase the number of particles and, simultaneously, the total energy. The question is: do the asymptotic distributions in both ensembles coincide? The answer is contained in the following beautifully simple fact that can be formulated, in the current terms, as follows.

Theorem 1. *Consider the sequence of the normalized Lebesgue measures on the Euclidean spheres* $S^{n-1}_{r_n} \subset \mathbb{R}^n$ *of radius* $r_n = c\sqrt{n}$, $c > 0$, *and the limit of spaces*

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^n \subset \cdots \subset \mathbb{R}^\infty.$$

Then the weak limit of these measures is the standard Gaussian measure μ *which is the infinite product of the identical Gaussian measures on the line with zero mean and variance* c^2. *It is clear that the sequence of Gibbs measures has the same weak limit.*

Thus the infinite-dimensional ensemble that is the limit of both canonical ensembles in the above-described sense exists.

Proof. The weak convergence of a sequence of measures is, by definition, the convergence of the corresponding sequence of finite-dimensional distributions for any finite collection of linear functionals. In its turn, it is sufficient for this that the distribution of a single (arbitrary) functional converge; for instance, one can consider the functional that takes the first coordinate of a vector. As a result, the question reduces to the following calculation. One should find the limiting distribution of the projection of the Lebesgue measure on the sphere $S^{n-1}_{r_n}$ onto the first coordinate. The density relative to the Lebesgue measure of the projection of the (normalized) such measure is $C_n(r_n^2 - x^2)^{\frac{n-2}{2}}$. After an evident renormalization, as $r/\sqrt{n} \to \theta > 0$, we get the density $C\exp(-\theta x^2)$, $\theta > 0$, of the Gaussian measure as a limit. □

The same result can be obtained in a number of different ways. For example, one can use the Fourier transform and consider the asymptotic behavior of Bessel functions. For the sphere S^{n-1}, set $\nu = (n-2)/2$. From [35], formula 3.771.8

$$\int_0^r (r^2 - x^2)^{\nu-1/2} \exp(itx)dx = C \cdot \left(\frac{2r}{t}\right)^{\nu} J_\nu(tr),$$

where $C = \frac{\sqrt{\pi}/2}{\Gamma(\nu+1/2)}$, and the formula giving the asymptotical behavior of the Bessel function $J_\nu(.)$ as its argument and number ν tend to infinity, it follows that

$$\lim_{n\to\infty} \int_{S^{n-1}_{r_n}} \exp(it\omega)d\Omega_n(\omega) = \exp(-\theta t^2)$$

as $r/\sqrt{n} = r/\sqrt{2(\nu+1)} \to \theta > 0$, where Ω_n is the normalized Lebesgue measure on the sphere $S^{n-1}_{r_n}$. Thus the sequence of the Fourier transforms tends to the Fourier transform of the Gaussian measure, and the weak convergence of the measure follows. We give an analog of this very proof in the situation in question replacing Fourier transform with Laplace transform.

2.2 Comments

1. A more serious comprehension of the latter calculation is the following. This demonstration can be viewed as the derivation of the empirical distribution of the first (and then any) coordinate of the vector in the space $\mathbb{R}^\infty$ relative to an a priory unknown spherically invariant measure. Indeed, it follows from the general ergodic and martingale convergence theorems (see the so-called ergodic method in [12, 13]) that the limit of such functional is the limit of its empirical distributions for any probability Borel ergodic measure in the space $\mathbb{R}^\infty$ that is invariant under the action of all finite-dimensional orthogonal (in the l^2 sense) groups (and, consequently, under the whole infinite-dimensional orthogonal group $O(\infty)$ in l^2). But the thing is that we do not know in the beginning what set of vectors constitutes the set of "almost all" vectors relative to the measure we are looking for, and so we do not know what orbits to take. However, the theorems cited imply that taking all the orbits we will not miss any invariant ergodic measure. It turns out that in our case it suffices to take orbits having the form

$$(\underbrace{x, x, \dots x}_{n}, 0, 0 \dots),$$

these and only these orbits give all necessary measures, the other sequences of orbits do not have nontrivial limits. This is the manifestation of the fact that the average square of the norm of such vector relative to the Gaussian measure grows proportionally to n, and consequently there are no ergodic measures except the Gaussian ones. It is clear that the knowledge of the distributions of all (in our case, one) linear functionals defines the measure completely.

It immediately follows that the general spherically invariant measure is a mixture of Gaussian measures with various dispersions, i.e., the general form of the characteristic functional of a spherically invariant measure is the following: $\int_0^\infty \exp(-cx^2)dm(c)$. Hence the Schoenberg theorem follows which states that all indecomposable positive definite normalized functions of the norm of a vector in an infinite-dimensional Hilbert space have the form $\phi(h) = \exp(-||h||^2)$. This fact, which is essentially one of the versions of the ergodic theorem (or the martingale theorem), makes it possible to describe all invariant measures, not only in this particular example but also in the general case, by choosing in a special way the orbits of the subgroups that approximate the given group. This is essentially what we do in the example of noncompact Cartan subgroups, where we also describe all invariant measures.

2. A more delicate fact, which we will use below, is that the action of the whole infinite-dimensional orthogonal group O^∞ in the space $\mathbb{R}^\infty$ should be meant only in the sense that every orthogonal operator $g \in O^\infty$ is defined, and acts leaving the Gaussian measure invariant, on a *certain* measurable linear subspace of total measure (it can be easily constructed using, for example, the spectral decomposition of g in l^2) that *depends on the operator*, but a common *linear measurable subspace* where all orthogonal operators were defined simultaneously does not exist, as was proved in [11]. It was also shown recently in [36] that no measurable set of total measure exist where all the elements of the group O^∞ were defined simultaneously. [3] This gives an example of the group action that does not admit an individual measurable realization. It is well known that in the case of locally compact groups a measurable realization always exists.

3. One can define a measure invariant under arbitrary group possessing a dense subgroup that is a union of an increasing sequence of compact or locally compact subgroups in a similar manner (this is the ergodic method of the description of invariant measures, characters, etc.) We choose an orbit for any subgroup from the given sequence of groups and take an invariant measure on the orbit. Then we look for all cases when these measures on the orbits weakly converge. The ergodic theorem or the martingale convergence theorem guarantee that the list of invariant measures thus obtained is complete. The case of compact groups is simpler. For the Maxwell-Poincaré– case, the orbits are n-dimensional spheres of radius $c\sqrt{n}$, and the Lebesgue measures on them weakly converge to the Gaussian measure. Exactly in the same way, changing the spheres and embedding maps, one can obtain any Gaussian measure in the infinite-dimensional space, since they all are linearly isomorphic. For example, the Wiener measure can be constructed in this manner. We will use the described technique for noncompact groups in what follows.

4. Some remarks of historical character. The above calculation can be found in many books and papers. Most commonly, it is called Poincaré's Lemma, or even the Maxwell Theorem [5], (make sense to mention also the name L.Boltzmann - Maxwell-Boltzmann distribution). Yet a number of authors [7, 9] claim that they could not find this lemma nowhere in the papers by Poincaré. E. Borel quotes it many times [2, 3]; however, he does not mention Poincaré in this connection while abundantly quoting him on many other occasions [1]. D. Strook, G. McKean and M. Yor [8] showed me a paper [4] (1866) by the German mathematician F. Mehler where on can already find this calculation; It seems that E. Borel did not know about this work. In

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[3] It was not mentioned in [36] that the absence of common *linear subspace* was proved in [11].

fact, there is a theorem in [4] that the generating function of spherical harmonics converges, as its index increases, to the generating function of Hermite polynomials. This evidently implies our modest fact (and even the convergence of all the moments of the distributions); however, the geometrical picture that is the essence of the method remains concealed in this general theorem. H. McKean informed me that, among the others, M. Kac mentioned H. Poincaré as the author of this statement. See also the recent preprint by P. Cartier [6]. One can guess that H. Poincaré mentioned this method of obtaining Maxwell's distribution in his lectures but has not written it down: the fact that he was aware of this calculation can be seen from his lectures [1]. Thus, according to the principle expressed by many authors (some of whom, following this very principle, attribute the principle itself to V. I. Arnold) which states that the names ascribed by the later generations to theories, theorems, lemmas rarely belong to the true discoverers of these theories etc., we continue to call the statement in question Maxwell-Poincaré's Lemma, taking a risk to violate the (possibly erroneous) tradition.

In the present paper we show that in another, non-compact, sigma-finite version, the analogous asymptotic method brings us not to the Gaussian measure, but to a no less remarkable infinite-dimensional measure. It appeared earlier in representation theory of the current group [15] and, as it turned out later, is closely related to the Lévy gamma process. We will describe it in various aspects but will show what is the most natural way to discover it using geometric approach.

3 Orbital measures on a Cartan subgroup and the limit of their Laplace transforms

3.1 Orbits of the Cartan subgroups

Instead of $(n-1)$-dimensional spheres S_r^{n-1} of radius r in Maxwell-Poincaré's Lemma, we consider the hypersurfaces

$$M_r^{n-1} = \left\{(x_1, \dots x_n) : \prod_{k=1}^{n} x_k = r^n > 0;\ x_k > 0\right\}$$

in $\mathbb{R}^n$. The number r wil be called the *radius* of the hypersurface and will be specified later. On this hypersurface M_r^{n-1} (for all r), the group $SDiag_+(n, \mathbb{R})$ of positive diagonal matrices with determinant one, i.e., the positive part of the Cartan subgroup of the group $SL(n, \mathbb{R})$, acts freely and transitively. Therefore, an invariant sigma-finite measure m_n, which is finite on any bounded set, is defined on the hypersurface; this measure is the image of the Haar measure on $SDiag_+(n, \mathbb{R})$. In the sequel, it is important that when the radius is multiplied by a positive number, the invariant measure also changes being multiplied by the nth power of this number, though it remains an image of the Haar measure. Our aim, as in the Maxwell-Poincaré's Lemma, to find under what conditions the sequence of the measure spaces $(M_r^{n-1},\ m_n)$ has a limit in some sense and to study the properties of the limiting measure. The difference with the spherical case are rather important. First of all, in our case the measure m_n is not a probability measure any more but only a sigma-finite one. Second, the symmetry group is commutative while in the spherical case it is the group $SO(n)$. All this brings us to a different interpretation of the weak limit. In particular, the manifolds M_r^{n-1} are embedded into the space of distributions, not into the space of sequences $(\mathbb{R}^\infty)$ as in the case of spheres.

Notice that the positivity property of the coordinates x_k and of the group will be lifted in the sequel and we will consider the whole group $SDiag(n, \mathbb{R})$; however, the main point of the problem will clear up already in this particular case.

3.2 Embedding of the orbits into the cone

Consider the cone $K(X) = K$ of measurable positive step functions on the interval $X = [0, 1]$ with the Lebesgue measure (we take the interval only for the sake of simplicity, one can replace X by an arbitrary measure space isomorphic to the interval, see below). We embed the hypersurface M_r^{n-1} into K sending each vector $(x_1, \dots x_n)$ to the function taking the values x_k on the intervals $[\frac{k}{n}, \frac{k+1}{n})$. The image of the manifold M_r^{n-1} in K is the orbit of the constant function $\phi(t) = r^{1/n}$ relative to the corresponding piecewise linear action of the group $SDiag(n, \mathbb{R})$ on the space of step functions. The topology on the cone K will be defined, for instance, as a usual weak topology. In other words, we introduce the duality of the cone K and itself by the form $\langle f, g\rangle = \int_X f(x)g(x)dx$, and consider the weak topology corresponding to this duality. The cone K is not complete in this topology. We carry the measure m_n from the hypersurface M_r^{n-1} to its image and thus get a sigma-finite measure $\mu_{n,r}$ on the cone K that is concentrated on the set of the functions that take constant positive values on the intervals $[\frac{k}{n}, \frac{k+1}{n})$, with the product of all values equal to r^n, where r depends on n in general.

3.3 Weak convergence: definitions

Consider real Borel finite or sigma-finite measures on K which take finite values on precompact (= relatively compact) sets in K. We will introduce a notion of weak convergence in itself for Borel measures. This can be done in a traditional way defining the convergence of measures as the convergence of the integrals on a certain class of functions or sets. Minor difficulties arise as a result of the infiniteness of measures. However, we adopt here, for the sake of brevity, a more direct and convenient way. In what follows, we restrict ourself only with those measures μ on the cone K for which the *Laplace transform* $\widehat{\mu}$ (or the characteristic functional) *is defined for every step function* $f \in K$:

$$\widehat{\mu}(f) \equiv \int_K \exp\Big\{-\int_X f(x)g(x)dx\Big\}d\mu(g) < \infty,$$

and, in accordance with this notion, we assume the following definition.

Definition 1. *A sequence of sigma-finite Borel measures μ_n on the cone K is said to weakly converge in itself if, for any step function $f \in K$, the sequence $\lim_n \widehat{\mu}_n(f)$ converges; we say that the sequence μ_n converges to a measure μ if the functional $\lim_n \widehat{\mu}_n(f)$ is the Laplace transform of some measure μ that is concentrated on the cone K itself, not on its completion.*

For finite measures, this definition coincides with the usual one. In the case of Maxwell-Poincaré's Lemma the sequence of measures converges to the Gaussian measure. The technical difference is that in our case we cannot use the Fourier transform because the integrals diverge, we use the Laplace transform instead.

3.4 The limit of the Laplace transforms and the direct description of the convergence of distributions

Theorem 2. *Let the radius of our hypersurface be equal to $r_n = \frac{\theta}{n}$, $\theta > 0$. Then the sequence of measures $\mu_{n,r_{n,\theta}} \equiv \mu_{n,\theta}$ on the cone K weakly converges in itself. More precisely, the sequence of the Laplace transforms of the measures $\mu_{n,\theta}$ on the orbits converges to the functional*

$$\lim_n \widehat{\mu}_{n,\theta}(f) = \exp\Big\{-\theta\int_X \ln f(x)dx\Big\}.$$

This theorem is an analog of Maxwell-Poincaré's Lemma with the only difference that in the lemma, the limiting measure is immediately identifiable with the Gaussian measure. In our case the measure whose Laplace transform is the right-hand side of the relation remains to be described.

Proof. The proof goes along the lines of the proof of Maxwell-Poincaré's Lemma in the version that uses Fourier transform (see above); here we use Laplace transform instead. In essence, the following computation exactly repeats the deduction of the formula for the characteristic functional of the law for the Lévy process starting from the characteristic function of the infinitely divisible law that determines this process:

$$\mathbb{E}_\Psi \exp i\langle f, \xi\rangle = \exp\Big\{-\int \ln\psi\big(f(x)\big)dx\Big\},$$

where $\xi(.)$ is the trajectory of the process, $f(.)$ a function, and $\langle f, \xi \rangle = \int_X f(x) d\xi(x)$; the Lévy process is defined by an infinitely divisible distribution α (for example, on the half-line), and $\psi(.)$ is the Fourier transform of the measure α. An essential difference is that we consider sigma-finite infinitely divisible measures $\Gamma(\theta)^{-1} t^{\theta-1} dt$ on the half-line, in particular, the Lebesgue measure, and their Laplace transforms ($1/\lambda$ for the Lebesgue measure). It should be noted that the Haar measure on $SDiag_+$ is infinitely divisible. For simplicity, consider $\theta = 1$. Let $f(x)$ be a fixed step function on X with positive values $f_1, \dots f_m$. The step function $y(x)$ taking the values $(y_1 \dots y_n)$ on the intervals, with $n = ms$ steps, will vary as $s \to \infty$. Recall that $M_n = \big\{ y = (y_1, \dots y_n) : \prod_k y_k = n^{-n} \big\}$. We have:

$$\lim_{n\to\infty} \ln \Big\{ \int_{M_n} \exp\Big(-\sum_k f_k y_k \Big) dm_n(y) \Big\} = \lim_{n\to\infty} \frac{1}{n} \ln \Big\{ \int_{M_n} \exp\Big(-\sum_k f_k \widehat{y}_k \Big) dm'_n(y) \Big\} =$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln \Big\{ \int_{H_n} \exp\Big(-\sum_k f_k \exp x_k \Big) dx_1 \dots dx_n \Big\} = \lim_{n\to\infty} \frac{1}{n} \ln \frac{1}{\prod_{k=1}^n f_k}$$

$$= \lim_{n\to\infty} \frac{1}{n} \Big\{ -\sum_k \ln f_k \Big\} = -\int_X \ln f(x)\, dx,$$

where $H_n = \big\{ x = (x_1, \dots x_n) : \sum_k x_k = 0 \big\}$. Passing from the first expression to the second one, we used the change of variables $y_k = n\widehat{y}_k$, so that $\prod_k \widehat{y}_k = 1$, and the measure m_n is replaced by the measure m'_n, which invokes the appearance of the factor $\frac{1}{n}$ in front of the logarithm. In this passage, the property equivalent to the infinite divisibility of measures is used: a measure on the orbit can be represented as an n-fold convolution with itself of the proportional measure on the orbit; this fact follows from the invariance and uniqueness (up to a factor) of the Haar measure.[4] The passage from the second expression to the third one consists in the change of variables $\widehat{y}_k = \exp x_k$, $k = 1 \dots n$. The concluding expression in the computation is defined for positive measurable functions $f \in K$ with finite integral of the logarithm; therefore, according to our definition, we can conclude that the sequence of measures weakly converges. It should be particularly pointed out that when passing from the integral to the product $[\prod_{k=1}^n f_k]^{-1}$ (the last but one equality), we replaced the integration over the hyperplane H_n by the integration over the whole space $\mathbb{R}^n$. The validity of such change is based upon the application of the ergodic method; we discussed this in detail in the first comment to Maxwell-Poincaré's Lemma, where we explained the replacement of the integration over one orbit by the integration over the whole space. The difference is that here the method is applied to the sequence of noncompact groups. We will not dwell on this. □

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[4] In other words, we extract the nth root of the initial Laplace transform of the measure on M_n, which corresponds to a renormalization of the infinitely divisible Haar measure.

Remark. Unfortunately, the author failed to find the asymptotical behavior of the integral over the hyperplane H_n by a direct computation using asymptotical properties of cylindrical functions. It would be sufficient for our purposes to prove only the following intriguing identity ($\lambda > 0$):

$$\lim_{n\to\infty} \Big\{ \int_{H_n=\left\{x:\,\sum_{k=1}^n x_k=0\right\}} \exp\big(-\lambda \sum_{k=1}^{n} \exp x_k \big)\, dx_1 \dots dx_n \Big\}^{1/n} = 1/\lambda.$$

Integrating over the whole space $\mathbb{R}^n$ of the integrand multiplied by the function $\exp\sum_k x_k$ (which is equal to one on H_n), we get $1/\lambda$ (without passing to the limit). However, these formula (and more general ones) can be proved without computations using probabilistic arguments.

A different proof of weak convergence of the measures, which leads to the same limiting functional, is outlined below.

We obtained the formula for the values of the characteristic functional of step functions $f(\cdot)$:

$$\Phi_\theta(f) = \lim_{n\to\infty} \int_K \left[\exp\Big(-\int_X f(x)y(x)\,dx \Big) \right] d\mu_{n,\theta} = \exp\left\{ -\theta \int_X \ln f(x)\,dx \right\},\ \theta > 0.$$

It can be continued to the class of all measurable functions f with finite integral of the logarithm in a natural way.

Corollary 1. *The obtained characteristic functionals Φ_θ are invariant under the multiplication of the argument by an arbitrary measurable nonnegative function $a(\cdot)$ with zero integral of its logarithm:*

$$\Phi(a\cdot f) = \exp\left\{ -\theta \int_X \ln a(x) f(x)\,dx \right\}$$
$$= \exp\left\{ -\theta \left(\int_X \ln a(x)\,dx + \int_X \ln f(x)\,dx \right) \right\} = \exp\left\{ -\theta \int_X \ln f(x)\,dx \right\};$$

it is multiplied by a constant if the integral $\int_X \ln a(x)\,dx$ is finite.

Thus the sigma-finite measure whose Laplace transform is Φ_θ is invariant (correspondingly, projective invariant) under the multiplication operators M_a by the functions a with zero (correspondingly, finite) integral $\int_X \ln a(x)dx$.

We note that the most direct way to establish the weak convergence of the measures on orbits is to compute the distributions of the system of functionals; this brings us to the weak distribution considered in the next section.

Denote by m_d the Lebesgue measure in the space $\mathbb{R}^d$, let $\theta > 0$. The following lemma actually is equivalent to the claim of theorem 2.

Proposition 1.

$$\lim_{n\to\infty} m_{n-1}\Big\{(x_1, x_2 \dots x_n) : \sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} \exp(\theta x_k) \le r_n \cdot t\Big\} = Ct^\theta$$

where $r_n = \exp(An)$ and A, C are constants independent of n.

4 Description of the Lebesgue measures $\mathcal{L}_\theta^+$ and of the Poisson–Dirichlet measure.

4.1 Measures $\mathcal{L}_\theta^+$ as weak distributions.

Now we proceed to the description of the measures which have been described indirectly so far and which are our main object. We need to prove that, in some completion of the cone K, there exists a one-parameter family of measures $\mathcal{L}_\theta$ with the following remarkable Laplace transform:

$$\int_K \exp(-\langle f, g\rangle) d\mathcal{L}_\theta^+(g) = \Phi_\theta(f) \equiv \exp\left(-\theta \int_X \ln f(x) dx\right),$$

$\theta > 0$, and to explain what set supports it. For $\theta = 1$, this measure $\mathcal{L}_1^+$ is the one that should be called *the multiplicative Lebesgue measure in the infinite-dimensional space.* All these measures are supported by some completion of the cone K, whereas the cone itself has measure zero for all θ.

First, we describe these measures in a way this is done for weak distributions, namely, by means of coordinated families of finite-dimensional sigma-finite measures. For that, we restrict our characteristic functional Φ_θ to the finite-dimensional cone of step functions that are constant on the elements of a given finite partition ξ of the set X, $X = \cup_{k=1}^n F_k$, and take the inverse Laplace transform. As a result of this direct computation, we obtain some sigma-finite measures $L_{\theta,\xi}$ in $\mathbb{R}^n$ whose densities are described as follows.

Proposition 2. *The density of the measure $L_{\theta,\xi}$ with respect to the Lebesgue measure is*

$$\frac{dL_{\theta,\xi}}{dx}(x_1, \dots x_n) = \prod_{k=1}^{n} \frac{1}{\Gamma(\theta m_k)} x_k^{\theta m_k - 1}, \; x_k > 0, \quad k = 1, \dots, n$$

(here m_k is the Lebesgue measure of the set F_k, $\Gamma(\cdot)$ is the Euler Gamma).

See [16], and also [19], where the measures $\mathcal{L}_\theta^+$ were defined in a different way.

Proof. The formula is checked using the standard formulas for the integrals of gamma distributions. ☐

We note that the consistency of the measures relative to the refinement of the partitions cannot be interpreted in the sense of projections of finite-dimensional spaces, as for finite measures: this is impossible since the projections are infinite. A dual description is involved instead: the Laplace transforms of all finite-dimensional distributions are the restrictions to finite-dimensional subspaces of a single functional. The two interpretations of the consistency are equivalent in the case of probability measures. Specifically, in the case where $\theta = 1$, all these finite-dimensional measures are the Lebesgue measures with consistent normalization (say, on the unit cubes).

This description is an analog of a pre-measure, or a weak distribution in an infinite-dimensional vector space, and does not present an explicit description of the measure itself. However, it helps to see that the corresponding measure (we will see that it exists) is an analog of the measure generated by the process with independent nonnegative values, yet a sigma-finite one. We will give a direct description of such measures.

4.2 Direct description of the measures $\mathcal{L}_\theta^+$ using the Poisson–Dirichlet measures

Consider another cone

$$D_+ = \Big\{ \xi = \sum c_i \delta_{x_i},\, x_i \in X,\, c_i > 0,\, \sum c_i < \infty \Big\}$$

of all positive finite (non-normalized) measures with countable support in the space X. If X is a segment, such a measure may be regarded as a monotone step function with countable number of jumps whose sum is finite. In stochastic processes, probability measures on such a space are called subordinators. We would prefer to regard the elements of D_+ as positive discrete measures, i.e., the positive linear combinations of delta functions, the more so because the previous interpretation is possible only on a segment.

There is a natural coupling between the space D_+ and the cone K: each step function $f = \sum f_k \chi_{F_k}$ defines a functional on D_+:

$$\langle f, \xi \rangle = \sum_k f_k \cdot \Big(\sum_{i: x_i \in F_k} c_i \Big).$$

Therefore, the cone D_+ lies in the weak completion of the cone K; we will not use this later. We define the measures $\mathcal{L}_\theta^+$ on the cone D_+ in a direct way and show that they are the continuations of the above-defined weak distributions on the cone K to true sigma-additive sigma-finite measures.

To do this, we describe the cone D_+ in a more convenient and direct way. Namely, consider the family Σ_∞ of decreasing (in a nonstrict way) series with nonnegative summands and finite nonzero sums. This family constitutes a blunted cone (without the vertex) with an infinite-dimensional simplex Σ_1 of the monotone nonnegative series summing to one as a base. Note that $\Sigma_\infty = \Sigma_1 \times \mathbb{R}_+$. Let X^∞ be the direct product of a countable number of copies of the space X. We take the product

$$\Sigma_\infty \times X^\infty = \Sigma_1 \times \mathbb{R}_+ \times X^\infty$$

and identify it with D_+ using the map T that sends the pair made up by the series $\{c_1 \geq c_2 \ldots\} \in \Sigma_\infty$ and the sequence $\{x_1, x_2, \ldots\} \in X^\infty$ to a discrete measure as follows:

$$T\Big(\{c_k\}, \{x_k\}\Big) = \sum_k c_k \cdot \delta_{x_k} \in D_+.$$

It is clear that T is a bijection between the product

$$\Sigma_\infty \times X^\infty$$

and the space D_+.

Next we describe the measures on D_+ as the T-images of some canonical measures. Take a product measure m^∞ (a Bernoulli measure) on X^∞ (it does not depend on θ). We consider a one-parameter family of probability *Poisson–Dirichlet* measures PD_θ, $\theta > 0$, on the simplex Σ_1, see [20]; we discuss them below and in Appendix 1. The most significant of them, the proper Poisson–Dirichlet measure, corresponds to $\theta = 1$. Finally, we introduce the measures on the half-line $\mathbb{R}_+$ defined by the density $dL_\theta = \frac{t^{\theta-1}}{\Gamma(\theta)} dt$, $\theta > 0$, relative to the Lebesgue measure; it is the Lebesgue measure on the half-line if $\theta = 1$.

A useful notation for the measure on the cone Σ_∞ of monotone convergent positive series is

$$PDC_\theta = PD_\theta \times L_\theta.$$

The measures PDC might be called the "poissonization" of the Poisson–Dirichlet measures (or the conic Poisson–Dirichlet measures), in contrast to the usual measures $PD(\theta)$ concentrated on the simplex Σ_1. It seems that the sigma-finite measures PDC_θ have not been considered so far.

Definition 2. *The measure $\mathcal{L}_\theta^+$ on the cone D_+ is defined as the T-image of the product of measures:*

$$\mathcal{L}_\theta^+ = T\Big(PDC(\theta) \times m^\infty\Big).$$

It is clear that these measures are sigma-finite, sigma-additive and finite on compact sets. The following theorem identifies the measure $\mathcal{L}_\theta^+$ and the measure with Laplace transform equal to the above-computed functional. To be precise, we prove that this measure corresponds to the weak distribution introduced above and computed in Proposition 2. Further, this implies that the weak distribution in question leads to the measure with the given Laplace transform and therefore, by Theorem 2, these measures are the weak limits of the measures on the orbits.

Theorem 3.

$$\int_{D_+} \exp\{-\langle f,\xi\rangle\}\, d\mathcal{L}_\theta^+(\xi) = \Phi_\theta(f) \equiv \exp\Big\{-\int_X \ln f(x)\,dx\Big\}.$$

Thus the measures $\mathcal{L}_\theta^+$ *are the weak limits of the measures on the positive parts of the Cartan subgroups.*

Proof. We use the following remarkable property of the conic Poisson–Dirichlet measures supported by the cone Σ_∞.

Theorem 4. *Consider an arbitrary random partition of the set of positive integers* $\mathbb{N}$ *into a finite number* r *of subsets. In other words, we ascribe each positive integer, independently of the others, to one of the* r *subsets with equal probability* $1/r$*. Then the joint distribution of* r *partial sums over these sets of a random series (distributed according the measure* $PDC(\theta)$*) is the product measure* $\underbrace{L_\theta \times \cdots \times L_\theta}_{r}$ *in* $\mathbb{R}_+^r$*.*

We do not prove this characteristic property of the measures $PDC(\theta)$ here. The corresponding property of the measures $PD(\theta)$, with the multiple product of measures replaced by the Lebesgue measure on the r-dimensional simplex, follows from the results in [10] about the relation between these measures and the Lévy processes defined by stable laws; however, it can be deduced directly from the definitions of these measures (see Appendix). In the sequel, we use only this characteristic property of the measures $PDC(\theta)$; it shows that the operations on the measures $PD(\theta)$ are closely connected with the admissible independence of the the terms of the series. It immediately follows from this property that the measure $\mathcal{L}_\theta^+$ is a continuation of the weak distribution described in the previous section, and thus it has the Laplace transform we need. $\square$

The most profound properties of the measures $\mathcal{L}_\theta^+$, including their invariance relative to multiplication operators, are evidently related to the properties of the Poisson–Dirichlet measures. On the contrary, the Poisson–Dirichlet measures can be defined via the measures $\mathcal{L}_\theta^+$ as the projections onto a simplex (ore a cone).

4.3 Relationship with the gamma process, and a different definition of the measures $\mathcal{L}_\theta^+$

Gamma distribution on the half-line $[0, \infty)$ is the distribution with density $\frac{t^{\theta-1}e^{-t}dt}{\Gamma(\theta)}$ relative to the Lebesgue measure. This infinitely divisible distribution generates the Lévy process y_θ with characteristic functional

$$\chi_\theta(f) = \exp\Big\{ -\theta \int \ln\big(1 + f(x)\big)\, dx \Big\}.$$

The realizations of this process, with probability one, are discrete positive measures with countable support on X, i.e., countable linear combinations $\sum c_k \delta_{x_k}, x_k \in X, c_k > 0$, $k = 1, 2, \dots$, with finite total charge $\sum_k c_k < \infty$. The distribution of this charge (i.e., of the sums $\sum_k c_k$) is the gamma distribution. The law of this process will be denoted by $\mathcal{G}_\theta$.

Theorem 5. *The measure $\mathcal{L}_\theta^+$ is absolutely continuous relative to the measure generated by the gamma process χ_θ, with density $\frac{d\mathcal{L}_\theta}{d\mathcal{G}_\theta}(\xi) = \exp\big\{ \sum_k c_k \big\}$, where $\xi = \sum_k c_k$. This density is not integrable, due to the infiniteness of the sigma-finite measure $\mathcal{L}_\theta$.*

Corollary 2. *The measure $\mathcal{G}_\theta$ is quasi-invariant relative to the multiplication by functions with finite integral of the logarithm.*

Note that in [16, 19], the statement of this theorem was the definition of the measures $\mathcal{L}_\theta$, thus all properties of $\mathcal{L}_\theta$ were deduced from the properties of $\mathcal{G}_\theta$. For instance, the invariance relative to the multipliers was deduced from the quasi-invariance of the measure $\mathcal{G}_\theta$ and the type of the density. Here we choose an opposite and more natural line (though the proof of the quasi-invariance of the measure $\mathcal{G}_\theta$ was established in [16, 19] without difficulty): we use the weak approximation by finite invariant measures and their relation, important on its own, with the Poisson–Dirichlet measures. Moreover, the remarkable and characterizing properties of the gamma process find a natural explanation under this approach.

It was shown in [32] that the sigma-finite measure $\mathcal{L}_\theta$ may be treated as a derivative of the infinite-dimensional distribution of the Lévy processes according to the parameter α of stable laws at the point $\alpha = 0$ (see also [19]). At the same time, to obtain the distribution of the gamma process in a similar way, a passage to the (weak) limit as $\alpha \to 0$ with simultaneous renormalization of the measures is also needed. Thus the measure $\mathcal{L}_\theta$ is absolutely continuous relative to the distribution of the gamma process, but it is more natural to regard it as a derivative with respect to α. This fact is undoubtedly deeply related with the representation theory of the group of the $SL(2, \mathbb{R})$-currents since the state corresponding to the ground representation, which lies in the base of the construction of the irreducible representation of the current group (the canonical state), is the exponent of the derivative of the spherical function corresponding to the complemented series, with respect to the parameter, taken at the end point (see [16]). This is not a formal resemblance since the above-indicated realization of the representation is constructed using the measure $\mathcal{L}_1$ that is a derivative with respect to the same parameter. The relation of stable laws spherical functions of the complemented series is doubtless. All this suggests the comparison of the Wiener measure corresponding to $\alpha = 2$ with the measure $\mathcal{L}_1$ corresponding, as was indicated, to $\alpha = 0$: these values are the ends of the segment $[0, 2]$ whose points parameterize stable laws. The symmetry group of these two measures is an infinite-dimensional group of linear transformations in both cases: the group of orthogonal operators in the Hilbert space in the case of the Wiener measure, and the commutative group of multipliers in the case of the group of measure preserving transformations. Stable laws form a sort of deformation joining these two laws; their symmetry groups (already essentially nonlinear) are not described yet. We may conjecture that they constitute a nonlinear deformation similar to the homotopy between the orthogonal group and the diagonal one.

The symmetrized gamma process induced by the symmetric gamma distribution $\frac{|t|^{\theta-1}e^{-|t|}dt}{2\Gamma(\theta)}$ on the line is similarly related to the measures $\mathcal{L}_\theta$ introduced below in Subsection 5.1.

4.4 An additive version of the Lebesgue measure in the infinite-dimensional space

The measures $\mathcal{L}_\theta^+$ constructed above were invariant under the action of the multiplication operators. It is more habitual to regard the finite-dimensional Lebesgue measure as a unique (up to a factor) shift-invariant measure. By taking logarithms of the elements of the support of the measure constructed, one can transform them into shift-invariant measures.

Consider the cone K^+, see Sect. 3, of positive step functions on X and the measures on it. We pass from the multiplicative notation of the actions of the multipliers to the additive one, i.e., we take logarithms of the elements of K and of the multipliers. Then the cone turns into the vector space V of step functions and the finite-dimensional Cartan groups $SDiag_+(n, \mathbb{R})$ into the vector spaces of dimension $n-1$ that act on V additively. V

We come to the following, probably more transparent, situation. The map Log transforms the space D_+ of discrete positive measures of finite variation into a vector space, namely, the space $E(X) = \left\{ \sum_k b_k \delta_{x_k},\, x_k \in X;\, \sum_k \exp(-b_k) < \infty \right\}$ of discrete sigma-finite (signed) measures on the segment X:

$$Log : D_X \mapsto E(X) \qquad Log\Big(\sum_k c_k \cdot \delta_{x_k}\Big) = -\sum_k \log(c_k) \cdot \delta_{x_k};$$

it is clear that the sequences b_k must grow to infinity fast enough. This space is the support of the measure $\bar{\mathcal{L}}_\theta \equiv Log\mathcal{L}_\theta^+$ that is the image of the measure $\mathcal{L}_\theta^+$ under the logarithmic map. The topology on $E(X)$ is also defined as the image of the topology on D_+ under the map

Log. The measures $\bar{\mathcal{L}}_\theta$ are infinite, sigma-finite and finite on the compact sets in $E(X)$. Consider the following action of the vector space $L^1_{\mu,0}(X) = \{f \in L^1_\mu(X) : \int f = 0\} \subset L^1$ on the space $E(X)$:

$$T_f(\sum_k b_k \delta_{x_k}) = \sum_k [b_k + f(x_k)]\delta_{x_k}$$

. Both spaces are the spaces of measures: of absolutely continuous and, correspondingly, countable signed measures. Therefore, $G(X)$ is also a Banach space of measures. We restrict ourselves with the measure $\bar{\mathcal{L}}_(1)$ in $E(X)$, which will be regarded as a measure in a wider Banach space $G(X)$.

Theorem 6. *The Banach space $L^1_{\mu,0}(X)$ acts by the operators $T_f, f \in L^1_{\mu,0}(X)$ on the space $E(X)$ leaving the measure $\bar{\mathcal{L}}_(1)$ invariant. More precisely, for any element $f \in L^1_{\mu,0}(X)$ a set E_f of total $\bar{\mathcal{L}}_(1)$-measure exists such that for all $\omega \in E_f, \omega \equiv \sum_k b_k\delta_{x_k}$, the image $(T_f)(\omega) \equiv \sum_k [b_k + f(x_k)\delta_{x_k}]$ lies in $E(X)$ and T_f leaves invariant the measure $\bar{\mathcal{L}}_(1)$.*

The theorem follows from the theorem proved in Sect. 4.2 about the invariance of the multiplicative action, i.e., about the conservation of the measure under the multiplication by a function with zero integral of the logarithm. The invariance under the shifts by arbitrary elements of the space $L^1_\mu(X)$ can be obtained when one takes the direct product of the measure constructed and the Lebesgue measure on the line of constants.

Thus, we have defined a Banach space and a Borel sigma-finite measure on it which is invariant under the translations by any elements of some infinite-dimensional closed subspace. This is the circumstance that allows us to call this measure an infinite-dimensional additive Lebesgue measure.

The map *Log* allows us to analyze the properties of the measure $\bar{\mathcal{L}}_(1)$ using the properties of the Poisson–Dirichlet measure $PDC(1)$. The remark in the theorem about the choice of the set of total measure is essential (see comments in Sect. 2.2 concerning Maxwell-Poincaré's Lemma). Recall that $f \in L^1$ is not an individual function but a class of coinciding $\mod 0$ functions. Therefore, the action by shifts must be understood in the following sense. Take an individual function $\widehat{f}$ in the class f which is defined on some set $A_{\widehat{f}} \subset X$ of total measure and single out those $\omega \in \Phi(X)$ for which x_k, $k = 1, 2, \ldots$, are in $A_{\widehat{f}}$. Then the formula

$$T_f\Big(\sum_k b_k \cdot \delta_{x_k}\Big) = \sum_k \big(b_k + f(x_k)\big) \cdot \delta_{x_k}$$

determines such an action: the shift of the coefficients in the configuration ω by the values of the function f at the corresponding points. This formula makes sense and is well defined relative to the change of values $\mod 0$: if $f = f' \mod 0$, then $T_f = T_{f'} \mod 0$. Nevertheless, there is no set on which all the shifts would be defined simultaneously. The reason is somewhat different from that in the Maxwell-Poincaré's Lemma example. Here the action itself for a fixed element $f \in L_1$ is defined as a class of mod 0 coinciding transformations. It is interesting that, in addition, the group of shifts is commutative. This is an algebraic example of an action of a commutative group with invariant measure which does not admit a simultaneous individualization (of the pointwise action) of all the elements in the group. See our comments about invariant measures in Appendix.

5 Properties and applications of the measures introduced

5.1 Removing the positivity condition

Up to now, we assumed nonnegativity of the parameters of orbits and groups, i.e., the positive part $SDiag_+$ of the Cartan subgroup, the positivity of the step functions forming the cone K and of the multipliers $a(\cdot)$ acting on them, the positivity of the series forming the simplex Σ_1 and the cone Σ_∞, measures on the half-line (L_θ), and so on. The measures $\mathcal{L}_\theta^+$ we constructed were defined on the cone D_+ of discrete positive measures on the space X.

It is not difficult to lift the positivity restriction and to extend all the definitions and statements to the real parameter case. We mention the evident changes. The whole Cartan subgroup $SDiag \subset SL(n,\mathbb{R})$ replaces its positive part $SDiag_+$; its entire orbit $M_n = \{(x_1, \dots x_n) : \quad |\prod x_k| = r > 0\}$ is considered (the condition $x_k > 0$ is lifted); the cone K is replaced by the vector space of all step functions and, finally, we consider the multipliers with zero or finite integral of the *modulus* of their logarithm $\int_X \ln |a(x)|\, dx$ instead of the $\int_X \ln a(x)\, dx$. The measure space is the family of all absolutely convergent series with decreasing moduli of their members[5] instead of the cone Σ_∞ of decreasing positive convergent series, etc. All the proofs and constructions remain unaltered, the only essential change worth noting concerns the construction of the measures (Sect. 5.4). As to the definition of weak distributions, in all places where the measures on the half-line $\mathbb{R}_+$ or on the cone $\mathbb{R}_+^n$ were considered, one must extend them to $\mathbb{R}$ or $\mathbb{R}^n$ using the multiplication of the cones by 2^n vectors $\varepsilon_1 \dots \varepsilon_n$, where $\varepsilon_k = \pm 1$, with the uniform measure on them. The extension of the measure $PD(\theta)$ from the simplex of positive monotone series summing to one to the octahedron O_1 of all absolutely convergent series with decreasing moduli summing to one is made in the same way: one takes the direct product of the Poisson–Dirichlet measure and the uniform (Haar) measure on the family of infinite sequences of numbers ± 1. Then we take the space D of all discrete measures (charges) of finite variation on X instead of the

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[5] The family of the series where there are members of equal moduli has zero measure for all the measures considered, thus the ordering is defined unambiguously on the set of total measure.

cone D_+. The isomorphism of the space D and the product

$$O_1 \times \mathbb{R} \times X^\infty$$

is constructed using the extension of the map T:

$$T\big(\{c_k\}, \{x_k\}\big) = \sum_k c_k \cdot \delta_{x_k} \in D_+,$$

with the only difference that c_k may be positive or negative numbers with finite sum - $\sum |c_k| < \infty$. We denote the measures obtained on D by $\mathcal{L}_\theta$, $\theta > 0$ (omitting the subscript $+$). The measure $\mathcal{L}_1$ corresponding to $\theta = 1$ is called the infinite-dimensional Lebesgue measure. As above, the following principal result is true.

Theorem 7. *The Lebesgue measure $\mathcal{L}_1$ is the weak limit of measures on complete orbits, and its characteristic functional has the form*

$$\int_D \exp\big\{-\langle f, \xi\rangle\big\} d\mathcal{L}_1(\xi) = \exp\Big\{-\int_X \ln|f(x)dx|\Big\}.$$

The analogous formula can be written in the case of the measures $\mathcal{L}_\theta$:

$$\int_D \exp\big\{-\langle f, \xi\rangle\big\} d\mathcal{L}_\theta(\xi) = \exp\Big\{-\theta\int_X \ln|f(x)dx|\Big\}.$$

The further properties of these measures will be discussed in the next section. We note that the difference between the positive and the signed versions are not important, and the theorems about the invariance and uniqueness are proved in the general case in the same way as in the positive one.

5.2 Invariance and uniqueness

Proposition 3. *The above-constructed measures $\mathcal{L}_\theta$ in the vector space D*

1) *are invariant relative to the group $\mathcal{M}$ of multipliers M_a by the functions $a \in L^0$ with zero integral $\int_X \ln|a(x)|\,dx$; they are also projectively invariant, i.e., are multiplied by the constant $\exp\int_X \ln|a(x)|\,dx$ if this integral is finite;*

2) *are invariant relative to the group $\mathfrak{A}(X)$ of all transformations that leave the measure m on X invariant.*

Both propositions follow directly from the definition of these measures. It follows that the measures $\mathcal{L}_\theta$ are invariant relative the crossed product $\mathfrak{A}(X) \leftthreetimes \mathcal{M}$.

It is easy to show (see [19]) that the action of the group $\mathcal{M}$, and even of the crossed product, on $(D, \mathcal{L}_\theta)$ is ergodic.

Proposition 4. *The list of the measures invariant and ergodic relative to the group* $\mathfrak{A}(X) \leftthreetimes \mathcal{M}$ *is exhausted by the measures* $\mathcal{L}_\theta, \theta > 0$.

The measure $\mathcal{L}_\theta$ is concentrated on countable linear combinations of the delta functions with absolutely convergent series of coefficients. The distribution of the sum of the coefficients is the Lebesgue measure on the line. The property of the measures $\mathcal{L}_\theta^+$ expressed in Theorem 4 also holds.

Recall that on the space of countable discrete real measures (or on the space of countable linear combinations of delta-measures), there exists an ergodic equivalence relation: the equivalence class consists of the measures with the same support. This equivalence relation is ergodic in the case of the measure $\mathcal{L}_\theta$. In other words, the corresponding partition into the classes is absolutely nonmeasurable. It is, in essence, the partition into the orbits (mod 0) of the multiplier group action.

It is interesting that the measures whose support consists of discrete measures has so large infinite-dimensional group of linear symmetries. For comparison, the support of the white noise, which also has a large symmetry group (see above), consists of distributions rather than measures.

5.3 Application to the current group for the group $SL(2, \mathbb{R})$

The main application of the measures constructed is in the current group representations. This is how they were first discovered in [15]: the L^2 spaces with respect to these measures are the natural Hilbert spaces where the representations of the current groups can be implemented. Here the invariance of the measure relative to the multiplications by the elements of the infinite-dimensional diagonal subgroup is used; this fact generalizes the classical result about the representations of the group $SL(2, \mathbb{R})$, namely, the possibility to extend the representations from the parabolic subgroup to the Cartan involution, and consequently to the whole current group.

Consider the group of lower triangular matrices with determinant one and elements in the space of real functions with integrable logarithm of the modulus.

$$\begin{pmatrix} a(\cdot) & 0 \\ b(\cdot) & a(\cdot)^{-1} \end{pmatrix}$$

Note that this group, together wit the involution

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

generates the whole group $SL(2, \mathcal{F})$, where $\mathcal{F} = \Big\{ f : \int_X \ln |f(x)| dx < \infty \Big\}$.

Theorem 8. *Consider the Hilbert space $L^2(D, \mathcal{L}_\theta)$ of complex square-integrable functions on the space D with measure $\mathcal{L}_\theta$.*

The unitary operators

$$(U_{a,b}F)(\xi) = \exp\Big\{i \sum c_k b(x_k) + \int_X \ln |a(x)dx|\Big\} F(M_a^2 \xi),$$

where $\xi = \sum_k c_k \delta_{x_k} \in D$, M_a is the operator of multiplication by the function a, define an irreducible unitary representation of the above group of lower triangular matrices that extends to an irreducible representation of the group $SL(2, \mathcal{F})$. This representation also extends to a unitary representation of the group $\mathfrak{A}(X)$ of transformations of X that leave the measure m invariant.

The correctness of the definition and the unitariness of the operators is the consequence of the fact that the measures $\mathcal{L}_\theta$ are projectively invariant relative to the group of multipliers, the remaining properties are proved directly. The formulas that define the involution are given in [14], however the principal possibility to extend the representation to the group $SL(2, \mathcal{F})$ had been proved in [15] still before the measures $\mathcal{L}_\theta$ were discovered. Also note that for all $\theta > 0$, the representations are equivalent; therefore, it suffices to consider only the Lebesgue measure, i.e., the case $\theta = 1$. The mentioned commutative model of the representation of the current group $SL(2, \mathcal{F})$ is a direct continual analog of the classical representation of the group $SL(2, \mathbb{R})$ in the space $L^2(\mathbb{R})$ of functions on the line (or the projective line) with the Lebesgue measure: the line is replaced, in a sense, by the continual product of lines, the space D, and the Lebesgue measure on the line by the Lebesgue measure in the space D introduced here. It is interesting that the space $L^2(D, \mathcal{L})$ has the structure of metric factorization, i.e., of a continual tensor product of the L^2 spaces, but this metric factorization is not isomorphic to the Gaussian, i.e., the Fock factorization, but is isomorphic to the latter as a Hilbert factorization (see [33]).

5.4 Many dimensional generalization of the Poisson–Dirichlet measures and the representations of the current groups of the groups $SO(n, 1)$

We considered the measures in the space D of countable real linear combinations of delta measures on the space X so far. For applications, it is important to broaden the range of the coefficients and pass to the vector delta measures. We denote by $D^n(X) \equiv D^n$ the vector space of countable linear combinations $\sum_K c_k \delta_{x_k}$ with coefficients in the Euclidean space $\mathbb{R}^n$ that satisfy the following two conditions:

1) $\sum_k ||c_k|| < \infty$;

2) The space D^n is invariant under the action of the pointwise action of the orthogonal group $SO(n-1)$ and the homothety group in $\mathbb{R}^n$, i.e., it is invariant with respect to the current group with coefficients in the group $SO(n-1) \times \mathbb{R}^*$. In other words, given a linear combination $\sum_k c_k \cdot \delta_{x_k} \in D^n$, the linear combination $\sum_k \varepsilon_k \cdot g_k(c_k) \cdot \delta_{x_k}$, with $\varepsilon_k \in \mathbb{R}^*, g_k \in SO(n-1)$, $k = 1, 2, \dots$, is also in D^n.

The topology in D^n is defined in the usual way. Note that the direct product $\Sigma^n \times X^\infty$, where Σ^n is the set of the convergent vector series with members decreasing in the Euclidean norm, is an everywhere dense thick set in the space D^n. A bijection between $\Sigma^n \times X^\infty$ and a dense subset of D^n is constructed as in the case $n = 1$: to an arbitrary linear combination $\sum_k c_k \delta_{x_k} \in D^n$, where all $||c_k||$ are different, we assign a decreasing in the norm permutation of the sequence c_k and the corresponding permutation of x_k. Let T denote the converse map (defined in the obvious way).

An analog of the measure $\mathcal{L}_\theta$ in the case $n > 1$ was defined in [17, 18] by analogy with the case $n = 1$. First, we define the vector gamma process with characteristic functional

$$\Phi(f) = \exp\Big\{ -\theta \int \ln\big(1 + ||f(x)||^2\big)\, dx \Big\},$$

with subsequent introduction of a density. The geometry of the measure (asymptotic approach) as well as Poisson–Dirichlet measures are in no way used under this approach.

Here we define these measures using geometric point of view and applying again an analog of the Poisson–Dirichlet measures. A direct analog of the Poisson–Dirichlet measures as measures on the convergent series hardly exists in the case $n > 1$: it is not clear what does positivity mean, and thus there is no analog of the simplex of the series. However, there is an analog of the conic Poisson–Dirichlet measures which we introduce using the characteristic property of these measures given in Theorem 4. After that the sigma-finite measures can be defined in the same way as in the case $n = 1$. We restrict ourselves with the case where $\theta = 1$, for brevity.

We define a generalized (conic) Poisson–Dirichlet measure PDC^n in the space D^n as a measure in the space of convergent vector series with members decreasing in the Euclidean norm that have the following property: for any partition of the members of the series independently into an arbitrary finite number r of classes (see Theorem 4) the joint distribution of the r-dimensional vector composed of the sums of these members over the classes is the r-dimensional Lebesgue measure. It follows from the definition that these measures are spherically (i.e., in the sense of $SO(n-1)^X$) invariant. The uniqueness of such measure is verified exactly as in the one-dimensional case. The measure $\mathcal{L}_1^n$ on the vector space D^n is defined as the T-image of the product $PDC^n \times m^\infty$ of measures. The correctness of the definition follows from the fact that the PDC^n-measure of the family of the series that have at least two members with equal norms is zero.

Theorem 9. *1. The measure $\mathcal{L}_1^n$ is sigma-finite and take finite values on compact sets.*

2. The Laplace transform of the measure $\mathcal{L}_1^n$ is the functional

$$\Phi(f) = \exp\Big\{ - \int_X \ln ||f(x)||\, dx \Big\}.$$

3. Thus the measure is invariant under the action (by the pointwise multiplication) of the elements $a(\cdot)$ of the group of measurable currents with coefficients in $SO(n-1) \times \mathbb{R}^*$ satisfying the condition

$$\int \ln ||a(x)||\, dx = 0.$$

Moreover, it is invariant relative to all changes of the variable x that leave invariant the measure m.

4. There is a natural representation in the Hilbert space $L^2(D^n(X), \mathcal{L}_1^n)$ of the current group composed by the elements of $O(n,1)^X$ with finite integral of the modulus of the current.

The items 1-3 are proved as in Sections 3-4 for $n = 1$. As to the proof of item 4, see [17, 18]. We only note that the action of the subgroup of the commutative unipotent currents is realized by the operators of the multiplication of the functionals $h(\cdot) \in L^2$ by the exponent of a linear functional. The action of the subgroup of compact currents $SO(n-1)^X$ and of the homotheties is described above: it is the action on the argument of the functional $h(\cdot)$, and this model generalizes the one given in the previous Subsection 5.3. A similar definition of the Poisson–Dirichlet measures and of the Lebesgue measures in the infinite-dimensional Hilbert space is also possible. The details will be given in a forthcoming paper.

A Appendix

A.1 On the Poisson–Dirichlet measures on the space of positive series

The Poisson–Dirichlet measure $D(\theta)$ received widespread interest in the 70s on several reasons (see [20], [29] [22]). They are used in combinatorics, partition theory, population genetics, etc. Here we touch upon the three most spectacular occurrences of these measures. A deep analysis of the measure $D(1)$ and of an interesting Markov chain related to it appeared in the 70s in papers [21, 22, 24]. Although these papers are mentioned sometimes (however, insufficiently, in our opinion), the deep analysis and the ideas developed in them, in particular, the reduction to a stationary Markov chain, did not develop further for the time being.

1. The stick breaking process. Consider a sequence of independent identically distributed random variables $\xi_1, \xi_2, \dots$ on the unit interval with the Lebesgue measure. We break the interval into parts putting the points

$$x_1 = \xi_1,\, x_2 = \xi_2(1-\xi_1),\, \dots,\, x_n = \xi_n\biggl(1 - \sum_{k=1}^{n-1} \xi_k\biggr), \dots$$

one by one, so that the interval is finally broken into a countable number of parts. The corresponding measure on the family of positive series summing to one is sometimes called the Ewens measure. One gets the Poisson–Dirichlet measure $PD(1)$ from it by passing to the variational series: each of the initial series is rearranged using the (random) permutation in the decreasing order of its members. If the Lebesgue distribution of the variables ξ_k is replaced by the distribution with density $\frac{1}{\Gamma(\theta)} t^{\theta-1}$ (relative to the Lebesgue measure), then the same procedure leads us to the measure $PD(\theta)$.

2. The limiting distribution of the cycle lengths in a random permutation ([22], see also [29] and references therein). Consider the symmetric group S_n and assign to each permutation in it the vector of the lengths of its cycles normalized by the coefficient n, in the descending order, i.e., a point in the simplex $\Sigma_n = \{(x_1 \dots x_n) : \sum_k x_k = 1\}$. Denote by μ_n the image in Σ_n, under this map, of the uniform measure on the group S_n and embed the simplices Σ_n into the infinite-dimensional simplex Σ_∞. The sequence of the measures μ_n weakly converges to the measure $PD(1)$. The measures $PD(\theta)$ are obtained using the same procedure if one replaces the uniform measure on S_n with the measure defined by the density proportional to the $(\theta-1)$th power of the number of cycles.

3. The limiting distribution of the prime divisors of positive integers [28, 24, 29, 30].

Consider the expansion of positive integers into the product of primes arranged in the descending order,

$$n = p_1 \cdot p_2 \dots p_k, \quad p_1 \geq \dots \geq p_k > 1,$$

and take the vector $\left(\frac{\ln p_1}{\ln n}, \dots, \frac{\ln p_k}{\ln n}\right) \in \Sigma_1$. If we take the first N positive integers and the uniform distribution on them, then we obtain a measure on the simplex, and the sequence of such measures is weakly convergent in Σ_∞ to the measure $PD(1)$.

Here many questions are left open. Undoubtedly, a mysterious universality of the measure $PD(1)$ is present in the additive problems of analytical number theory with infinite number of summands, and in combinatorics. The comprehension of this phenomenon advanced slowly and did not reach a satisfactory level so far.[6] The technical reason of the universality is that,

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[6] The German mathematician K. Dickman was the first to put, in 1930, the question on the distribution

as mentioned, the summands of a random series with respect to these measures have, in a sense, the maximal possible independence. A more accurate meaning of this statement is revealed when one passes from the random series to the Markov sequence of the quotients of the summands and the remaining sums, see [22]. This explanation is, however, insufficient for the understanding why such independence occurs in these and many other examples.[7]

The lifting of the measures $PD(\theta)$ (the "poissonization") from the simplex to the cone of positive monotone convergent series Σ_∞ with the conic Poisson–Dirichlet measure $PDC(\theta) = PD(\theta) \times L_\theta$ plays a no less important role: see its characteristic property (Theorem 4). This property can be proved directly; moreover, it is a consequence of the theorem in [10] which states that the measures $PD(\theta)$ are the measures on the set of the trajectory jumps of the gamma process with parameter θ, i.e., of the Lévy process constructed by means of the gamma distribution $\frac{1}{\Gamma(\theta)} t^{\theta-1} e^{-t} dt$ (see Subsect. 4.3).

Some other characteristic properties of these measures are known. One of them was used above, another is the recently proved in [27] author's conjecture (see an important preliminary result in [26]): the measure $PD(1)$ is a unique invariant measure on the simplex Σ_1 for the Markov chain generated by the merging and subdivision of the summands of the series. The Poisson–Dirichlet measures find applications also in representation theory of the infinite-dimensional symmetric group (see [37]). All these facts show a fundamental character of the Poisson–Dirichlet measures. These measures also play a role in combinatorics and in the problems concerning the series and partitions which may be compared to that of Gaussian measures in the theory of vector spaces. The multi-dimensional generalization of the Poisson–Dirichlet measures was treated in Subsect. 5.4.

A.2 Restrictions on the groups imposed by the invariance and quasi-invariance of measures

The fact that a Borel nonzero nonnegative finite or sigma-finite measure on a separable group that is left-invariant under all shifts exists only on locally compact groups is the clas-

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of the logarithm of the maximal prime divisor. In the 40s, V.L. Goncharov (who apparently did not know Dickman's work) studied the distribution of the maximal cycle length of the random permutation. The understanding of the identity of the two questions came only in the 80s.

[7] We can add to the discussion initiated by the letter by V.I. Arnold in [25] that the pioneering work [22] and paper [24] are tightly related. When the author was writing paper [24], he did not know about [28]; however, though short paper it was, [24] contained some statements that were new as compared to [28] and used the results of [22], including the functional equation for the Dickman–Goncharov density of the distribution. We note that the functional equations for these densities introduced in [22] and [24] are slightly different and are proved in a different way, but the solutions remain the same, as well as the statement about the invariant measure for the Markov operator. Thus the quoting of both papers in the reviews about the Poisson–Dirichlet measures is a necessity.

sical theorem by A. Weil [38]; it is "converse" to Haar's theorem about the existence of an invariant measure on locally compact groups. Its most simple and more recent proof uses representation theory. A slightly stronger result is that the same statement about the measures is true if they are only quasi-invariant relative to all (left) shifts. Therefore, in the case of non-locally compact groups, one can only ask about the (quasi-)invariance of the measure under the elements of some subgroup of admissible shifts. For any quasi-invariant measure on a non-locally compact group, this subgroup must have measure zero; however, this subgroup can be massive. For probability measures on groups, the subgroup of admissible shifts (with quasi-invariant measure) may be a Banach or a Hilbert infinite-dimensional space (for instance, the group of admissible shifts for the standard Gaussian measure in $\mathbb{R}^\infty$ is l^2). Numerous works of probabilistic or analytical character are devoted to this subject starting with the 1940s. Such measures, according to a rather improper tradition, are called quasi-invariant; nevertheless, this does not raise a confusion because there exist no "true" quasi-invariant measures (i.e., the measures for which the set of admissible shifts has positive measure). Of special interest are the quasi-invariant measures on non-Abelian infinite-dimensional groups, which remain still not adequately studied. They are needed for the development of the analysis and the representation theory of such groups, and their applications to theoretical physics (a groups of diffeomorphisms, current groups, automorphism groups of various structures).

If one wishes that nonnegative and nonzero measure were invariant, rather than quasi-invariant, with respect to the shifts by the elements of a non-locally compact group, then this measure must already be infinite. Only sigma-finite Borel measures that take finite values on compact sets are of interest for us. It is easy to present such examples with meagre group of admissible shifts. Here is one of them. Consider the infinite product m^∞ of the infinite number of copies of the Lebesgue measure m on the unit interval in the space of all real sequences $\mathbb{R}^\infty$, and a sigma-finite measure that is obtained using the shifts of this product measure by the finite integer-valued sequences. This measure is invariant under the translations by finite vectors in the space $\mathbb{R}_\infty$. However, this example is not very interesting due to the poor family of linear symmetries of the measure. The group of admissible shifts is merely the sum of finite-dimensional spaces here.

Our example in Subsect. 4.4 of an additive infinite-dimensional Lebesgue measure $Log\mathcal{L}^+$ is new and unexpected in this very aspect: *the group of admissible shifts that leave invariant some sigma-finite measure that is finite on compact sets is an infinite-dimensional Banach space* ($L^1(X)$). Moreover, this measure is concentrated on the set of countable linear combinations of delta functions. Possibly that in essence this example exhausts all the possibilities where the group of shifts is a Banach space. It is interesting, which non-Abelian complete infinite-dimensional groups can play the role of the group of admissible shifts. One may expect that the study of such examples would lead to interesting applications in the theory of infinite-dimensional integration.

A.3 The model of continuous tensor product which is associated with infinite dimensional Lebesgue measure

The measure $\mathcal{L}_1^n$ for all values of n gives new model of the continuous tensor product of the Hilbert space. Usually the right meaning of continuous tensor product plays Fock space (or exponent of Hilbert space). It is possible to substitute Fock space with another space L^2 over the law of Levi processes. Using the measure $\mathcal{L}_1$ we can give decomposition of the continuous tensor product onto direct integral with respect to $\mathcal{L}_1$ of the countable tensor product of Hilbert space. More precisely, it is possible to give exact interpretation of the left side of the formula (continuous tensor product)

$$\int_X^{\otimes} L^2(\mathbb{R};K)dm = \int_{D(X)}^{\oplus} \bigotimes_{i=1}^{\infty} H_{\xi_i} d\mathcal{L}_1(\xi),$$

using right side of this formula; - here X is an arbitrary Lebesgue space with finite measure m; the space $L^2(\mathbb{R};K)$ is a space of K-valued L^2-functions with respect to Lebesgue measure on $\mathbb{R}$ with some auxiliary Hilbert space K; $H_\lambda, \lambda \in \mathbb{R}_+$ is a family of Hilbert spaces which depend of real positive parameter λ, and related to the space K, and $\xi = \{\xi_i\}$ runs over the elements of the set of full $\mathcal{L}_1$-measure in the space $D(X)$. Thus this formula reduces (or gives definition) of the continuous tensor product (LHS) to the direct integral of countable tensor products (RHS). The role of measure $\mathcal{L}_1$ here is crucial, - we use the invariance and ergodicity of the measure $\mathcal{L}_1$ with respect to the group of multiplicators (see 5.2). One concrete example of such interpretation will be done in the paper [39] concerning to the representations of the current groups with coefficients in the groups $O(n,1)$ and $U(n,1)$.

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