

Domain Derivatives in Electromagnetic Scattering

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Within the integral equation approach we study the dependence of the solution to the electromagnetic scattering problem from a perfect conductor with respect to the obstacle. We study the differentiability properties of strongly singular and vector valued boundary integral operators in Hölder spaces. We prove that the solution to the scattering problem depends infinitely differentiable on the boundary of the obstacle. We give a characterization of the first derivative as a solution to a boundary value problem.

1. Introduction

In this paper we deal with the time harmonic electromagnetic scattering problem from a perfect conductor. This problem is one of the standard problems of mathematical physics and of practical interest in many fields of applied mathematics, as for example geophysical exploration, radar or biomedical imaging.

Consider a bounded obstacle D in \mathbb{R}^3 with sufficiently smooth boundary ∂D . Let the scattering operator \mathcal{S} map the domain D onto the scattered field E^s for a fixed incident electromagnetic field E^{in} . Especially in the framework of *inverse problems* it is interesting to study the solutions to the scattering problems *in dependence of the domain* of the scatterer. The inverse problem consists of looking for a solution of

$$E^s = \mathcal{S}(D) \tag{1}$$

given the scattered field E^s on an exterior domain or the farfield E^∞ of E^s . \mathcal{S} is non-linear and equation (1) is ill-posed, which causes the well known difficulties which arise in the solution of ill-posed problems as they are described for example in [1, 5, 12]. Here we deal with the *differentiability properties* of the mapping (1) and with the *computation of the derivatives*.

Using boundary integral equation methods to solve the scattering problem following Colton and Kress (cf. [5]) it is possible to derive a representation of \mathcal{S} consisting of vector potentials on the boundary ∂D and of weakly and strongly singular boundary integral operators. We briefly recall this method in section 2. We use section 3 to prove the infinite Fréchet differentiability and write down the Fréchet derivatives of the integral operators used in section 2 in dependence of the domain. They are considered as operators between the spaces of continuous, Hölder continuous and

Hölder continuously differentiable functions on ∂D . Using well known properties of the Fréchet derivative (mainly the chain rule and an inversion rule) it is then possible to obtain the infinite Fréchet differentiability and the Fréchet derivatives of the scattering operator \mathcal{S} .

This work substantially extends a method which was introduced by the author to prove differentiability properties of acoustic obstacle scattering problems (cf. [17, 19]). These ideas have also been adopted by Kress to solve the inverse scattering problem from a crack [9] and by Charalambopoulos to deal with the elastic scattering problem [3]. In section 4 we give a *characterization* of the derivative of E^s . The derivative is shown to be a solution of a special boundary value problem. The arguments are presented in a way that will give us the possibility to characterize higher derivatives in a subsequent paper, when the differentiability properties of integral operators in Hölder spaces of higher order are available [20].

The results can be considered to be the first step to apply Newton's method and *Quasi-Newton methods* to solve the inverse problem, but they seem to be interesting on their own right.

2. The solution to the scattering problem

We consider the scattering of a *time-harmonic electromagnetic wave* by a bounded obstacle D surrounded by a homogeneous medium with vanishing conductivity $\sigma = 0$. The obstacle is assumed to be perfectly conducting. In the following analysis we consider only the space dependent parts of the electric and magnetic fields which we consider to be normed in a suitable way. For a derivation of this approach from the full Maxwell equations we refer to [5, Part 6].

Consider the scattering of a given incoming wave $E^{\text{in}}, H^{\text{in}}$. We denote the scattered wave by E^s, H^s . Then the total wave $E = E^{\text{in}} + E^s, H = H^{\text{in}} + H^s$ must satisfy the *reduced Maxwell equations*

$$\text{curl } E - ikH = 0, \quad \text{curl } H + ikE = 0 \quad (2)$$

in the open exterior $\mathbb{R}^3 \setminus \bar{D}$ of D and the *perfect conductor boundary condition*

$$\nu \times E = 0 \quad (3)$$

on the boundary ∂D , where ν denotes the unit outer normal vector to the boundary ∂D . Here the wave number k is a constant given by

$$k^2 = \left(\varepsilon + \frac{i\sigma}{\omega} \right) \mu \omega^2 \quad (4)$$

with the electric permittivity ε , the magnetic permeability μ , the electric conductivity σ and the frequency $\omega > 0$ of the time harmonic waves. Since we are only interested in the time harmonic case we will leave out in general the supplement *reduced*. A solution E, H to the Maxwell equations whose domain of definition contains the exterior of some sphere is called *radiating* if it satisfies one of the *Silver-Müller radiation conditions*

$$\lim_{r \rightarrow \infty} (H \times x - rE) = 0 \quad (5)$$

or

$$\lim_{r \rightarrow \infty} (E \times x + rH) = 0, \quad (6)$$

where $r = |x|$ and where the limit is assumed to hold uniformly in all directions $x/|x|$. By

$$\Phi(x, y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,$$

we denote the *fundamental solution* of the *Helmholtz equation*

$$\Delta u + k^2 u = 0. \quad (7)$$

We use the notations $CT(\partial D)$ and $CT^{0,\alpha}(\partial D)$, $0 < \alpha \leq 1$ for the spaces of all continuous and uniformly Hölder continuous tangential fields equipped with the supremum norm and the Hölder norm, respectively. Let us consider the *electromagnetic obstacle scattering problem*: given a solution $E^{\text{in}}, H^{\text{in}}$ to the Maxwell equations in \mathbb{R}^3 representing an incident electromagnetic field, find a solution $E^s, H^s \in C^1(\mathbb{R}^3 \setminus \bar{D})$ to the Maxwell equations such that the scattered field E^s, H^s satisfies the Silver–Müller radiation condition and the total electric field $E = E^{\text{in}} + E^s$ satisfies the boundary condition $\nu \times E = 0$ on ∂D . Following Colton and Kress [5, Theorem 6.19] we look for a solution to the electromagnetic obstacle scattering problem using the *combined magnetic and electric dipole distribution*

$$\begin{aligned} E^s(x) &= \text{curl} \int_{\partial D} a(y) \Phi(x, y) \, ds(y) \\ &\quad + i\eta \text{curl} \text{curl} \int_{\partial D} \nu(y) \times (S_0^2 a)(y) \Phi(x, y) \, ds(y), \quad (8) \\ H^s(x) &= \frac{1}{ik} \text{curl} E^s(x), \quad x \in \mathbb{R}^3 \setminus \partial D, \end{aligned}$$

with a density $a \in T_d^{0,\alpha}(\partial D) := \{a \in CT^{0,\alpha}(\partial D) : \text{Div} a \in C^{0,\alpha}(\partial D)\}$ and a real coupling parameter $\eta \neq 0$. Here the operator S_0 is the acoustic single layer operator

$$(Sa)(x) := 2 \int_{\partial D} \Phi(x, y) a(y) \, ds(y), \quad x \in \partial D, \quad (9)$$

in the potential theoretic case where $k = 0$. Using the jump relations for vector potentials [4, section 2.6] we see that E^s, H^s solves the electromagnetic obstacle scattering problem provided the density solves the integral equation

$$(I + M + i\eta NS_0^2)a = -2\nu \times E^{\text{in}}. \quad (10)$$

The operator

$$(Ma)(x) := 2 \int_{\partial D} v(x) \times \operatorname{curl}_x \{a(y)\Phi(x, y)\} ds(y), \quad x \in \partial D, \tag{11}$$

is bounded from $CT(\partial D)$ into $CT^{0,\alpha}(\partial D)$,

$$(Nb)(x) := 2v(x) \times \operatorname{curl} \operatorname{curl} \int_{\partial D} v(y) \times b(y)\Phi(x, y) ds(y), \quad x \in \partial D, \tag{12}$$

is bounded from $C^{1,\alpha}(\partial D, \mathbb{R}^3)$ into $CT^{0,\alpha}(\partial D)$, S_0 is bounded from $C(\partial D, \mathbb{R}^3)$ into $C^{0,\alpha}(\partial D, \mathbb{R}^3)$ and S_0 is also bounded from $C^{0,\alpha}(\partial D, \mathbb{R}^3)$ into $C^{1,\alpha}(\partial D, \mathbb{R}^3)$ (cf. [4] or [7]). Colton and Kress establish in [5] the existence and the boundedness of the inverse of the operator $(I + M + i\eta NS_0^2)$ in $T_d^{0,\alpha}(\partial D)$ using the Riesz–Fredholm theory. With the same arguments using the mapping properties given above we obtain its invertibility and the boundedness of the inverse in $CT(\partial D)$. We are interested in the values of the scattered field on a set $G \subset \mathbb{R}^3 \setminus \bar{D}$, where we assume G to have an arbitrary positive distance from the domain D . We combine the potential (8) with the restriction operator to a linear bounded mapping $Q: CT(\partial D) \rightarrow C(G)$. Define $(Ra)(x) := v(x) \times a(x)$, $x \in \partial D$ for continuous vector fields a defined on ∂D to obtain the *representation*

$$\begin{aligned} E^s(x) &= -2Q(I + M + i\eta NS_0^2)^{-1} RE^{in}, \quad x \in G, \\ H^s(x) &= \frac{1}{ik} \operatorname{curl} E^s(x), \quad x \in G, \end{aligned} \tag{13}$$

for the solution to the scattering problem.

We want to study the properties of the mapping $\mathcal{S}: D \rightarrow E^s|_G$. For this we have to introduce a norm on a suitable set of domains. Let $D_0 \subset \mathbb{R}^3$ be a fixed and bounded reference domain with boundary of class C^2 and let ρ be a sufficiently small real parameter. Then for a vector field $r \in C^2(\partial D, \mathbb{R}^3)$ with norm $\|r\|_{C^2(\partial D, \mathbb{R}^3)} < \rho$ the set $\partial D_r := \{x + r(x), x \in \partial D_0\}$ is again the boundary of class C^2 of a bounded domain D_r . Therefore on the ball $B_\rho^2 := \{r \in C^2(\partial D, \mathbb{R}^3), \|r\|_{C^2(\partial D, \mathbb{R}^3)} < \rho\}$ the mapping $r \mapsto \mathcal{S}(D_r)$ is well defined. Note that there may be different vector fields r_1 and r_2 which define the same domain $D_{r_1} = D_{r_2}$. We consider the mapping $r \mapsto \mathcal{S}(D_r)$ as an *extension* of the mapping \mathcal{S} to the set of vector fields and denote this mapping by \mathcal{S}^{ex} . In the last section we also will use the notation $B_\rho^{2,\alpha} := \{r \in C^{2,\alpha}(\partial D, \mathbb{R}^3), \|r\|_{C^{2,\alpha}(\partial D, \mathbb{R}^3)} < \rho\}$ for the ball with radius ρ in the space of vector fields $C^{2,\alpha}(\partial D, \mathbb{R}^3)$.

To study the mapping $\mathcal{S}^{ex}: B_\rho^2 \rightarrow C(G)$, $r \mapsto E^s = \mathcal{S}^{ex}(r)$ we transform the potential and the boundary integral operators onto the reference boundary ∂D_0 . A vector field $\tilde{a} \in C(\partial D_r, \mathbb{R}^3)$ is transformed into $a := \mathcal{T}\tilde{a} \in C(\partial D_0, \mathbb{R}^3)$ by

$$(\mathcal{T}\tilde{a})(x) := \tilde{a}(x + r(x)), \quad x \in \partial D.$$

Note that for this transformation we have to know the vector field r rather than only the set ∂D_r . Integral operators of the form

$$(\tilde{K}\tilde{a})(x) = \int_{\partial D_r} \tilde{k}(x, y)\tilde{a}(y) ds(y), \quad x \in \partial D_r,$$

are transformed into K by

$$\begin{aligned} (Ka)(x) &:= \mathcal{F}(\tilde{K}(\mathcal{F}^{-1}a))(x) \quad x \in \partial D \\ &= \int_{\partial D_0} k(r, x, y)a(y)J_T(r, y)ds(y), \quad x \in \partial D, \end{aligned}$$

where $J_T(r, y)$ denotes the Jacobian of the mapping $x \mapsto x + r(x)$ and where we define $k(r, x, y) := k(x + r(x), y + r(y))$. By $v(r, x)$, $x \in \partial D$, we denote the unit outer normal vector to the boundary ∂D_r in the point $x + r(x)$. We use the abbreviations $D = D_0$ and $v(x) = v(0, x)$. In the following we use the same notation for the original and for the transformed functions and operators. Hence for the representation of \mathcal{S}^{ex} we obtain

$$\mathcal{S}^{\text{ex}}(r) = -2Q(r)(I + M(r) + i\eta N(r)S_0(r)^2)^{-1}R(r)E^{\text{in}}, \quad x \in G. \quad (14)$$

By $\text{BL}(X, Y)$ we denote the space of all bounded linear operators mapping a normed space X into a normed space Y . Our goal is to study the differentiability properties of the mapping $\mathcal{S}^{\text{ex}}: B_\rho^2 \mapsto C(G)$. For this we study each of the operators Q , M , N , S_0 and T in suitable operator spaces and then use functional analytic arguments to prove the infinite differentiability of \mathcal{S}^{ex} . To proceed in this way we have to modify the representation (14) for the following reason. For continuous tangential densities a we have for the kernel of the operator M

$$\begin{aligned} v(x) \times \text{curl}_x \{ \Phi(x, y)a(y) \} &= \text{grad}_x \Phi(x, y) \langle v(x) - v(y), a(y) \rangle \\ &\quad - a(y) \frac{\partial \Phi(x, y)}{\partial v(x)}. \end{aligned}$$

Therefore the operator M considered on the space of continuous tangential densities a has the same regularity properties as the kernel of the double layer potential. Since $R(r)E^{\text{in}} \in \text{CT}(\partial D_r)$ for the solution of the scattering problem the operator M can be considered in the space of tangential fields. But if we differentiate the operator R with respect to r , the derivative $(\partial R / \partial r)E^{\text{in}}$ is no longer tangential to ∂D_r . To handle this we consider the projection operator

$$P_0 b := (v \times b) \times v = b - \langle v, b \rangle v. \quad (15)$$

The operator P_0 stands for the orthogonal projection of a vector field b defined on ∂D onto the tangent plane of $\partial D = \partial D_0$. The inverse P_2 of the restriction of the operator P_0 to the space $\text{CT}(\partial D_r)$ is given by

$$P_2(r)a := a - v(0) \frac{1}{\langle v(0), v(r) \rangle} \langle v(r), a \rangle, \quad r \in B_\rho^2. \quad (16)$$

The operator P_2 can be extended in an obvious way as a projection operator $C(\partial D, \mathbb{R}^3) \rightarrow \text{CT}(\partial D_r)$. Define the operator $\hat{M}(r) := M(r)P_2(r)$. Then the operator \hat{M} is well defined for all continuous vector fields on the boundary ∂D . Consider an operator $A: \text{CT}(\partial D_r) \rightarrow \text{CT}(\partial D_r)$ such that $I + A$ is invertible. Then we have

$$\begin{aligned} (P_0(I + A)^{-1}P_2) &= (P_0(I + A)P_2)^{-1} \\ &= (I + P_0AP_2)^{-1} \end{aligned}$$

on $CT(\partial D)$. Therefore inserting the identity $I_{CT(\partial D)} = P_2 P_0$ and using the operator \hat{M} we derive from (14) the representation

$$\mathcal{S}^{ex}(r) = -2Q(r)P_2(r)(I + P_0\hat{M}(r) + i\eta P_0N(r)S_0(r)^2 P_2(r))^{-1} P_0R(r)E^{in}. \quad (17)$$

We will study the differentiability properties of the mappings

$$\begin{aligned} B_p^2 &\rightarrow \text{BL}(CT(\partial D), C(M)) &: r &\mapsto Q(r)P_2(r), \\ B_p^2 &\rightarrow \text{BL}(CT(\partial D), CT(\partial D)) &: r &\mapsto (I + P_0\hat{M}(r) + i\eta P_0N(r)S_0(r)^2 P_2(r))^{-1}, \\ B_p^2 &\rightarrow CT(\partial D) &: r &\mapsto P_0R(r)E^{in} \end{aligned} \quad (18)$$

in the next section.

3. Differentiability properties of the integral operators and of the scattering problem

For the properties of the Fréchet derivative of a non-linear mapping we refer to [2]. We just give a summary of our notations.

Let Y be a normed space, X be a Banach space and let $U \subset Y$ be an open set. A mapping $A : U \rightarrow X$ is called *Fréchet differentiable in $r_0 \in U$* , if there is a bounded linear mapping

$$\frac{\partial A}{\partial r}(r_0) \in \text{BL}(Y, X),$$

such that for the mapping $A_R(r_0)$ defined by

$$A_R(r_0, h) := A(r_0 + h) - A(r_0) - \frac{\partial A}{\partial r}(r_0, h) \quad (19)$$

for all sufficiently small $h \in Y$ there holds

$$A_R(h) = o(\|h\|).$$

The mapping

$$\frac{\partial A}{\partial r}(r_0)$$

is called the (*Fréchet*) *derivative* of A . The *chain rule* and the *product rule* for the Fréchet derivative are valid analogously to the finite dimensional case. Higher derivatives are defined inductively. The m th derivative for $m \in \mathbb{N}$ is a m -linear form on $Y \times \dots \times Y$. We use the abbreviation

$$\frac{\partial^m A}{\partial r^m}(r_0, h) := \frac{\partial^m A}{\partial r^m}(r_0, h, \dots, h).$$

We deal with the Fréchet differentiability of integral operators of the form

$$(A(r)\varphi)(x) := \int_{G_2} f(r, x, y)\varphi(y) d\sigma(y), \quad x \in G_1, r \in U. \quad (20)$$

Here G_1 and G_2 are subsets of \mathbb{R}^n , σ denotes a σ -finite measure on G_2 and $U \subset Y$ is a subset of a normed space Y . We assume the integral to exist in the sense of a Cauchy

principal value. Let $\mathcal{F}_1 \subset C(G_1)$ and $\mathcal{F}_2 \subset C(G_2)$ be normed linear spaces. For every fixed $r \in V$ and a suitable kernel A is a bounded linear operator $\mathcal{F}_2 \rightarrow \mathcal{F}_1$. We consider A as a mapping $U \rightarrow \text{BL}(\mathcal{F}_2, \mathcal{F}_1)$. The following theorem states that for suitable properties of the kernel f the differentiation of (20) can be done by the differentiation of the kernel f . The derivative of A is then given by the operator

$$(A^{(1)}(r, h)\varphi)(x) := \int_{G_2} \frac{\partial f}{\partial r}(r, h, x, y)\varphi(y) d\sigma(y), \quad x \in G_1, r \in U, h \in Y. \quad (21)$$

We use the notation $\Delta_G := \{(x, y), x = y, x \in G_1, y \in G_2\}$ and denote for any space Z the Ball with radius ε and centre x by $\Omega_\varepsilon(x) := \{y \in Z, \|x - y\| < \varepsilon\}$.

Theorem 1. *Let G_1, G_2 be subsets of \mathbb{R}^n , σ a (σ -finite) measure on G_2 , and $U \subset Y$ an open convex subset of a Banach space Y , $\mathcal{F}_1 \subset C(G_1)$ and $\mathcal{F}_2 \subset C(G_2)$ normed linear subspaces. Take $r_0 \in U$ and let $f: U \times ((G_1 \times G_2) \setminus \Delta_G) \rightarrow \mathbb{C}$ be a continuous function with the following properties:*

- (1) *For all fixed $x \in G_1, y \in G_2, x \neq y$ the function $f(\cdot, x, y): U \rightarrow \mathbb{C}$ is two times continuously Fréchet differentiable.*
- (2) *$f(r, x, \cdot): G_2 \setminus \{x\} \rightarrow \mathbb{C}$ and*

$$\frac{\partial f}{\partial r}(r_0, h, x, \cdot): G_2 \setminus \{x\} \rightarrow \mathbb{C}$$

are integrable in the sense of a Cauchy principal value for all $x \in G_1, r \in U, h \in Y$.

- (3) *$A(r)$ and $A^{(1)}(r_0, h)$ given by (20) and (21) are elements of $\text{BL}(\mathcal{F}_2, \mathcal{F}_1)$ for all $r \in U, h \in Y$. We have $A^{(1)}(r_0, h) = O(\|h\|)$.*
- (4) *For the second derivative we have for all $\varphi \in \mathcal{F}_2, r \in U, h \in \Omega_1(0)$*

$$\left\| \int_{G_2} \frac{\partial^2 f}{\partial r^2}(r, h, x, y)\varphi(y) d\sigma(y) \right\|_{\mathcal{F}_1} \leq c \|\varphi\|_{\mathcal{F}_2} \quad (22)$$

with a constant c , and the limit

$$\lim_{\varepsilon \rightarrow 0} \int_{G_2 \setminus \Omega_\varepsilon(x)} \frac{\partial^2 f}{\partial r^2}(r, h, x, y)\varphi(y) d\sigma(y) \quad (23)$$

exists uniformly for $r \in U, h \in \Omega_1(0) \subset Y$.

Then A is Fréchet differentiable in r_0 considered as a mapping $U \rightarrow \text{BL}(\mathcal{F}_2, \mathcal{F}_1)$, $r \mapsto A(r)$ and the derivative of A is given by

$$\frac{\partial A}{\partial r}(r_0, h) = A^{(1)}(r_0, h).$$

Remark. The theorem covers the case $G_1 = G_2$ and weakly singular or singular f as well as $G_1 \cap G_2 = \emptyset$ and continuous f . For \mathcal{F}_1 and \mathcal{F}_2 we can choose suitable Hölder spaces $C^{n_1, \alpha}(G_1)$ and $C^{n_2, \alpha}(G_2)$ or $C(G_1)$ and $C(G_2)$. The theorem is an extension of Theorem 3 of [17], where only weakly singular operators in the spaces of continuous functions are considered.

Proof. For all sufficiently small h we can assume $r_0 + th \in U \forall t \in [0, 1]$. Then there holds the decomposition:

$$f(r_0 + h, x, y) = f(r_0, x, y) + \frac{\partial f}{\partial r}(r_0, h, x, y) + f_R(r_0, h, x, y) \tag{24}$$

and we have for all $(x, y) \in G_1 \times G_2 \setminus \Delta_G$

$$f_R(r_0, h, x, y) = \int_0^1 (1 - t) \frac{\partial^2 f}{\partial r^2}(r_0 + th, h, x, y) dt.$$

Because of (22) and (23) with the help of Lebesgue's theorem we find

$$\begin{aligned} & \left\| \int_{G_2} f_R(r_0, h, x, y) \varphi(y) d\sigma(y) \right\|_{\mathcal{F}_1} \\ &= \left\| \int_{G_2} \int_0^1 (1 - t) \frac{\partial^2 f}{\partial r^2}(r_0 + th, h, x, y) dt \varphi(y) d\sigma(y) \right\|_{\mathcal{F}_1} \\ &= \left\| \lim_{\varepsilon \rightarrow 0} \int_{G_2 \setminus \Omega_\varepsilon(x)} \int_0^1 (1 - t) \frac{\partial^2 f}{\partial r^2}(r_0 + th, h, x, y) dt \varphi(y) d\sigma(y) \right\|_{\mathcal{F}_1} \\ &= \left\| \lim_{\varepsilon \rightarrow 0} \int_0^1 (1 - t) \int_{G_2 \setminus \Omega_\varepsilon(x)} \frac{\partial^2 f}{\partial r^2}(r_0 + th, h, x, y) \varphi(y) d\sigma(y) dt \right\|_{\mathcal{F}_1} \\ &= \left\| \int_0^1 (1 - t) \int_{G_2} \frac{\partial^2 f}{\partial r^2}(r_0 + th, h, x, y) \varphi(y) d\sigma(y) dt \right\|_{\mathcal{F}_1} \\ &\leq \left(\int_0^1 c \|\varphi\|_{\mathcal{F}_2} dt \right) \|h\|^2 \\ &\leq c \|\varphi\|_{\mathcal{F}_2} \|h\|^2. \end{aligned}$$

We know that all terms of equation (24) are integrable on G_2 and all integrals are well defined operators $\mathcal{F}_2 \rightarrow \mathcal{F}_1$. We can use the linearity of the integral to obtain

$$\begin{aligned} (A(r_0 + h)\varphi)(x) &= \int_{G_2} f(r_0 + h, x, y) \varphi(y) d\sigma(y) \\ &= \int_{G_2} f(r_0, x, y) \varphi(y) d\sigma(y) \\ &\quad + \int_{G_2} \frac{\partial f}{\partial r}(r_0, h, x, y) \varphi(y) d\sigma(y) + \int_{G_2} f_R(r_0, h, x, y) \varphi(y) d\sigma(y) \\ &= (A(r_0)\varphi)(x) + (A^{(1)}(r_0, h)\varphi)(x) + (A_R(r_0, h)\varphi)(x), \tag{25} \end{aligned}$$

where the operator A_R fulfills

$$\|(A_R(r_0, h)\varphi)(x)\|_{\mathcal{F}_1} \leq c \|\varphi\|_{\mathcal{F}_2} \|h\|^2.$$

Therefore A is Fréchet differentiable in r_0 considered as a mapping $U \rightarrow \text{BL}(\mathcal{F}_2, \mathcal{F}_1)$ and the Fréchet derivative is given by $A^{(1)}$. \square

We now show that the mappings given by (18) at the end of section 2 are infinitely differentiable. The kernel of the operator Q (see equation (8)) is infinitely differentiable

in r and each derivative is again continuous and bounded on $(x, y) \in G \times \partial D$. This can be seen by elementary calculations using the ingredients written down in [17], section 4. So for $a \in C(\partial D, \mathbb{R}^3)$ the operators

$$\begin{aligned} & (Q^{(m)}(r, h)a)(x) \\ & := \int_{\partial D} \frac{\partial^m}{\partial r^m} \{ \text{curl}_x(a(y)\Phi(x, y_r))J_T(r, y) \} (r, h) ds(y) \\ & \quad + i\eta \int_{\partial D} \frac{\partial^m}{\partial r^m} \{ \text{curl}_x \text{curl}_x(v(r, y) \times (S_0^2(r)a)(y)\Phi(x, y_r))J_T(r, y) \} (r, h) ds(y), \end{aligned} \quad (26)$$

$x \in G$, where we replace the kernel of Q by its m th Fréchet derivative, is well defined. We can check the assumptions of Theorem 1 for the operator $Q: B_\rho^2 \rightarrow \text{BL}(\text{CT}(\partial D, \mathbb{R}^3), C(G))$. We conclude that Q is Fréchet differentiable and that the Fréchet derivative of Q is given by $Q^{(1)}$. Inductively we derive with the help of Theorem 1 that Q is m times Fréchet differentiable for arbitrary $m \in \mathbb{N}$ and that the m th Fréchet derivative of the operator Q is given by the operator $Q^{(m)}$. Because of the infinite differentiability of the normal vector the operator P_2 is infinite differentiable and the derivative can be computed using the representation of [17] for the normal vector. The operator \hat{M} can be decomposed into

$$\begin{aligned} & (\hat{M}(r)a)(x) \\ & = 2 \int_{\partial D} \left\{ (\text{grad}_x \Phi)(r, x, y) \langle v(r, x) - v(r, y), a(y) \rangle \right. \\ & \quad - (\text{grad}_x \Phi)(r, x, y) \langle v(r, x) - v(r, y), v(y) \rangle \frac{\langle v(r, y), a(y) \rangle}{\langle v(y), v(r, y) \rangle} \\ & \quad - \langle v(r, x), (\text{grad}_x \Phi)(r, x, y) \rangle a(y) \\ & \quad \left. + v(y) \frac{\langle v(r, y), a(y) \rangle}{\langle v(y), v(r, y) \rangle} \langle v(r, x), (\text{grad}_x \Phi)(r, x, y) \rangle \right\} J_T(r, y) ds(y), \end{aligned} \quad (27)$$

$x \in \partial D$. Each of the integrals in (27) is weakly singular and so are the Fréchet derivatives of arbitrary order of the kernels. Analogously to Q we define the operators $\hat{M}^{(m)}$ where the kernel of \hat{M} is replaced by its m th Fréchet derivative. In the same way as for the operators S and K in [17] we can see that $\hat{M}: B_\rho^2 \rightarrow \text{BL}(\text{CT}(\partial D, \mathbb{R}^3), C(\partial D, \mathbb{R}^3))$ is Fréchet differentiable. Computing higher derivatives of the kernel by induction we obtain even the infinite differentiability of \hat{M} by repeating the arguments used to show the differentiability. The operator P_0 is a constant and therefore infinitely differentiable in r . The infinite differentiability of $R(r)E^{\text{in}}$ can be seen from the infinite differentiability of the normal vector $v(r)$ and the analyticity of the incident field E^{in} .

In the following we will present the main part of the proof for the differentiability of the operator

$$N: B_\rho^2 \mapsto \text{BL}(C^{1,\alpha}(\partial D, \mathbb{R}^3), C^{0,\alpha}(\partial D, \mathbb{R}^3)). \quad (28)$$

Using (2.86) of [4] we obtain

$$\begin{aligned} (N(r)b)(x) &:= 2v(r, x) \times k^2 \int_{\partial D} v(r, y) \times b(y) \Phi(r, x, y) J_T(r, y) ds(y) \\ &\quad + 2v(r, x) \times \int_{\partial D} \text{Div}_T(v(r, y) \\ &\quad \times b(y)) (\text{grad}_x \Phi)(r, x, y) J_T(r, y) ds(y), \quad x \in \partial D. \end{aligned} \tag{29}$$

The second integral is strongly singular and exists in the sense of a Cauchy principal value. Note that here we take the Cauchy principal value on the surface ∂D , not on ∂D_r . It can be shown that for the operator N these limits are the same cf. [18]. For a Hölder continuous function φ and for $j = 1, 2, 3$ define the operators

$$(T_j(r)\varphi)(x) := \int_{\partial D} (\text{grad}_x \Phi_0)_j(r, x, y) \varphi(y) ds(y), \quad x \in \partial D, \tag{30}$$

where Φ_0 denotes the fundamental solution Φ in the potential theoretic case $k = 0$. The operators T_j contain the strongly singular part of N . We can decompose N into a sum of integrals with weakly singular kernel and the operators T_j . We will show the infinite differentiability of the operators

$$T_j : B_\rho^2 \mapsto \text{BL}(C^{0,\alpha}(\partial D, \mathbb{R}^3), C^{0,\alpha}(\partial D, \mathbb{R}^3)) \tag{31}$$

and

$$S_0 : B_\rho^2 \mapsto \text{BL}(C(\partial D), C^{0,\alpha}(\partial D)). \tag{32}$$

Lemma 2. *The operators T_j and S_0 given by (31) and (32) are infinitely differentiable. The m th derivative of the operator T_j or S_0 , respectively, is given by the operator $T_j^{(m)}$ or $S_0^{(m)}$, where we replace the kernel by its m th Fréchet derivative.*

Proof. Consider the operator T_j . Using again section 4 of [17] we derive that the kernel of $T_j(r)$ is infinite differentiable in r for fixed $x, y \in \partial D$. By induction we compute the m th Fréchet derivative of the kernel. We obtain a linear combination of terms of the form

$$\begin{aligned} K(r, h, x, y) &:= \frac{h_j(x) - h_j(y)}{|x + r(x) - y - r(y)|^{k_1}} \langle x + r(x) - y - r(y), h(x) - h(y) \rangle^{k_2} \\ &\quad \times \langle h(x) - h(y), h(x) - h(y) \rangle^{k_3}, \quad x \neq y, \quad x, y \in \partial D, \end{aligned} \tag{33}$$

and

$$\begin{aligned} K(r, h, x, y) &:= \frac{(x + r(x) - y - r(y))_j}{|x + r(x) - y - r(y)|^{k_1}} \langle x + r(x) - y - r(y), h(x) - h(y) \rangle^{k_2} \\ &\quad \times \langle h(x) - h(y), h(x) - h(y) \rangle^{k_3}, \quad x \neq y, \quad x, y \in \partial D, \end{aligned} \tag{34}$$

where we have $k_1, \dots, k_3 \in \mathbb{N}_0$ with $k_1 - 2k_2 - 2k_3 = 3$ and $j \in \{1, 2, 3\}$. The kernels K are strongly singular. We have

Lemma 3. Take $\varphi \in C^{0,\alpha}(\partial D)$. Then the functions

$$W_\varepsilon(r, h, x) := \int_{\partial D \setminus \Omega_\varepsilon(x)} K(r, h, x, y) \varphi(y) ds(y), \quad x \in \partial D$$

converge uniformly for $(r, h, x) \in B_\rho^2 \times B_1^2 \times \partial D$ towards a limit $W(r, h, x)$. The functions $W(r, h, \cdot)$ are Hölder continuous on ∂D and we have

$$\|W(r, h, \cdot)\|_{C^{0,\alpha}(\partial D)} \leq \Lambda \|\varphi\|_{C^{0,\alpha}(\partial D)} \quad (35)$$

with a constant Λ independent of $(r, h) \in B_\rho^2 \times B_1$.

The proof of Lemma 3 can be obtained by a slight modification of the proof of Theorem 4.2 of [13], it is carried out in detail in [18]. Then according to Lemma 3 we know that for each $m \in \mathbb{N}_0$ we have $T_j^{(m)}(r, h) \in \text{BL}(C^{0,\alpha}(\partial D), C^{0,\alpha}(\partial D))$ and $T_j^{(m)}(r) = O(\|h\|^m)$ uniformly for $r \in B_\rho^2$. We have established all assumptions of Theorem 1 for the operators $T_j^{(m)}$, $m \in \mathbb{N}$, and therefore obtain the infinite differentiability of the operator T_j .

We now consider the operator S_0 . The Fréchet differentiability of the kernel is shown in [17], but in the same way easily its infinite differentiability can be proved. By induction we again compute the m th Fréchet derivative of the kernel. We obtain a linear combination of terms of the form

$$\frac{\langle h(x) - h(y), h(x) - h(y) \rangle^{k_1} \langle x + r(x) - y - r(y), h(x) - h(y) \rangle^{k_2}}{|x + r(x) - y - r(y)|^{k_3}} \cdot \frac{\partial^j J_T}{\partial r^j}(r, h, y), \quad (36)$$

$x \neq y$, $x, y \in \partial D$ with $j \leq \mu$, $2k_1 + 2k_2 - k_3 = -1$, $2k_1 + k_2 = \mu - j$. All terms are weakly singular. By straightforward calculation we compute the estimates (2.13) and (2.14) of Theorem 2.7 in [4] for the terms (36) uniformly for $(r, h) \in B_\rho^2 \times \Omega_1$. Then from Theorem 2.7 of [4] we derive that $S_0^{(m)}(r, h) \in \text{BL}(C(\partial D), C^{0,\alpha}(\partial D))$ and $S_0^{(m)}(r, h) = O(\|h\|^m)$ uniformly for $r \in B_\rho^2$. We have established the assumptions of Theorem 1 for the operators $S_0^{(m)}$, $m \in \mathbb{N}_0$, and obtain the infinite differentiability of S_0 , which ends the proof. \square

Lemma 4. The operator N defined by (28) is infinitely differentiable. The m th derivative of N is given by the operator $N^{(m)}$ where the kernel is replaced by its m th Fréchet derivative.

Proof. In Lemma 2 we have treated the strongly singular part T_j and one weakly singular part S_0 which occur in a suitable decomposition of N . We extend the surface divergence Div to the space $C^{1,\alpha}(\partial D_r)$ by composition with an orthogonal projection operator onto the tangent space of ∂D_r . Then the infinite differentiability of the extended surface Divergence $\text{Div}_T(v \times b)$ as a mapping $B_\rho^2 \rightarrow \text{BL}(C^{1,\alpha}(\partial D, \mathbb{R}^3), C^{0,\alpha}(\partial D))$ can be obtained by elementary considerations with the help of the relation (6.38) of [5]. We do not want to present here the estimates for the other weakly singular parts of the operator N . They are analogous to the calculations for S_0 . \square

To complete our analysis of the integral operators of section 2 we have to study the mapping properties of

$$S_0 : B_\rho^2 \rightarrow \text{BL}(C^{0,\alpha}(\partial D), C^{1,\alpha}(\partial D)), \quad r \mapsto S_0(r). \quad (37)$$

Lemma 5. *The operator S_0 given by (37) is infinitely differentiable. The m th Fréchet derivative is given by the operator $S_0^{(m)}$ for $m \in \mathbb{N}$.*

Proof. We show that for all $r \in B_\rho^2$, $h \in C^2(\partial D, \mathbb{R}^3)$ and $m \in \mathbb{N}_0$, the operator $S_0^{(m)}(r, h)$ is element of $\text{BL}(C^{0,\alpha}(\partial D), C^{1,\alpha}(\partial D))$ and the operator norm $\|S_0^{(m)}(r, h)\|$ is bounded for $(r, h) \in B_\rho^2 \times B_1$. Then we can use Theorem 1 to derive the statement. In view of Lemma 2 we only have to derive bounds for the Hölder norm of the surface gradient $\text{Grad} S_0^{(m)} \varphi$ on ∂D . First we show that for $\varphi \in C^{0,\alpha}(\partial D)$ the function $S_0^{(m)}(r, h)\varphi$ is differentiable on ∂D . Denote the kernel of $S_0^{(m)}$ by $K(x, y)$ and define

$$W_\varepsilon(x) := \int_{\partial D \setminus \Omega_\varepsilon(x)} K(x, y)\varphi(y) \, ds(y), \quad x \in \partial D, \quad \varepsilon > 0. \tag{38}$$

Let τ be a Hölder continuous unit tangential vector field in a neighbourhood V of a point x . Then estimating the differential quotient of W_ε we obtain

$$\begin{aligned} \frac{\partial W_\varepsilon}{\partial \tau}(x) &= \int_{\partial D \setminus \Omega_\varepsilon(x)} \frac{\partial K}{\partial \tau}(x, y)\varphi(y) \, ds(y) \\ &\quad - \int_{\partial D \cap \partial \Omega_\varepsilon(x)} \langle \nu_0(y), \tau(x) \rangle K(x, y) \, d\Gamma(y), \quad x \in \partial D, \end{aligned} \tag{39}$$

where $d\Gamma$ denotes the line element of the curve $\partial D \cap \partial \Omega_\varepsilon(x)$ and ν_0 is the unit normal vector to this curve in ∂D . By straightforward calculation we show that the limit $\varepsilon \rightarrow 0$ of the second integral vanishes. The kernel of the first integral in (39) can be written as a sum of products of Hölder continuous functions depending on x with terms of the form (33) and (34). From Lemma 3 we derive that

$$\frac{\partial W_\varepsilon}{\partial \tau}(x)$$

converges uniformly on ∂D towards a function

$$\frac{\partial W}{\partial \tau}(x).$$

Therefore the surface gradient of $S_0^{(m)}\varphi$ exists and it is a Hölder continuous function on ∂D . An application of Theorem 1 with the help of the estimates in the proof of Lemma 2 for the Hölder norm of $S^{(m)}\varphi$ yields the statement of the lemma. \square

We now collect all statements and obtain by functional analytic arguments the differentiability properties of the scattering problem.

Theorem 6. *The solution to the electromagnetic scattering problem depends infinitely differentiable on the boundary of the scatterer in the sense that the mapping*

$$\mathcal{S}^{\text{ex}}: B_\rho^2 \rightarrow C(G), r \mapsto E^s(r)$$

is infinitely differentiable. The derivatives can be obtained by differentiating the representation (17) using the product rule.

Proof. We derive the infinite differentiability of the operators (QP_2) , $(I + P_0\hat{M} + i\eta P_0NS_0^2P_2)$ and (P_0TE^{1n}) with the help of the product rule from the Lemmas 2–5 and

the considerations after Theorem 1. From the inversion rule (Theorem 2 of [17]) we obtain the infinite differentiability of the inverse of the second operator. Now the infinite differentiability of the mapping (17) can be obtained by again using the product rule. \square

4. Characterization of the first derivative of \mathcal{S}^{ex}

We want to characterize the first Fréchet derivative of \mathcal{S}^{ex} as a solution to a boundary value problem. For this we return to Theorem 6 and present the arguments in a slightly different form which can also be used to derive inductively characterizations of higher derivatives. For the characterization of higher derivative we need the differentiability properties of boundary integral operators in Hölder spaces of higher order. These results and the characterizations for higher derivatives can be found in [20]. Here we restrict ourselves to the characterization of the first derivative.

Theorem 7. *Let the domain D_0 have a boundary of class $C^{2,\alpha}$. Consider for each $r \in B_\rho^{2,\alpha}$ the solution $E^s(r), H^s(r)$ to the exterior Maxwell problem for the domain D_r with boundary values $g(r) \in CT^{1,\alpha}(\partial D)$, i.e. $E^s(r), H^s(r)$ solve the Maxwell equations in the open exterior of D_r , satisfy the Silver–Müller radiation condition and the boundary condition $v(r) \times E^s(r) = g(r)$ on ∂D_r . Assume that the mapping $B_\rho^{2,\alpha} \rightarrow C^{1,\alpha}(\partial D)$, $r \mapsto g(r)$ is Fréchet differentiable. Then the mapping $B_\rho^{2,\alpha} \rightarrow C(M)$, $r \mapsto E^s(r)$ is Fréchet differentiable. The Fréchet derivative*

$$\frac{\partial E^s}{\partial r}(0, h)$$

in the point $r = 0$ solves the exterior Maxwell problem with boundary values

$$g_1(x) := v(x) \times \frac{\partial E^s}{\partial r}(h, x)$$

given by

$$g_1(x) = P \frac{\partial g}{\partial r}(h, x) - P \left(\frac{\partial v}{\partial r}(h, x) \times E^s(x) \right) - v(x) \times \left(\sum_j \frac{\partial E^s}{\partial x_j}(x) \cdot h_j(x) \right), \quad (40)$$

$x \in \partial D$, where P is the orthogonal projection into the tangent plane of ∂D .

Remark. Note that for the computation of the boundary values of the derivative $\partial E^s / \partial r$ of the scattered field we have to work with boundaries of class $C^{2,\alpha}$.

Proof. Looking for the solution to the exterior Maxwell problem in the form (8) we proceed as described in section 2 and we obtain for the solution the representation

$$E^s(r) = 2Q(r)(I + P_0 \hat{M}(r) + i\eta P_0 N(r) S_0(r)^2 P_2(r))^{-1} P_0 g(r). \quad (41)$$

The Fréchet differentiability of E^s is proved in the same way as the differentiability of \mathcal{L}^{ex} . For the derivative we compute

$$\begin{aligned} \frac{\partial E^s}{\partial r} &= + 2 \frac{\partial(QP_2)}{\partial r} P_0(I + M + i\eta NS_0^2)^{-1} g \\ &\quad - 2Q(I + M + i\eta NS_0^2) \frac{\partial(P_0(M + i\eta NS_0^2)P_2)}{\partial r} P_0(I + M + i\eta NS_0^2)^{-1} g \\ &\quad + 2Q(I + M + i\eta NS_0^2)^{-1} P_2 \frac{\partial(P_0 g)}{\partial r}. \end{aligned} \tag{42}$$

The functions

$$\frac{\partial E^s}{\partial r} \quad \text{and} \quad \frac{\partial H^s}{\partial r} := \frac{1}{ik} \operatorname{curl} \frac{\partial E^s}{\partial r}$$

solve the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$, and satisfy the Silver Müller radiation conditions. This can be derived by elementary calculation from the potentials in (42). We want to compute the boundary values of $\partial E^s / \partial r$. For this we use $x_r^+ := x + r(x) + \nu(r, x) \cdot \tau$ and $x^r := x_0^r$.

I. We compute the boundary values $\nu \times Q(\dots)$ for the last term of (42) at the point $r = 0$. With

$$\lim_{r \rightarrow 0} \nu(x) \times (2Q(I + M + i\eta NS_0^2)^{-1} P_2] b)(x^r) = (P_2 b)(x), \quad x \in \partial D, \tag{43}$$

$P_2 P_0 = P_2$ and $P = P_0 = P_2(0)$ we obtain

$$\begin{aligned} \lim_{r \rightarrow 0} \nu(x) \times \left(2Q(I + M + i\eta NS_0^2)^{-1} P_2 \frac{\partial[P_0 g(r)]}{\partial r}(0, h) \right)(x^r) \\ = P \frac{\partial g(r)}{\partial r}(0, h, x), \quad x \in \partial D. \end{aligned} \tag{44}$$

II. Now we want to compute for the first two terms of equation (42) the limit

$$\begin{aligned} \lim_{r \rightarrow 0} \left\{ \nu(x) \times \left(2 \frac{\partial[QP_2]}{\partial r} [P_0(I + M + i\eta NS_0^2)^{-1} P_2] g \right)(x^r) \right. \\ \left. - \nu(x) \times \left(2[QP_2][P_0(I + M + i\eta NS_0^2)^{-1} P_2] \right. \right. \\ \left. \left. \frac{\partial P_0[M + i\eta NP_1 S_0^2] P_2}{\partial r} [P_0(I + M + i\eta NS_0^2)^{-1} P_2] g \right)(x^r) \right\} \\ = -P \left(\frac{\partial \nu}{\partial r}(h, x) \times E^s(x) \right) - \nu(x) \times \left(\sum_j \frac{\partial E^s}{\partial x_j}(x) \cdot h_j(x) \right), \quad x \in \partial D. \end{aligned} \tag{45}$$

With the help of the chain rule and the insertion of P for $\tau > 0$ we derive

$$\begin{aligned}
 v(x) \times \frac{\partial}{\partial r} \{QP_2 b\}(0, h, x_r^\tau) &= P v(x) \times \frac{\partial}{\partial r} \{QP_2 b\}(0, h, x_r^\tau) \\
 &= P \frac{\partial}{\partial r} \{v(r, x) \times [QP_2 b](x_r^\tau)\}(0, h) - P \left(\frac{\partial v}{\partial r}(0, h, x) \times [QP_2 b](x^\tau) \right) \\
 &\quad - P \left(v(x) \times \left(\sum_j \frac{\partial [QP_2 b]}{\partial x_j}(x^\tau) \cdot h_j(x) \right) \right) \\
 &\quad - P \left(v(x) \times \left(\sum_j \frac{\partial [QP_2 b]}{\partial x_j}(x^\tau) \cdot \frac{\partial v_j}{\partial r}(r, h, x) \cdot \tau \right) \right) \quad (46)
 \end{aligned}$$

for all vector fields $b \in C(\partial D)$. We set $b = 2P_0(I + M + i\eta NS_0^2)^{-1}g$, use $E^s = 2Q(I + M + i\eta NS_0^2)^{-1}g$ and obtain for the first term of (45)

$$\begin{aligned}
 v(x) \times \frac{\partial}{\partial r} \{QP_2 b\}(0, h, x^\tau) \\
 &= P \frac{\partial}{\partial r} \{v(r, x) \times [QP_2 b](x_r^\tau)\}(0, h) - P \left(\frac{\partial v}{\partial r}(0, h, x) \times E^s(x^\tau) \right) \\
 &\quad - v(x) \times \left(\sum_j \frac{\partial E^s}{\partial x_j}(x^\tau) \cdot h_j(x) \right) - v(x) \times \left(\sum_j \frac{\partial E^s}{\partial x_j}(x^\tau) \cdot \frac{\partial v}{\partial r}(0, h, x) \cdot \tau \right). \quad (47)
 \end{aligned}$$

Using the results of [7] it is possible to show that the solution to the exterior Maxwell problem is in $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ [18]. Therefore the limit of the last term of (47) for $\tau \rightarrow 0$ vanishes. To show equation (45) now because of (47) we only have to verify

$$\begin{aligned}
 \lim_{\tau \rightarrow 0} P \left\{ 2 \frac{\partial}{\partial r} \{v(r, x) \times [QP_2 b](x_r^\tau)\}(0, h) \right. \\
 \left. - \left(\frac{\partial [MP_2 + i\eta NS_0^2 P_2]}{\partial r}(0, h) b \right)(x_r) \right\} = 0. \quad (48)
 \end{aligned}$$

We separately study the two parts $Q = Q_1 + i\eta Q_2 S_0^2$ of the potential Q given by equation (8).

III. First we show

$$\lim_{\tau \rightarrow 0} P \left\{ 2 \frac{\partial}{\partial r} \{v(r, x) \times [Q_1 P_2 b](x_r^\tau)\}(0, h) - \left(\frac{\partial [MP_2]}{\partial r}(0, h) b \right)(x) \right\} = 0. \quad (49)$$

We define

$$\begin{aligned}
 z(r, \tau, x) := 2 \int_{\partial D} \left\{ (\text{grad}_x \Phi)(x_r^\tau, y_r) \langle v(r, x) - v(r, y), b(y) \rangle \right. \\
 \left. - (\text{grad}_x \Phi)(x_r^\tau, y_r) \langle v(r, x) - v(r, y), v(y) \rangle \frac{\langle v(r, y), b(y) \rangle}{\langle v(y), v(r, y) \rangle} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & - \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle b(y) \\
 & + v(y) \frac{\langle v(r, y), b(y) \rangle}{\langle v(y), v(r, y) \rangle} \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle \Big\} J_T(r, y) ds(y)
 \end{aligned} \tag{50}$$

for $\tau > 0$ and $x \in \partial D$ and obtain by elementary calculations

$$2v(r, x) \times [Q_1 P_2 b](x_r^i) = z(r, \tau, x), \quad x \in \partial D, \quad \tau > 0. \tag{51}$$

The second part of (49) is given by the Fréchet derivative of (27). Now (27) reads

$$[MP_2 b](x) = z(r, 0, x), \quad x \in \partial D. \tag{52}$$

We have to show the continuity of

$$P_2 \frac{\partial}{\partial r} \{z(r, \tau, x)\} (0, h) \quad \text{for } \tau \rightarrow 0.$$

For all terms we can proceed in the same way as in the parts III and IV of the proof of Theorem 11 of [19]. Here we do not want to write down all details but only give some hints for the treatment. The third term of z is treated in [19]. For the first two terms of z we again reach the aim with the help of

$$\text{grad}_x \Phi = -\text{grad}_y \Phi = -\text{Grad}_y \Phi - v \frac{\partial}{\partial v} \Phi,$$

the use of U , defined in [19] and Lemma 3 of [17]. For the last term of z consider the decomposition

$$\begin{aligned}
 & 2 \int_{\partial D} v(y) \frac{\langle v(r, y), b(y) \rangle}{\langle v(y), v(r, y) \rangle} \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle J_T(r, y) ds(y) \\
 & = 2v(x) \frac{\langle v(r, x), b(x) \rangle}{\langle v(x), v(r, x) \rangle} \int_{\partial D} \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle J_T(r, y) ds(y) \\
 & \quad + 2 \int_{\partial D} \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle \\
 & \quad \times \left\{ v(y) \frac{\langle v(r, y), b(y) \rangle}{\langle v(y), v(r, y) \rangle} - v(x) \frac{\langle v(r, x), b(x) \rangle}{\langle v(x), v(r, x) \rangle} \right\} J_T(r, y) ds(y).
 \end{aligned} \tag{53}$$

The continuity of the Fréchet derivative of the last term of (53) is a consequence of Lemma 3 of [17]. We have $P(v \cdot \lambda) = 0$ for all $\lambda \in \mathbb{R}$. If we use the Fréchet differentiability of the first term of (53) we obtain

$$\begin{aligned}
 & P \frac{\partial}{\partial r} \left\{ 2v(x) \frac{\langle v_r(r, x), b(x) \rangle}{\langle v(x), v(r, x) \rangle} \int_{\partial D} \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle J_T(r, y) ds(y) \right\} \\
 & = P \left(v(x) \frac{\partial}{\partial r} \left\{ 2 \frac{\langle v(r, x), b(x) \rangle}{\langle v(x), v(r, x) \rangle} \right. \right. \\
 & \quad \left. \left. \times \int_{\partial D} \langle v(r, x), (\text{grad}_x \Phi)(x_r^i, y_r) \rangle J_T(r, y) ds(y) \right\} \right) = 0.
 \end{aligned}$$

Then collecting all statements we derive (49).

IV. Now we show

$$\lim_{\tau \rightarrow 0} P \left(\frac{\partial}{\partial r} \{2v(r, x) \times [Q_2 S_0^2 P_2 b](x_r^\tau)\}(0, h) - \frac{\partial [NS_0^2 P_2]}{\partial r}(0, h)b \right) = 0. \quad (54)$$

For $\tau \geq 0$ we define

$$v_1(r, \tau, x) := 2k^2 v(r, x) \times \int_{\partial D} (S_0^2 P_2 b)(y) \Phi(x_r^\tau, y_r) J_T(r, y) ds(y), \quad x \in \partial D$$

and

$$v_2(r, \tau, x) := 2v(r, x) \times \int_{\partial D} \text{Div}_T(v(r) \times S_0^2 P_2 b)(y) (\text{grad}_x \Phi)(x_r^\tau, y_r) J_T(r, y) ds(y), \quad x \in \partial D.$$

With the help of (6.40) and Theorem 3.3 of [5] we obtain

$$v(r, x) \times (Q_2 S_0^2 P_2 b)(x_r^\tau) = v_1(r, \tau, x) + v_2(r, \tau, x), \quad x \in \partial D \quad (55)$$

for $\tau > 0$ and

$$(NS_0^2 P_2 b)(r, x) = v_1(r, 0, x) + v_2(r, 0, x), \quad x \in \partial D. \quad (56)$$

We have to show the continuity

$$P \frac{\partial}{\partial r} \{v_1(r, \tau, x) + v_2(r, \tau, x)\}(0, h) \quad \text{for } \tau \rightarrow 0.$$

The continuity of the single layer part

$$\frac{\partial}{\partial r} \{v_1(r, \tau, x)\}(0, h)$$

is shown in Part III of Theorem 7 of [17]. We decompose once more $v_2 = v_3 + v_4$ with

$$v_3(r, \tau, x) := 2 \text{Div}_T(v(r) \times (S_0^2 P_2 b))(x) v(r, x) \times \int_{\partial D} (\text{grad}_x \Phi)(x_r^\tau, y_r) J_T(r, y) ds(y)$$

and

$$v_4(r, \tau, x) := 2v(r, x) \times \int_{\partial D} \left\{ \text{Div}_T(v(r) \times (S_0^2 P_2 b))(y) - \text{Div}_T(v(r) \times (S_0^2 P_2 b))(x) \right\} (\text{grad}_x \Phi)(x_r^\tau, y_r) J_T(r, y) ds(y).$$

The continuity

$$\frac{\partial}{\partial r} \{v_4(r, \tau, x)\}(r, h) \quad \text{for } \tau \rightarrow 0$$

can be obtained analogous to the first two terms of the expression z . We have

$$v_3(r, \tau, x) = 2 \operatorname{Div}_T(v(r) \times (S_0^2 P_2 b))(x) \{v(r, x) \times U_r(x_r^\tau) + v(r, x) \times V_r(x_r^\tau)\},$$

where U_r and V_r are defined in Part III of Theorem 11 of [19]. The continuity of

$$\frac{\partial}{\partial r} \{v(r, x) \times U_r(x_r^\tau)\}(0, h)$$

for $\tau \rightarrow 0$ has already been shown. We use

$$v(r, x) \times V_r(x_r^\tau) = \int_{\partial D} v(r, x) \times (v(r, x) - v(r, y)) \frac{\partial \Phi}{\partial v(r, y)}(x_r^\tau, y_r) J_T(r, y) \, ds(y)$$

to derive the continuity of

$$\frac{\partial}{\partial r} \{v(r, x) \times V_r(x_r^\tau)\}(0, h) \quad \text{for } \tau \rightarrow 0.$$

Collecting all terms we now obtain (54), which ends the proof. □

Now we obtain the characterization of the Fréchet derivative of the solution to the scattering problem.

Corollary 8. *Let the domain D have a boundary ∂D of class $C^{2,\alpha}$. The Fréchet derivative of \mathcal{S}^{ex} at the point $r = 0$ with argument $h \in C^{2,\alpha}(\partial D, \mathbb{R}^3)$ is a solution to the exterior Maxwell problem for the domain D with boundary values*

$$\begin{aligned} &v(x) \times \frac{\partial \mathcal{S}^{\text{ex}}(r)}{\partial r}(h, x) \\ &= -P \left(\frac{\partial v}{\partial r}(x, h) \times E(x) \right) - v(x) \times \left(\sum_j \frac{\partial E}{\partial x_j}(x) \cdot h_j(x) \right), \quad x \in \partial D, \end{aligned} \quad (57)$$

where P denotes the orthogonal projection into the tangent plane and $E = E^{\text{in}} + E^{\text{s}}$ is the solution to the original scattering problem for the domain D .

Proof. Take $g(r, x) := -v(r, x) \times E^{\text{in}}(x + r(x))$, $x \in \partial D$, in the preceding theorem. □

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