

## Domain walls of gauged supergravity, M-branes, and algebraic curves

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### Abstract

We provide an algebraic classification of all supersymmetric domain wall solutions of maximal gauged supergravity in four and seven dimensions, in the presence of non-trivial scalar fields in the coset  $SL(8, \mathbb{R})/SO(8)$  and  $SL(5, \mathbb{R})/SO(5)$  respectively. These solutions satisfy first-order equations, which can be obtained using the method of Bogomol'nyi. From an eleven-dimensional point of view they correspond to various continuous distributions of M2- and M5-branes. The Christoffel-Schwarz transformation and the uniformization of the associated algebraic curves are used in order to determine the Schrödinger potential for the scalar and graviton fluctuations on the corresponding backgrounds. In many cases we explicitly solve the Schrödinger problem by employing techniques of supersymmetric quantum mechanics. The analysis is parallel to the construction of domain walls of five-dimensional gauged supergravity, with scalar fields in the coset  $SL(6, \mathbb{R})/SO(6)$ , using algebraic curves or continuous distributions of D3-branes in ten dimensions. In seven dimensions, in particular, our classification of domain walls is complete for the full scalar sector of gauged supergravity. We also discuss some general aspects of D-dimensional gravity coupled to scalar fields in the coset  $SL(N, \mathbb{R})/SO(N)$ .

# 1 Introduction

Recent years have seen an increasing interest in gauged and ungauged supergravities in various dimensions following the conjectured duality between gauge theories and string/M-theory [1, 2, 3]. The AdS/CFT correspondence offers the possibility to understand strongly coupled gauge theories from a dual supergravity description. The relevant backgrounds have Poincaré invariance along the boundary directions and are asymptotic to Anti-de Sitter (AdS) space. The deviations from the AdS geometry in the interior correspond either to the broken phase of the theory due to non-zero vacuum expectation values of scalar fields [1], [4]-[12] or to deformations of the conformal field theory [13]-[21].

In this paper we shall be concerned only with the first possibility. In particular, we will construct a large class of solutions of  $D = 7$  and  $D = 4$  supergravity that are dual to the  $(2, 0)$  theories in six dimensions [22] and the three-dimensional theories with sixteen supersymmetries [23] on the Coulomb branch. These theories correspond to the world volume theories of parallel M5- and M2-branes respectively. In contrast to the conformal cases, the solutions include also non-zero scalar fields, which, although they vanish at the boundary, become large in the interior. These solutions can be lifted to eleven dimensions describing the gravitational field of distributions of a large number of M2- or M5-branes. The location of the branes is directly related to scalar Higgs expectation values on the field theory side. Using such solutions, the spectrum of scalar and graviton excitations as well as the expectation values of Wilson loops can be calculated, shedding new light on the AdS/CFT correspondence. Such investigations have been carried out for supergravity duals of  $\mathcal{N} = 4$  SYM, sometimes with surprising findings [8, 9, 10]. Interestingly, many of these backgrounds arise as limits of charged AdS black holes in the lower-dimensional gauged supergravity theories or as extremal limits [5, 6, 11] of rotating brane solutions [24, 5, 25, 26]; they have null singularities near the continuous distributions in the higher dimensional backgrounds. Such singularities arise generically in the flows from conformal to non-conformal theories and it would be interesting to understand them better from the field theory side.<sup>1</sup> Furthermore, these geometries can be viewed as examples of consistent truncations in various dimensions. The embedding of lower-dimensional into higher-dimensional supergravities has been worked out explicitly only for a couple of cases including the  $S^4$  and  $S^7$  compactifications of eleven-dimensional supergravity [28] to  $D = 7$  [29] and  $D = 4$  [30] gauged supergravity, respectively. More examples of explicit Kaluza–Klein Ansätze which relate gauged supergravities to ten/eleven-dimensional supergravity have been worked out in the recent papers [31].

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<sup>1</sup>Recently, supergravity duals of supersymmetric gauge theories with eight supersymmetries were constructed and a mechanism for resolving a certain type of space-time singularity was proposed [27]. However, the situation in [27] is different because, in  $\mathcal{N} = 2$  SYM, the classical superconformal point at the origin of the Coulomb branch is removed by quantum effects, whereas the Coulomb branch of  $\mathcal{N} = 4$  SYM is uncorrected. Therefore, it is not expected that a similar mechanism is at work for backgrounds with sixteen supersymmetries.

An additional motivation for investigating solutions of gauged supergravity in detail, as well as the spectra associated to quantum fluctuations, stems from the possibility to apply them to scenarios that view our world as a membrane embedded non-trivially in a higher dimensional non-factorizable space-time [32]. This old idea has been revived recently in relation to the mass hierarchy problem [33] (see also [34]). In the latter work a slice of the  $AdS_5$  space, where our four dimensional world is embedded, was cut out, thus resulting in a normalizable graviton zero mode. However, it turns out that there exists in addition a continuum of massive graviton modes with no mass gap separating them from the massless one. It was shown in [33] that these modes have negligible effect to Newton's law. However, it is quite desirable in this context to find ways to model possible modifications of Newton's law, since any deviations from it at the sub-millimeter scale have not been ruled out by the present day experiments (see, for instance, [35]). It is known, on the other hand, that there are Yukawa-type modifications to Newton's law in theories that possess a mass gap separating massless from massive graviton modes. Besides the obvious phenomenological advantage of such models, the existence of a mass gap leads to a well-defined effective field theory of the standard model of particles plus the massless graviton. This mass gap should be independent of the details of the slicing or at least be practically insensitive to them. Toy models possessing such desired features have already been constructed in [36] in the context of five-dimensional gauged supergravity. In this sense, many of the models, that we will describe in the present paper, as well as many of the models in [11], can also be used to further pursue these ideas.

This paper is organized as follows: In section 2 we describe the bosonic sector of gauged supergravity in  $D$  dimensions. The non-zero scalars take values in the coset space  $SL(N, \mathbb{R})/SO(N)$  common to gauged supergravities in any dimension. We restrict to vacua with  $(D - 1)$ -dimensional Poincaré invariance and find that for certain values of  $D$  and  $N$  the equations can be cast in first-order form. These Bogomol'nyi-type equations are equivalent to Killing spinor equations and these solutions preserve sixteen supersymmetries. In section 3 we present solutions in arbitrary dimensions that preserve part of the  $SO(N)$  isometries and examine some of their general features. We explain how these solutions, for the cases that correspond to gauged supergravity, can be lifted to string/M-theory, thus showing that they are consistent truncations of ten- or eleven-dimensional supergravity respectively. These higher-dimensional backgrounds arise in the AdS/CFT correspondence as supergravity duals of the field theories living on D3, M2 or M5 branes on the Coulomb branch. In section 4 we give a general discussion of the scalar field equations and graviton fluctuations in these backgrounds, and the mass spectrum of operators in the dual field theory. Our discussion is facilitated by the fact that the corresponding equations can be cast into one-dimensional Schrödinger equations with appropriately chosen potentials. We also make contact with the theory of supersymmetric quantum mechanics and outline the necessary elements that will be used for solving explicitly the Schrödinger problem in some cases of current interest. Section 5 is devoted to a detailed discussion of the differential equation for the conformal factor of the background metrics written in conformally flat form. We present a complete clas-

sification of solutions with different unbroken isometries in terms of irreducible algebraic curves. The problem of solving the domain-wall equations is then essentially reduced to the uniformization of the relevant curves and the inversion of the corresponding functions. In Section 6 we treat in detail distributions of M2-branes, which correspond to curves with genus  $g \leq 1$ . We present their uniformization and obtain explicit expressions for the conformal factor and the Schrödinger potential of the equation for the scalar and graviton field fluctuations, whenever this is possible in analytic form. In almost all cases of genus zero, the spectrum can be found exactly, whereas for the other models we utilize the WKB approximation to determine the mass gap and spacing in the spectra. In section 7 we repeat the same analysis for distributions of M5-branes. We also include, as application, the calculation of vacuum expectation values of Wilson surface operators in the six-dimensional  $(0, 2)$  theories on the Coulomb branch using the eleven-dimensional backgrounds. Section 8 contains a summary of the Lamé equation and its generalizations, which arise in the study of scalar and graviton fluctuations in the background of domain walls associated to elliptic functions. Elements of supersymmetric quantum mechanics are also used to expose connections between different elliptic potentials, and discuss briefly some features of their exact spectrum beyond the WKB approximation. Finally, we end with a short discussion in section 9 and list some open problems.

The present paper generalizes previous work by two of the authors [11], where all domain-wall solutions of five-dimensional gauged supergravity with non-trivial scalar fields in the coset  $SL(6, \mathbb{R})/SO(6)$  were classified in terms of algebraic curves. In various places we include for completeness, but with no further explanation, results on the  $D = 5$ ,  $N = 6$  theory, which correspond to continuous distributions of D3-branes in ten-dimensional type-IIB supergravity.

## 2 Gauged supergravity and first-order equations

We consider the sector of gauged supergravity theories with non-trivial fields in the coset space  $SL(N, \mathbb{R})/SO(N)$ . This is in general only a subset of a larger coset space and is common to gauged supergravity theories in any dimension. To be specific we will study the following Lagrangian in  $D$  dimensions

$$\mathcal{L} = \frac{1}{4}\mathcal{R} - \frac{1}{2}\sum_{I=1}^{N-1}(\partial\alpha_I)^2 - P(\alpha_I) , \quad (2.1)$$

although we will be mainly interested in the two cases  $(D, N) = (4, 8)$  and  $(D, N) = (7, 5)$  that correspond to a sector of the four- and seven-dimensional gauged supergravities [30, 29]. Another case of interest, corresponding to a sector of the five-dimensional gauged supergravity [37, 38], is  $(D, N) = (5, 6)$ . All other fields including the fermions are zero. In this subsector the scalar potential  $P$  can be expressed in terms of a symmetric  $N \times N$  matrix  $M = S^T S$ , where  $S$  is an element of  $SL(N, \mathbb{R})$ . Using an  $SO(N)$  rotation

matrix,  $M$  can be diagonalized

$$M = \text{diag}\{e^{2\beta_1}, \dots, e^{2\beta_N}\}, \quad \sum_{i=1}^N \beta_i = 0. \quad (2.2)$$

In addition, since the determinant of  $M$  is 1, it depends on  $N - 1$  independent fields. We parametrize the fields  $\beta_i$  by the  $N - 1$  independent scalars  $\alpha_I$  in the following way

$$\beta_i = \sum_{I=1}^{N-1} \lambda_{iI} \alpha_I, \quad (2.3)$$

where  $\lambda_{iI}$  is an  $N \times (N - 1)$  matrix. The rows of this matrix correspond to  $N$  weights of the fundamental representation of  $SL(N)$ , which obey the following normalizations:

$$\sum_{I=1}^{N-1} \lambda_{iI} \lambda_{jI} = 2\delta_{ij} - \frac{2}{N}, \quad \sum_{i=1}^N \lambda_{iI} \lambda_{iJ} = 2\delta_{IJ}, \quad \sum_{i=1}^N \lambda_{iI} = 0. \quad (2.4)$$

Then the potential takes the form

$$P = -\frac{g^2}{32} [(\text{tr}M)^2 - 2\text{tr}(M^2)]. \quad (2.5)$$

The equations of motion that follow from varying the action (2.1) with respect to the metric  $G_{MN}$  and the scalars  $\alpha_I$  are

$$\begin{aligned} \frac{1}{4} \mathcal{R}_{MN} - \frac{1}{2} \sum_{I=1}^{N-1} \partial_M \alpha_I \partial_N \alpha_I - \frac{1}{D-2} G_{MN} P &= 0, \\ \partial_M (\sqrt{-G} G^{MN} \partial_N \alpha_I) - \sqrt{-G} \frac{\partial P}{\partial \alpha_I} &= 0. \end{aligned} \quad (2.6)$$

For the applications we have in mind we need the metric to exhibit  $(D - 1)$ -dimensional Poincaré invariance, namely

$$\begin{aligned} ds^2 &= e^{2A(z)} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) = \\ &= e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2, \end{aligned} \quad (2.7)$$

where the relation between the coordinates  $z$  and  $r$  is such that  $dr = -e^A dz$ . Furthermore we assume that the scalars depend only on  $z$  and we concentrate on solutions arising from first order equations.

The traditional way would be to consider the Killing spinor first-order equations in the three different cases, when (2.1) corresponds to a sector of gauged supergravity, as it was done for  $(D, N) = (5, 6)$  in [8]. That would leave half of the supersymmetries unbroken. However, there is an alternative way to arrive at first-order field equations using a method à la Bogomol'nyi [39]-[42] that will be useful for further generalizations.

For this task we have to plug the ansatz (2.7) into the action and rewrite the potential in terms of the *prepotential*

$$W = -\frac{1}{4}\text{tr}M = -\frac{1}{4}\sum_{i=1}^N e^{2\beta_i} . \quad (2.8)$$

We find

$$P = \frac{g^2}{8} \left( \sum_{I=1}^{N-1} \left( \frac{\partial W}{\partial \alpha_I} \right)^2 - 4 \left( 1 - \frac{2}{N} \right) W^2 \right) . \quad (2.9)$$

The resulting one-dimensional effective action

$$S = \int dr e^{(D-1)A} \left( \sum_I \left( \frac{d\alpha_I}{dr} \right)^2 - \frac{1}{2}(D-1)(D-2) \left( \frac{dA}{dr} \right)^2 + 2P \right) , \quad (2.10)$$

can be written as a functional, which is a complete square plus a boundary term

$$S = \int dr e^{(D-1)A} \left( \sum_I \left( \frac{d\alpha_I}{dr} - \frac{g}{2} \partial_I W \right)^2 - \frac{(D-1)(D-2)}{2} \left( \frac{dA}{dr} + \frac{g}{(D-2)} W \right)^2 \right) + 2g(D-2)e^{(D-1)A}W \Big|_{r=-\infty}^{\infty} , \quad (2.11)$$

provided that the number of scalar fields and the dimension of space-time are related as

$$N = 4 \frac{D-2}{D-3} . \quad (2.12)$$

This relation will be used to simplify various expressions in the rest of the paper.

Then, from (2.11) we can read off the first-order differential equations:

$$\frac{dA}{dr} = -\frac{g}{D-2}W , \quad \frac{d\alpha_I}{dr} = \frac{g}{2} \frac{\partial W}{\partial \alpha_I} . \quad (2.13)$$

Note that (2.12) has integer solutions only for the values of  $N$  and  $D$  that were mentioned at the beginning of this section, namely  $(D, N) = (5, 6)$ ,  $(D, N) = (7, 5)$  and  $(D, N) = (4, 8)$ . In these three cases there exists a maximally supersymmetric solution of the equations of motion (2.6) that preserves all 32 supercharges, in which all scalar fields are set to zero and the metric is just  $AdS_D$ . In these cases the potential in (2.5) becomes  $P = -g^2 N(N-2)/32$  and equals by definition to the negative cosmological constant. This defines the mass scale  $g$  that we have already used. The associated length scale  $R$  defined by  $g = 2/R$  will also be used in this paper. The same  $AdS_D$  space is obtained for general values of  $D$  and  $N$  when all scalars are set to zero, but then, there is no notion of supersymmetry.

### 3 The general solution

We begin this section with the construction of the most general solution of the non-linear system of equations within the  $D$ -dimensional ansatz (2.7) that preserves Poincaré

invariance in the embedded  $(D-1)$ -dimensional space-time. To keep our considerations as general as possible we will assume that the scalar field potential is given by (3.1), but we will not assume from the very beginning that the relation between the number of scalars and the dimensionality of the space-time (2.12) holds. After discussing some general properties of the corresponding configurations, we concentrate on those cases where the relation (2.12) holds, and therefore the solutions originate from gauged supergravity in the appropriate number of dimensions. Then, we lift our solutions to eleven or ten dimensions in the context of eleven- or ten-dimensional supergravity. Our analysis is based on an analogous treatment of the case  $(D, N) = (5, 6)$  that was developed in reference [11].

### 3.1 Solutions in arbitrary dimensions

It is possible to find the most general solution of the coupled system of equations (2.13). As we have already mentioned, we will consider the potential arising from expression (2.9) after using (2.12). We find that

$$P = \frac{g^2}{8} \left( \sum_{I=1}^{N-1} \left( \frac{\partial W}{\partial \alpha_I} \right)^2 - 2 \frac{D-1}{D-2} W^2 \right). \quad (3.1)$$

Unless otherwise specified, we will no longer restrict  $N$  and  $D$  to obey (2.12). The resulting theory is that of  $N-1$  scalars coupled to  $D$ -dimensional gravity with the interaction potential (3.1). However, even though it no longer represents, for arbitrary values of  $N$  and  $D$ , a sector of some gauged supergravity theory, it still admits interesting solutions arising from the first-order Bogomol'nyi equations (2.13).

In order to proceed further, we first compute the evolution of the auxiliary scalar fields  $\beta_i$ . Using (2.3) and (2.13) we find

$$\beta'_i = 2 \frac{D-2}{N} A' + \frac{g}{2} e^{2\beta_i + A}, \quad i = 1, 2, \dots, N, \quad (3.2)$$

where the prime denotes derivative with respect to the argument  $z$ . It is easy to integrate these  $N$  decoupled first-order equations for the  $\beta_i$ 's. For further convenience we reparametrize the function  $A(z)$  in terms of an auxiliary function  $F(zg^2)$  as follows

$$e^A = g(-F')^{\frac{N}{4(D-2)+N}}, \quad (3.3)$$

where the prime denotes here derivative with respect to the argument  $zg^2$ . The minus sign we have included in this definition implies that, for consistency, the function  $F$  should be monotonously decreasing with  $z$ . Then, according to this ansatz we find that the general solution for the  $D$ -dimensional metric (2.7) is

$$ds^2 = g^2 f^{\frac{1}{2(D-2)}} \eta_{\mu\nu} dx^\mu dx^\nu + g^{-2} f^{-\frac{2}{N}} dF^2, \quad (3.4)$$

with

$$f = \prod_{i=1}^N (F - b_i), \quad (3.5)$$

and the solution for the scalar fields in (3.2) is given by

$$e^{2\beta_i} = \frac{f^{1/N}}{F - b_i}, \quad i = 1, 2, \dots, N. \quad (3.6)$$

The  $b_i$ 's are  $N$  constants of integration, which can be ordered as

$$b_1 \geq b_2 \geq \dots \geq b_N, \quad (3.7)$$

without loss of generality. Also, since the sum of the  $\beta_i$ 's is zero, we find that the function  $F$  has to satisfy the differential equation

$$(-F')^\Delta = \prod_{i=1}^N (F - b_i) = f, \quad \Delta = \frac{4(D-2)N}{4(D-2) + N}. \quad (3.8)$$

If we assume that our models arise from gauged supergravity, and therefore the relation (2.12) between the number of scalar fields  $N$  and the dimensionality of space-time  $D$  holds, the metric (3.4) will become

$$ds^2 = g^2 f^{\frac{1}{2(D-2)}} \eta_{\mu\nu} dx^\mu dx^\nu + g^{-2} f^{-\frac{D-3}{2(D-2)}} dF^2, \quad (3.9)$$

whereas the scalar fields are still given by (3.6). The function  $F(zg^2)$  is determined by solving the equation

$$(F')^4 = \prod_{i=1}^N (F - b_i) = f, \quad (3.10)$$

since the constant  $\Delta$  equals 4 in all three cases  $(D, N) = (4, 8), (5, 6)$  and  $(7, 5)$ . Although the relation (2.12) guarantees that our models originate from gauged supergravity, it should be emphasized that it is not the only possibility that results into an integer value for  $\Delta$ . As an example, choosing  $D = 5$  and  $N = 4$  we obtain  $\Delta = 3$ . Another interesting case, but with non-integer  $\Delta$ , arises for  $D = 3$  and  $N = 2$ : it has  $\Delta = 4/3$  and describes three-dimensional gravity coupled to a single scalar in the presence of a negative cosmological constant (since the potential (3.1) turns out to be constant).

So far, we have presented our general solution in a coordinate system where  $F$  is viewed as the independent variable.<sup>2</sup> If we insist on presenting the solution in a conformally flat form, as given by the first line in (2.7), the differential equation (3.8) needs to be solved to obtain  $F(zg^2)$ . This will be studied in detail in section 5, using the theory of algebraic curves and their uniformization, as it is a necessary step for investigating the Schrödinger equations that arise for the massless scalar and graviton fluctuations.

If the constants  $b_i$  are all equal, our solution becomes nothing but  $AdS_D$  (with radius  $2(D-2)R/N$ ) with all scalar fields turned off to zero. In the opposite case, when all constants  $b_i$  are different from one another, there is no continuous subgroup of  $SO(N)$  preserved by our solution. If we let some of the  $b_i$ 's coincide, we restore various continuous

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<sup>2</sup>To compare this with the results of [11] for the values  $D = 5$  and  $N = 6$ , we should replace the function  $F(z/R^2)$  and the constants  $b_i$  used in [11] by  $4F(zg^2)$  and  $4b_i$ , respectively.



subgroups of  $SO(N)$  accordingly. Imposing the reality condition on the scalars in (3.6) restricts the values of  $F$  to be larger than the maximum of the constants  $b_i$ ; then, according to the ordering in (3.7), this means that  $F \geq b_1$ . For  $F \gg b_1$  the scalars tend to zero and  $f \simeq F^N$ , in which case the metric in (2.7) approaches  $AdS_D$  (with radius  $2(D-2)R/N$ ) as expected. In the conformally flat form of the metric, where  $z$  is viewed as the independent variable, we have  $e^A \sim 1/z$  and therefore  $z = 0$  is taken as the origin of the  $z$ -coordinate. Hence, in solving the differential equation (3.10), we will choose the constant of integration so that in the limit  $F \gg b_1$  we have the asymptotic behaviour  $F \sim 1/z^{4(D-2)/N}$ .<sup>3</sup> For intermediate values of  $F$  we have a flow in the  $D$ -dimensional space spanned by all scalar fields  $\beta_i$ . In general we may have  $b_1 = b_2 = \dots = b_n$ , with  $n \leq N$ , when  $b_1$  is  $n$ -fold degenerate. In this case, the solution preserves an  $SO(n)$  subgroup of  $SO(N)$  and the flow is actually taking place in  $N - n$  dimensions. On the other hand, let us consider the case when  $F$  approaches its lower value  $b_1$ . Then, the scalars in (3.6) are approaching

$$e^{2\beta_i} \simeq \left\{ \begin{array}{ll} f_0^{1/N} (F - b_1)^{(n-N)/N}, & \text{for } i = 1, 2, \dots, n \\ \frac{f_0^{1/N}}{b_1 - b_i} (F - b_1)^{n/N}, & \text{for } i = n + 1, \dots, N \end{array} \right\}, \quad (3.11)$$

where  $f_0 = \prod_{i=n+1}^N (b_1 - b_i)$ . Consequently, we have a one-dimensional flow in this limit since the scalar fields  $\beta_i$  can be expressed in terms of a single (canonically normalized) scalar  $\alpha$  as

$$\begin{aligned} \vec{\beta} &\simeq \sqrt{\frac{2}{Nn(N-n)}} (n - N, \dots, n - N, n, \dots, n) \alpha, \\ \alpha &\simeq \frac{1}{2} \sqrt{\frac{n(N-n)}{2N}} \ln(F - b_1). \end{aligned} \quad (3.12)$$

When our solutions correspond to gauged supergravities and the relation (2.12) applies, it is also useful to find the limiting form of the metric (3.9) when  $F \rightarrow b_1$ . Changing the variable to  $\rho$  as

$$F = b_1 + \left( (1 - n/N) f_0^{1/N} g \rho \right)^{\frac{N}{N-n}}, \quad (3.13)$$

the metric (3.9) becomes, for  $\rho \rightarrow 0^+$

$$ds^2 \simeq d\rho^2 + \left( (1 - n/N)^n f_0 g^{(D-3)(N-n)+n} \right)^{\frac{2}{(D-3)(N-n)}} \rho^{\frac{2n}{(D-3)(N-n)}} \eta_{\mu\nu} dx^\mu dx^\nu. \quad (3.14)$$

Hence, at  $\rho = 0$  (or equivalently at  $F = b_1$ ) there is a naked singularity that can be interpreted, as we will see later in a higher-dimensional context, as the location of a distribution of M5-, M2- or D3-branes for  $(D, N) = (7, 5)$ ,  $(4, 8)$  and  $(5, 6)$  respectively. It will also be seen that this singularity is time-like for  $n = 1, 2, 3$  and null for  $4 \leq n < N$ .

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<sup>3</sup>Using (3.6) we can show that the scalar fields go to zero as  $\alpha_I \sim z^{D-3}$ , when (2.12) is obeyed. This is consistent with the interpretation that these scalars parametrize states in the Coulomb branch of the  $N = 4$  SYM theory and they do not correspond to explicit mass deformations.

### 3.2 M-theory branes

It is possible to lift our seven- and four-dimensional solutions with metric and scalars given by (3.9) and (3.6) with  $(D, N) = (7, 5)$  and  $(4, 8)$ , respectively, to supersymmetric solutions of eleven-dimensional supergravity, where only the metric and the three-form are turned on. The eleven-dimensional solutions will correspond to the gravitational field of a large number of M5-branes and M2-branes in the field theory limit with a special continuous distribution of branes in the transverse to the brane space. It shows that they are true compactifications of eleven-dimensional supergravity on  $S^4$  or  $S^7$ , respectively; this is also what is expected on general grounds [43]. The method we follow is the same as that used in [11] to lift the five-dimensional solution with  $(D, N) = (5, 6)$  to a supersymmetric solution of the ten-dimensional type-IIB supergravity representing distributions of D3-branes in the field theory limit. A brief summary of the main results will also be included for completeness in this case.

The higher-dimensional metrics for the various distributions of branes have the form

$$\text{M5-brane :} \quad ds^2 = H_0^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{2/3} (dy_1^2 + dy_2^2 + \dots + dy_5^2) , \quad (3.15)$$

$$\text{M2-brane :} \quad ds^2 = H_0^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{1/3} (dy_1^2 + dy_2^2 + \dots + dy_8^2) , \quad (3.16)$$

and

$$\text{D3-brane :} \quad ds^2 = H_0^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{1/2} (dy_1^2 + dy_2^2 + \dots + dy_6^2) . \quad (3.17)$$

In all cases  $H_0$  is a harmonic function in the  $N$ -dimensional space  $\mathbb{R}^N$  transverse to the brane parametrized by the  $y_i$  coordinates. However, instead of being asymptotically flat, the metrics (3.15)–(3.17) will become asymptotically  $AdS_D \times S^{N-1}$  for large radial distances, with  $D$  and  $N$  taking their appropriate values. The radius of the sphere is always  $R$ , whereas, in agreement with our previous normalization, it is  $\frac{1}{2}(D-3)R$  for  $AdS_D$ . Under these conditions, the higher-dimensional solutions break half of the maximum of 32 supersymmetries (see, for instance, [44]). It is only for coinciding branes, when the metric is exactly  $AdS_D \times S^{N-1}$ , that the maximal number of supersymmetries is preserved and the backgrounds are presumably exact vacua.

We proceed further by first performing the coordinate change

$$y_i = 2g^{(D-5)/2} (F - b_i)^{1/2} \hat{x}_i , \quad i = 1, 2, \dots, N , \quad (3.18)$$

where the  $\hat{x}_i$ 's define a unit  $N$ -sphere. It can be shown that the  $N$ -dimensional flat metric in the transverse part of the brane metric (3.15)–(3.17) can be written as

$$\sum_{i=1}^N dy_i^2 = g^{D-5} \sum_{i=1}^N \frac{\hat{x}_i^2}{F - b_i} dF^2 + 4g^{D-5} \sum_{i=1}^N (F - b_i) d\hat{x}_i^2 . \quad (3.19)$$

The harmonic function  $H_0$  is determined by comparing the massless scalar equation for the eleven- and ten-dimensional metrics (3.15)–(3.17), with the same equation arising

using the D-dimensional metric (3.9). In both cases one makes the ansatz that the solution does not depend on the sphere coordinates, i.e.  $\Phi = e^{ik \cdot x} \phi(z)$ . Since the solutions for the scalar  $\Phi$  should be the same in any consistent truncation of the theory, the resulting second-order ordinary differential equations should be identical. A comparison allows to determine the function  $H_0$  as follows

$$H_0^{-1} = \frac{4}{R^4} f^{1/2} \sum_{i=1}^N \frac{y_i^2}{(F - b_i)^2} . \quad (3.20)$$

The coordinate  $F$  is determined in terms of the transverse coordinates  $y_i$  as a solution of the algebraic equation

$$\sum_{i=1}^N \frac{y_i^2}{F - b_i} = 4g^{D-5} . \quad (3.21)$$

Note that the algebraic equation (3.21) for  $F$  cannot be solved analytically for general choices of the constants  $b_i$ . However, this becomes possible when some of the  $b_i$ 's coincide in such a way that the degree of the corresponding algebraic equations is reduced to 4 or less. We also note that  $H_0$  as defined in (3.20) and (3.21) is indeed a harmonic function in  $\mathbb{R}^N$ . The proof was made in [11] for the case  $N = 6$ , but even if  $N$  is kept general all intermediate steps are essentially the same. We note that one may use as independent variables, instead of the  $y_i$ 's, spherical coordinates since the constraint (3.21) is then automatically satisfied. For D3-branes, this was done for the various cases of interest in [11]. For M2- and M5-branes we expect to recover the metrics found in [12]. Various subcases were also considered before in [5, 6, 8, 9] in connection with the Coulomb branch of gauge theories at strong coupling.

Brane solutions that are asymptotically flat are obtained by replacing  $H_0$  in (3.15)–(3.17) by  $H = 1 + H_0$ . Then, in this context, the length parameter  $R$  has a microscopic interpretation using the eleven-dimensional Planck scale  $l_P$  or the string scale  $\sqrt{\alpha'}$  and the string coupling  $g_s$ , and the (large) number of branes  $N_b$ . For M5-branes we have  $R^3 = \pi N_b l_P^3$ , for M2-branes  $R^6 = 32\pi N_b l_P^6$ , and for D3-branes  $R^4 = 4\pi N_b g_s \alpha'^2$ .

## 4 The spectrum of fluctuations

In this section we investigate the problem of solving the differential equations that arise for the massless scalar field as well as for the graviton fluctuations in our general D-dimensional background metrics (3.9). After presenting some general features, we make contact with supersymmetric quantum mechanics, which will prove useful for making further progress in computing the exact spectrum later in sections 6, 7 and 8.

### 4.1 Generalities

Within the AdS/CFT correspondence [1, 2, 3] (assuming (2.12)), the solutions and eigenvalues of the massless scalar equation have been associated, on the gauge theory side,

with the spectrum of the operator  $\text{Tr}F^2$ , whereas those of the graviton fluctuations polarized in the directions parallel to the brane, with the energy momentum tensor  $T_{\mu\nu}$  [45, 2, 3, 46]. A priori these two spectra are different. However, it can be shown, similarly to the case of five-dimensional gauged supergravity investigated in [36], that the two spectra and the corresponding eigenfunctions coincide. Hence, in what follows,  $\Phi$  will denote either a massless scalar field or any component of the graviton tensor field.

We proceed further by making the following ansatz for the solution

$$\Phi(x, z) = \exp(ik \cdot x) \exp\left(-\frac{D-2}{2}A\right) \Psi(z) , \quad (4.1)$$

which represents plane waves propagating along the  $(D-2)$ -brane with an amplitude function that is  $z$ -dependent. The mass-square is defined as  $M^2 = -k \cdot k$ . Then the equation for  $\Psi$  can be cast into a time-independent Schrödinger equation for a wave function  $\Psi(z)$  as

$$-\Psi'' + V\Psi = M^2\Psi , \quad (4.2)$$

with a potential given by

$$V = \frac{(D-2)^2}{4}A'^2 + \frac{D-2}{2}A'' . \quad (4.3)$$

It should be noted and will be further discussed later in this section, that this potential is of the form that appears in supersymmetric quantum mechanics and therefore the spectrum is non-negative [47, 48].<sup>4</sup> However, this does not guarantee that the massless mode is normalizable. In our case,  $\Psi_0 \sim \exp((D-2)A/2)$  and it is clearly not square-normalizable due to its behaviour near  $z = 0$ , namely  $\Psi_0 \sim 1/z^{(D-2)/2}$ . Using the general formulae (3.3) and (3.8) it turns out that the potential takes the form

$$V = \frac{f^{2/\Delta}}{4R^4} \left[ \left(1 + \frac{8}{\Delta}\right) \left(\sum_{i=1}^N \frac{1}{F-b_i}\right)^2 - 8 \sum_{i=1}^N \frac{1}{(F-b_i)^2} \right] . \quad (4.4)$$

We note that in deriving (4.4) from (4.3) the relation (2.12) has not been used.

The expression (4.4) for the potential depends, of course, on the variable  $z$  through the function  $F(zg^2)$ . Even without any knowledge of the explicit  $z$ -dependence of the potential, we may deduce some general properties about the spectrum in the various cases of interest. In general,  $F$  takes values between the maximum of the constants  $b_i$  (which according to the ordering made in (3.7) is taken to be  $b_1$ ) and  $+\infty$ . When  $F \rightarrow +\infty$  (or equivalently  $z \rightarrow 0^+$ ), the space approaches  $AdS_D$  and the potential becomes

$$V \simeq \frac{D(D-2)}{4} \frac{1}{z^2} , \quad \text{as } z \rightarrow 0^+ , \quad (4.5)$$

and hence it is unbounded from above. Let us now consider the behaviour of the potential close to the other end, namely when  $F \rightarrow b_1$ . Consider the general case where  $b_1$  appears

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<sup>4</sup>In the context of five-dimensional gauged supergravity this has been first hinted in [11] and explicitly noted in [49].

$n$  times, as in the corresponding discussion made at the end of subsection 3.1. Using (4.4) we find that the potential behaves (including the subscript  $n$  to distinguish among various cases) as

$$V_n \simeq \frac{f_0^{2/\Delta}}{4R^4} \left( (1 + 8/\Delta)n^2 - 8n \right) (F - b_1)^{\frac{2n}{\Delta} - 2}, \quad \text{as } F \rightarrow b_1, \quad (4.6)$$

with  $f_0$  being a constant given, as before, by  $f_0 = \prod_i (b_1 - b_i)$ . Hence, for  $\Delta < n \leq N$ , the potential goes to zero and the spectrum is continuous. For  $n = \Delta$  the potential approaches a constant value, which is given by  $V_{\Delta, \min} = \frac{1}{4R^4} \Delta^2 f_0^{2/\Delta}$ . Therefore, although the spectrum is continuous, it does not start from zero, but there is a mass gap whose squared value is given by the minimum of the potential. Hence, for  $\Delta \leq n \leq N$  the range of  $z$  necessarily extends to  $+\infty$ , i.e.  $0 \leq z < \infty$ , with  $z = \infty$  corresponding to a null naked singularity (except in the AdS case where  $n = N$ ). If  $\Delta < n < N$  we have that  $(F - b_1)^{1-n/\Delta} \simeq (n/\Delta - 1)g^2 f_0^{1/\Delta} z$  near  $F = b_1$ , and therefore the potential behaves as

$$\begin{aligned} \Delta < n < N : \quad V_n &\simeq \frac{C_{n,\Delta}}{z^2}, \quad \text{as } z \rightarrow \infty, \\ C_{n,\Delta} &= \frac{1}{64} \left( \frac{n(\Delta + 4) - 4\Delta}{\Delta - n} \right)^2 - \frac{1}{4}. \end{aligned} \quad (4.7)$$

For  $n = \Delta$  we have instead that  $F - b_1 \simeq e^{-f_0^{1/n} g^2 z}$ , as  $z \rightarrow \infty$  with the potential reaching the constant value that we have mentioned above. For  $n < \Delta$  the potential goes to either  $+\infty$  or  $-\infty$ , as  $F \rightarrow b_1$  and therefore the spectrum cannot be continuous but discrete. Therefore there should be a maximum value for  $z$ , denoted by  $z_{\max}$ , that is determined by solving the algebraic equation  $F(z_{\max} g^2) = b_1$ . The value of  $z = z_{\max}$  corresponds to a time-like naked singularity. Then, using the relation  $(F - b_1)^{1-n/\Delta} \simeq (1 - n/\Delta)g^2 f_0^{1/\Delta} (z_{\max} - z)$  near  $F = b_1$ , we find that

$$\begin{aligned} n < \Delta : \quad V_n &\simeq \frac{C_{n,\Delta}}{(z - z_{\max})^2}, \quad \text{as } z \rightarrow z_{\max}^-, \\ C_{n,\Delta} &= \frac{1}{64} \left( \frac{n(\Delta + 4) - 4\Delta}{\Delta - n} \right)^2 - \frac{1}{4}. \end{aligned} \quad (4.8)$$

One may worry that there are cases where the potential goes to  $-\infty$  at  $z = z_{\max}$  with a coefficient  $C_{n,\Delta}$  that is smaller than  $-1/4$ ; then, it is well known from elementary quantum mechanics, that the spectrum might be unbounded from below. However, this does not happen because we are dealing with supersymmetric quantum mechanics and the spectrum is bounded from below by zero. In addition, the coefficient  $C_{n,\Delta}$  in (4.8) is manifestly greater or equal than  $-1/4$ ; the limiting value  $-1/4$  is reached for  $n = 4\Delta/(\Delta + 4)$ . The qualitative analysis of the spectrum we just presented agrees with that performed for  $(D, N) = (5, 6)$  in [11] and for  $(D, N) = (7, 5)$  and  $(D, N) = (4, 8)$  in [12].

As we see later, solving the Schrödinger equation (4.2) and determining the spectrum is a non-trivial problem, except for a few particular cases where explicit calculations can

be carried out in detail. However, when the spectrum is discrete, as for  $n < \Delta$ , we may use the approximate method of WKB. Since the quantum mechanical potentials are supersymmetric, WKB is expected to be an excellent approximation, not only for high quantum numbers, but also for low ones [48]. Moreover, if the potential turns out to be shape-invariant, the WKB approximation is also exact [48]. It is convenient at this point to use  $F$  as an independent variable instead of  $z$ . Then, changing the dependent variable in (4.2) as  $\Psi = e^{(D-2)A/2}\phi$ , we find the equation

$$g^4 \partial_F f^{\frac{1}{4} + \frac{1}{\Delta}} \partial_F \phi + M^2 f^{\frac{1}{4} - \frac{1}{\Delta}} \phi = 0 . \quad (4.9)$$

Using well-developed methods for studying this type of differential equations (see, for instance, [25]), the spectrum is found to be given approximately by

$$M_m^2 = \frac{\pi^2}{z_{\max}^2} m \left( m + \frac{D-3}{2} + \frac{|(n/4-1)\Delta+n|}{2(\Delta-n)} \right) + \mathcal{O}(m^0) , \quad m = 1, 2, \dots , \quad (4.10)$$

where, as usual,  $z_{\max}$  is the maximum value of  $z$  and  $\Delta$  is the constant defined in (3.8). It turns out that the validity of the WKB approximation requires that the inequality  $n < \Delta$  be satisfied. This is also consistent with the fact that the potential should be unbounded at the end points  $F = +\infty$  and  $F = b_1$ .

## 4.2 Relation to supersymmetric quantum mechanics

We have already mentioned that the potential (4.3) has the form encountered in supersymmetric quantum mechanics [47, 48]. Let us make this relationship more explicit by first recalling that in supersymmetric quantum mechanics, two potentials are supersymmetric partners of one another provided that there is a superpotential  $W(z)$  so that

$$V_1(z) = W^2 - W' , \quad V_2(z) = W^2 + W' . \quad (4.11)$$

Then, in terms of the ladder operators

$$a = \frac{d}{dz} + W(z) , \quad a^\dagger = -\frac{d}{dz} + W(z) , \quad (4.12)$$

the eigenstates of the two Hamiltonians  $H_1, H_2$  are related to each other as

$$\psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} a \psi_{n+1}^{(1)} , \quad \psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} a^\dagger \psi_n^{(2)} , \quad (4.13)$$

whereas for the energy levels, in the case that the spectra of the two partner potentials are discrete, we have the relation

$$E_n^{(2)} = E_{n+1}^{(1)} , \quad n = 0, 1, 2, \dots , \quad (4.14)$$

with  $E_0^{(1)} = 0$ . Notice that if  $\psi_{n+1}^{(1)}$  of  $H_1$  is normalized then the wave-function  $\psi_n^{(2)}$  of  $H_2$  will be also normalized and vice-versa. Thus, on general grounds, knowing the spectrum

of  $H_1$  one can construct the spectrum of  $H_2$ ; conversely, from the spectrum of  $H_2$  one can deduce the spectrum of  $H_1$  apart from the ground state with  $E_0^{(1)} = 0$  which is not paired. This relation is true only for the case of unbroken supersymmetry. When supersymmetry is broken, there is a 1-1 pairing of all eigenstates of  $H_1$  and  $H_2$  and the relations become modified by replacing  $\psi_{n+1}^{(1)}$  with  $\psi_n^{(1)}$  and  $E_{n+1}^{(1)}$  with  $E_n^{(1)}$  in the equations above. Then, the potentials  $V_1$  and  $V_2$  have degenerate positive ground state energies. We also recall that when two partner potentials have continuum spectra the corresponding reflection and transmission probabilities are identical.

The Schrödinger potential (4.3) that arose in the study of quantum fluctuations on domain wall backgrounds has indeed the form  $V_2(z)$  with superpotential

$$W(z) = \frac{D-2}{2}A'(z) . \quad (4.15)$$

Then, using (3.3) and (3.8) we find that the partner potential  $V_1(z)$  in (4.11) takes the form

$$V_1 = \frac{f^{2/\Delta}}{4R^4} \left[ \left(1 - \frac{8}{\Delta}\right) \left(\sum_{i=1}^N \frac{1}{F-b_i}\right)^2 + 8 \sum_{i=1}^N \frac{1}{(F-b_i)^2} \right] . \quad (4.16)$$

which is analogous to the form (4.4) for  $V_2$ . Specific examples of this relation will be considered in detail in later sections.

## 5 Algebraic classification

The underlying mathematical structure for solving the differential equation

$$(F'(z))^4 = (F(z) - b_1)(F(z) - b_2) \cdots (F(z) - b_N) , \quad (5.1)$$

with arbitrary moduli  $b_i$ , is that of the Christoffel–Schwarz transformation in complex analysis. This transformation is familiar from electrostatics, where one applies the technique to find the electric potential for a given distribution of charges. From this point of view, it is not surprising that the ansatz we made for constructing static domain walls in theories of gauged supergravity amounts to solving a similar mathematical problem. It is useful to think of the variable  $z$  as being complex, whereas  $F$  takes values in the complex upper-half plane. Of course, appropriate restrictions have to be made at the end in order to ensure the reality of the variable  $z$  and hence the reality of our domain wall solutions. As we will see in detail, the solutions are characterized by the uniformization of Riemann surfaces, which are naturally associated to the Christoffel–Schwarz transformation. Hence, the explicit derivation of the Schrödinger potential  $V(z)$  requires, for all practical purposes, going through such a mathematical framework, apart from its own value in providing a systematic classification of all domain-wall solutions in terms of algebraic curves. Also note that the variable  $F$  is a function  $F(zg^2)$ , but for simplicity we set  $g = 1$  in the following; this parameter can be easily reinstated at the end by appropriate scaling in  $z$ . Here, we do not assume any particular ordering of  $b_i$ .

We will treat the Christoffel–Schwarz transformation in a unified way for all three cases of interest, namely  $(D, N) = (4, 8)$  (M2-branes),  $(D, N) = (5, 6)$  (D3-branes), and  $(D, N) = (7, 5)$  (M5-branes), since there is a hierarchy of algebraic curves within this transformation that depends on the isometry groups of the distributions of branes. It is useful to start with  $N = 8$  and consider an octagon in the complex  $z$ -plane, which is mapped onto the upper-half plane via a Christoffel–Schwarz transformation

$$\frac{dz}{dF} = (F - b_1)^{-\varphi_1/\pi} (F - b_2)^{-\varphi_2/\pi} \cdots (F - b_8)^{-\varphi_8/\pi} . \quad (5.2)$$

This transformation maps the vertices of the octagon to the points  $b_1, b_2, \dots, b_8$  on the real axis of the complex  $F$ -plane, whereas its interior is mapped onto the entire upper-half  $F$ -plane. The variables  $\varphi_i$  denote the exterior (deflection) angles of the octagon at the corresponding vertices, which are constrained by geometry to satisfy the relation  $\varphi_1 + \varphi_2 + \cdots + \varphi_8 = 2\pi$ . We proceed by making the canonical choice of angles  $\varphi_1 = \varphi_2 = \cdots = \varphi_8 = \pi/4$ , in which case we arrive at the differential equation that relates  $dz$  and  $dF$ :

$$\left( \frac{dz}{dF} \right)^4 = (F - b_1)^{-1} (F - b_2)^{-1} \cdots (F - b_8)^{-1} , \quad (5.3)$$

which is the equation we have to solve for the case of M2-branes.

It is convenient at this point to introduce complex algebraic variables

$$x = F(z) , \quad y = F'(z) , \quad (5.4)$$

which cast the above differential equation into the form of an algebraic curve

$$y^4 = (x - b_1)(x - b_2) \cdots (x - b_8) . \quad (5.5)$$

This defines a Riemann surface of genus  $g = 9$ ,<sup>5</sup> when any two moduli are not equal, as follows by direct application of the Riemann–Hurwitz relation, which is standard in algebraic geometry. The task now is to uniformize the algebraic curve by finding another complex variable, call it  $u$ , so that  $x = x(u)$  and  $y = y(u)$ , which resolves the problem of multi-valuedness of the algebraic equation above; the corresponding Riemann surface is pictured geometrically by gluing four sheets together along their branch cuts. Then, following the definition of  $x$  and  $y$  in terms of  $F(z)$  and its  $z$ -derivative, one applies the chain rule in order to obtain the function  $z(u)$  by integration of the resulting first-order equation

$$\frac{dz}{du} = \frac{1}{y(u)} \frac{dx(u)}{du} . \quad (5.6)$$

Finally, by inverting the result we obtain the function  $u(z)$ , which yields  $F(z)$ , and hence the conformal factor of the corresponding domain wall solutions, as well as the Schrödinger potential  $V(z)$  for the graviton and scalar field fluctuations in these backgrounds. Of course, there is an integration constant that appears in the function  $z(u)$ ,

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<sup>5</sup>We will use the symbol  $g$  to denote the genus of a Riemann surface. The fact that the same notation has already been used for a mass scale should not create a confusion to the reader.



but this can be fixed by requiring that the asymptotic behaviour of the domain walls approach the *AdS* geometry as  $z \rightarrow 0$ . We also note for completeness that there is a discrete symmetry  $x \leftrightarrow -x$ ,  $b_i \leftrightarrow -b_i$  that leaves invariant the form of the algebraic curve. It can be employed in order to set  $F$  bigger or equal to the maximum value of the moduli  $b_i$  instead of being smaller or equal to the minimum value, thus insuring that  $z \rightarrow 0$  corresponds to  $F \rightarrow +\infty$  instead of  $-\infty$ .

The whole procedure is straightforward, but in practice it turns out to be cumbersome when the moduli parameters  $b_i$  take general values. After all, the uniformization of a genus-9 Riemann surface and the explicit derivation of the function  $u(z)$  is a formidable task. Matters simplify considerably when one imposes some isometry that effectively reduces the genus of the algebraic curve as certain moduli are allowed to coalesce. In general we will have models for each continuous subgroup of the maximal isometry group  $SO(8)$ , in which case the algebraic curve takes the irreducible form

$$y^m = (x - b_1)^{a_1} (x - b_2)^{a_2} \cdots (x - b_n)^{a_n} , \quad (5.7)$$

where the integer exponents (with  $n \leq 8$ ) satisfy the relation  $a_1 + a_2 + \cdots + a_n = 2m$ . We present below in table 1 all Riemann surfaces that classify the domain-wall solutions of four-dimensional gauged supergravity with non-trivial scalar fields in the coset  $SL(8, \mathbb{R})/SO(8)$  by giving their genus according to the Riemann–Hurwitz relation, their irreducible form (since in certain cases the exponents have common factors and the curve might be reducible when written in its original form), as well as the corresponding isometry groups that determine the geometrical distribution of M2-branes in eleven dimensions. We have 22 models in total, which are listed.

| <i>Genus</i> | <i>Irreducible Curve</i>                                    | <i>Isometry Group</i>             |
|--------------|---|-----------------------------------|
| 9            | $y^4 = (x - b_1)(x - b_2) \cdots (x - b_7)(x - b_8)$        | None                              |
| 7            | $y^4 = (x - b_1)(x - b_2) \cdots (x - b_6)(x - b_7)^2$      | $SO(2)$                           |
| 6            | $y^4 = (x - b_1)(x - b_2) \cdots (x - b_5)(x - b_6)^3$      | $SO(3)$                           |
| 5            | $y^4 = (x - b_1) \cdots (x - b_4)(x - b_5)^2(x - b_6)^2$    | $SO(2) \times SO(2)$              |
| 4            | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^2(x - b_5)^3$   | $SO(2) \times SO(3)$              |
| 3            | $y^4 = (x - b_1) \cdots (x - b_4)(x - b_5)^4$               | $SO(4)$                           |
|              | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^5$              | $SO(5)$                           |
|              | $y^4 = (x - b_1)(x - b_2)(x - b_3)^3(x - b_4)^3$            | $SO(3) \times SO(3)$              |
|              | $y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^2(x - b_5)^2$ | $SO(2) \times SO(2) \times SO(2)$ |
| 2            | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^2(x - b_4)^3$          | $SO(2) \times SO(2) \times SO(3)$ |
| 1            | $y^4 = (x - b_1)(x - b_2)(x - b_3)^6$                       | $SO(6)$                           |
|              | $y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^4$            | $SO(2) \times SO(4)$              |
|              | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^5$                     | $SO(2) \times SO(5)$              |
|              | $y^4 = (x - b_1)^2(x - b_2)^3(x - b_3)^3$                   | $SO(2) \times SO(3) \times SO(3)$ |
|              | $y^2 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)$                | $SO(2)^4$                         |
| 0            | $y^4 = (x - b_1)(x - b_2)^7$                                | $SO(7)$                           |
|              | $y = (x - b)^2$   | $SO(8)$ (maximal)                 |
|              | $y^2 = (x - b_1)(x - b_2)^3$                                | $SO(2) \times SO(6)$              |
|              | $y^4 = (x - b_1)(x - b_2)^3(x - b_3)^4$                     | $SO(3) \times SO(4)$              |
|              | $y^4 = (x - b_1)^3(x - b_2)^5$                              | $SO(3) \times SO(5)$              |
|              | $y = (x - b_1)(x - b_2)$                                    | $SO(4) \times SO(4)$              |
|              | $y^2 = (x - b_1)(x - b_2)(x - b_3)^2$                       | $SO(2) \times SO(2) \times SO(4)$ |

Table 1: Curves and symmetry groups of domain walls for M2-branes.

We will see later that, for the models with low genus (0 or 1), the uniformization can be carried out in all detail and it is possible in many cases to arrive at explicit expressions for the exact form of the Schrödinger potential.

It is interesting to note that the classification of domain walls of five-dimensional gauged supergravity with non-trivial scalar fields in the coset  $SL(6, \mathbb{R})/SO(6)$  follows immediately from above by restricting our attention to models with an  $SO(2)$  isometry factor in the symmetry group. It is known that in this case the classification reduces to the list of all algebraic curves [11]

$$y^4 = (x - b_1)(x - b_2) \cdots (x - b_6) , \quad (5.8)$$

depending on the values of the six moduli  $b_i$ . But such curves can be viewed as special cases of the  $N = 8$  curves when  $b_7 = b_8 = \infty$ . Therefore, by comparison with table 1 we list in table 2 all domain walls of five-dimensional gauged supergravity, which correspond to various continuous distributions of D3-branes in ten dimensions. We have 11 models in total, which are listed.

| <i>Genus</i> | <i>Irreducible Curve</i>                                | <i>Isometry Group</i>             |
|--------------|---|-----------------------------------|
| 7            | $y^4 = (x - b_1)(x - b_2) \cdots (x - b_5)(x - b_6)$    | None                              |
| 5            | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)(x - b_5)^2$ | $SO(2)$                           |
| 4            | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^3$          | $SO(3)$                           |
| 3            | $y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^2$        | $SO(2) \times SO(2)$              |
| 2            | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^3$                 | $SO(2) \times SO(3)$              |
| 1            | $y^4 = (x - b_1)(x - b_2)(x - b_3)^4$                   | $SO(4)$                           |
|              | $y^4 = (x - b_1)(x - b_2)^5$                            | $SO(5)$                           |
|              | $y^4 = (x - b_1)^3(x - b_2)^3$                          | $SO(3) \times SO(3)$              |
|              | $y^2 = (x - b_1)(x - b_2)(x - b_3)$                     | $SO(2) \times SO(2) \times SO(2)$ |
| 0            | $y^2 = (x - b_1)(x - b_2)^2$                            | $SO(2) \times SO(4)$              |
|              | $y^2 = (x - b)^3$                                       | $SO(6)$ (maximal)                 |

Table 2: Curves and symmetry groups of domain walls for D3-branes.

Finally, the algebraic classification of all domain-wall solutions of seven-dimensional gauged supergravity with non-trivial scalar fields in the coset  $SL(5, \mathbb{R})/SO(5)$  (which by the way provides the full scalar sector in this case) follows by considering all Riemann surfaces of the form

$$y^4 = (x - b_1)(x - b_2) \cdots (x - b_5) , \quad (5.9)$$

for various values of the five moduli  $b_i$ . As before, these surfaces can be viewed as special cases of the  $N = 8$  algebraic curves where three of the moduli are taken to infinity, i.e.  $b_6 = b_7 = b_8 = \infty$ , whereas the remaining are free to vary. Put differently, we may compose the list of all domain walls that correspond to various continuous distributions of M5-branes in eleven dimensions by considering all  $N = 8$  models with a  $SO(3)$  isometry factor. Thus, we have 7 models in total, which are listed.

| <i>Genus</i> | <i>Irreducible Curve</i>                              | <i>Isometry Group</i> |
|--------------|---|-----------------------|
| 6            | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)(x - b_5)$ | None                  |
| 4            | $y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^2$        | $SO(2)$               |
| 3            | $y^4 = (x - b_1)(x - b_2)(x - b_3)^3$                 | $SO(3)$               |
| 2            | $y^4 = (x - b_1)(x - b_2)^2(x - b_3)^2$               | $SO(2) \times SO(2)$  |
| 1            | $y^4 = (x - b_1)^2(x - b_2)^3$                        | $SO(2) \times SO(3)$  |
| 0            | $y^4 = (x - b_1)(x - b_2)^4$                          | $SO(4)$               |
|              | $y^4 = (x - b)^5$                                     | $SO(5)$ (maximal)     |

Table 3: Curves and symmetry groups of domain walls for M5-branes.

In this latter case, the invariance of the curves under the discrete symmetry  $x \leftrightarrow -x$ ,  $b_i \leftrightarrow -b_i$  is not present any more because the algebraic equations contain an odd number of factors.

## 6 Distributions of M2-branes

In this section we treat the distributions of M2-branes with isometries that correspond to genus 0 and 1 and present the uniformization of the associated algebraic curves. We have 12 such models of low genus, including the  $SO(8)$  model of  $AdS_4$  space. When it is possible, we determine the conformal factor of the metrics (written in conformally flat form), as well as the Schrödinger potentials corresponding to the equation for the scalar field and graviton fluctuations. There are several cases where we can explicitly solve the Schrödinger equation and find the spectrum by means of elementary methods, otherwise we will use the WKB approximation for the computations.

We begin first with the model  $SO(2) \times SO(2) \times SO(2) \times SO(2)$ , which is governed by a  $g = 1$  algebraic curve, and study certain limits for which some of the moduli are let to coincide and the genus is reduced to 0. These cases are  $SO(4) \times SO(2) \times SO(2)$ ,  $SO(4) \times SO(4)$  and  $SO(6) \times SO(2)$  as well as the trivial  $SO(8)$  model. We also present other  $g = 0$  models, which cannot be obtained as degenerate limits of the above, namely the models with symmetry group  $SO(7)$  and  $SO(3) \times SO(5)$ . For completeness we also present the uniformization of the remaining  $g = 1$  cases, as well as the remaining  $g = 0$  model with symmetry  $SO(3) \times SO(4)$ , which unfortunately cannot be explicitly brought to a conformally flat frame by expressing their uniformizing parameter as a function of  $z$ ,  $u(z)$ , in closed form.

### 6.1 $SO(2) \times SO(2) \times SO(2) \times SO(2)$

The algebraic curve can be taken from table 1 and corresponds to a  $g = 1$  Riemann surface

$$y^2 = (x - 1/\lambda_1)(x - 1/\lambda_2)(x - 1/\lambda_3)x , \quad (6.1)$$

with  $b_i = 1/\lambda_i$ . We set  $b_4 = 0$  by a shift of  $x$  and assume the ordering  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 = +\infty$ . Using the birational transformations

$$y = Yx^2/(2\sqrt{\lambda_1\lambda_2\lambda_3}) , \quad X = -1/x , \quad (6.2)$$

we obtain

$$Y^2 = 4(X + \lambda_1)(X + \lambda_2)(X + \lambda_3) . \quad (6.3)$$

Finally, we can bring the curve to its Weierstrass form

$$w^2 = 4v^3 - g_2v - g_3 , \quad (6.4)$$

by letting  $X = v - (\lambda_1 + \lambda_2 + \lambda_3)/3$ ,  $Y = w$ . The coefficients turn out to be

$$\begin{aligned} g_2 &= \frac{2}{9} \left( (\lambda_1 + \lambda_2 - 2\lambda_3)^2 + (\lambda_2 + \lambda_3 - 2\lambda_1)^2 + (\lambda_3 + \lambda_1 - 2\lambda_2)^2 \right) , \\ g_3 &= \frac{4}{27} (\lambda_1 + \lambda_2 - 2\lambda_3)(\lambda_2 + \lambda_3 - 2\lambda_1)(\lambda_3 + \lambda_1 - 2\lambda_2) . \end{aligned}$$

This elliptic curve can be uniformized in the standard way using the Weierstrass functions<sup>6</sup>

$$v = \wp(u) , \quad w = \wp'(u) . \quad (6.5)$$

In general, the two half-periods of the Weierstrass function are given by

$$\omega_1 = \frac{K(k)}{\sqrt{e_1 - e_3}} , \quad \omega_2 = \frac{iK(k')}{\sqrt{e_1 - e_3}} , \quad (6.6)$$

where  $K$  is the complete elliptic integral of the first kind with modulus  $k$  and complementary modulus  $k'$  given as

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} , \quad k'^2 = 1 - k^2 = \frac{e_1 - e_2}{e_1 - e_3} . \quad (6.7)$$

Here  $e_1$ ,  $e_2$  and  $e_3$  are the values of the Weierstrass function at the half-periods, i.e.  $\wp(\omega_1) = e_1$ ,  $\wp(\omega_2) = e_3$  and  $\wp(\omega_1 + \omega_2) = e_2$ , which are expressed in terms of the parameters  $\lambda_i$  as

$$e_i = -\lambda_i + \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) , \quad i = 1, 2, 3 . \quad (6.8)$$

So, we finally have

$$F(zg^2) = x = -\frac{1}{\wp(u) - (\lambda_1 + \lambda_2 + \lambda_3)/3} , \quad (6.9)$$

and

$$F'(zg^2) = y = \frac{1}{2}(\lambda_1\lambda_2\lambda_3)^{-1/2} \frac{\wp'(u)}{(\wp(u) - (\lambda_1 + \lambda_2 + \lambda_3)/3)^2} . \quad (6.10)$$

Next we compute  $dx/du = 2(\lambda_1\lambda_2\lambda_3)^{1/2}y$  and  $dx/dz = g^2y$  and find

$$u = \frac{g^2}{2}(\lambda_1\lambda_2\lambda_3)^{-1/2}z + c , \quad (6.11)$$

where the integration constant  $c$  is given implicitly by the relation

$$\wp(c) = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) . \quad (6.12)$$

The conformal factor is

$$e^{2A} = -\frac{g^2}{2\sqrt{\lambda_1\lambda_2\lambda_3}} \frac{\wp'(u)}{(\wp(u) - \wp(c))^2} . \quad (6.13)$$

The constant  $c$  in (6.12) was chosen in such a way that  $e^{2A} \sim 1/z^2$  as  $z \rightarrow 0$  and the space becomes  $AdS_4$ . Also, according to the conventions followed in this paper we choose the branch  $\wp'(c) < 0$ , since  $c$  is otherwise determined up to a sign; then, the function  $F$  in (6.9) indeed approaches  $+\infty$  as  $z \rightarrow 0$ . Using the equation for  $c$  and certain identities involving the Weierstrass function, the conformal factor can be brought to the form

$$e^{2A} = \frac{g^2}{4\lambda_1\lambda_2\lambda_3} (\wp(u - c) - \wp(u + c)) . \quad (6.14)$$

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<sup>6</sup>Throughout the rest of the paper we will make use of elliptic functions as well as of other special functions following the conventions of [50, 51, 52].

Finally, the Schrödinger potential (4.3) can be written as

$$V(z) = \frac{g^4}{4\lambda_1\lambda_2\lambda_3} (2\wp(u+c) + 2\wp(u-c) - \wp(2u)) . \quad (6.15)$$

The structure of the spectrum is understood using the results of section 4. For generic non-identical values of  $\lambda_i$  the spectrum is discrete and the range of  $0 \leq z \leq z_{\max}$  is finite, where  $z_{\max}$  is found by solving the equation  $F(z_{\max}g^2) = 1/\lambda_1$ . We find, in particular, that

$$z_{\max} = \frac{2\sqrt{\lambda_1\lambda_2\lambda_3}}{g^2}(\omega_1 - c) , \quad (6.16)$$

where  $\omega_1$  and  $c$  are given by (6.6) and (6.12). Then, using (4.10) with  $D = 4$ ,  $\Delta = 4$  and  $n = 2$ , we determine the spectrum within the WKB approximation

$$M_m^2 = \frac{\pi^2}{z_{\max}^2} m \left( m + \frac{1}{2} \right) + \mathcal{O}(m^0) , \quad m = 1, 2, \dots . \quad (6.17)$$

Note that the model we have just considered corresponds to the supersymmetric (extremal) limit of the most general rotating M2-brane solution [24] of eleven-dimensional supergravity. Physically, the parameters  $b_i$  are set equal (up to a factor) to the four different rotational parameters that appear in that general solution, and therefore are naturally associated to the Cartan subgroup of  $SO(8)$ . This identification is analogous to the model of a D3-brane distribution with  $SO(2) \times SO(2) \times SO(2)$  isometry, which corresponds to the supersymmetric limit of the most general rotating D3-brane solution [25] of type-IIB supergravity, as shown in [11].

We will return again to this model in section 8, together with other potentials that arise in elliptic solutions of gauged supergravities, and apply the rules of supersymmetric quantum mechanics to simplify the calculation of the exact spectrum. It will turn out that the partner potential is formulated as a Lamé problem with half-integer characteristic  $n = 1/2$ .

## 6.2 $SO(4) \times SO(2) \times SO(2)$

The corresponding algebraic curve of genus 0 is  $y^2 = (x-b_1)(x-b_2)(x-b_3)^2$ , according to table 1. It can be brought to the unicursal form  $v = w$  by the birational transformation

$$x = \frac{b_1vw - b_2}{vw - 1} , \quad y = (b_2 - b_1) \frac{v}{(vw - 1)^2} (b_2 - b_3 - (b_1 - b_3)vw) . \quad (6.18)$$

Introducing a uniformizing parameter  $u$ , so that  $u = v = w$ , we find after some computation

$$dz = \frac{2du}{b_2 - b_3 - (b_1 - b_3)u^2} , \quad (6.19)$$

which can be easily integrated by elementary functions; the final result depends on the sign of  $(b_1 - b_3)(b_2 - b_3)$ .

Alternatively, the result can be obtained as the limit  $\lambda_1 = \lambda_2$  of the  $SO(2) \times SO(2) \times SO(2) \times SO(2)$  model. Using some limiting properties of the Weierstrass functions, we find that the conformal factor is

$$e^{2A} = \frac{2}{R^2 \lambda_1 \lambda_3} (\lambda_3/\lambda_1)^{1/2} (\lambda_3/\lambda_1 - 1)^{3/2} \frac{\sinh 2\sqrt{\lambda_3 - \lambda_1} u}{(\cosh^2 \sqrt{\lambda_3 - \lambda_1} u - \lambda_3/\lambda_1)^2}, \quad (6.20)$$

and that the Schrödinger potential takes the trigonometric form

$$V = 4 \frac{\lambda_3 - \lambda_1}{R^4 \lambda_1^2 \lambda_3} \left( 4 \frac{\lambda_3/\lambda_1 \cosh 2\sqrt{\lambda_3 - \lambda_1} u - \cosh^2 \sqrt{\lambda_3 - \lambda_1} u}{(\cosh^2 \sqrt{\lambda_3 - \lambda_1} u - \lambda_3/\lambda_1)^2} + 1 - \frac{1}{\sinh^2 2\sqrt{\lambda_3 - \lambda_1} u} \right), \quad (6.21)$$

where

$$u = \frac{g^2 z}{2\lambda_1 \sqrt{\lambda_3}} + c, \quad c = \frac{\sinh^{-1}(\sqrt{\lambda_3/\lambda_1 - 1})}{\sqrt{\lambda_3 - \lambda_1}}. \quad (6.22)$$

The spectrum is continuous (according to the ordering of our parameters  $\lambda_1 = \lambda_2 < \lambda_3$ ), and  $0 \leq z < \infty$ . This example corresponds to  $n = 4$ , which we have already discussed on general grounds in section 4. Hence, there is a finite mass gap given by the value of the potential at  $z = \infty$ . We find, in particular, that

$$M_{\text{gap}}^2 = 4 \frac{\lambda_3 - \lambda_1}{\lambda_1^2 \lambda_3 R^4}. \quad (6.23)$$

Alternatively, one may consider the limit  $\lambda_2 = \lambda_3$  of the model with  $SO(2) \times SO(2) \times SO(2) \times SO(2)$  symmetry. It turns out that the result is equivalently described by considering the analytic continuation of (6.21) and (6.25) by taking  $\lambda_3 < \lambda_1$ . Then, the conformal factor becomes

$$e^{2A} = \frac{2}{R^2 \lambda_1 \lambda_3} (\lambda_3/\lambda_1)^{1/2} (1 - \lambda_3/\lambda_1)^{3/2} \frac{\sin 2\sqrt{\lambda_1 - \lambda_3} u}{(\cos^2 \sqrt{\lambda_1 - \lambda_3} u - \lambda_3/\lambda_1)^2}, \quad (6.24)$$

and the Schrödinger potential is

$$V = 4 \frac{\lambda_1 - \lambda_3}{R^4 \lambda_1^2 \lambda_3} \left( 4 \frac{\cos^2 \sqrt{\lambda_1 - \lambda_3} u - \lambda_3/\lambda_1 \cos 2\sqrt{\lambda_1 - \lambda_3} u}{(\cos^2 \sqrt{\lambda_1 - \lambda_3} u - \lambda_3/\lambda_1)^2} - 1 - \frac{1}{\sin^2 2\sqrt{\lambda_1 - \lambda_3} u} \right), \quad (6.25)$$

where

$$u = \frac{g^2 z}{2\lambda_1 \sqrt{\lambda_3}} + c, \quad c = \frac{\sin^{-1}(\sqrt{1 - \lambda_3/\lambda_1})}{\sqrt{\lambda_1 - \lambda_3}}. \quad (6.26)$$

The spectrum is discrete and the range of  $z$  is  $0 \leq z \leq z_{\text{max}}$ , where

$$z_{\text{max}} = \frac{2\lambda_1 \sqrt{\lambda_3} \sin^{-1}(\sqrt{\lambda_3/\lambda_1})}{g^2 \sqrt{\lambda_1 - \lambda_3}}. \quad (6.27)$$

Using the general results found in (4.10) (with  $D = 4$ ,  $\Delta = 4$ ,  $n = 2$ ), we obtain within the WKB approximation the spectrum

$$M_m^2 = \frac{\pi^2}{z_{\text{max}}^2} m \left( m + \frac{1}{2} \right) + \mathcal{O}(m^0), \quad m = 1, 2, \dots \quad (6.28)$$

The exact solution: It might seem hard to find the explicit solution of the Schrödinger equation with potentials (6.21) and (6.25). However, we will show that the explicit solution can be readily obtained by applying techniques of supersymmetric quantum mechanics, thus going beyond the WKB approximation. We consider for simplicity only the case when the spectrum is discrete and the potential is given by (6.25). It turns out that the partner potential to (6.25) is given by

$$V_1 = \frac{4(\lambda_1 - \lambda_3)}{R^4 \lambda_1^2 \lambda_3} \left( \frac{3}{\sin^2 2\sqrt{\lambda_1 - \lambda_3} u} - 1 \right), \quad (6.29)$$

using the notation of section 4, where  $u$  is given by (6.26). If we change variables for the wave-functions of the potential  $V_1$  to

$$\begin{aligned} x &= \cos^2 \sqrt{\lambda_1 - \lambda_3} u, & 0 \leq x \leq \frac{\lambda_3}{\lambda_1} < 1, \\ \Psi^{(1)} &= x^{3/4} (1-x)^{3/4} Y(x), \end{aligned} \quad (6.30)$$

the new independent variable  $Y(x)$  satisfies the hypergeometric equation with parameters  $a = 3/2 + \mu$ ,  $b = 3/2 - \mu$  and  $c = 2$  in the standard notation, where the constant  $\mu$  is defined as

$$\mu = \frac{1}{2} \sqrt{1 + \frac{R^4 \lambda_1^2 \lambda_3}{4(\lambda_1 - \lambda_3)} M^2}. \quad (6.31)$$

Hence, we may write down the solution for the wave-function  $\Psi^{(1)}$  that is regular at  $x = 0$ , as

$$\Psi^{(1)} \sim x^{3/4} (1-x)^{3/4} F\left(\frac{3}{2} + \mu, \frac{3}{2} - \mu, 2; x\right). \quad (6.32)$$

Demanding that it vanishes at the end point located at  $x = \lambda_3/\lambda_1 < 1$ , yields the quantization condition for the spectrum

$$F\left(\frac{3}{2} + \mu_m, \frac{3}{2} - \mu_m, 2; \frac{\lambda_3}{\lambda_1}\right) = 0, \quad m = 1, 2, \dots \quad (6.33)$$

It is not possible to find a closed formula for the exact spectrum except for the particular case when  $\lambda_1 = 2\lambda_3$ . Then, we have  $0 \leq x \leq \frac{1}{2}$  and making use of the identity

$$F\left(\frac{3}{2} + \mu_m, \frac{3}{2} - \mu_m, 2; \frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{4} + \frac{\mu_m}{2}\right) \Gamma\left(\frac{5}{4} - \frac{\mu_m}{2}\right)}, \quad (6.34)$$

we see that the condition (6.33) (with  $\lambda_3/\lambda_1 = 1/2$ ) is satisfied provided that  $\mu_m = 2m + \frac{1}{2}$ , where  $m = 1, 2, \dots$ . Finally, by employing (6.31), we find that the mass spectrum is given by

$$\lambda_1 = 2\lambda_3 : \quad M_m^2 = \frac{16}{R^4 \lambda_3^2} m(m + \frac{1}{2}), \quad m = 1, 2, \dots \quad (6.35)$$

One may easily check that it coincides with the WKB result (6.28) (after setting  $\lambda_1 = 2\lambda_3$ ), which is also exact in this case. The situation is similar to the distribution of



D3-branes with  $SO(2) \times SO(4)$  symmetry for which the exact spectrum [8, 9] coincides with what has been found using the WKB approximation [25]. This is a characteristic property of shape-invariant potentials [48]. We also note presently that the solution (6.32) can be expressed in terms of Jacobi polynomials as

$$\lambda_1 = 2\lambda_3 : \quad \Psi_m^{(1)} \sim x^{3/4}(1-x)^{3/4} P_{2m-1}^{(1,1)}(2x-1), \quad 0 \leq x \leq \frac{1}{2}, \quad m = 1, 2, \dots \quad (6.36)$$

For the generic case, where the constant parameters  $\lambda_1$  and  $\lambda_3$  satisfy the condition  $\lambda_1 > \lambda_3$ , but otherwise they remain unrelated, one may solve (6.33) in the asymptotic regime of large  $\mu$ . Using the fact that for large  $\mu$

$$F\left(\frac{3}{2} + \mu, \frac{3}{2} - \mu, 2; \frac{\lambda_3}{\lambda_1}\right) \sim \frac{\Gamma(\mu - 1/2)}{\Gamma(\mu + 1/2)} \mu^{-1/2} (\sin \varphi)^{-3/2} \sin \mu \varphi (1 + \mathcal{O}(\mu^{-1})), \quad (6.37)$$

where the angle  $\varphi$  is determined by  $\cos \varphi = 1 - 2\lambda_3/\lambda_1$ , we see that the spectrum is given to leading order by the solutions of the simple trigonometric equation  $\sin \mu \varphi = 0$ . It follows easily that the result agrees with the leading term in the WKB formulae (6.28).

Finally, note that the wave-functions  $\Psi_m^{(2)}$  of the corresponding partner potential  $V_2 = V$  have a form that is more complicated than (6.32). They are obtained using (4.13), where for the lowering operator (4.12) we have  $W(z) = A'(z)$  with  $A(z)$  given by (6.24). The result is

$$\begin{aligned} \Psi_m^{(2)} \sim & \frac{(x-x^2)^{1/4}}{\lambda_1 x - \lambda_3} \left[ 8(\lambda_3 - 2\lambda_3 x + \lambda_1 x^2) F\left(\frac{3}{2} + \mu, \frac{3}{2} - \mu, 2, x\right) + \right. \\ & \left. + (4\mu^2 - 9)(x-x^2)(\lambda_1 x - \lambda_3) F\left(\frac{5}{2} + \mu, \frac{5}{2} - \mu, 3, x\right) \right], \quad (6.38) \end{aligned}$$

with  $x$  being defined in (6.30); at the two endpoints  $\Psi_m^{(2)}$  vanishes.

### 6.3 $SO(4) \times SO(4)$

This model is described by an algebraic curve with  $g = 0$  and can easily be obtained by taking the limit  $\lambda_3 \rightarrow \infty$  in the above expressions. We find that the conformal factor for the metric is

$$e^{2A} = \frac{1}{R^2 \lambda_1^2} \frac{1}{\sinh^2\left(\frac{2z}{R^2 \lambda_1}\right)}, \quad (6.39)$$

whereas the potential is

$$V = \frac{4}{R^4 \lambda_1^2} \left( 1 + \frac{2}{\sinh^2\left(\frac{2z}{R^2 \lambda_1}\right)} \right). \quad (6.40)$$

In this case we have  $0 \leq z < \infty$ , and the spectrum is continuous with a mass gap

$$M_{\text{gap}}^2 = \frac{4}{R^4 \lambda_1^2}. \quad (6.41)$$

Alternatively, the same result is obtained by direct uniformization of the irreducible genus 0 curve,  $y = (x - b_1)(x - b_2)$  (see table 1). Letting  $v = y/(x - b_1)$ ,  $w = x - b_2$ , we arrive at the standard form  $v = w$ , which is uniformized by a complex parameter  $u$  as

$$x = u + b_2, \quad y = u(u + b_2 - b_1) \quad (6.42)$$

and so the final result reads

$$z(u) = \frac{1}{b_2 - b_1} \ln \left( \frac{u}{u + b_2 - b_1} \right); \quad u(z) = \frac{b_1 - b_2}{1 - e^{(b_1 - b_2)z}} \quad (6.43)$$

choosing the integration constant to be zero. Then, we have

$$y = \frac{(b_1 - b_2)^2}{4 \sinh^2 \frac{1}{2}(b_1 - b_2)z}, \quad (6.44)$$

which is equivalent to the expression for the conformal factor above, after introducing the appropriate scale.

In this model we can also determine the eigenfunctions exactly. It turns out that the wave-functions are expressed in terms of hypergeometric functions, using a characteristic parameter  $q \neq 0$ , as

$$\begin{aligned} \Psi = & \frac{1}{\sqrt{y^2 - 1}} \left[ C_1 y^{1+q} F \left( -\frac{1}{2} - \frac{q}{2}, -\frac{q}{2}, 1 - q; \frac{1}{y^2} \right) + \right. \\ & \left. C_2 y^{1-q} F \left( -\frac{1}{2} + \frac{q}{2}, \frac{q}{2}, 1 + q; \frac{1}{y^2} \right) \right], \end{aligned} \quad (6.45)$$

where

$$y = \cosh \left( \frac{2z}{R^2 \lambda_1} \right), \quad q = \sqrt{1 - \frac{M^2 R^4 \lambda_1^2}{4}}, \quad (6.46)$$

with constant coefficients  $C_1$  and  $C_2$ . A solution valid for  $q = 0$  can also be written down following [50, 51], but it will not be needed for the present purposes. Because of the mass gap (6.41), the parameter  $q$  is purely imaginary. This provides an orthonormalizability in the Dirac sense (with the use of a  $\delta$ -function), since the solution (6.45) becomes an incoming and an outgoing plane wave for  $z \rightarrow \infty$ . On the other hand, examining the behaviour of (6.45) near  $z = 0$ , we require the coefficient of the most singular term to vanish. It yields the following condition,

$$C_1 \frac{\Gamma(1 - q)}{\Gamma\left(\frac{3}{2} - \frac{q}{2}\right) \Gamma\left(1 - \frac{q}{2}\right)} + C_2 \frac{\Gamma(1 + q)}{\Gamma\left(\frac{3}{2} + \frac{q}{2}\right) \Gamma\left(1 + \frac{q}{2}\right)} = 0. \quad (6.47)$$

Note that the supersymmetric partner of the potential (6.40) is just a constant (given by  $V_1 = \frac{4}{R^4 \lambda_1^2}$ ) and therefore, according to the results of supersymmetric quantum mechanics, the potential (6.40) is reflectionless [48].

## 6.4 $SO(2) \times SO(6)$

In this case the irreducible algebraic curve is given by

$$y^2 = (x - b_1)(x - b_2)^3 . \quad (6.48)$$

Then, the birational transformation

$$x = \frac{wvb_2 - b_1}{wv - 1} , \quad y = -w \frac{(b_1 - b_2)^2}{(wv - 1)^2} , \quad (6.49)$$

brings it to the standard unicursal form  $v = w$ , which is uniformized as usual by introducing a complex parameter  $u$ ,  $u = v = w$ . We may easily find the coordinate that brings the four-dimensional metric to a conformally flat form,

$$z = -\frac{2(u - 1)}{g^2(b_1 - b_2)} , \quad (6.50)$$

with conformal factor given by

$$e^{2A} = \frac{8(2 - g^2(b_1 - b_2)z)}{g^2 z^2 (4 - g^2(b_1 - b_2)z)^2} . \quad (6.51)$$

In turn, the Schrödinger potential becomes

$$V(z) = \frac{2}{z^2} - \frac{g^4(b_1 - b_2)^2}{4(2 - g^2(b_1 - b_2)z)^2} + \frac{2g^4(b_1 - b_2)^2}{(4 - g^2(b_1 - b_2)z)^2} . \quad (6.52)$$

For  $b_1 < b_2$  we have  $0 \leq z < \infty$  and the spectrum is continuous with no gap, which is in agreement with our general discussion. Otherwise, for  $b_2 < b_1$ , we find  $0 \leq z \leq z_{\max} = 2/(g^2(b_1 - b_2))$  and the spectrum is discrete. Using (4.10) with  $D = 4$ ,  $\Delta = 4$  and  $n = 2$ , we find within the WKB approximation that the spectrum is

$$M_m^2 = \frac{4\pi^2(b_1 - b_2)^2}{R^4} m \left( m + \frac{1}{2} \right) + \mathcal{O}(m^0) , \quad m = 1, 2, \dots . \quad (6.53)$$

Note for completeness that (6.51) and (6.52) can also be obtained from (6.20) and (6.21) and from (6.24) and (6.25) by simply taking the limit  $\lambda_3 \rightarrow \lambda_1$ .

Finally, let us mention that the wave-functions and the spectrum can be determined exactly using, as before, techniques of supersymmetric quantum mechanics. The same is true for the other two cases that will be discussed in subsections 6.5 and 6.6 below. Since there are certain similarities in all three cases of interest, we choose to present their analysis based on supersymmetric quantum mechanics all together, and uniformly, in subsection 6.7.

## 6.5 $SO(3) \times SO(5)$

The algebraic curve of this model with genus 0 is given by

$$y^4 = (x - b_1)^3(x - b_2)^5, \quad (6.54)$$

which by means of the following birational transformation

$$x = \frac{w^3 v b_1 - b_2}{w^3 v - 1}, \quad y = -w^4 v \frac{(b_1 - b_2)^2}{(w^3 v - 1)^2}, \quad (6.55)$$

can be brought to the standard form  $w = v$ , which is uniformized (as usual) by a parameter  $u$ ,  $u = v = w$ . The coordinate choice that brings the four-dimensional metric to conformally flat form is

$$z = \frac{4(u - 1)}{g^2 (b_1 - b_2) u}, \quad (6.56)$$

and the corresponding conformal factor is

$$e^{2A} = \frac{1024 (4 - g^2 b z)^3}{g^2 z^2 (8 - g^2 b z)^2 (32 - g^2 b z (8 - g^2 b z))^2}, \quad (6.57)$$

with  $b = b_1 - b_2$ .

The Schrödinger potential turns out to be

$$V(z) = \frac{2}{z^2} + \frac{g^4 b^2}{4} \left( \frac{3}{(4 - g^2 b z)^2} + \frac{8}{(8 - g^2 b z)^2} - \frac{512}{(32 - g^2 b z (8 - g^2 b z))^2} + \frac{16}{32 - g^2 b z (8 - g^2 b z)} \right). \quad (6.58)$$

For  $b_1 < b_2$  the spectrum is continuous without a gap, in which case we have  $0 \leq z < \infty$ . For  $b_1 > b_2$ , we find  $0 \leq z \leq z_{\max} = 4/(g^2 b)$  and the spectrum is discrete; within the WKB approximation, it is given by the simple expression

$$M_m^2 = \frac{\pi^2 b^2}{R^4} m \left( m + \frac{3}{2} \right) + \mathcal{O}(m^0), \quad m = 1, 2, \dots, \quad (6.59)$$

where (4.10) has been used with  $D = 4$ ,  $\Delta = 4$ ,  $n = 3$ .

## 6.6 $SO(7)$

The corresponding  $g = 0$  algebraic curve is given by

$$y^4 = (x - b_1)^7(x - b_2) \quad (6.60)$$

and by means of the birational transformation

$$x = \frac{w^3 v b_1 - b_2}{w^3 v - 1}, \quad y = -w \frac{(b_1 - b_2)^2}{(w^3 v - 1)^2}, \quad (6.61)$$

it can be brought to the unicursal form  $v = w$ , which is again uniformized as  $u = v = w$ . The coordinate that brings the four-dimensional metric to conformally flat form is

$$z = -\frac{4(1-u^3)}{3g^2(b_1-b_2)} \quad (6.62)$$

and the conformal factor is given by

$$e^{2A} = \frac{g^2 b^2 \left(1 - \frac{3}{4} g^2 b z\right)^{\frac{1}{3}}}{\left(-1 + \left(1 - \frac{3}{4} g^2 b z\right)^{\frac{4}{3}}\right)^2}. \quad (6.63)$$

The potential of the Schrödinger equation of this model is

$$V(z) = \frac{g^4 b^2 \left(-5 + 42 \left(1 - \frac{3}{4} g^2 b z\right) \left(1 - \frac{3}{4} g^2 b z\right)^{\frac{1}{3}} + 91 \left(1 - \frac{3}{4} g^2 b z\right)^{\frac{8}{3}}\right)}{64 \left(-1 + \frac{3}{4} g^2 b z + \left(1 - \frac{3}{4} g^2 b z\right)^{\frac{7}{3}}\right)^2}, \quad (6.64)$$

with  $b = b_2 - b_1$ . For  $b_1 > b_2$  we have  $0 \leq z < \infty$  and the spectrum is continuous without a mass gap. On the other hand, for  $b_1 < b_2$ , we have  $0 \leq z \leq z_{\max} = 4/(3g^2b)$  and the spectrum is discrete given, within the WKB approximation, by

$$M_m^2 = \frac{9\pi^2 b^2}{R^4} m \left(m + \frac{5}{6}\right) + \mathcal{O}(m^0), \quad m = 1, 2, \dots, \quad (6.65)$$

where (4.10) has been used with  $D = 4$ ,  $\Delta = 4$ ,  $n = 1$ .

## 6.7 Some exact results

We are in a position to apply techniques of supersymmetric quantum mechanics, as in subsection 6.2 above, in order to find the explicit solution of the Schrödinger equation with potentials (6.52), (6.58) and (6.64) and determine the associated spectra in a uniform way; all three quantum mechanical problems will be treated at once.

First consider the case where the spectrum is discrete. Let us change the variable to  $x$ ,  $z = z_{\max} - x$ , where  $z_{\max}$  is defined as in the appropriate subsections and  $0 \leq x \leq z_{\max}$ . Then, by the rules of supersymmetric quantum mechanics, the partner potential to our problem is given by

$$V_1(x) = \frac{\nu^2 - 1/4}{x^2}, \quad (6.66)$$

where  $\nu = 1, 2, \pm 2/3$  for the three models that correspond to  $SO(2) \times SO(6)$ ,  $SO(3) \times SO(5)$  and  $SO(7)$ , respectively. Both signs  $\pm 2/3$  were chosen for the  $SO(7)$  model for reasons that will be indicated towards the end of this subsection. The solution of the Schrödinger equation which is regular at  $x = 0$  (for  $\nu = 1, 2, 2/3$ ) is given in terms of Bessel functions as

$$\Psi^{(1)} \sim x^{1/2} J_\nu(Mx). \quad (6.67)$$

For  $\nu = -2/3$  the wave-function diverges at  $x = 0$ , but it is not strong enough to make it non-integrable.<sup>7</sup> Imposing the condition that the wave-function vanishes at  $x = z_{\max}$ , which also ensures the Hermiticity of the Hamiltonian, we find the mass spectrum in terms of the zeros of the Bessel function,

$$J_\nu(M_m z_{\max}) = 0, \quad m = 1, 2, \dots \quad (6.68)$$

Then, the states in (6.67), which correspond to the different solutions of (6.68), constitute a complete set of states. The undetermined overall constant in (6.67) can be found, as usual, by demanding the orthonormalizability condition  $\int_0^{z_{\max}} dx \Psi_n^{(1)} \Psi_m^{(1)} = \delta_{n,m}$  and using the fact that

$$\int_0^{z_{\max}} dx x J_\nu(M_m x) J_\nu(M_n x) = \frac{z_{\max}^2}{2} J_{\nu+1}(M_m z_{\max}) \delta_{m,n}. \quad (6.69)$$

Note that the conditions (6.68) and  $\nu > -1$  are crucial for the validity of this equation.

An asymptotic expression for the eigenvalues  $M_m$  can be found, which is valid for large values of the argument  $M_m z_{\max}$  of  $J_\nu$ . Using standard formulae from the theory of Bessel functions we arrive at the result

$$M_m^2 = \frac{\pi^2}{z_{\max}^2} m \left( m + \nu - \frac{1}{2} \right) + \mathcal{O}(m^0), \quad m = 1, 2, \dots \quad (6.70)$$

For  $\nu = 1$  and  $\nu = 2$ , it agrees with the WKB formulae (6.53) and (6.59). However, the WKB formula (6.65) is reproduced for the value  $\nu = -2/3$  (by first shifting  $m$  by one unit) instead of  $\nu = 2/3$ . Actually, for  $\nu = 2/3$ , we obtain (6.65) with the number  $5/6$  replaced by  $1/6$ . This ambiguity in the spectrum remains unresolved even for the partner wave-functions  $\Psi^{(2)}$  that we are interested in computing afterall; these are obtained using (4.13), where for the lowering operator in (4.12) we have  $W(z) = A'(z)$ , with  $A(z)$  given by (6.51), (6.57) and (6.63) for all three different cases respectively.

We find, in particular, for the wave-functions that

$$\begin{aligned} \nu = 1 : \quad \Psi_m^{(2)} &\sim x^{1/2} \left( \frac{2x}{z_{\max}^2 - x^2} J_1(M_m x) + M_m J_0(M_m x) \right), \\ \nu = 2 : \quad \Psi_m^{(2)} &\sim x^{1/2} \left( \frac{4x^3}{z_{\max}^4 - x^4} J_2(M_m x) + M_m J_1(M_m x) \right), \\ \nu = \pm \frac{2}{3} : \quad \Psi_m^{(2)} &\sim \frac{x^{1/2}}{z_{\max}^{4/3} - x^{4/3}} \left( x^{4/3} J_{\pm 5/3}(M_m x) + z_{\max}^{4/3} J_{\mp 1/3}(M_m x) \right). \end{aligned} \quad (6.71)$$

It is crucial to note here that for all three type of wave-functions above,  $\Psi_m^{(1)}$  vanish at the end point  $x = z_{\max}$ . This can be shown by first expanding the wave-function

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<sup>7</sup> In fact, the general criteria developed in [53] render the propagation of quantum test particles in such a space-time geometry as unphysical, and hence should be related to unphysical vacuum expectation values of the scalar fields. Indeed, it was shown in [12] that, in this case, the density of the M2-brane distribution has a negative component which is physically unacceptable.

around  $x = z_{\max}$ , using properties of the Bessel functions, and then observe that the coefficient of the divergent part in the expansion is proportional to  $J_\nu(M_m z_{\max})$ , and hence vanishes due to (6.68); the constant part vanishes identically. Note also that the asymptotic behaviour of the wave-function near  $x = 0$  is  $\Psi_m^{(1)} \sim x^{1/6}$  for both  $\nu = 2/3$  and  $\nu = -2/3$ . Hence, there is a priori no reason to dismiss either one of the two values  $\nu = 2/3$  or  $\nu = -2/3$ . It remains unclear to us, at least for the moment being, what is the extra condition one should impose in order to exclude one of these two values. As soon as this becomes possible, the spectrum corresponding to the distribution of M2-branes with  $SO(7)$  symmetry will be determined unambiguously in the discrete case.

Considering the same potentials (6.52), (6.58) and (6.64), but for continuous spectrum, we find in all three cases that the parameter  $z_{\max}$  becomes negative. By changing variable to  $x$ , where  $z = x - |z_{\max}|$ , we note that the appropriate partner potential is still given by (6.66), but with  $|z_{\max}| \leq x < \infty$ . Since the point  $x = 0$  is not contained in the range of  $x$  we should admit both independent solutions of the corresponding Bessel equation, namely  $J_\nu$  and  $N_\nu$ , following the standard nomenclature. Demanding that the wave-functions vanish at the endpoint  $x = |z_{\max}|$ , determines their relative coefficient. Hence, the wave-function is given by

$$\Psi^{(1)} \sim N_\nu(M|z_{\max}|)J_\nu(Mx) - J_\nu(M|z_{\max}|)N_\nu(Mx) , \quad (6.72)$$

whereas the spectrum is continuous with no mass gap.

## 6.8 $SO(3) \times SO(4)$

Last, but not least, we consider the remaining curve of genus 0

$$y^4 = (x - b_1)(x - b_2)^3(x - b_3)^4 , \quad (6.73)$$

which can be brought to the unicursal form using the birational transformation

$$x = \frac{b_1 - b_2 v^3 w}{1 - v^3 w} , \quad y = \frac{(b_1 - b_2)v}{1 - v^3 w} \left( \frac{b_1 - b_2 v^3 w}{1 - v^3 w} - b_3 \right) \quad (6.74)$$

and hence uniformized by setting  $u = v = w$ . Assuming for definiteness that  $b_1 \neq b_2$ , we may proceed to solve the differential equation (5.6) in order to obtain the corresponding function  $z(u)$ . Taking the limits  $b_1 \rightarrow b_3$  or  $b_2 \rightarrow b_3$  yields the models  $SO(3) \times SO(5)$  and  $SO(7)$  respectively, which we have already discussed. For general  $b_i$ 's we arrive at

$$\begin{aligned} \alpha < 0 : \quad z &= \frac{1}{\sqrt{2}g^2(b_1 - b_3)q^3} \left[ 2 \tan^{-1} \left( \frac{\sqrt{2}qu}{1 - q^2 u^2} \right) + \right. \\ &\quad \left. + \ln \left( \frac{1 - \sqrt{2}qu + q^2 u^2}{1 + \sqrt{2}qu + q^2 u^2} \right) \right] + \text{const.} , \quad (6.75) \\ \alpha > 0 : \quad z &= \frac{1}{g^2(b_1 - b_3)q^3} \left[ -2 \tan^{-1} qu + \ln \left( \frac{1 + qu}{1 - qu} \right) \right] + \text{const.} , \end{aligned}$$

with

$$\alpha = \frac{b_2 - b_3}{b_1 - b_3}, \quad q = |\alpha|^{\frac{1}{4}}. \quad (6.76)$$

Unfortunately, it is not possible to invert the relations and find  $u(z)$  in closed form, and so they will not be pursued any further.

## 6.9 More $g = 1$ surfaces

In the following we present the uniformization of the remaining models with genus 1 in a unifying way. These cases are

$$\begin{aligned} (i) \quad & y^4 = (x - a)(x - b)(x - c)^6 && : SO(6), \\ (ii) \quad & y^4 = (x - a)(x - b)(x - c)^2(x - d)^4 && : SO(2) \times SO(4), \\ (iii) \quad & y^4 = (x - a)(x - c)^2(x - b)^5 && : SO(2) \times SO(5), \\ (iv) \quad & y^4 = (x - c)^2(x - a)^3(x - b)^3 && : SO(2) \times SO(3) \times SO(3). \end{aligned}$$

Using birational transformations, they can be brought into the same form

$$(X - c)^2 Y^4 = (X - a)(X - b), \quad (6.77)$$

where in each case we consider the following:

$$\begin{aligned} (i) \quad & X = x, \quad Y = \frac{y}{(x - c)^2}, \\ (ii) \quad & X = x, \quad Y = \frac{y}{(x - c)(x - d)}, \\ (iii) \quad & X = x, \quad Y = \frac{y}{(x - b)(x - c)}, \\ (iv) \quad & X = x, \quad Y = \frac{(x - a)(x - b)}{y}. \end{aligned}$$

Then, the birational transformations are employed

$$\begin{aligned} X &= c \frac{v^2 + \frac{k}{2} \frac{2ab - c(a+b)}{c(a-b)} v + \frac{k^2}{16}}{v^2 + \frac{k}{2} \frac{a+b-2c}{a-b} v + \frac{k^2}{16}}, \\ Y &= \sqrt{\frac{b-c}{a-c} \frac{w}{2v}}, \quad \text{with } k^2 = \frac{(a-b)^2(a-c)}{(b-c)^3}, \end{aligned} \quad (6.78)$$

to bring the common form (6.77) into the Weierstrass normal form of the curve

$$w^2 = 4v^3 - g_2 v - g_3; \quad g_2 = \frac{k^2}{4}, \quad g_3 = 0. \quad (6.79)$$



Note that in all cases we have set  $x = X$ , with  $X$  given by (6.78) above, whereas for  $y$  we have to treat each transformation separately. Explicit calculation shows that  $y$  is equal to

$$\begin{aligned}
(i) \quad & \frac{(a-c)^3}{\sqrt{(a-c)(b-c)}} \frac{vw}{2} \frac{1}{\left(v^2 + \frac{k}{2} \frac{a+b-2c}{a-b} v + \frac{k^2}{16}\right)^2}, \\
(ii) \quad & \frac{(a-c)^2}{\sqrt{(a-c)(b-c)}} \frac{w}{2} \frac{(a-d)v + (c-d)\sqrt{\frac{b-c}{a-c}}(v+k/4)^2}{\left(v^2 + \frac{k}{2} \frac{a+b-2c}{a-b} v + \frac{k^2}{16}\right)^2}, \\
(iii) \quad & \text{same as in (ii) setting } d = b, \\
(iv) \quad & -\frac{(a-c)^2}{(b-c)} \frac{2v(v+k/4)^2}{w} \frac{(a-b)v + (c-b)\sqrt{\frac{b-c}{a-c}}(v+k/4)^2}{\left(v^2 + \frac{k}{2} \frac{a+b-2c}{a-b} v + \frac{k^2}{16}\right)^2}
\end{aligned}$$

As usual, in all four models we have to solve the differential equation (5.6) in order to determine the function  $u(z)$ .

Using  $v = \wp(u)$  and  $w = \wp'(u)$  we find the following results in each case separately:

$$\begin{aligned}
(i) \quad & z = \frac{4}{a-c} \left( \zeta(u) + \frac{1}{4} \frac{\wp'(u)}{\wp(u)} \right) + \text{const.} \\
(ii) \quad & z = -\frac{8(b-c)^3}{(c-d)(a-b)^2} \frac{1}{\sqrt{(a-c)(b-c)}} \left[ \frac{1}{4} k^2 u + \right. \\
& \quad \left. + \frac{v_- \wp'(a_+)}{v_+ - v_-} \left( \log \frac{\sigma(u - a_+)}{\sigma(u + a_+)} + 2\zeta(a_+)u \right) - \right. \\
& \quad \left. - \frac{v_+ \wp'(a_-)}{v_+ - v_-} \left( \log \frac{\sigma(u - a_-)}{\sigma(u + a_-)} + 2\zeta(a_-)u \right) \right] + \text{const.}, \\
(iii) \quad & z = -\frac{4}{a-b} \left( 2\zeta(u) + \frac{\wp'(u)}{\wp(u) - k/4} \right) + \text{const.} \\
(iv) \quad & z = -\frac{2}{b-c} u + \text{const.},
\end{aligned} \tag{6.80}$$

where  $v_{\pm}$  are the two roots of the equation

$$(v + k/4)^2 + \frac{a-d}{c-d} \sqrt{\frac{a-c}{b-c}} v = 0 \tag{6.81}$$

and  $a_{\pm}$  are defined by  $v_{\pm} = \wp(a_{\pm})$ . In the case (ii) above we assume that  $c \neq d$ , and generically  $a_{\pm}$  differ from the half-periods of the corresponding Riemann surface. Although case (i) corresponds to taking  $c = d$ , there is no smooth limit of (ii) that yields the expression (i). In the case (iii) above we have  $v_+ = v_-$ , and so the derivation has to be performed separately without taking (ii) in the limit  $b = d$ .

It is rather unfortunate that we can not invert the relations and find  $u(z)$  in closed form for these models, apart from the case (iv). Hence, they will not be discussed any more. We only leave case (iv) as an exercise for the interested reader to explore it further.

## 7 Distributions of M5-branes

In this section we treat the distributions of M5-branes with isometries that correspond to genus 1 and 0 and present the uniformization of the associated algebraic curves in as much the same way as for the M2-branes. There are only two models to consider apart from the  $AdS_7$  space, namely the distribution with isometry group  $SO(2) \times SO(3)$  that corresponds to a genus 1 algebraic curve, and that with isometry group  $SO(4)$  corresponding to a genus 0 algebraic curve. Unfortunately there is no case where we could find the exact spectrum in terms of known functions and therefore we will only resort to the WKB approximation for the computations.

### 7.1 $SO(2) \times SO(3)$

Consider the genus 1 curve of table 3

$$y^4 = (x - a)^2(x - b)^3 . \quad (7.1)$$

With the aid of the birational transformation

$$x = \frac{b - a w^2}{4 v^3} + a , \quad y = (b - a) \frac{w}{v} \left( \frac{w^2}{4v^3} - 1 \right) , \quad (7.2)$$

it can be brought into the Weierstrass form

$$w^2 = 4v^3 - \frac{a - b}{4} v , \quad (7.3)$$

which is uniformized using the Weierstrass functions defined for  $g_2 = \frac{a-b}{4}$  and  $g_3 = 0$ . Since

$$x = b + \frac{(a - b)^2}{16\wp(u)^2} \equiv F(zg^2) , \quad y = \frac{(a - b)^2 \wp'(u)}{16\wp(u)^3} \equiv F'(zg^2) , \quad (7.4)$$

we obtain

$$u = c - \frac{g^2 z}{2} , \quad (7.5)$$

where  $c$  is an integration constant (to be fixed by the asymptotic conditions). Then, using (3.3), we find the conformal factor

$$e^{2A} = g^2 F'(zg^2)^{\frac{2}{5}} . \quad (7.6)$$

Note that the two half-periods of the torus  $\omega_1, \omega_2$  can be computed using (6.6) and the fact that in our model the modulus and its complement are equal to each other,  $k = k' = 1/\sqrt{2}$ , since  $g_3 = 0$ . We have

$$\omega_1 = -i\omega_2 = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}(a - b)^{1/4}} . \quad (7.7)$$

Hence, for  $a > b$ ,  $\omega_1$  is real and  $\omega_2$  is purely imaginary, whereas for  $a < b$  they are complex conjugate of each other. The constant of integration  $c$  is determined by requiring that

the space becomes  $AdS_7$  for  $z \rightarrow 0$ . We find, in particular, that  $c = \omega_1 + \omega_2$ , which is complex (or real) if  $a > b$  (or  $a < b$ ).

We also find the following Schrödinger potential, using the rescaling factor  $g = 2/R$ ,

$$V(z) = \frac{1}{R^4} \left[ 35\wp \left( \frac{2z}{R^2} \right) + 3\wp \left( \frac{2z}{R^2} + \omega_1 + \omega_2 \right) - \wp \left( \frac{2z}{R^2} + \omega_1 \right) - \wp \left( \frac{2z}{R^2} + \omega_2 \right) \right]. \quad (7.8)$$

Note that  $0 \leq z < z_{\max}$ , where  $z_{\max} = \frac{R^2}{2}(\omega_1 + \omega_2)$  if  $a < b$  and  $z_{\max} = \frac{R^2}{2}\omega_1$  if  $a > b$ . The spectrum is discrete and, within the WKB approximation, it is given by

$$a < b: \quad M_m^2 = \frac{16\pi^3\sqrt{b-a}}{\Gamma(1/4)^4 R^4} m(m+3) + \mathcal{O}(m^0), \quad m = 1, 2, \dots, \quad (7.9)$$

$$a > b: \quad M_m^2 = \frac{32\pi^3\sqrt{a-b}}{\Gamma(1/4)^4 R^4} m(m+2) + \mathcal{O}(m^0), \quad m = 1, 2, \dots, \quad (7.10)$$

where (4.10) has been used with  $D = 7$ ,  $\Delta = 4$  and  $n = 3$  (or  $n = 2$ ) for  $a < b$  (or  $a > b$ ). We will see later, in the context of supersymmetric quantum mechanics, that the partner potential is related to the potential of the  $SO(3) \times SO(3)$  model of D3-branes in five dimensions.

## 7.2 $SO(4)$

Consider next the genus 0 curve

$$y^4 = (x-a)(x-b)^4. \quad (7.11)$$

The birational transformation

$$x = \frac{1}{v^3 w} + a, \quad y = \frac{1}{v} \left( \frac{1}{v^3 w} + a - b \right), \quad (7.12)$$

brings it into the unicursal form  $v = w$ , which can be uniformized with a complex parameter  $u$ , as  $v = w = u$ . Consequently, we arrive at

$$dz = -\frac{4}{g^2} \frac{du}{(a-b)u^4 + 1}, \quad (7.13)$$

which yields upon integration the following cases:

$$\begin{aligned} a = b: \quad z &= -\frac{4u}{g^2} + \text{const.}, \quad \text{which gives } AdS_7, \\ a < b: \quad z &= -\frac{1}{g^2 \sqrt[4]{b-a}} \left[ \ln \left( \frac{1 + u \sqrt[4]{b-a}}{1 - u \sqrt[4]{b-a}} \right) + 2 \tan^{-1}(u \sqrt[4]{b-a}) \right] + \text{const.}, \\ a > b: \quad z &= -\frac{1}{g^2 \sqrt{2} \sqrt[4]{a-b}} \left[ \ln \left( \frac{1 + u \sqrt{2} \sqrt[4]{a-b} + u^2 \sqrt[4]{a-b}}{1 - u \sqrt{2} \sqrt[4]{a-b} + u^2 \sqrt[4]{a-b}} \right) + \right. \\ &\quad \left. + 2 \tan^{-1} \left( \frac{u \sqrt{2} \sqrt[4]{a-b}}{1 - u^2 \sqrt[4]{a-b}} \right) \right] + \text{const.} \end{aligned} \quad (7.14)$$

Apart from the maximally symmetric model that corresponds to  $a = b$ , these relations cannot be inverted to yield  $u(z)$  in closed form.

The nature of the spectrum depends crucially on the sign of  $a - b$ . Using our general formulae we find that for  $a < b$  we have  $0 \leq z < \infty$  and that the spectrum is continuous with a mass gap given by

$$a < b : \quad M_{\text{gap}}^2 = 4 \frac{\sqrt{b-a}}{R^4} . \quad (7.15)$$

For  $a > b$  we have  $0 \leq z \leq z_{\text{max}} = \sqrt{2}\pi/g^2(a-b)^{1/4}$  and the spectrum is discrete. It is approximated by the WKB formulae (4.10) with  $D = 7$ ,  $\Delta = 4$  and  $n = 1$  as

$$a > b : \quad M_m^2 = \frac{8\sqrt{a-b}}{R^4} m(m + \frac{7}{3}) + \mathcal{O}(m^0) , \quad m = 1, 2, \dots . \quad (7.16)$$

Finally we would like to mention the relation of the models we have presented in this section to the most general solution of rotating M5-brane [26] of eleven-dimensional supergravity. The latter, besides the usual Poincaré invariance along the brane, has also an  $SO(2) \times SO(2)$  symmetry group corresponding to the Cartan subgroup of  $SO(5)$ . Hence, in the extremal limit it will correspond to a supersymmetric solution associated with an algebraic curve of genus 2, as can be seen from the appropriate entry in table 3. The two independent parameters in the equation of the algebraic curve are related to the rotational parameters of this rotating M5-brane solution. The genus 1 model with symmetry  $SO(2) \times SO(3)$  corresponds to the particular limit when one rotational parameter is set equal to zero. The case with  $a < b$  corresponds to the rotating solution with Lorentzian signature, whereas for  $a > b$  it corresponds to the same solution, but with the time and angular parameters analytically continued so that the metric remains real but its signature becomes Euclidean. The associated spectra are given in the WKB approximation by (7.9) and (7.10).<sup>8</sup> The genus 0 model with symmetry group  $SO(4)$  corresponds to letting the two angular parameters become equal. We note that this is not equivalent to setting one of them zero and keeping the other finite. Then,  $a < b$  describes the Euclidean solution, whereas  $a > b$  describes the Lorentzian. The corresponding spectra are described by (7.15) and (7.16) respectively.

### 7.3 Wilson surfaces

We would like to calculate the vacuum expectation values of Wilson surface operators in the six-dimensional  $(0, 2)$  theories on the Coulomb branch. It was shown in [55] that the AdS/CFT correspondence could be used to compute Wilson surface observables [56] of  $(0, 2)$  theories in the limit of a large number  $N_b$  of M5-branes. The Wilson area operator in the supergravity picture is defined by requiring that a membrane ends at

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<sup>8</sup>In fact (7.10) coincides with the supersymmetric limit of the WKB formula given in [54] for the masses of  $0^{++}$  glueballs using rotating M5-branes with one rotational parameter, in a supergravity approach to QCD<sub>4</sub> [2].

the boundary of  $AdS_7 \times S^4$  on the surface that defines the operator. We will consider Wilson surfaces corresponding to a pair of parallel strings on the boundary using the prescription of [55] in the special backgrounds constructed in sections 3.1, 7.1 and 7.2. Wilson loops turned out to be useful tools for learning about the physics of gauge theories in the study of supergravity duals of four-dimensional theories on the Coulomb branch. It is interesting that complete screening was found with an associated screening length suppressed by  $1/\sqrt{g_s N_b}$  compared to what is expected from field theory considerations at weak coupling [8, 9].

In the conformal limit, this calculation was performed in [55] leading to the result

$$E = -\frac{8\sqrt{\pi}\Gamma\left(\frac{2}{3}\right)^3 N_b}{\Gamma\left(\frac{1}{6}\right)^3 L^2}, \quad (7.17)$$

where  $E$  denotes the energy per unit length or tension between the infinitely long strings as function of their separation  $L$ . This can easily be generalized to backgrounds of the form (3.15), (3.18)-(3.20) with  $f$  defined as in equation (3.5). We will impose the minor restriction  $b_1 = \dots = b_n = b$  and  $b_{n+1} = \dots = b_5 = 0$ , thus breaking the  $SO(5)$  symmetry to  $SO(n) \times SO(5-n)$ . Furthermore, we choose the orientation of the Wilson surface on the deformed  $S^4$  to be constant and to lie in the subspace spanned by  $y_{n+1}, \dots, y_5$  (see also (3.18)).

Minimizing the membrane action in these backgrounds with the orientation chosen as above, yields the following integrals for the length and the energy (for  $1 \leq n \leq 4$ )

$$L = \frac{R^2}{2} \int_{F_0}^{\infty} dF \frac{\sqrt{h(F_0)g(F)}}{\sqrt{h(F)(h(F) - h(F_0))}}, \quad (7.18)$$

$$E = \frac{1}{\pi^2 R} \int_{F_0}^{\infty} dF \frac{\sqrt{h(F)g(F)}}{\sqrt{h(F) - h(F_0)}} - \frac{1}{\pi^2 R} \int_b^{\infty} dF \sqrt{g(F)}, \quad (7.19)$$

$$h = (F - b)^{\frac{n}{2}} F^{\frac{3-n}{2}}, \quad g = F^{-1}, \quad (7.20)$$

with  $F_0 \geq b$  being the minimal value of  $F$  that the Wilson surface reaches. In general these integrals cannot be expressed in terms of known functions and so we will only present here some numerical results. For  $F \gg b$ , or equivalently for small separations  $L \ll R^2/\sqrt[4]{b}$ , the behaviour of the potential is as in (7.17) and goes to zero faster for larger separations. From a certain distance  $L > L_{\max}$ , and further on, there does not exist a minimal surface connecting the two strings on the boundary. Instead, a configuration of two separated surfaces hanging straight into the interior of the geometry is energetically preferred. This means that the potential is screened for large separation; a phenomenon that was also observed in four-dimensional superconformal theories on the Coulomb branch [8, 9]. The maximal distance at which the string breaks can be determined numerically

$$L_{\max} = c_n \frac{R^2}{\sqrt[4]{b}} \quad (7.21)$$

with  $c_1 \sim 0.71$ ,  $c_2 \sim 0.69$ ,  $c_3 \sim 0.70$  and  $c_4 \sim 0.78$ . For  $n = 1, 2, 3$  the length reaches its maximum  $L_{\max}$  at  $F_0 > b$  and becomes zero as  $F_0 \rightarrow b$ . At  $L = L_{\max}$  the energy

is larger than zero and the split configuration is preferred. For  $n = 4$ ,  $L_{\max}$  is reached exactly when  $F_0 \rightarrow b$ ; at this point the potential tends smoothly to zero and will remain there even if the separation is increased. This is the phenomenon of complete screening that was also found in the context of some special continuous distributions of D3-branes in [8, 9]. This phenomenon occurs in cases where the mass spectrum is continuous with a mass gap.

## 8 Comments on Lamé equations

In this section we summarize some results on the Lamé equation, and its various generalizations, which arise in the study of quantum fluctuations for the scalar and graviton fields in the background of domain walls associated to elliptic functions. We have already seen that in many cases the Schrödinger potential has the common form

$$V(u) = \lambda(\lambda+1)\wp(u) + \mu(\mu+1)\wp(u+\omega_1) + \nu(\nu+1)\wp(u+\omega_1+\omega_2) + \kappa(\kappa+1)\wp(u+\omega_2), \quad (8.1)$$

for various choices of the coefficients  $\lambda$ ,  $\mu$ ,  $\nu$  and  $\kappa$ ; they are all constrained to satisfy the Hermiticity bound  $\geq -1/2$ . We have found, in particular, the following list of examples:

- (i) D3-branes with  $SO(2) \times SO(2) \times SO(2)$ :  $\lambda = \frac{3}{2}$ ,  $\mu = \nu = \kappa = -\frac{1}{2}$ ,
- (ii) D3-branes with  $SO(3) \times SO(3)$ :  $\lambda = \nu = \frac{3}{2}$ ,  $\mu = \kappa = \frac{1}{2}$ ,
- (iii) M5-branes with  $SO(2) \times SO(3)$ :  $\lambda = \frac{5}{2}$ ,  $\nu = \frac{1}{2}$ ,  $\mu = \kappa = -\frac{1}{2}$ .

The first two cases were derived in [11], where emphasis was placed on analyzing domain wall solutions of five-dimensional gauged supergravity.<sup>9</sup> There, the expressions for the conformal factor were found to be

$$e^{2A(z)} = \left(\frac{\wp'(u)}{2R^3}\right)^{2/3} \quad \text{and} \quad \left(\frac{\wp'(u)}{4R\wp(u)}\right)^2, \quad (8.2)$$

with  $u = z/R^2$  and  $u = z/(2R^2)$ , respectively. The third example was discussed here in section 7.1, and has  $u = 2z/R^2$ . The parameter  $u$  assumes real values from 0 to  $\omega_1$  (real semi-period) for the cases (i), (ii) and (iii) (with  $a > b$ ), whereas for the case (iii) (with  $a < b$ )  $u$  takes real values from 0 to  $\omega_1 + \omega_2$ . In the context of supersymmetric quantum mechanics [48] the potential (i) is mapped to a potential with coefficients  $\lambda = 1/2$ ,  $\mu = \nu = \kappa = 0$ , which is simpler to study. The potentials (ii) and (iii) (with  $a > b$ ) also turn out to be related to each other via supersymmetric quantum mechanics, and the details are worth exposing.

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<sup>9</sup>Actually, for the  $SO(3) \times SO(3)$  model of D3-branes, the Schrödinger potential was originally presented in another form in [11], but that is equivalent to the potential (ii) because of special identities of the underlying Riemann surface with  $g_3 = 0$ .

More precisely, we find that the supersymmetric partner potentials corresponding to the  $SO(2) \times SO(2) \times SO(2)$  model in  $D = 5$  are given respectively by the pair

$$V_1(u) = 3\wp(2u) , \quad V_2(u) = 4\wp(u) - \wp(2u) , \quad (8.3)$$

making use of the identity

$$\begin{aligned} \wp(2u) &= -2\wp(u) + \left( \frac{\wp''(u)}{2\wp'(u)} \right)^2 \\ &= \frac{1}{4} (\wp(u) + \wp(u + \omega_1) + \wp(u + \omega_2) + \wp(u + \omega_1 + \omega_2)) . \end{aligned} \quad (8.4)$$

So, by rescaling  $u$  by a factor of 2, setting  $\tilde{u} = 2u$ , we obtain a partner Schrödinger problem in  $\tilde{u}$  with potential  $V_1(\tilde{u}) = \lambda(\lambda + 1)\wp(\tilde{u})$  having  $\lambda = 1/2$  as advertised. For later use we drop the tilde and still use the variable  $u$ , but with a range from 0 to  $2\omega_1$ . According to the previous general discussion, the ground state energy of  $H_1$  is expected to be zero, which will also be encountered later using a direct approach.

The Schrödinger potentials appearing in the examples (ii) and (iii) (with  $a > b$ ) also have supersymmetric partners within the same class. We find, in particular, that the supersymmetric partner potential of the model (ii) has  $\lambda = \nu = 1/2$ ,  $\mu = \kappa = 3/2$ , whereas a similar analysis for the model (iii) yields a potential with  $\lambda = \nu = 3/2$ ,  $\mu = \kappa = 1/2$ . We do not observe important simplifications occurring in the form of the potential in any of the two cases. An interesting observation is that the supersymmetric partner of the potential (iii) is (ii), and we believe that there is a deeper reason for this finding. Note at this point that, generically, any two supersymmetric partner potentials are related to each other by simply changing sign in the superpotential  $W \leftrightarrow -W$ . For domain wall solutions this would mean that the conformal factor reverses, since  $A \leftrightarrow -A$ , and so this transformation could only be of mathematical interest in relating different spectra. In physical terms, the transformation  $W \leftrightarrow -W$  cannot be used to map one domain wall solution to another because it fails to preserve the AdS boundary condition imposed on the conformal factor for  $z \rightarrow 0$ . However, it may happen in certain cases (as above) that there are two different superpotentials with the correct asymptotic behaviour as  $z \rightarrow 0$  which yield the same supersymmetric partners; indeed, the potential (ii) is the  $V_1$  partner of a potential  $V_2$  given by the model (iii), due to special identities on Riemann surfaces with  $g_3 = 0$ , but conversely this is not so because  $W \leftrightarrow -W$  does not relate the model (ii) and (iii).

Another notable relation concerns the supersymmetric partner of the elliptic potential for the  $SO(2) \times SO(2) \times SO(2) \times SO(2)$  model of M2-branes in four-dimensional gauged supergravity. Actually, in this case we find that the two partner potentials

$$V_1 = 3\wp(2u) , \quad V_2 = 2\wp(u + c) + 2\wp(u - c) - \wp(2u) \quad (8.5)$$

are connected by supersymmetry, and so by reinstating the overall scaling factor and constant shift that relates the uniformizing parameter to the Schrödinger variable (now

called  $u$ ), we obtain again the Lamé potential with  $\lambda = 1/2$  but in the range from  $2c$  to  $2\omega_1$ . This concludes the presentation of some qualitative results on the Schrödinger equation of quantum fluctuations on elliptic backgrounds.

An interesting problem that remains unsolved is the exact evaluation of the full spectrum of the Schrödinger equation in this class of potentials. So far we have relied on semi-classical approximation methods to get a feeling about the spectrum and the existence of a mass gap. Unfortunately, the exact result is very hard to find, even in some simple cases, and involves transcendental equations in rather implicit form. Nevertheless, it is quite instructive to highlight some special results in order to appreciate the degree of difficulty one faces in the general case. We note that the family of potentials under investigation are indeed a natural generalization of Lamé's potential  $n(n+1)\wp(u)$  by adding terms located at all four corners of the parallelogram  $u, u+\omega_1, u+\omega_2, u+\omega_1+\omega_2$  in the complex domain of a genus 1 Riemann surface. The Lamé potential was originally introduced in order to describe the analogue of spherical harmonics for the solutions of the Laplace equation in three dimensions with ellipsoidal symmetry. As such,  $n$  can only take (positive) integer values. However, generalizations were also considered, more than a century ago, for half-integer and other values of the parameter  $n$ . The main results in this direction go back to Hermite, Brioschi and Halphen (see, for instance, [57] and [58]), but also Darboux apparently studied some aspects of the general potential (8.1) in its Jacobi form. Half-integer values of the coefficients are particularly interesting for the examples we have at hand, but we will be able to say something explicit only for the case  $\lambda = 1/2$  and  $\mu = \nu = \kappa = 0$ , which is related to the  $SO(2) \times SO(2) \times SO(2)$  model of five-dimensional gauged supergravity or to the  $SO(2) \times SO(2) \times SO(2) \times SO(2)$  model in four dimensions.

We proceed by considering the Lamé equation

$$\left(-\frac{d^2}{du^2} + n(n+1)\wp(u)\right)\Psi(u) = E\Psi(u), \quad (8.6)$$

where  $E$  are the energy levels of the corresponding one-dimensional quantum mechanical problem and  $\Psi(u)$  are normalized wave functions that vanish (typically) at  $u = 0$  modulo the real period  $2\omega_1$ . Of course, when  $n$  is a positive integer, the solutions are easily described using appropriate ratios of the Weierstrass sigma-function, namely

$$\begin{aligned} \Psi_1(u) &= e^{\alpha u} \frac{\sigma(u - \beta_1)\sigma(u - \beta_2) \cdots \sigma(u - \beta_n)}{\sigma^n(u)}, \\ \Psi_2(u) &= e^{-\alpha u} \frac{\sigma(u + \beta_1)\sigma(u + \beta_2) \cdots \sigma(u + \beta_n)}{\sigma^n(u)}, \end{aligned} \quad (8.7)$$

where the constants  $\alpha$  and  $\beta_i$  are determined by substituting the ansatz into Lamé's equation. Using these two (in general independent) solutions one can construct regular solutions by taking suitable linear combinations of them and obtain a transcendental equation for the energy eigenvalues.

On the other hand, if  $n$  is not an integer the solutions will be difficult to describe, even in a formal sense. Some simplifications occur when  $n$  is half of an odd positive



integer. It is known in this case, using the substitution

$$\Psi = (\wp'(z))^{-n} \Phi(z) , \quad \text{where } z = u/2 , \quad (8.8)$$

that Lamé's equation transforms into the differential equation

$$\frac{d^2\Phi}{dz^2} - 2n \frac{\wp''(z)}{\wp'(z)} \frac{d\Phi}{dz} + 4(n(2n-1)\wp(z) + E)\Phi = 0 . \quad (8.9)$$

Then, according to results obtained by Brioschi and Halphen more than a century ago, a formal solution can be written as

$$\Phi(z) = \sum_{r=0}^{\infty} c_r (\wp(z) - e_2)^{a-r} , \quad (8.10)$$

provided that

$$(a - 2n)(a - n + 1/2) = 0 , \quad (8.11)$$

and that the following recursive relations are satisfied:

$$\begin{aligned} (a-r-2n)(a-r-n+1/2)c_r + (3e_2(a-r+1)(a-r-2n+1) + e_2n(2n-1) + E)c_{r-1} \\ = (e_1 - e_2)(e_2 - e_3)(a-r+2)(a-r-n+3/2)c_{r-2} . \end{aligned} \quad (8.12)$$

Here,  $e_1$ ,  $e_2$  and  $e_3$  denote the three roots of the cubic curve in its Weierstrass form. Actually, when  $n$  is half of an odd positive integer, there is a solution expressible in finite form

$$\Phi(z) = \sum_{r=0}^{n-1/2} c_r (\wp(z) - e_2)^{2n-r} , \quad (8.13)$$

which corresponds to  $a = 2n$ , and provides discrete energy levels  $E$  by solving the recursive relations with  $c_{n+1/2} = 0$ ; they are all real for curves having real roots  $e_i$ . Otherwise, for solutions expressible as an infinite sum, the energy levels remain arbitrary by these general considerations alone.

The simplest case to consider has  $n = 1/2$ . It turns out that a solution expressible in finite form has only one term:

$$\Phi(z) = c_0 (\wp(z) - e_2) , \quad (8.14)$$

with energy  $E = 0$  and  $c_0$  arbitrary. This yields one of the two independent solutions of Lamé's equation with zero energy, call it  $\Psi_1(u)$ , whereas the other is obtained by employing the general formula that relates any two solutions with the same energy  $E$ :

$$\Psi_2(u) = C\Psi_1(u) \int \frac{du}{\Psi_1^2(u)} , \quad (8.15)$$

where  $C$  is another constant. An explicit calculation shows that the most general solution with  $E = 0$  is given in terms of two integration constants  $A$ ,  $B$  by

$$\Psi(u) = \frac{1}{\sqrt{\wp'(u/2)}} (A\wp(u/2) + B) . \quad (8.16)$$

Having established this, one may impose the regularity of the physical solution at  $u = 0$ , namely  $\Psi(0) = 0$ , and set the coefficient  $A = 0$ ; otherwise the wave function will diverge as  $1/\sqrt{u}$  for  $u \rightarrow 0$ . For  $u = 2\omega_1$ , however, the wave-function blows up and hence it is not normalized.

For  $n = 1/2$ , but with  $E \neq 0$ , the formal solution has an infinite number of terms. According to the general discussion, we find after some calculation (working with either  $a = 0$  or  $a = 1$ ) the result

$$\Phi(z) = C \left( \frac{1}{E} + \sum_{k=1}^{\infty} \frac{d_k}{(\wp(z) - e_2)^k} \right), \quad (8.17)$$

where  $C$  is an arbitrary constant and

$$\begin{aligned} d_1 &= -\frac{1}{2}, & d_2 &= \frac{1}{2} \left( \frac{E}{6} + e_2 \right), \\ d_3 &= -\frac{1}{12} \left( (e_1 - e_2)(e_2 - e_3) + 9 \left( \frac{E}{6} + e_2 \right) \left( \frac{E}{18} + e_2 \right) \right), \\ d_4 &= \frac{3}{20} \left( (e_1 - e_2)(e_2 - e_3) \left( \frac{7E}{36} + 2e_2 \right) + 9 \left( \frac{E}{6} + e_2 \right) \left( \frac{E}{18} + e_2 \right) \left( \frac{E}{36} + e_2 \right) \right), \end{aligned} \quad (8.18)$$

and so on. Unfortunately, even in this simplest case with  $n = 1/2$ , it is very difficult to solve the recursive relations and find all coefficients  $d_k$  in closed form in order to sum up the infinite series of terms. In any case, this procedure yields one formal solution of Lamé's equation with  $E \neq 0$ ,  $\Psi_1(u)$ , whereas the other (independent) formal solution of the same energy,  $\Psi_2(u)$ , can be obtained according to the general formula above. However, the integration that determines  $\Psi_2(u)$  in terms of  $\Psi_1(u)$  cannot be performed explicitly unless one knows first how to sum up the infinite series of terms for  $\Phi(z)$ .

Note that for  $n = 1/2$  the formal solution  $\Psi_1(u)$  appears to be regular at  $u = 0$ , whereas at  $u = 2\omega_1$  (where  $\wp(u/2) = e_1$ ) this is not guaranteed. In fact, since the coefficients  $d_k$  are polynomial functions of the energy  $E$ , one must demand that the resulting “energy series” converge. Since  $\wp'(\omega_1) = 0$ , we demand i.e.

$$\frac{1}{E} + \sum_{k=1}^{\infty} \frac{d_k(E)}{(e_1 - e_2)^k} = 0. \quad (8.19)$$

This is certainly a non-trivial constraint on the allowed energy bands when  $E \neq 0$ , which also depend on the relative size of the  $a$ - and  $b$ -cycles of the Riemann surface, i.e. the differences  $e_2 - e_3$  and  $e_2 - e_1$  that appear in the “energy sum”. However, we are not able at present to find the complete solution to the problem in closed (even transcendental) form.

## 9 Conclusions

We have investigated in detail the structure of domain wall solutions in theories of gauged supergravity in diverse dimensions by considering the effect of non-trivial scalar fields tak-

ing values in the coset space  $SL(N, \mathbb{R})/SO(N)$ . The presentation was kept quite general to cover, where possible, aspects of domain wall solutions in theories of  $D$ -dimensional gravity by turning off the effect of any other fields, such as gauge fields and fermions. Special emphasis has been placed on two cases, namely  $(D, N) = (4, 8)$  and  $(7, 5)$ , which arise by compactification of eleven-dimensional supergravity on  $S^7$  and  $S^4$  respectively. The effect of the scalar fields in four (or seven) dimensions is related to deformations of the round spheres in the compactifying space, thus breaking the isometry group  $SO(8)$  (or  $SO(5)$ ) into appropriate symmetry subgroups. In fact, we were able to give an algebraic classification of all such cases using the Christoffel–Schwarz transformation, which arises in the solution of the first-order Bogomol’nyi-type equations for the conformal factor and the scalar fields of these models. As a result, for  $D = 4$ , we found a hierarchy of 22 solutions starting from an algebraic curve of genus 9 corresponding to completely broken isometry group. When some cycles shrink to zero size, by letting some moduli coalesce, the symmetry group is enhanced, whereas the genus of the Riemann surface is lowered accordingly. Among the genus 0 models there is the  $AdS_4$  space with no scalar fields, having maximal isometry group  $SO(8)$ , while in all other cases  $AdS_4$  is reached only asymptotically. Similarly, for  $D = 7$ , we found 7 models in total, which can be classified starting from a genus 6 algebraic curve with no isometry and proceeding all the way down to genus 0, where the  $AdS_7$  space arises with maximal isometry group  $SO(5)$ . All other models in the list admit subgroups of  $SO(5)$  as isometries and approach  $AdS_7$  only asymptotically due to the presence of non-trivial scalar fields.

A geometrical picture of our solutions in four and seven dimensions is provided in terms of distributions of M2- and M5-branes in eleven dimensions. The analysis has been carried out in detail choosing suitable harmonic functions that describe continuous distributions of branes in eleven dimensions, and is in accordance with the geometric deformation of seven and four dimensional spheres induced by the non-trivial moduli of the underlying algebraic curves. The resulting picture resembles the construction of domain walls of five-dimensional gauged supergravity with non-trivial scalar fields in the coset  $SL(6, \mathbb{R})/SO(6)$ , where suitable continuous distributions of D3-branes were considered in ten-dimensional type-IIB supergravity. An interesting problem that remains open for further study is the possible effect of dualities on the structure of domain-wall (and other) solutions in various dimensions. For example, M2- and M5-branes are related to each other via electric-magnetic duality, and hence various distributions of them in eleven dimensions should yield strong-weak coupling relations among solutions of gauged supergravities in lower dimensions. Also, D3-branes appear in a sequence of dualities between extended objects in higher dimensions, using both S- and T-dualities, and so the variety of domain walls in  $D = 4, 5$  and 7 dimensions (where consistent truncations of supergravity are known to exist) ought to be interrelated. Of course, it will be interesting to formulate this in the algebro-geometric context provided by the Christoffel–Schwarz transformation, where there is a universal moduli space of the genus 9 Riemann surface for M2-branes  $y^4 = (x - b_1)(x - b_2) \cdots (x - b_8)$ . It has been noted earlier that the Riemann surfaces of D3-branes arise as special cases by taking two of the moduli to infinity, while

for M5-branes there are three moduli taken to infinity. Thus, it is quite natural to expect that the modular transformations of the underlying algebraic curves have a natural interpretation in higher dimensions as S- and T-dualities among branes and their continuous distributions thereof. We hope to return to this issue in a separate publication.

Another topic that has been investigated in detail concerns the analytic form of fluctuations for the graviton and scalar fields in our domain-wall backgrounds. Since the quantitative analysis of this problem amounts to solving a one-dimensional quantum mechanical problem with potential  $V(z)$  that behaves as  $1/z^2$  for  $z \rightarrow 0$  (the asymptotic AdS region), the right identification of the variable  $z$  and the associated Schrödinger potential  $V(z)$  become crucial for extracting the spectrum. We found that the variable  $z$  appears naturally in the Christoffel–Schwarz transformation, which is a complex transformation that maps the interior of a closed polygon in the  $z$ -plane onto the upper-half  $F$ -plane, while the variable  $F$  is more appropriate for the brane description of our configurations in higher dimensions. Thus, the uniformization of the associated algebraic curves is a necessary step in order to derive the exact form of the potential, in each case of interest, and, consequently, to determine its spectrum. We were able to complete this mathematical task for the models with Riemann surfaces of genus 0 and 1 and derive the potentials in closed form for most of these cases. Subsequently, using techniques of supersymmetric quantum mechanics, which ensure that the spectrum is non-negative, we were able to compute the spectrum exactly for many models and make estimates using the WKB approximation for many others. The elliptic models exhibit an interesting class of potentials within the family of generalized Lamé potentials, which can be studied analytically only in certain cases.

It might seem surprising that we have encountered domain wall solutions with very little or even no isometry in the classification in terms of Riemann surfaces. Of course, we are unable to perform the uniformization and find explicit expressions for the conformal factor of their metrics and their scalar fields in general, since this has only been done for models of low genus and hence bigger isometry groups. Nevertheless, this situation can be perfectly accommodated in supersymmetric theories, where all the domain-wall solutions leave half of the supersymmetries unbroken. It is rather instructive to compare this situation with the geometry of four-dimensional hyper-Kähler manifolds and their possible isometry groups. Recall that all hyper-Kähler manifolds are supersymmetric (preserving half of the supersymmetries) and as such they admit three independent complex structures  $I, J, K$ , hence a whole sphere of them, since  $aI + bJ + cK$  will be a complex structure if  $a^2 + b^2 + c^2 = 1$ . The group  $SO(3)$  acts naturally on the space of complex structures as rotations; but this does not necessarily imply that all hyper-Kähler manifolds are  $SO(3)$  symmetric. Although many examples (like Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin) have an  $SO(3)$  isometry group, there are others with less isometry or none; for example Dancer’s manifold has only an  $SO(2)$  isometry, whereas K3 commonly used in compactifications of string theory has no isometries at all. In this sense, our domain-wall solutions to gauged supergravities in four, five and seven dimensions provide useful tools for developing a deeper understanding of the consistent

truncations of eleven-dimensional supergravity, and they can be rather exotic. Also, the associated Bogomol'nyi bounds and their possible description using contour integrals on the underlying Riemann surfaces pose some interesting mathematical questions for the future.

Finally, there is the conceptual issue of relating  $\mathcal{R}$ -symmetry to the isometry group that remains unbroken by the geometric structure of our solutions, which we believe is worth emphasizing. In our approach, the  $\mathcal{R}$ -symmetry is spontaneously broken by giving vacuum expectation values to the scalar fields of the theory that are charged under this symmetry. Then, the  $\mathcal{R}$ -symmetry is not a symmetry anymore, but relates different vacua of the theory. Generically, this procedure also breaks conformal invariance, thus breaking half of the 32 supersymmetries independently of the amount of symmetry that is left unbroken. In supergravity this can be understood as follows: for generic values of the real parameters  $b_i$  in the master curve  $y^4 = (x-b_1)(x-b_2)\cdots(x-b_8)$ , the gauge symmetry (related to the  $\mathcal{R}$ -symmetry in field theory), is spontaneously broken. In fact, for any given choice of the parameters, the curve is not invariant under  $SO(8)$ . However, its form is preserved since  $SO(8)$  acts naturally on  $b_1, b_2, \dots, b_8$  in its fundamental representation and rotates any given choice of moduli  $b_i$  into another. This is different from the situation where the background describes a flow from the maximally supersymmetric theory to a conformal theory with less supersymmetry. In the latter case the theory is perturbed by adding suitable deformations to the action, which explicitly break some or all of the supersymmetries; at the IR fixed point the geometry is again  $AdS_D$  and the  $\mathcal{R}$ -symmetry is related to the number of supersymmetries. This means, in particular, that the isometry group contains a factor that is equal to the  $\mathcal{R}$ -symmetry of the field theory.

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