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DOMATICALLY COCRITICAL GRAPHS

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In this paper we shall present some results on graphs which are cocritical with respect to the domatic number. We consider finite undirected graphs without loops and multiple edges.

A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$. This concept was introduced by E. J. Cockayne and S. T. Hedetniemi [1]. It is a well-defined concept, because for every graph there exists at least one domatic partition, namely, the partition consisting of one class.

A graph G is called domaticaly cocritical, if any graph obtained from G by joining a pair of its non-adjacent vertices by an edge has the domatic number greater than that of G. The investigation of domatically cocritical graphs was suggested by the author of this paper at the Czechoslovak Conference on Graph Theory in Pardubice in 1980.

Note that any complete graph fulfils the condition of the definition trivially; it includes no pairs of non-adjacent vertices. Hence complete graphs will be also considered as domatically cocritical graphs.

A graph G is called domatically full, if $d(G) = \delta(G) + 1$, where $\delta(G)$ is the minimum degree of a vertex of G. (This concept was introduced by the authors of [1] who have also proved that $d(G) \leqq \delta(G) + 1$ for every graph G.)

Theorem 1. *Let G be a finite non-complete undirected graph. Then the following two assertions are equivalent:*

(i) *G is simultaneously domatically full and domatically cocritical and its domatic number is d.*

(ii) *G is obtained from a complete graph K_n with n vertices, where $n \geqq d$, by adding a new vertex and joining it by edges with exactly $d - 1$ vertices of K_n.*

Proof. (i) $\Rightarrow$ (ii). Let G be domatically full and domatically cocritical. As it is domatically full, there exists a vertex u of G of the degree $d - 1$, where d is the domatic number of G. Suppose that there exist vertices v, w of $V(G) - \{u\}$ which are not adjacent. Then by joining v with w by an edge we obtain a graph G' in which u has also the degree $d - 1$. Therefore $\delta(G') \leqq d - 1$ and $d(G') \leqq d$, which is a contradiction with the assumption that G is domatically cocritical. Hence the subgraph of G induced by the set $V(G) - \{u\}$ must be a complete graph. As the degree of u is $d - 1$, it is adjacent to exactly $d - 1$ vertices of this subgraph.

(ii) $\Rightarrow$ (i). Let G fulfil (ii). Let $v_1, \ldots, v_n$ be the vertices of K_n, let u be the newly added vertex and let u be adjacent to all the vertices v_i for $i = 1, \ldots, d - 1$. The sets $\{v_1\}, \ldots, \{v_{d-1}\}, \{u, v_d, \ldots, v_n\}$ are dominating sets in G which are pairwise disjoint, therefore they form a domatic partition of G with d classes. As the degree of u is $d - 1$, the domatic number of G is equal to d. Each pair of non-adjacent vertices in G consists of the vertex u and one of the vertices $v_d, \ldots, v_n$. Let k be an integer, $d \leqq k \leqq n$, let G_k be the graph obtained from G by joining the vertices u, v_k by an edge. Then the sets $\{v_1\}, \ldots, \{v_{d-1}\}, \{v_k\}, V(G) - \{v_1, \ldots, v_{d-1}, v_k\}$ are dominating sets in G_k, they are pairwise disjoint and their union is $V(G)$. Hence they form a domatic partition of G_k with $d + 1$ classes and $d(G_k) = d + 1 > d(G)$. As k was chosen arbitrarily, the graph G is domatically cocritical. As the degree of u is $d - 1$, this graph is also domatically full.

Corollary 1. *A graph with the domatic number* 1 *is domatically cocritical if and only if it consists of two connected components, one of which is formed by an isolated vertex and the other is a complete graph.*

Proof. It was proved by E. J. Cockayne and S. T. Hedetniemi that every graph with the domatic number 1 contains an isolated vertex, i.e. it is domatically full. Thus the assertion follows immediately from Theorem 1.

Proposition 1. *Let G be a domatically cocritical graph, let u, v be non-adjacent vertices of G, let G' be the graph obtained from G by joining u with v by an edge. Then $d(G') = d(G) + 1$ and every domatic partition $\mathscr{D}$ of G' with $d(G')$ classes has the property that the vertices u, v belong to different classes of $\mathscr{D}$.*

Proof. As G is domatically cocritical, $d(G') > d(G)$. Suppose that $d(G') \geqq d(G) + 2$. Then there exists a domatic partition $\mathscr{D}_0$ of G' with $d(G) + 2$ classes. (Note that from each domatic partition we can obtain domatic partitions of smaller cardinalities by taking unions of some classes.) Among the classes of $\mathscr{D}_0$ there are at least $d(G)$ classes which contain neither u nor v; they are dominating sets in both G' and G. The union of the class containing u with the class containing v is also a dominating set in G, therefore there exists a domatic partition of G with $d(G) + 1$ classes, which is a contradiction. Therefore $d(G') = d(G) + 1$. Now let $\mathscr{D}$ be a domatic partition of G' with $d(G) + 1$ classes. Suppose that u, v belong to the same class C

of $\mathscr{D}$. Then C, being dominating in G', is dominating also in G and so are other classes of $\mathscr{D}$, therefore $\mathscr{D}$ is a domatic partition of G, too, which is a contradiction.

A graph G is called domatically critical, if for any graph G' obtained from G by deleting an edge, $d(G') < d(G)$ holds.

Theorem 2. *Let G be a finite non-complete undirected graph. Then the following two conditions are equivalent*:

(i) *G is obtained from a complete graph K_n, where $n \geqq 3$, by deleting an edge*;

(ii) *G is simultaneously domatically critical and domatically cocritical.*

Proof. (i) ⇒ (ii). Let G be obtained from a complete graph K_n by deleting an edge, $n \geqq 3$. Let $u_1, \ldots, u_n$ be the vertices of G and let u_1, u_2 be non-adjacent in G. The sets $\{u_1, u_2\}, \{u_3\}, \ldots, \{u_n\}$ form a domatic partition of G with $n - 1$ classes, therefore $d(G) \geqq n - 1$. It cannot be greater; otherwise each one-element subset of $V(G)$ would have to be dominating, which is not true. Therefore $d(G) = n - 1$. Let G' be a graph obtained from G by deleting an edge. Then in G' at least three vertices have degrees less than $n - 1$ and the one-element sets formed by these vertices are not dominating in G. Therefore in each domatic partition of G' at most $n - 3$ classes may have the cardinality 1. As the sum of cardinalities of all classes is n, the number of classes cannot be greater than $n - 2$ and $d(G') \leqq n - 2 < d(G)$ and G is domatically critical. The unique edge which can be added to G is u_1u_2; after adding it a complete graph with n vertices is obtained and its domatic number is $n > d(G)$. The graph G is domatically cocritical.

(ii) ⇒ (i). Let G be simultaneously domatically critical and domatically cocritical. Suppose that (i) does not hold, i.e. that in G there are at least two pairs of non-adjacent vertices. Let $\{u, v\}$ be one of them. Let G' be the graph obtained from G by adding the edge uv. According to Proposition 1 the graph G' has the domatic number $d(G) + 1$ and any domatic partition of G' with $d(G) + 1$ classes has the property that u, v belong to its different classes. Thus let $\mathscr{D} = \{D_1, \ldots, D_{d+1}\}$, where $d = d(G)$, be a domatic partition of G'; without loss of generality let $u \in D_d$, $v \in D_{d+1}$. If we denote $D_i^* = D_i$ for $i = 1, \ldots, d - 1$ and $D_d^* = D_d \cup D_{d+1}$, then $\mathscr{D}^* = \{D_1^*, \ldots, D_d^*\}$ is a domatic partition of G with d classes. As $\mathscr{D}$ is a domatic partition of G', any vertex of $V(G) - D_d^*$ is adjacent to at least one vertex of D_d and to at least one vertex of D_{d+1}, hence to at least two vertices of D_d^*. Now let $\{x, y\}$ be a pair of non-adjacent vertices of G different from the pair $\{u, v\}$. No two vertices of D_d^* are adjacent; otherwise in the graph obtained from G by deleting the edge joining them, D^* would be also a domatic partition and the domatic number of this graph would be $d(G)$, which would be a contradiction with the domatic criticality of G. Thus the vertices of D_d^* are adjacent only to vertices of the classes $D_1, \ldots, D_{d-1}$. If some vertex of D_d^* is adjacent to only one vertex of any one of the classes $D_1, \ldots, D_{d-1}$, then its degree is $d - 1$ and the graph G is domatically full and has the structure described in Theorem 1. As there are at least two pairs of non-adjacent vertices in G, we have $|V(G)| \geqq d + 2$. One of the sets $V(G) - \{u\}$, $V(G) - \{v\}$ induces a complete subgraph of G; without loss of generality let it be $V(G) - \{u\}$. Then u is the vertex of the degree $d - 1$ and one of the vertices x, y coincides with u; without loss of generality let $y = u$. Then v and x, being both in $V(G) - \{u\}$, are adjacent to one another and none of them is adjacent to u. If we delete the edge vx from G, we obtain a graph which has evidently the domatic number d, hence G is not domatically critical, which is a contradiction. Now suppose that there exists a vertex $a \in D_d^*$ which is adjacent to at least two vertices of some of the sets $D_1, \ldots, D_{d-1}$; without loss of generality let this set be D_1. Let b, c be two vertices of D_1 adjacent to a. As $D_d^* = D_d \cup D_{d+1}$ and $D_d \cap D_{d+1} = \emptyset$, assume without loss of generality that $a \in D_d$. As $\mathscr{D}$ is a domatic partition of G', the vertex c must be adjacent to a vertex $f \in D_{d+1}$, thus to a vertex of D_d^* different from a. If we delete the edge ac from G, then we obtain a graph in which $\mathscr{D}^*$ is a domatic partition, because c is adjacent to $f \in D_d$ and a is adjacent to $b \in D_1$ and the adjacency of other pairs of vertices remains unchanged. This is again a contradiction with the assumption that G is domatically critical. We have proved that (i) must hold.

The Zykov sum $G_1 \oplus G_2$ of two vertex-disjoint graphs G_1, G_2 is the graph obtained from these graphs by joining each vertex of G_1 with each vertex of G_2 by an edge.

Proposition 2. *Let G_0 be a domatically cocritical graph, let $d(G_0) = d$. Let K_n be a complete graph with n vertices which is vertex-disjoint with G_0. Then $G_0 \oplus K_n$ is a domatically cocritical graph and $d(G_0 \oplus K_n) = d + n$.*

The proof is left to the reader.

Now we shall study the complements of forests. A forest is a graph, all of whose connected components are trees. A graph consisting of one isolated vertex will be also considered a tree.

Proposition 3. *Let G be the complement of a forest consisting of a star and k isolated vertices. Then G is domatically cocritical and $d(G) = k + 1$.*

This is an immediate consequence of Theorem 1.

Proposition 4. *Let G be the complement of a forest consisting of k connected components isomorphic to K_2 and of l isolated vertices. Then G is domatically cocritical and $d(G) = k + l$.*

Proof. The vertex set of any connected component of the complement $\bar{G}$ of G is a dominating set in G, therefore $d(G) \geqq k + l$. Suppose that $d(G) = d > k + l$. The number of vertices of G is $2k + l$, hence at least $2d - 2k - l$ classes of a domatic partition of G with d classes must be one-element sets. This implies that G contains at least $2d - 2k - l$ vertices which are adjacent to all other vertices. But there are only l such vertices and $l < 2d - 2k - l$, which is a contradiction. Hence $d(G) = k + l$. By joining a non-adjacent pair of vertices of G by an edge a graph G' is obtained which is the complement of a forest consisting of $k - 1$ connected components isomorphic to K_2 and $l + 2$ isolated vertices. Thus $d(G') = k + l + 1 = d(G) + 1$ and G is domatically cocritical.

Theorem 3. *Let G be the complement of a forest, one of whose connected components is a tree with at least three edges not being a star. Then G is not domatically cocritical.*

Proof. Let T be the mentioned tree. As T is not a star, there exist two vertices u, v of T which are adjacent in T and none of them is a terminal vertex of T. Let G' be the graph obtained from G by joining the vertices u, v by an edge. Suppose that G is domatically critical and denote $d(G) = d$. Then $d(G') = d + 1$ and there exists a domatic partition $\mathscr{D}$ of G' with $d + 1$ classes such that u, v belong to different classes of $\mathscr{D}$. Let $D_1 \in \mathscr{D}$, $D_2 \in \mathscr{D}$, $u \in D_1$, $v \in D_2$. In G the sets D_1, D_2 are not both dominating, otherwise $\mathscr{D}$ would also be a domatic partition of G. Therefore either u is adjacent in G to no vertex of D_2, or v is adjacent in G to no vertex of D_1. Without loss of generality suppose that u is adjacent in G to no vertex of D_2, i.e. D_2 is a subset of the set $\Gamma(u)$ of vertices which are adjacent to u in T. If there exists a vertex $z \in D_1 - \{u\}$ different from u and adjacent in G neither to v nor to any vertex of $D_1 - \{u\}$, then z is adjacent to v in T and $D_1 - \{u\}$ is the subset of $\Gamma(z) - \{v\}$, where $\Gamma(z)$ is the set of vertices adjacent to z in T. Let w be a vertex of D_2 it is adjacent to u in T. Put $D_1^* = (D_1 - \{u\}) \cup \{w\}$, $D_2^* = (D_2 - \{w\}) \cup \{u\}$. If $x \in V(G) - D_1^*$ and x is adjacent to no vertex of $D_1 - \{u\}$ in G, then it is adjacent to w in G otherwise there would be a circuit in T. Thus D_1^* is dominating in G. If $y \in V(G) - D_2^*$ and y is adjacent to no vertex of $D_2 - \{w\}$, then it is adjacent to u, otherwise there would be a triangle in T. Therefore also D_2^* is a dominating set in G. If we substitute D_1 by D_1^* and D_2 by D_2^* in $\mathscr{D}$, we obtain a domatic partition of G with $d + 1$ classes, which is a contradiction. If u is adjacent in G to no vertex of $D_1 - \{u\}$, then $D_1 - \{u\}$ is a subset of $\Gamma(u) - \{v\}$, where $\Gamma(u)$ is the set of all vertices adjacent to u in T. Let $t \in D_1 - \{u\}$ and put $D_1^{**} = (D_1 - \{t\}) \cup \{v\}$, $D_2^{**} = (D_2 - \{v\}) \cup \{t\}$. If $x \in V(G) - D_1$ and x is adjacent to no vertex of $D_1 - \{t\}$ in G, then x is adjacent to v in G otherwise there would be a triangle in T. If $y \in V(G) - D_2$ and y is adjacent to no vertex of $D_2 - \{v\}$ in G, then it is adjacent to t in G; otherwise there would be a circuit in T. Thus again D_1^{**}, D_2^{**} are dominating sets in G and we may substitute D_1, D_2 by them in $\mathscr{D}$ to obtain a domatic partition of G with $d + 1$ classes. Finally, let each vertex from $V(G) - D_1$ and also the vertex u be adjacent in G to a certain vertex of $(D_1 - \{u\}) \cup \{v\}$. Then $D_1^{***} = (D_1 - \{u\}) \cup \{v\}$ is dominating in G. The set $D_2^{***} = (D_2 - \{v\}) \cup \{u\}$ is also dominating in G, because any vertex different from u and adjacent to no vertex of $D_2 - \{v\}$ is adjacent to u; otherwise there would be a triangle in T. Analogously as above we may obtain a domatic partition of G with $d + 1$ classes, which is a contradiction with the assumption that $d(G) = d$.

Proposition 5. *Let G be a graph whose complement $\bar{G}$ consists of two connected components with the same number of vertices greater than two. Then G is not domatically cocritical.*

Proof. Let C_1, C_2 be the mentioned connected components of G. Choose a bijective mapping φ of the vertex set $V(C_1)$ of C_1 onto the vertex set $V(C_2)$ of C_2. Let $x \in V(C_1)$. Then x is adjacent in G to all vertices of C_2 and $\varphi(x)$ is adjacent in G to all vertices of C_1; therefore $\{x, \varphi(x)\}$ is a dominating set in G for each $x \in V(C_1)$. Such sets form a partition of G with $\frac{1}{2}n$ classes, where n is the number of vertices of G. Hence $d(G) \geqq \frac{1}{2}n$. If we had $d(G) > \frac{1}{2}n$, at least one class of any domatic partition of G with $d(G)$ classes would have the cardinality one and this would imply the existence of an isolated vertex in $\bar{G}$. We have proved that $d(G) = \frac{1}{2}n$. If we join two non-adjacent vertices of G by an edge, we obtain a graph G' in whose complement at most one vertex is isolated. Therefore in any domatic partition of G' at most one class can have the cardinality one and thus the domatic number of G' cannot exceed $\frac{1}{2}n$.

Proposition 6. *Let G be a domatically cocritical graph with n vertices. Then*

(a) $d(G) = n$ *if and only if G is a complete graph;*

(b) $d(G) = n - 1$ *if and only if G is obtained from a complete graph by deleting one edge;*

(c) $d(G) = n - 2$ *if and only if G is obtained from a complete graph by deleting two edges;*

(d) $d(G) = n - 3$ *if and only if G is obtained from a complete graph by deleting three edges not forming a triangle;*

(e) $d(G) = n - 4$ *if and only if the complement of G is isomorphic to one of the graphs in Figs. 1–4 or is obtained from such a graph by adding isolated vertices.*

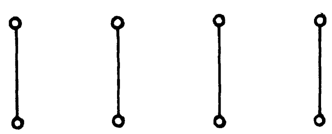

Fig. 1.

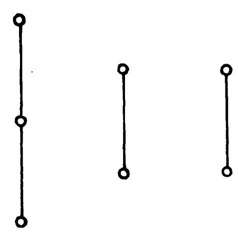

Fig. 2.

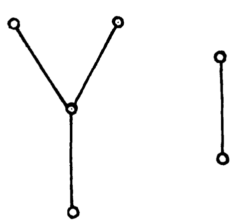

Fig. 3.

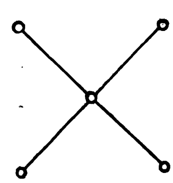

Fig. 4.

Proof. The assertion (a) is evident. The assertion (b) follows immediately from Theorem 2. The assertion (c) follows from Theorem 2 and Propositions 3 and 4. Similarly, it is clear that if G is obtained from a complete graph by deleting three edges not forming a triangle, then G is domatically cocritical and $d(G) = n - 3$. If we delete three edges forming a triangle from a complete graph, we obtain a graph in which there exists a domatic partition with one class of the cardinality three and all other classes of the cardinality one, therefore its domatic number is $n - 2$. Any graph obtained from a complete graph by deleting more than three edges is a spanning subgraph of a graph obtained from the same complete graph by deleting three edges not forming a triangle, therefore it cannot be domatically cocritical with the domatic number $n - 3$. Thus the assertion (d) is proved and we can prove also the assertion (e) in an analogous way.

Theorem 1 shows that for each positive integer d there exists a domatically cocritical graph with n vertices and with the domatic number d whose complement has $n - d$ edges. This is evidently also the minimal possible number of edges of the complement of a domatically cocritical graph with the domatic number d. We suggest the following problem.

Problem. *Does there exist a domatically cocritical graph G whose complement has more than $n - d(G)$ edges, where n is the number of vertices of G?*

Reference

[1] *E. J. Cockayne, S. T. Hedetniemi:* Towards a theory of domination in graphs. Networks *7* (1977), 247–261.

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