Bohdan Zelinka

Domatically critical graphs

Czechoslovak Mathematical Journal, Vol. 30 (1980), No. 3, 486–489

Persistent URL: `http://dml.cz/dmlcz/101697`

Czechoslovak Mathematical Journal, 30 (105) 1980, Praha

DOMATICALLY CRITICAL GRAPHS

BOHDAN ZELINKA, Liberec

(Received November 12, 1978)

In this paper we shall study domatically critical graphs. This study was proposed by E. J. COCKAYNE [1]. We consider finite undirected graphs without loops and multiple edges.

A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that each vertex of $V(G) - D$ is adjacent to at least one vertex of D. A domatic partition is a partition of $V(G)$ into pairwise disjoint dominating sets. The domatic number $d(G)$ of G is the maximal cardinality of a domatic partition of G. (The domatic number is to be distinguished from the domination number.)

A graph G is called domatically critical, if after deleting an arbitrary edge from G a graph with a smaller domatic number than that of G is obtained.

We shall prove a theorem.

Theorem 1. *Let G be a domatically critical graph with the domatic number $d(G) = d$. Then the vertex set $V(G)$ of G is the union of d pairwise disjoint sets $V_1, V_2, \ldots, V_d$ with the property that for any two distinct numbers i, j from the numbers $1, 2, \ldots, d$ the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ is a bipartite graph on the sets V_i, V_j, all of whose connected components are stars.*

Remark. A graph consisting of one edge with its end vertices is considered a star; a graph consisting of one isolated vertex is not.

Proof. As G has the domatic number d, there exists a domatic partition $\{V_1, V_2, \ldots, V_d\}$. Suppose that there exist two vertices of the same class of this partition which are joined by an edge e and let G' be the graph obtained from G by deleting e. Then each V_i, being a dominating set in G, is a dominating set also in G', because all edges joining vertices of V_i with vertices of $V(G) - V_i$ in G exist in G' as well. Therefore $\{V_1, V_2, \ldots, V_d\}$ is a domatic partition of G' and $d(G') \geqq d$; the graph G is not critical, which is a contradiction. We have proved that all sets $V_1, V_2, \ldots, V_d$ are independent in G; therefore each G_{ij} is a bipartite graph on the sets V_i, V_j. Now let i, j be two distinct numbers from the numbers $1, 2, \ldots, d$. As V_i

is a dominating set in G and $V_i \cap V_j = \emptyset$, each vertex of V_j must be adjacent to at least one vertex of V_i. Analogously each vertex of V_i must be adjacent to at least one vertex of V_j. Therefore in G_{ij} each vertex has the degree at least 1. If two vertices of a degree greater than 1 are adjacent in G_{ij}, then after deleting the edge joining them again each vertex of G_{ij} has a degree at least 1 in the resulting graph and in the graph obtained in this way from G the sets V_i, V_j (and obviously also all V_k for $i \neq k \neq j$) are domating sets, this graph has the domatic number d and G is not critical. Therefore each edge of G_{ij} is incident with a vertex of degree 1 in G_{ij} and each connected component of G_{ij} is a star.

A graph G is called indominable, if its vertex set can be partitioned into independent dominating sets. We have a corollary.

Corollary. *Every domatically critical graph is indominable.*

We express a conjecture.

Conjecture. *Every graph having the structure described in Theorem 1 is domatically critical.*

E. J. Cockayne and S. T. Hedetniemi [2] have proved that $d(G) \leqq \varrho(G) + 1$, where $\varrho(G)$ is the minimal degree of a vertex of G. If the equality $d(G) = \varrho(G) + 1$ holds, the graph G is called domatically full. We shall prove a theorem concerning regular domatically full graphs.

Theorem 2. *A regular domatically full graph G with n vertices and with a domatic number d exists if and only if d divides n; such a graph is also domatically critical. Its structure is the following: The vertex set $V(G) = \bigcup_{i=1}^{d} V_i$, $V_i \cap V_j = \emptyset$, $|V_i| = n/d$ and the subgraph G_{ij} of G induced by $V_i \cup V_j$ is regular of degree 1 (for $i = 1, \ldots, d$; $j = 1, \ldots, d$; $i \neq j$).*

Proof. Suppose that there exists a regular domatically full graph G with n vertices and with the domatic number d. As G is regular and domatically full, each vertex of G has degree $d - 1$. After deleting an arbitrary edge from G a graph G' is obtained in which two vertices have degree $d - 2$; hence $d(G') \leqq d - 1$ and G is domatically critical. This implies that G has the structure described in Theorem 1. Consider an integer i such that $1 \leqq i \leqq d$. Each vertex $x \in V_i$ must be adjacent to at least one vertex of V_j for each $j \in \{1, \ldots, d\} - \{i\}$. As these sets are pairwise disjoint, for each $j \neq i$ there exists exactly one edge joining x with a vertex of V_j. Therefore in each G_{ij} all vertices have degree 1. As G_{ij} is a bipartite graph on the sets V_i, V_j it is a complete matching of these sets and $|V_i| = |V_j|$. As i, j were chosen arbitrarily, all classes of the partition $\{V_1, V_2, \ldots, V_d\}$ have equal cardinalities and $|V_i| = n/d$ for each $i = 1, \ldots, d$. This is possible only if n/d is an integer, i.e. if d divides n. Therefore a regular domatically full graph with n vertices and with the domatic number d exists only if d divides n, and if it exists, it has the described structure. On the other hand, if d divides n, then obviously there exists a graph G with the described structure (e.g. the graph with n/d connected components which are all isomorphic to K_d). Thus let G be a graph with the described structure. Then it is evidently regular of degree $d - 1$. Its domatic number is at least d, because there exists a domatic partition $\{V_1, V_2, \ldots, V_d\}$. The inequality $d(G) \leqq \varrho(G) + 1$ implies that this domatic number cannot be greater than d, therefore it is equal to d and G is domatically full.

We shall now solve Problem 9 from [1]. An indivisible dominating set in a graph G is such a dominating set in G which is not a union of two distinct dominating sets. The least cardinality of a partition of the vertex set of G into indivisible dominating sets is called *the adomatic number of* G and denoted by $ad(G)$. (This is an analogue of the achromatic number of a graph.) Obviously $ad(G) \leqq d(G)$. Problem 9 in [1] is the following:

Do there exist vertex partitions into indivisible dominating sets of all orders between $ad(G)$ and $d(G)$?

The answer is negative.

Theorem 3. *To each positive integer n there exists a graph G for which*

$$d(G) - ad(G) = n$$

holds and which has the property that each partition of its vertex set into indivisible dominating sets has the cardinality either $d(G)$, or $ad(G)$.

Proof. Let G be the complete bipartite graph on sets A, B such that $|A| = |B| = n + 2$. The set A is evidently a dominating set in G. If A' is a proper subset of A, then no vertex of $A - A'$ is adjacent to a vertex of A', therefore A' is not a dominating set in G and A is an indivisible dominating set in G. Analogously B is an indivisible dominating set in G. Each two-element set $\{a, b\}$, where $a \in A$, $b \in B$, is a dominating set in G, because each vertex of $A - \{a\}$ is adjacent to b and each vertex of $B - \{b\}$ is adjacent to a. Evidently neither $\{a\}$ nor $\{b\}$ is a dominating set in G, therefore $\{a, b\}$ is a indivisible dominating set in G. Now let D be an indivisible dominating set in G. If $D \cap A = \emptyset$, then $D < B$. As shown above, D cannot be a proper subset of B, therefore $D = B$. Analogously $D \cap B = \emptyset$ implies $D = A$. If $D \cap A \neq \emptyset$, $D \cap B \neq \emptyset$, then D is the union of the sets $\{a, b\}$ for all $a \in D \cap A$ and all $b \in D \cap B$. As all these sets are indivisible dominating sets and D is also an indivisible dominating set, we must have $D = \{a, b\}$ for some $a \in A$ and $b \in B$. We have proved that each indivisible dominating set in G is equal either to A, or to B, or to some set $\{a, b\}$, where $a \in A$, $b \in B$. Each partition of the vertex set of G into indivisible dominating sets either is $\{A, B\}$, or consists of two-elements sets $\{a, b\}$ with the property that the edges joining these pairs $\{a, b\}$ form a complete matching of G. Therefore the cardinality of such a partition is either $ad(G) = 2$ or $d(G) = n + 2$ and the assertion is proved.

References

[1] *Cockayne, E. J.:* Domination of undirected graphs — a survey. In: Theory and Applications of Graphs, Proc. Michigan 1976, ed. by Y. Alavi and D. R. Lick. Springer Verlag Berlin—Heidelberg—New York 1978, pp. 141—147.

[2] *Cockayne, E. J. - Hedetniemi, S. T.:* Towards a theory of domination in graphs. Networks 7 (1977), 247—261.

Author's address: 460 01 Liberec 1, Komenského 2, ČSSR (katedra matematiky VŠST).