Dominance–based Rough Interval–valued Fuzzy Set in Incomplete Fuzzy Information System

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Abstract— **The fuzzy rough set is a fuzzy generalization of the classical rough set. In the traditional fuzzy rough model, the set to be approximated is a fuzzy set. This paper deals with an incomplete fuzzy information system with interval-valued decision by means of generalizing the rough approximation of a fuzzy set to the rough approximation of an interval-valued fuzzy set. Since all condition attributes are considered as criteria in such incomplete fuzzy information system, the interval-valued fuzzy set is approximated by using the information granules, which are constructed on the basis of a dominance relation. By the proposed rough approximation, the "at least" and "at most" decision rules can be generated from the incomplete fuzzy information system with interval-valued decision. To obtain the optimal "at least" and "at most" decision rules, the concepts of the lower and upper approximate reducts, the relative lower and upper approximate reducts for an object are proposed in the incomplete fuzzy information system with interval-valued decision. The judgement theorems and discernibility matrixes associated with these reducts are also obtained. Some numerical examples are employed to substantiate the conceptual arguments.**

Index Terms— **incomplete fuzzy information system, interval-valued fuzzy set, dominance relation, rough set theory, knowledge reduction, decision rule**

I. INTRODUCTION

Rough set theory [27]–[33], after a rocky start in the last stage of twentieth century, both in theoretic investigations and practical applications, has received more and more attentions by many researchers all over the world. In recent years, the rough set theory has been demonstrated to be useful in many fields such as Artificial Intelligence, Automatic knowledge Acquisition, Data Mining, Pattern Recognition and so on.

In the traditional rough set model, the lower and upper approximations were introduced with reference to an indiscernibility relation [27] (reflexive, symmetric, transitive), which is assumed to be an equivalence relation. Such approximations can only be used to deal with the information system in which the values of attributes are assumed to be nominal data, i.e. symbols. In many practical applications, however, the situations may be more complex because the complicated or mixed data. Therefore, how to expand the classical rough set model in complex information systems has become a necessity.

Presently, many generalizations of the rough set models, have been proposed in different types of the information systems. For example, by considering the unknown values in the information system (i.e. incomplete information system), many researchers have proposed different types of binary relations (similarity relation [36]–[38], tolerance relation [20], [23], limited tolerance relation [40] and so on) for classification purpose and constructing of the rough approximations [11], [16]–[21], [23], [24], [34], [37], [38], [40], [43], [44]. By considering the linguistic terms (i.e. fuzzy sets) of the attributes values, the rough set model can also be generalized to different fuzzy environments, i.e. fuzzy rough approaches [4]–[6], [10], [15], [26], [39], [42], [46]. Moreover, since the original rough set approach is not able to discover inconsistencies coming from consideration of criteria, that is, attributes with preference-ordered domains (scales), such as product quality, market share, and debt ratio, Greco et al. have proposed an extension of the Classic Rough Sets Approach, which is called the Dominance-based Rough Sets Approach (DRSA) [1]–[3], [7], [8], [11]–[14], [34], [45]. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. Greco et al. also generalized the DRSA to the fuzzy environment in Ref. [10].

In the incomplete information system, the set to be approximated is a crisp subset of the universe, which is induced from the partition determined by the decision attributes (decision class). In the DRSA, the sets to be approximated are upward and downward unions of the decision classes. On the other hand, in the fuzzy rough model, the set to be approximated is tend to be a fuzzy set on the universe of discourse. Moreover, it should be noticed that by generalizing the fuzzy rough approach, Ref. [9] proposed an extension of the fuzzy rough set model which is used to approximate an interval-valued fuzzy set. However, such approximations of the interval-valued fuzzy set are only constructed in Pawlak's approximate space (indiscernibility relation is used for classification purpose).

From discussion above, the purpose of this paper is to investigate a complex information system, which is called the incomplete fuzzy information system with interval-valued decision. Such a system has the following four

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characteristics:

- It is a *fuzzy* system because it formulates of a problem with fuzzy samples (samples containing fuzzy representations);
- It is an *incomplete* information system because some objects have unknown values on some of the condition attributes;
- All condition attributes in the incomplete fuzzy information system with interval-valued decision are considered as *criteria*;
- The set to be approximated in the incomplete fuzzy information system with interval-valued decision is an *interval-valued fuzzy set*.

Obviously, the incomplete fuzzy information system with interval-valued decision is a generalization of the incomplete and fuzzy information systems. By assuming that the unknown values in such system are just "missed", but they do exist [18], [19], an expanded dominance relation is used for classifying objects. The lower and upper approximations of the interval-valued fuzzy set are then presented, which are generalizations of the dominance-based fuzzy rough set proposed by Greco in Ref. [10]. By the lower approximation of the interval-valued fuzzy set, one can induce the "at least" decision rules, while by using the upper approximation of the interval-valued fuzzy set, the "at most" decision rules hidden in the information system can be unravelled.

Since knowledge reduction is one of the central problems in the rough set theory, based on the proposed rough approximations in this paper, we further propose four types of knowledge reductions, the lower (upper) approximate reducts and the relative lower (upper) approximate reducts for an object in the universe. The lower (upper) approximate reducts are minimal subsets of the condition attributes, which preserve the lower (upper) approximations of the interval-valued fuzzy set. The relative lower (upper) approximate reducts for an object in the universe, are minimal subsets of the condition attributes, which preserve the membership values of the lower (upper) approximations of the interval-valued fuzzy set for such object. Thus, by the relative lower (upper) approximate reducts for an object in the universe, one can obtain the optimal "at least" ("at most") decision rules supported by such object.

To facilitate our discussion, we first present the concepts of fuzzy information system and dominance-based fuzzy rough set in Section 2. We then propose the rough approximations in the incomplete fuzzy information system with interval-valued decision in Section 3. The concepts of the lower and upper approximate reducts, the relative lower and upper approximate reducts for an object are laid out in Section 4. We also present the practical approaches to compute these four types of reducts. We then summarize our paper in Section 5.

II. Dominance–based rough set model in fuzzy information system

Definition 1: A fuzzy set $\widetilde{F}$ defined on an universe U may be given as

$$\widetilde{F} = \{< x, \mu_{\widetilde{F}}(x) >: x \in U\} \tag{1}$$

where $\mu_{\widetilde{F}} : U \to [0,1]$ is the membership function of $\widetilde{F}$. The membership value $\mu_{\widetilde{F}}(x)$ describes the degree of belongingness of $x \in U$ in $\widetilde{F}$.

A fuzzy information system represents the formulation of a problem with fuzzy samples (samples containing fuzzy representations). A fuzzy information system can be denoted by a pair $\mathscr{I} =< U, AT >$ where U is a non-empty finite set of objects, it is called the universe, AT is a non-empty finite set of attributes.

A fuzzy decision table is a fuzzy information system $\mathscr{D} =< U, AT \cup d >$, where $d \notin AT$. d is an attribute called a decision, and AT is termed the condition attributes set.

In a fuzzy decision table $\mathscr{D}$, if $A \subseteq AT$ and $A = \{a_1, \cdots, a_m\}$ is the set of condition attributes, d is the decision attribute, then we consider an universe of discourse U and $m+1$ fuzzy sets, denoted by $\widetilde{a_1}, \cdots, \widetilde{a_m}$ and $\widetilde{d}$, defined on U by means of membership functions $\mu_{\widetilde{a_i}} : U \to [0,1]$, $i \in \{1, \cdots, m\}$ and $\mu_{\widetilde{d}} : U \to [0,1]$. $\mu_{\widetilde{a_i}}(x)$ and $\mu_{\widetilde{d}}(x)$ are used to represent the values of the object x with respect to the condition attribute a_i and the decision attribute d respectively.

Suppose that we want to approximate knowledge contained in $\widetilde{d}$ by using knowledge about $\{\widetilde{a_1}, \cdots, \widetilde{a_m}\}$. Then, the lower approximation of the fuzzy set $\widetilde{d}$ given the information on $\widetilde{a_1}, \cdots, \widetilde{a_m}$ is a fuzzy set $\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{d})$, whose membership value for each $x \in U$, denoted by $\mu_{\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x)$, is defined as [10]:

$$\mu_{\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x) = \inf_{z \in D_A^{\uparrow}(x)}\{\mu_{\widetilde{d}}(z)\}; \tag{2}$$

where for each $x \in U$, $D_A^{\uparrow}(x)$ is a non-empty set such that

$$\begin{aligned} D_A^{\uparrow}(x) &= D^{\uparrow}_{\{\widetilde{a_1},\cdots,\widetilde{a_m}\}}(x) \\ &= \{y \in U : \mu_{\widetilde{a_i}}(y) \geq \mu_{\widetilde{a_i}}(x), i \in \{1, \cdots, m\}\}, \end{aligned}$$

$D_A^{\uparrow}(x)$ is the set of objects dominating x in terms of the set of condition attributes A.

The lower approximation membership value $\mu_{\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x)$ can be interpreted as an "at least" decision rule:

$$\mu_{\widetilde{a_1}}(y) \geq \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \geq \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge$$
$$\mu_{\widetilde{a_m}}(y) \geq \mu_{\widetilde{a_m}}(x) \to \mu_{\widetilde{d}}(y) \geq \mu_{\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x).$$

Similarity, the upper approximation of $\widetilde{d}$ given the information on $\widetilde{a_1}, \cdots, \widetilde{a_m}$ is a fuzzy set $\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{d})$, whose membership value for each $x \in U$, denoted by $\mu_{\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x)$, is defined as [10]:

$$\mu_{\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x) = \sup_{z \in D_A^{\downarrow}(x)}\{\mu_{\widetilde{d}}(z)\}; \tag{3}$$

where for each $x \in U$, $D_A^{\downarrow}(x)$ is a non-empty set such that

$$D_A^{\downarrow}(x)=D^{\downarrow}_{\{\widetilde{a_1},\cdots,\widetilde{a_m}\}}(x) =\big\{y \in U : \mu_{\widetilde{a_i}}(y) \le \mu_{\widetilde{a_i}}(x), i \in \{1,\cdots,m\}\big\},$$

$D_A^{\downarrow}(x)$ is the set of objects dominated by x in terms of the set of condition attributes A.

The upper approximation membership value $\mu_{\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x)$ can be interpreted as an "at most" decision rule:

$$\mu_{\widetilde{a_1}}(y) \le \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \le \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge \mu_{\widetilde{a_m}}(y) \le \mu_{\widetilde{a_m}}(x) \rightarrow \mu_{\widetilde{d}}(y) \le \mu_{\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})}(x).$$

$\big[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d}), \overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})\big]$ is referred to as a pair of rough set of the fuzzy set $\widetilde{d}$ by using knowledge about $\{\widetilde{a_1},\cdots,\widetilde{a_m}\}$ in terms of the dominance principle. For more details about properties of $[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d}),\ \overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{d})]$, we refer the readers to Ref. [10].

III. DOMINANCE-BASED ROUGH SET APPROACH TO INCOMPLETE FUZZY INFORMATION SYSTEM WITH INTERVAL-VALUED DECISION

A. Rough Approximation of Interval-valued Fuzzy Set

In this section, what will be discussed is a complex decision table which is called the Incomplete Fuzzy Information System with Interval-valued Decision (IFISID). Such a decision table is still denoted without confusion by $\mathscr{D} =< U, AT \cup d >$. However, it should be noticed that the incomplete fuzzy information system with interval-valued decision is different from the traditional fuzzy decision table because the following reasons:

- Precise values of some objects on the fuzzy attributes are not known, i.e. unknown values. In this paper, the special symbol "*" is used to express the unknown value. Moreover, we assume here that the unknown value is just "missed", but it does exist. By such explanation, the unknown value "*" is considered as to be comparable with any real value in the domain of the corresponding attribute.
- The set to be approximated in the IFISID is not a fuzzy set, but an interval-valued fuzzy set. The membership function of such interval-valued fuzzy set is $\mu_{\widetilde{[d]}} : U \rightarrow \mathcal{I}[0,1]$ where $\mathcal{I}[0,1]$ is the the set of all closed subintervals of the interval $[0, 1]$.

Since the existence of unknown values, the traditional dominance relation should be generalized.

Definition 2: [34] Let $\mathscr{D}$ be an IFISID, $A = \{a_1,\cdots,a_m\} \subseteq AT$, the dominance relation in terms of A is defined as:

$$D(A) = \big\{(x, y) \in U^2 : \mu_{\widetilde{a_i}}(x) \ge \mu_{\widetilde{a_i}}(y) \vee \mu_{\widetilde{a_i}}(x) = * \vee \mu_{\widetilde{a_i}}(y) = *\big\} \quad (4)$$

where $i \in \{1,\cdots,m\}$.

Different from the traditional dominance relation proposed by Greco in Ref. [12], the dominance relation $D(A)$ is reflexive but–in general–does not need to be symmetric or transitive. Thus, $D(A)$ is a binary relation which satisfies

$$D(A) = \bigcap_{a_i} D(\{a_i\}), i \in \{1,\cdots,m\}; \quad (5)$$

$$A_1 \subseteq A_2 \Rightarrow D(A_1) \supseteq D(A_2). \quad (6)$$

By $D(A)$, one can define the following two sets for each $x \in U$:

- the set of objects *may* dominate x in terms of the set of condition attributes A, i.e.

$$D_A^{\uparrow *}(x) = D^{\uparrow *}_{\{\widetilde{a_1},\cdots,\widetilde{a_m}\}}(x) = \{y \in U : (y, x) \in D(A)\}, \quad (7)$$

- the set of objects *may* be dominated by x in terms of the set of condition attributes A, i.e.

$$D_A^{\downarrow *}(x) = D^{\downarrow *}_{\{\widetilde{a_1},\cdots,\widetilde{a_m}\}}(x) = \{y \in U : (x, y) \in D(A)\}. \quad (8)$$

Since by the decision attribute d, the set to be approximated is an interval-valued fuzzy set $\widetilde{[d]}$, $\forall x \in U$, let us denote $\mu^-_{\widetilde{[d]}}(x)$ and $\mu^+_{\widetilde{[d]}}(x)$ by the lower and upper limits of the object x with respect to the decision attribute d respectively with the condition $\mu^-_{\widetilde{[d]}}(x) \le \mu^+_{\widetilde{[d]}}(x)$. Moreover, $\forall x, y \in U$, let us denote by

- $\mu_{\widetilde{[d]}}(x) = \mu_{\widetilde{[d]}}(y) \Leftrightarrow \mu^-_{\widetilde{[d]}}(x) = \mu^-_{\widetilde{[d]}}(y), \mu^+_{\widetilde{[d]}}(x) = \mu^+_{\widetilde{[d]}}(y)$;
- $\mu_{\widetilde{[d]}}(x) \le \mu_{\widetilde{[d]}}(y) \Leftrightarrow \mu^-_{\widetilde{[d]}}(x) \le \mu^-_{\widetilde{[d]}}(y), \mu^+_{\widetilde{[d]}}(x) \le \mu^+_{\widetilde{[d]}}(y)$;
- $\mu_{\widetilde{[d]}}(x) < \mu_{\widetilde{[d]}}(y) \Leftrightarrow \mu_{\widetilde{[d]}}(x) \le \mu_{\widetilde{[d]}}(y), \mu_{\widetilde{[d]}}(x) \ne \mu_{\widetilde{[d]}}(y)$.
- The complementary of $\widetilde{[d]} = [\mu^-_{\widetilde{[d]}}(x), \mu^+_{\widetilde{[d]}}(x)]$ is denoted by $\widetilde{[d]}^C$ where $\widetilde{[d]}^C = [1-\mu^+_{\widetilde{[d]}}(x), 1-\mu^-_{\widetilde{[d]}}(x)]$.

Similar to the fuzzy set theory [47], the operators $\subseteq, \cap, \cup$ of the interval-valued fuzzy sets are defined as follows. Suppose that $\widetilde{[d_1]}$, $\widetilde{[d_2]}$ are two different interval-valued fuzzy sets induced by two different decisions d_1 and d_2, then

- $\widetilde{[d_1]} \subseteq \widetilde{[d_2]} \Leftrightarrow \mu_{\widetilde{[d_1]}}(x) \le \mu_{\widetilde{[d_2]}}(x)$ for each $x \in U$;
- $\mu_{\widetilde{[d_1]}\cap\widetilde{[d_2]}}(x) = \big[\min\{\mu^-_{\widetilde{[d_1]}}(x), \mu^-_{\widetilde{[d_2]}}(x)\}, \min\{\mu^+_{\widetilde{[d_1]}}(x), \mu^+_{\widetilde{[d_2]}}(x)\}\big]$;
- $\mu_{\widetilde{[d_1]}\cup\widetilde{[d_2]}}(x) = \big[\max\{\mu^-_{\widetilde{[d_1]}}(x), \mu^-_{\widetilde{[d_2]}}(x)\}, \max\{\mu^+_{\widetilde{[d_1]}}(x), \mu^+_{\widetilde{[d_2]}}(x)\}\big]$.

Definition 3: Let $\mathscr{D}$ be an IFISID, $A = \{a_1,\cdots,a_m\} \subseteq AT$, the lower approximation of the interval-valued fuzzy set $\widetilde{[d]}$ given the information on $\widetilde{a_1},\cdots,\widetilde{a_m}$ is an interval-valued fuzzy set $[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]$, whose membership value for each $x \in U$, denoted by $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)$, where

$$\begin{aligned}&\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)\\&= \left[\mu^-_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x), \mu^+_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)\right]\\&= \left[\inf_{z\in D_A^{\uparrow *}(x)}\{\mu^-_{\widetilde{[d]}}(z)\}, \inf_{z\in D_A^{\uparrow *}(x)}\{\mu^+_{\widetilde{[d]}}(z)\}\right], \quad (9)\end{aligned}$$

the upper approximation of the interval-valued fuzzy set $\widetilde{[d]}$ given the information on $\widetilde{a}_1,\cdots,\widetilde{a_m}$ is an interval-valued fuzzy set $[\overline{App}(\widetilde{a}_1,\cdots,\widetilde{a_m},\widetilde{[d]})]$, whose membership value for each $x \in U$, denoted by $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)$, where

$$\begin{aligned}&\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)\\&= \left[\mu^-_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x), \mu^+_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)\right]\\&= \left[\sup_{z\in D_A^{\downarrow *}(x)}\{\mu^-_{\widetilde{[d]}}(z)\}, \sup_{z\in D_A^{\downarrow *}(x)}\{\mu^+_{\widetilde{[d]}}(z)\}\right]. (10)\end{aligned}$$

The pair $\left[[\underline{App}(\widetilde{a}_1,\cdots,\widetilde{a_m},\widetilde{[d]})], [\overline{App}(\widetilde{a}_1,\cdots,\widetilde{a_m},\widetilde{[d]})]\right]$ is referred to as the rough approximation of the interval-valued fuzzy set $\widetilde{[d]}$ by using the knowledge about $\{\widetilde{a_1},\cdots,\widetilde{a_m}\}$, i.e. rough interval-valued fuzzy set in terms of the dominance principle in the incomplete environment.

Remark 1:

- If for each $x \in U$, $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) = \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)$, then the interval-valued fuzzy set $\widetilde{[d]}$ is definable in the IFISID. Otherwise, it is undefinable.
- If $\widetilde{[d]}$ is an ordinary fuzzy set on universe U, then $[\underline{App}(\widetilde{a}_1,\cdots,\widetilde{a_m},\widetilde{[d]})]$ and $[\overline{App}(\widetilde{a}_1,\cdots,\widetilde{a_m},\widetilde{[d]})]$ would degenerate to be the ordinary lower and upper approximate fuzzy sets in terms of the dominance principle in the incomplete environment.

By the lower and upper approximations of the interval-valued fuzzy set $\widetilde{[d]}$, one can induce the corresponding decision rules for each training example $x \in U$ such that

- "at least" decision rules:

$$\mu_{\widetilde{a_1}}(y) \geq \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \geq \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge \mu_{\widetilde{a_m}}(y) \geq \mu_{\widetilde{a_m}}(x) \rightarrow \mu_{\widetilde{[d]}}(y) \geq \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x);$$

- "at most" decision rules:

$$\mu_{\widetilde{a_1}}(y) \leq \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \leq \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge \mu_{\widetilde{a_m}}(y) \leq \mu_{\widetilde{a_m}}(x) \rightarrow \mu_{\widetilde{[d]}}(y) \leq \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x).$$

In this paper, the above two types of decision rules are referred to as the *initial* "at least" and "at most" decision rules derived from the IFISID.

Example 1: To demonstrate the IFISID, let us consider data in Table 1, which describes a small training set with fuzzy objects. The universe of discourse is $U = \{x_1, x_2, \cdots, x_{10}\}$. $AT = \{a_1, a_2, a_3, a_4\}$ is the set of condition attributes and d is the decision attribute, which are used to describe such ten objects. In Table 1, the set to be approximated is an interval-valued fuzzy set such that

TABLE I.
AN EXAMPLE OF INCOMPLETE FUZZY INFORMATION SYSTEM WITH INTERVAL-VALUED DECISION.

U	a_1	a_2	a_3	a_4	d
x_1	0.9	*	0.2	0.7	[0.5, 0.7]
x_2	0.9	0.2	0.2	0.1	[0.8, 1.0]
x_3	0.1	0.1	0.1	0.9	[0.0, 0.3]
x_4	0.0	0.9	*	0.8	[0.2, 0.5]
x_5	0.1	0.1	1.0	0.8	[0.4, 0.7]
x_6	*	0.2	0.9	0.1	[0.3, 0.6]
x_7	0.0	0.1	0.9	0.2	[0.0, 0.2]
x_8	0.9	0.9	0.1	1.0	[0.6, 0.9]
x_9	0.8	0.4	1.0	1.0	[0.9, 1.0]
x_{10}	0.0	1.0	1.0	*	[0.1, 0.4]

$\widetilde{[d]} = \frac{[0.5,0.7]}{x_1} + \frac{[0.8,1.0]}{x_2} + \frac{[0.0,0.3]}{x_3} + \frac{[0.2,0.5]}{x_4} + \frac{[0.4,0.7]}{x_5} + \frac{[0.3,0.6]}{x_6} + \frac{[0.0,0.2]}{x_7} + \frac{[0.6,0.9]}{x_8} + \frac{[0.9,1.0]}{x_9} + \frac{[0.1,0.4]}{x_{10}}$.

By Definition 3, we obtain the following lower and upper approximations of $\widetilde{[d]}$:

$[\underline{App}(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4, \widetilde{[d]})] = \frac{[0.5,0.7]}{x_1} + \frac{[0.3,0.6]}{x_2} + \frac{[0.0,0.3]}{x_3} + \frac{[0.1,0.4]}{x_4} + \frac{[0.4,0.7]}{x_5} + \frac{[0.1,0.4]}{x_6} + \frac{[0.0,0.2]}{x_7} + \frac{[0.6,0.9]}{x_8} + \frac{[0.9,1.0]}{x_9} + \frac{[0.1,0.4]}{x_{10}}$,

$[\overline{App}(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4, \widetilde{[d]})] = \frac{[0.8,1.0]}{x_1} + \frac{[0.8,1.0]}{x_2} + \frac{[0.0,0.3]}{x_3} + \frac{[0.3,0.6]}{x_4} + \frac{[0.4,0.7]}{x_5} + \frac{[0.8,1.0]}{x_6} + \frac{[0.0,0.2]}{x_7} + \frac{[0.6,0.9]}{x_8} + \frac{[0.9,1.0]}{x_9} + \frac{[0.3,0.6]}{x_{10}}$.

By the above results, we can derive the following initial decision rules from Table 1:

"at least" decision rules:

r_1: $\mu_{\widetilde{a_1}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_2}}(y) \geq * \wedge \mu_{\widetilde{a_3}}(y) \geq 0.2 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.7 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.5, 0.7]$ // supported by x_1

r_2: $\mu_{\widetilde{a_1}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.2 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.2 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.3, 0.6]$ // supported by x_2

r_3: $\mu_{\widetilde{a_1}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.9 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.0, 0.3]$ // supported by x_3

r_4: $\mu_{\widetilde{a_1}}(y) \geq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \geq * \wedge \mu_{\widetilde{a_4}}(y) \geq 0.8 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.1, 0.4]$ // supported by x_4

r_5: $\mu_{\widetilde{a_1}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \geq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.8 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.4, 0.7]$ // supported by x_5

r_6: $\mu_{\widetilde{a_1}}(y) \geq * \wedge \mu_{\widetilde{a_2}}(y) \geq 0.2 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.1, 0.4]$ // supported by x_6

r_7: $\mu_{\widetilde{a_1}}(y) \geq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.2 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.0, 0.2]$ // supported by x_7

r_8: $\mu_{\widetilde{a_1}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_4}}(y) \geq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.6, 0.9]$ // supported by x_8

r_9: $\mu_{\widetilde{a_1}}(y) \geq 0.8 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.4 \wedge \mu_{\widetilde{a_3}}(y) \geq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \geq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.9, 1.0]$ // supported by x_9

r_{10}: $\mu_{\widetilde{a_1}}(y) \geq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \geq 1.0 \wedge \mu_{\widetilde{a_3}}(y) \geq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \geq * \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.1, 0.4]$ // supported by x_{10}

"at most" decision rules:

r_1': $\mu_{\widetilde{a_1}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_2}}(y) \leq * \wedge \mu_{\widetilde{a_3}}(y) \leq 0.2 \wedge \mu_{\widetilde{a_4}}(y) \leq 0.7 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.5, 0.7]$ // supported by x_1

r_2': $\mu_{\widetilde{a_1}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.2 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.2 \wedge \mu_{\widetilde{a_4}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.3, 0.6]$ // supported by x_2

r_3': $\mu_{\widetilde{a_1}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_4}}(y) \leq 0.9 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.0, 0.3]$ // supported by x_3

r'_4: $\mu_{\widetilde{a_1}}(y) \leq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \leq * \wedge \mu_{\widetilde{a_4}}(y) \leq 0.8 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.1, 0.4]$ // supported by x_4

r'_5: $\mu_{\widetilde{a_1}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \leq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \leq 0.8 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.4, 0.7]$ // supported by x_5

r'_6: $\mu_{\widetilde{a_1}}(y) \leq * \wedge \mu_{\widetilde{a_2}}(y) \leq 0.2 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.1, 0.4]$ // supported by x_6

r'_7: $\mu_{\widetilde{a_1}}(y) \leq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \leq 0.2 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.0, 0.2]$ // supported by x_7

r'_8: $\mu_{\widetilde{a_1}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_4}}(y) \leq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.6, 0.9]$ // supported by x_8

r'_9: $\mu_{\widetilde{a_1}}(y) \leq 0.8 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.4 \wedge \mu_{\widetilde{a_3}}(y) \leq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \leq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.9, 1.0]$ // supported by x_9

r'_{10}: $\mu_{\widetilde{a_1}}(y) \leq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \leq 1.0 \wedge \mu_{\widetilde{a_3}}(y) \leq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \leq * \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.1, 0.4]$ // supported by x_{10}

B. Properties of Rough Interval-valued Fuzzy Set

Theorem 1: Let $\mathscr{D}$ be an IFISID, then we have the following properties:

1) Contraction and extension:

$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})] \subseteq \widetilde{[d]} \subseteq [\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})]; \quad (11)$$

2) Monotone (with the monotone of the interval-valued fuzzy set)

$$\widetilde{[d_1]} \subseteq \widetilde{[d_2]} \Rightarrow$$
$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})] \subseteq [\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})],$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})] \subseteq [\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})];$$

3) Monotone (with the monotone of the condition attributes)

$$\{a_1, \cdots, a_m\} \subseteq \{a_1, \cdots, a_n\} \Rightarrow$$
$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})] \subseteq [\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_n}, \widetilde{[d]})],$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})] \supseteq [\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_n}, \widetilde{[d]})];$$

4) Multiplication and addition

$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]} \cap \widetilde{[d_2]})] =$$
$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})] \cap [\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})],$$
$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]} \cup \widetilde{[d_2]})] \supseteq$$
$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})] \cup [\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})],$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]} \cup \widetilde{[d_2]})] =$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})] \cup [\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})],$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]} \cap \widetilde{[d_2]})] \subseteq$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})] \cap [\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})];$$

5) Complement

$$[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]}^C)] = [\overline{App}(\widetilde{a_1}^C, \cdots, \widetilde{a_m^C}, \widetilde{[d]})]^C,$$
$$[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]}^C)] = [\underline{App}(\widetilde{a_1}^C, \cdots, \widetilde{a_m^C}, \widetilde{[d]})]^C,$$

where $\widetilde{a_i}^C$ ($i \in \{1, \cdots, m\}$) is the complementary of the fuzzy set $\widetilde{a_i}$ such that for each $x \in U$

$$\mu_{\widetilde{a_i}^C}(x) = \begin{cases} 1 - \mu_{\widetilde{a_i}}(x) & : \quad \mu_{\widetilde{a_i}}(x) \neq * \\ & : \quad \text{otherwise} \end{cases}$$

Proof:

1) Suppose that $\{a_1, \cdots, a_m\} = A$. Since $D(A)$ is reflexive, we have $x \in D_A^{\uparrow *}(x)$. Thus

$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d]}}(z)\} \leq \mu^-_{\widetilde{[d]}}(x),$$
$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d]}}(z)\} \leq \mu^+_{\widetilde{[d]}}(x),$$

hold, from which we can conclude that

$$\mu_{[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})]}(x) \leq \mu_{\widetilde{[d]}}(x). \quad (12)$$

Similarity, it is not difficult to prove that

$$\mu_{\widetilde{[d]}}(x) \leq \mu_{[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})]}(x). \quad (13)$$

2) Suppose that $\{a_1, \cdots, a_m\} = A$. By $\widetilde{[d_1]} \subseteq \widetilde{[d_2]}$, for each $z \in D_A^{\uparrow *}(x)$, we have $\mu_{\widetilde{[d_1]}}(z) \leq \mu_{\widetilde{[d_2]}}(z)$, from which we obtain that

$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d_1]}}(z)\} \leq \inf_{z \in D_A^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d_2]}}(z)\},$$
$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d_1]}}(z)\} \leq \inf_{z \in D_A^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d_2]}}(z)\},$$

i.e.

$$\mu_{[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})]}(x) \leq \mu_{[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})]}(x) \quad (14)$$

holds. Similarity, it is not difficult to prove that

$$\mu_{[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_1]})]}(x) \leq \mu_{[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d_2]})]}(x). \quad (15)$$

3) Suppose that $A_1 = \{a_1, \cdots, a_m\} \subseteq A_2 = \{a_1, \cdots, a_n\}$. By Definition 2 we have $D_{A_1}^{\uparrow *}(x) \supseteq D_{A_2}^{\uparrow *}(x)$ and $D_{A_1}^{\downarrow *}(x) \supseteq D_{A_2}^{\downarrow *}(x)$ for each $x \in U$. Thus

$$\inf_{z \in D_{A_1}^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d]}}(z)\} \leq \inf_{z \in D_{A_2}^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d]}}(z)\},$$
$$\inf_{z \in D_{A_1}^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d]}}(z)\} \leq \inf_{z \in D_{A_2}^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d]}}(z)\},$$

hold, from which we can conclude that

$$\mu_{[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})]}(x) \leq \mu_{[\underline{App}(\widetilde{a_1}, \cdots, \widetilde{a_n}, \widetilde{[d]})]}(x). \quad (16)$$

Similarity, it is not difficult to prove that

$$\mu_{[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_m}, \widetilde{[d]})]}(x) \geq \mu_{[\overline{App}(\widetilde{a_1}, \cdots, \widetilde{a_n}, \widetilde{[d]})]}(x). \quad (17)$$

4) Suppose that $\{a_1, \cdots, a_m\} = A$. For each $x \in U$, by the properties of the interval-valued fuzzy set, we have

$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d]_1} \cap \widetilde{[d]_2}}(z)\} = \min\{\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d]_1}}(z)\},$$
$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^-_{\widetilde{[d]_2}}(z)\}\},$$
$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d]_1} \cap \widetilde{[d]_2}}(z)\} = \min\{\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d]_1}}(z)\},$$
$$\inf_{z \in D_A^{\uparrow *}(x)} \{\mu^+_{\widetilde{[d]_2}}(z)\}\},$$

from which we can conclude that

$$\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]}\cap\widetilde{[d_2]})]}(x)=\min\{\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]})]}(x),\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_2]})]}(x)\}.$$

Other formulas can be proved analogously.

5) For each $x\in U$,

$$\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]}^C)]}(x)$$
$$=\big[\inf_{z\in D_A^{\uparrow *}(x)}\{\mu^-_{\widetilde{[d]}^C}(z)\},\inf_{z\in D_A^{\uparrow *}(x)}\{\mu^+_{\widetilde{[d]}^C}(z)\}\big]$$
$$=\big[\inf_{z\in D_A^{\uparrow *}(x)}\{1-\mu^+_{\widetilde{[d]}}(z)\},\inf_{z\in D_A^{\uparrow *}(x)}\{1-\mu^-_{\widetilde{[d]}}(z)\}\big].$$

Suppose that $\{a_1,\cdots,a_m\}=A$, thus,

$$D_A^{\uparrow *}(x)=D^{\uparrow *}_{\{\widetilde{a_1},\cdots,\widetilde{a_m}\}}(x)$$
$$=\{y\in U:\mu_{\widetilde{a_i}}(y)\geq\mu_{\widetilde{a_i}}(x)\vee\mu_{\widetilde{a_i}}(x)=*\vee\mu_{\widetilde{a_i}}(y)=*\}$$
$$=\{y\in U:1-\mu_{\widetilde{a_i}}(y)\leq 1-\mu_{\widetilde{a_i}}(x)\vee\mu_{\widetilde{a_i}}(x)=*\vee\mu_{\widetilde{a_i}}(y)=*\}$$
$$=D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)$$

where $i\in\{1,\cdots,m\}$, then

$$\inf_{z\in D_A^{\uparrow *}(x)}\{1-\mu^+_{\widetilde{[d]}}(z)\}=1-\sup_{z\in D_A^{\uparrow *}(x)}\{\mu^+_{\widetilde{[d]}}(z)\}=1-\sup_{z\in D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)}\{\mu^+_{\widetilde{[d]}}(z)\},$$
$$\inf_{z\in D_A^{\uparrow *}(x)}\{1-\mu^-_{\widetilde{[d]}}(z)\}=1-\sup_{z\in D_A^{\uparrow *}(x)}\{\mu^-_{\widetilde{[d]}}(z)\}=1-\sup_{z\in D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)}\{\mu^-_{\widetilde{[d]}}(z)\},$$

hold, from which we can conclude that

$$[\inf_{z\in D_A^{\uparrow *}(x)}\{\mu^-_{\widetilde{[d]}^C}(z)\},\inf_{z\in D_A^{\uparrow *}(x)}\{\mu^+_{\widetilde{[d]}^C}(z)\}]$$
$$=\big[1-\sup_{z\in D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)}\{\mu^+_{\widetilde{[d]}}(z)\},1-\sup_{z\in D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)}\{\mu^-_{\widetilde{[d]}}(z)\}]$$
$$=\big[\sup_{z\in D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)}\{\mu^-_{\widetilde{[d]}}(z)\},\sup_{z\in D^{\downarrow *}_{\{\widetilde{a_1}^C,\cdots,\widetilde{a_m}^C\}}(x)}\{\mu^+_{\widetilde{[d]}}(z)\}\big]^C,$$

i.e.

$$\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]}^C)]}(x)=1-\mu_{[\overline{App}(\widetilde{a_1^C},\cdots,\widetilde{a_m^C},\widetilde{[d]})]}(x).$$

Similarity, it is not difficult to prove that

$$[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]}^C)]=[\underline{App}(\widetilde{a_1}^C,\cdots,\widetilde{a_m^C},\widetilde{[d]})]^C.$$

■

Definition 4: Let $\mathscr{D}$ be an IFISID,

- If $[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]})]=[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_2]})]$, then the interval-valued fuzzy sets $\widetilde{[d_1]}$ and $\widetilde{[d_2]}$ are referred to as lower approximate equal, which is denote by $\widetilde{[d_1]}=_L\widetilde{[d_2]}$;
- If $[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]})]=[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_2]})]$, then the interval-valued fuzzy sets $\widetilde{[d_1]}$ and $\widetilde{[d_2]}$ are referred to as upper approximate equal, which is denote by $\widetilde{[d_1]}=_U\widetilde{[d_2]}$;
- If $\widetilde{[d_1]}=_L\widetilde{[d_2]}$ and $\widetilde{[d_1]}=_U\widetilde{[d_2]}$, then $\widetilde{[d_1]}$ and $\widetilde{[d_2]}$ are referred to as rough equal, which is denote by $\widetilde{[d_1]}=_R\widetilde{[d_2]}$.

Theorem 2: Let $\mathscr{D}$ be an incomplete fuzzy information system with interval decision, we have

$$\widetilde{[d_1]}=_L\widetilde{[d_2]}\Leftrightarrow(\widetilde{[d_1]}\cap\widetilde{[d_2]})=_L\widetilde{[d_1]},\widetilde{[d_2]};$$
$$\widetilde{[d_1]}=_U\widetilde{[d_2]}\Leftrightarrow(\widetilde{[d_1]}\cup\widetilde{[d_2]})=_U\widetilde{[d_1]},\widetilde{[d_2]};$$

Proof: By 4) of Theorem 1 and Definition 4, we have

$$\widetilde{[d_1]}=_L\widetilde{[d_2]}\Leftrightarrow$$
$$[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]})]=[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_2]})]\Leftrightarrow$$
$$[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]})]\cap[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_2]})]=$$
$$[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]}\cap\widetilde{[d_2]})]=$$
$$[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_1]})]=[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d_2]})]\Leftrightarrow$$
$$(\widetilde{[d_1]}\cap\widetilde{[d_2]})=_L\widetilde{[d_1]},\widetilde{[d_2]}.$$

Similar to the above progress, it is not difficult to prove that $\widetilde{[d_1]}=_U\widetilde{[d_2]}\Leftrightarrow(\widetilde{[d_1]}\cup\widetilde{[d_2]})=_U\widetilde{[d_1]},\widetilde{[d_2]}$. ■

IV. KNOWLEDGE REDUCTIONS OF ROUGH INTERVAL-VALUED FUZZY SET

One fundamental aspect of rough set theory involves the search for particular subsets of attributes, which provide the same information for classification or some other purposes as the full set of the condition attributes. Such subsets are called reducts. In recent years, many types of knowledge reductions [22], [25], [41], [42], [48], [49] have been proposed based on different types of rough set models. In the following, based on the rough approximation of the interval-valued fuzzy set proposed in the above section, we will propose the following four types of knowledge reductions.

Definition 5: Let $\mathscr{D}$ be an IFISID, $A=\{\widetilde{a_1},\cdots,\widetilde{a_m}\}\subseteq AT=\{\widetilde{a_1},\cdots,\widetilde{a_n}\}$,

1) If $[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]=[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]$, then A is referred to as a lower approximate consistent attributes set of $\mathscr{D}$; if A is a lower approximate consistent attributes set of $\mathscr{D}$ and no proper subset of A is the lower approximate consistent attributes set of $\mathscr{D}$, then A is referred to as the lower approximate reduct of $\mathscr{D}$;
2) If $[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]=[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]$, then A is referred to as a upper approximate consistent attributes set of $\mathscr{D}$; if A is a upper approximate consistent attributes set of $\mathscr{D}$ and no proper subset of A is the upper approximate consistent attributes set of $\mathscr{D}$, then A is referred to as the upper approximate reduct of $\mathscr{D}$;

3) $\forall x \in U$, if $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) = \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$, then A is referred to as a relative lower approximate consistent attributes set for x in $\mathscr{D}$; if A is a relative lower approximate consistent attributes set for x in $\mathscr{D}$ and no proper subset of A is the relative lower approximate consistent attributes set for x in $\mathscr{D}$, then A is referred to as the relative lower approximate reduct for x in $\mathscr{D}$;
4) $\forall x \in U$, if $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) = \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$, then A is referred to as a relative upper approximate consistent attributes set for x in $\mathscr{D}$; if A is a relative upper approximate consistent attributes set for x in $\mathscr{D}$ and no proper subset of A is the relative upper approximate consistent attributes set for x in $\mathscr{D}$, then A is referred to as the relative upper approximate reduct for x in $\mathscr{D}$.

By the above definition, we can see that the lower (upper) approximate consistent attributes sets of $\mathscr{D}$ are subsets of the condition attributes, which preserve the lower (upper) approximations of the interval-valued fuzzy set $\widetilde{[d]}$; the lower (upper) approximate reducts of $\mathscr{D}$ are *minimal* subsets of the condition attributes, which preserve the lower (upper) approximations of the interval-valued fuzzy set $\widetilde{[d]}$. The sets of the lower (upper) approximate reducts of $\mathscr{D}$ are denoted by Red^L (Red^U).

The relative lower (upper) approximate consistent attributes sets for x in $\mathscr{D}$ are subsets of the condition attributes, which preserve the lower (upper) approximate membership values of the interval-valued fuzzy set $\widetilde{[d]}$ for x; the relative lower (upper) approximate reducts for x in $\mathscr{D}$ are *minimal* subsets of the condition attributes, which preserve the membership values of the lower (upper) approximate interval-valued fuzzy set $\widetilde{[d]}$ for x. The sets of the relative lower (upper) approximate reducts for x in $\mathscr{D}$ are denoted by $Red^L(x)$ ($Red^U(x)$).

Suppose that $\mathscr{D}$ is an IFISID, $A = \{a_1, \cdots, a_m\} \subseteq AT = \{a_1, \cdots, a_n\}$, $\forall x \in U$,

$r_x : \mu_{\widetilde{a_1}}(y) \geq \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \geq \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge \mu_{\widetilde{a_n}}(y) \geq \mu_{\widetilde{a_n}}(x) \rightarrow \mu_{\widetilde{[d]}}(y) \geq \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$

is the initial "at least" decision rule supported by x, then it is not difficult to observe that:

- If A is a relative lower approximate consistent attributes set for x in $\mathscr{D}$, then the rule
 $r_x^{'} : \mu_{\widetilde{a_1}}(y) \geq \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \geq \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge \mu_{\widetilde{a_m}}(y) \geq \mu_{\widetilde{a_m}}(x) \rightarrow \mu_{\widetilde{[d]}}(y) \geq \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)$
 is a *simplified* "at least" decision rule supported by x in $\mathscr{D}$;
- If A is a relative lower approximate reduct for x in $\mathscr{D}$, then the rule
 $r_x^{'} : \mu_{\widetilde{a_1}}(y) \geq \mu_{\widetilde{a_1}}(x) \wedge \mu_{\widetilde{a_2}}(y) \geq \mu_{\widetilde{a_2}}(x) \wedge \cdots \wedge \mu_{\widetilde{a_m}}(y) \geq \mu_{\widetilde{a_m}}(x) \rightarrow \mu_{\widetilde{[d]}}(y) \geq \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)$
 is an *optimal* "at least" decision rule supported by x in $\mathscr{D}$.

Similarity, it is not difficult to obtain the *simplified* and *optimal* "at most" decision rules which are supported by x in $\mathscr{D}$.

Reducts' computation can also be translated into the computation of prime implicants of a Boolean function. It has been shown by Skowron and Rauszer [35] that the problem of finding reducts may be solved as a case in Boolean reasoning. We will generalize this approach to compute the above four types of reducts in the IFISID.

Definition 6: Let $\mathscr{D}$ be an IFISID, $AT = \{a_1, a_2, \cdots, a_n\}$ is the set of condition attributes, $\forall x, y \in U$, denote

$$D_{AT}^L = \{(x,y) \in U^2 : \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) > \mu_{\widetilde{[d]}}(y)\},$$
$$D_{AT}^U = \{(x,y) \in U^2 : \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) < \mu_{\widetilde{[d]}}(y)\},$$

define

$$D_{AT}^L(x,y) = \begin{cases} \{a_i \in AT : (y,x) \notin D(a_i)\}: & (x,y) \in D_{AT}^L \\ AT: & \text{otherwise} \end{cases}$$

$$D_{AT}^U(x,y) = \begin{cases} \{a_i \in AT : (x,y) \notin D(a_i)\}: & (x,y) \in D_{AT}^U \\ AT: & \text{otherwise} \end{cases}$$

where $1 \leq i \leq n$, $D_{AT}^L(x,y)$ and $D_{AT}^H(x,y)$ are referred to as the lower and upper approximate discernibility sets for pair of the objects (x, y) respectively, $\mathbf{D_{AT}^L} = \{D_{AT}^L(x,y) : (x,y) \in D_{AT}^L\}$ and $\mathbf{D_{AT}^U} = \{D_{AT}^U(x,y) : (x,y) \in D_{AT}^U\}$ are referred to as the lower and upper approximate discernibility matrixes of $\mathscr{D}$ respectively.

Theorem 3: Let $\mathscr{D}$ be an IFISID, $A = \{a_1, a_2, \cdots, a_m\} \subseteq AT = \{a_1, a_2, \cdots, a_n\}$, then we have

1) A is the lower approximate consistent attributes sets of $\mathscr{D} \Leftrightarrow A \cap D_{AT}^L(x,y) \neq \emptyset$, $\forall (x,y) \in D_{AT}^L$;
2) A is the upper approximate consistent attributes sets of $\mathscr{D} \Leftrightarrow A \cap D_{AT}^U(x,y) \neq \emptyset$, $\forall (x,y) \in D_{AT}^U$;
3) $\forall x \in U$, A is the relative lower approximate consistent attributes sets for x in $\mathscr{D} \Leftrightarrow A \cap D_{AT}^L(x,y) \neq \emptyset$, $\forall y \in U \wedge (x,y) \in D_{AT}^L$;
4) $\forall x \in U$, A is the relative upper approximate consistent attributes sets for x in $\mathscr{D} \Leftrightarrow A \cap D_{AT}^U(x,y) \neq \emptyset$, $\forall y \in U \wedge (x,y) \in D_{AT}^U$.

Proof:

1) "$\Rightarrow$": Suppose that $\exists (x,y) \in D_{AT}^L$ such that $A \cap D_{AT}^L(x,y) = \emptyset$, then by Definition 6 we have $(y,x) \in D(A)$, $y \in D_A^{\uparrow *}(x)$. By Definition 3, we obtain that $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) \leq \mu_{\widetilde{[d]}}(y)$. Since A is the lower approximate consistent attributes set of $\mathscr{D}$, i.e. $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) = \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$ for each $x \in U$, we obtain $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) \leq \mu_{\widetilde{[d]}}(y)$, which contradicts that $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) > \mu_{\widetilde{[d]}}(y)$ because $(x,y) \in D_{AT}^L$.
 "$\Leftarrow$": Since $A \subseteq AT$, by 3) of Theorem 1, we obtain that $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) \leq$

$\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$ for each $x \in U$. Suppose that A is not the lower approximate consistent attributes set of $\mathscr{D}$, then there must be $x \in U$ such that $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) < \mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$, from which we can conclude that there must be $y \in U$ where $\mu_{[\underline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) > \mu_{\widetilde{[d]}}(y)$ such that $(y,x) \in \overline{D(A)}$, i.e. $(x,y) \in D^L_{AT}$ and $A \cap D^L_{AT}(x,y) = \emptyset$. From discussion above, we can draw the following conclusion: $\forall x,y \in U$, if $(x,y) \in D^L_{AT}$ and $A \cap D^L_{AT}(x,y) \neq \emptyset$, then A is the lower approximate consistent attributes set of $\mathscr{D}$.

2) "$\Rightarrow$": Suppose that $\exists (x,y) \in D^U_{AT}$ such that $A \cap D^U_{AT}(x,y) = \emptyset$, then by Definition 6 we have $(x,y) \in D(A)$, $y \in D^{\downarrow *}_A(x)$. By Definition 3, we obtain that $\mu_{\widetilde{[d]}}(y) \leq \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x)$. Since A is the upper approximate consistent attributes set of $\mathscr{D}$, i.e. $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) = \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$ for each $x \in U$, we obtain $\mu_{\widetilde{[d]}}(y) \leq \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$, which contradicts that $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) < \mu_{\widetilde{[d]}}(y)$ because $(x,y) \in D^U_{AT}$.
"$\Leftarrow$": Since $A \subseteq AT$, by 3) of Theorem 1, we obtain that $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) \geq \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$ for each $x \in U$. Suppose that A is not the upper approximate consistent attributes set of $\mathscr{D}$, then there must be $x \in U$ such that $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_m},\widetilde{[d]})]}(x) > \mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x)$, from which we can conclude that there must be $y \in U$ where $\mu_{[\overline{App}(\widetilde{a_1},\cdots,\widetilde{a_n},\widetilde{[d]})]}(x) < \mu_{\widetilde{[d]}}(y)$ such that $(x,y) \in \overline{D(A)}$, i.e. $(x,y) \in D^U_{AT}$ and $A \cap D^U_{AT}(x,y) = \emptyset$. From discussion above, we can draw the following conclusion: $\forall x,y \in U$, if $(x,y) \in D^U_{AT}$ and $A \cap D^U_{AT}(x,y) \neq \emptyset$, then A is the upper approximate consistent attributes set of $\mathscr{D}$.

3) The proofs of 3) and 4) are similar to the proofs of 1) and 2) respectively. ■

Definition 7: Let $\mathscr{D}$ be an IFISID, define

$$\Delta_L = \bigwedge_{(x,y)\in U^2} \bigvee D^L_{AT}(x,y) = \bigwedge_{(x,y)\in D^L_{AT}} \bigvee D^L_{AT}(x,y); \quad (18)$$

$$\Delta_U = \bigwedge_{(x,y)\in U^2} \bigvee D^U_{AT}(x,y) = \bigwedge_{(x,y)\in D^U_{AT}} \bigvee D^U_{AT}(x,y); \quad (19)$$

$$\Delta_L(x) = \bigwedge_{y\in U} \bigvee D^L_{AT}(x,y) = \bigwedge_{(x,y)\in D^L_{AT}} \bigvee D^L_{AT}(x,y); \quad (20)$$

$$\Delta_U(x) = \bigwedge_{y\in U} \bigvee D^U_{AT}(x,y) = \bigwedge_{(x,y)\in D^U_{AT}} \bigvee D^U_{AT}(x,y); \quad (21)$$

Δ_L and Δ_H are referred to as the lower and upper approximate discernibility functions of $\mathscr{D}$ respectively, $\Delta_L(x)$ and $\Delta_H(x)$ are referred to as the relative lower and upper approximate discernibility functions for x in $\mathscr{D}$ respectively.

By using Boolean reasoning techniques, we can obtain the following Theorem 4 from Theorem 3 immediately.

Theorem 4: Let $\mathscr{D}$ be an IFISID, $A \subseteq AT$, then

1) A is the lower (upper) approximate reduct of $\mathscr{D}$ if and only if $\bigwedge A$ is a prime implicant of the lower (upper) approximate discernibility function Δ_L (Δ_U);
2) $\forall x \in U$, A is the relative lower (upper) approximate reduct for x in $\mathscr{D}$ if and only if $\bigwedge A$ is a prime implicant of the relative lower (upper) approximate discernibility function $\Delta_L(x)$ ($\Delta_U(x)$).

Example 2: Following Example 1, computing all of the optimal decision rules in Table 1.

By Definition 6, we have

$D^L_{AT} = \{(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_1,x_6),(x_1,x_7),(x_1,x_{10}),(x_2,x_3),(x_2,x_4),(x_2,x_7),(x_2,x_{10}),(x_3,x_7),(x_4,x_3),(x_4,x_7),(x_5,x_3),(x_5,x_4),(x_5,x_6),(x_5,x_7),(x_5,x_{10}),(x_6,x_3),(x_6,x_7),(x_8,x_1),(x_8,x_3),(x_8,x_4),(x_8,x_5),(x_8,x_6),(x_8,x_7),(x_8,x_{10}),(x_9,x_1),(x_9,x_2),(x_9,x_3),(x_9,x_4),(x_9,x_5),(x_9,x_6),(x_9,x_7),(x_9,x_8),(x_9,x_{10}),(x_{10},x_3),(x_{10},x_7)\}$.

By Definition 7, we obtain that $\Delta_L = a_1 \wedge a_2 \wedge a_3 \wedge a_4$. By Theorem 4, the set of attributes $\{a_1,a_2,a_3,a_4\}$ is the lower approximate reduct of Table 1, i.e. no condition attribute is redundant in Table 1 for preserving the lower approximation of $\widetilde{[d]}$.

By Definition 7, we can also obtain the following results:

$Red^L(x_1) = \{\{a_1,a_4\}\}$;
$Red^L(x_2) = \{a_1\}$;
$Red^L(x_3) = \{a_1,a_4\}$;
$Red^L(x_4) = \{a_2\}$;
$Red^L(x_5) = \{\{a_1,a_3\}\}$;
$Red^L(x_6) = \{a_2\}$;
$Red^L(x_7) = AT$;
$Red^L(x_8) = \{\{a_1,a_4\}\}$;
$Red^L(x_9) = \{\{a_1,a_3\}\}$;
$Red^L(x_{10}) = \{a_2,a_3\}$.

By these relative lower approximate reducts, we can generate all of the optimal "at least" decision rules from Table 1:

R_1: $\mu_{\widetilde{a_1}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.7 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.5, 0.7]$ // supported by $Red^L(x_1)$

R_2: $\mu_{\widetilde{a_1}}(y) \geq 0.9 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.3, 0.6]$ // supported by $Red^L(x_2)$

R_3: $\mu_{\widetilde{a_1}}(y) \geq 0.1 \vee \mu_{\widetilde{a_4}}(y) \geq 0.9 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.0, 0.3]$ // supported by $Red^L(x_3)$

R_4: $\mu_{\widetilde{a_2}}(y) \geq 0.9 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.1, 0.4]$ // supported by $Red^L(x_4)$

R_5: $\mu_{\widetilde{a_1}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \geq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.4, 0.7]$ // supported by $Red^L(x_5)$

R_6: $\mu_{\widetilde{a_2}}(y) \geq 0.2 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.1, 0.4]$ // supported by $Red^L(x_6)$

R_7: $\mu_{\widetilde{a_1}}(y) \geq 0.0 \wedge \mu_{\widetilde{a_2}}(y) \geq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \geq 0.2 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.0, 0.2]$ // supported by $Red^L(x_7)$

R_8: $\mu_{\widetilde{a_1}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_4}}(y) \geq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.6, 0.9]$ // supported by $Red^L(x_8)$

R_9: $\mu_{\widetilde{a_1}}(y) \geq 0.8 \wedge \mu_{\widetilde{a_3}}(y) \geq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.9, 1.0]$ // supported by $Red^L(x_9)$

R_{10}: $\mu_{\widetilde{a_2}}(y) \geq 1.0 \vee \mu_{\widetilde{a_3}}(y) \geq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \geq [0.1, 0.4]$ // supported by $Red^L(x_{10})$

Similarity, we obtain that $\Delta_U = a_1 \wedge a_2 \wedge a_3$. By Theorem 4, the set of attributes $\{a_1, a_2, a_3\}$ is the upper approximate reduct of Table 1. Moreover,

$Red^U(x_1) = \{a_3, a_4\}$;
$Red^U(x_2) = \{a_2, a_3, a_4\}$;
$Red^U(x_3) = \{\{a_2, a_3\}\}$;
$Red^U(x_4) = \{a_1\}$;
$Red^U(x_5) = \{\{a_1, a_2\}\}$;
$Red^U(x_6) = \{a_2, a_3, a_4\}$;
$Red^U(x_7) = \{\{a_1, a_2\}, \{a_2, a_4\}\}$;
$Red^U(x_8) = \{a_3\}$;
$Red^U(x_9) = AT$;
$Red^U(x_{10}) = \{a_1\}$.

Thus, we can generate the following optimal "at most" decision rules from Table 1:

R_1': $\mu_{\widetilde{a_3}}(y) \leq 0.2 \vee \mu_{\widetilde{a_4}}(y) \leq 0.7 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.5, 0.7]$ // supported by $Red^U(x_1)$

R_2': $\mu_{\widetilde{a_2}}(y) \leq 0.2 \vee \mu_{\widetilde{a_3}}(y) \leq 0.2 \vee \mu_{\widetilde{a_4}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.3, 0.6]$ // supported by $Red^U(x_2)$

R_3': $\mu_{\widetilde{a_2}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.0, 0.3]$ // supported by $Red^U(x_3)$

R_4': $\mu_{\widetilde{a_1}}(y) \leq 0.0 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.1, 0.4]$ // supported by $Red^U(x_4)$

R_5': $\mu_{\widetilde{a_1}}(y) \leq 0.1 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.4, 0.7]$ // supported by $Red^U(x_5)$

R_6': $\mu_{\widetilde{a_2}}(y) \leq 0.2 \vee \mu_{\widetilde{a_3}}(y) \leq 0.9 \vee \mu_{\widetilde{a_4}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.1, 0.4]$ // supported by $Red^U(x_6)$

R_7': $\mu_{\widetilde{a_2}}(y) \leq 0.1 \wedge (\mu_{\widetilde{a_1}}(y) \leq 0.0 \vee \mu_{\widetilde{a_4}}(y) \leq 0.2) \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.0, 0.2]$ // supported by $Red^U(x_7)$

R_8': $\mu_{\widetilde{a_3}}(y) \leq 0.1 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.6, 0.9]$ // supported by $Red^U(x_8)$

R_9': $\mu_{\widetilde{a_1}}(y) \leq 0.8 \wedge \mu_{\widetilde{a_2}}(y) \leq 0.4 \wedge \mu_{\widetilde{a_3}}(y) \leq 1.0 \wedge \mu_{\widetilde{a_4}}(y) \leq 1.0 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.9, 1.0]$ // supported by $Red^U(x_9)$

R_{10}': $\mu_{\widetilde{a_1}}(y) \leq 0.0 \rightarrow \mu_{\widetilde{[d]}}(y) \leq [0.1, 0.4]$ // supported by $Red^U(x_{10})$

V. CONCLUSION

In recent years, how to expand the traditional rough set model in different types of complex information systems playing an important role in the development of the rough set theory. In this paper, we have developed a general framework for the study of the dominance–based fuzzy rough set in the incomplete fuzzy information system with interval-valued decision. In our approach, the rough approximation of the interval-valued fuzzy set is constructed on the basis of an expanded dominance relation. Such rough approximation is a generalization of the dominance-based fuzzy rough set in the fuzzy environment. Based on the proposed rough approximation of the interval-valued fuzzy set, we also propose four types of the knowledge reductions, lower and upper approximate reducts, relative lower and upper approximate reducts for an object. By the relative lower and upper approximate reducts for an object, one can induce optimal "at least" and "at most" decision rules which are supported by such object in the information system.

For further research, the proposed approach can be extended to more general and complex information systems such as the information system with interval-valued domains of the condition attributes. On the other hand, the rough approximation of the interval-valued fuzzy set in the incomplete environment with some other explanations of the unknown values (e.g. the unknown value is a non-existing one) are exciting areas to be explored. We will study these issues in our future works.

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