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DOMINANT MODULES AND FINITE LOCALIZATIONS

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Colby and Rutter [6] proved that a ring R contains an essential left socle and an essential right socle and has a two sided semi-simple maximal (complete) quotient ring if and only if R is a QF-3 ring with zero right singular ideal. Utilizing the fact that a left perfect ring contains a minimal dense right ideal, Storrer [29] proved that a left and right artinian ring R has a two sided quasi-Frobenius maximal quotient ring if and only if R is a QF-3 ring whose minimal dense right ideal is projective. This paper was motivated by the desire to obtain a common generalization of these results.

Kato's [13] notion of a dominant module proved useful in this connection and the first section of this paper is devoted to results concerning dominant modules. With regard to the problem posed above, the most relevant are:

A finitely generated projective left module is a dominant left module if and only if its trace ideal is a minimal dense right ideal.

The minimal faithful left module over a left QF-3 ring is a dominant left module.

This section also contains results not directly related to our principal objective.

Also useful in this connection is Silver's [27] concept of a finite right localization. In the second section, we characterize those rings R whose maximal ring of right quotients is a finite right localization of R and belongs to one of the following classes of rings: right S-rings, semi-simple rings, right self injective right cogenerator rings, and right self injective rings. If the maximal ring of right quotients of a left perfect ring belongs to any of the first three of these classes, it is necessarily a finite right localization of R. Thus in most cases the results of this section generalize results of Storrer [29]. In some instances they sharpen Storrer's results even in case R is left perfect. The above results are used to prove that the following statements are equivalent:

R has a two sided maximal quotient ring which is a cogenerator ring and projective both as a left and a right R-module.

R contains a minimal dense left and a minimal dense right ideal and has a two sided maximal quotient ring which is a cogenerator ring.

R is a QF-3 ring whose minimal dense right ideal is projective.

This equivalence yields easily the results of Colby and Rutter [6] and Storrer [29] cited at the beginning of this discussion.

Throughout this paper all rings will be assumed to have an identity and all modules will be unital. R will always denote a ring. If M is an R-module the notation ${}_RM$ (respectively, M_R) will often be used to indicate that M is a left (respectively, right) module. Module homomorphisms will be written opposite the scalars with which they commute. Terminology not defined here may be found in [16] or [28]. Although we shall try to review briefly relevant facts concerning the maximal ring of right quotients as the need arises, much more complete information can be found in [16] and [28].

1. **Dominant modules.** Throughout this paper P will denote a finitely generated projective left R-module, $E = \mathrm{End}_R({}_RP)$ the endomorphism ring of ${}_RP$ and $({}_RP)^* = {}_E\mathrm{Hom}_R(P, R)_R$. The *trace ideal* of P is $T = \sum_f (P)f$ where $f \in P^*$. It is an idempotent ideal of R such that $TP = P$. If K is a right ideal of R with $KP = P$, then $K \supset T$. Furthermore, the module ${}_RP$ is faithful if and only if the left annihilator of T in R is zero. The module P_R^* is finitely generated and projective with $(P_R^*)^* \cong {}_RP$. Moreover, E is the endomorphism ring of P_R^* and T is its trace ideal. (See [1, § 1] and [2, Chapter II].)

Kato [13] called a faithful finitely generated projective left R-module P a *dominant* module provided each simple right E-module is isomorphic to a submodule of P_E.

PROPOSITION 1.1. *If ${}_RP$ is faithful, ${}_RP$ is a dominant module if and only if each simple homomorphic image U_R of P_R^* is isomorphic to a minimal right ideal of R.*

PROOF. By [26, Theorem 2.2] $\mathrm{Hom}_R(P^*, U)$ is a simple right E-module. Furthermore, $\mathrm{Hom}_R(P^*, U)_E \cong U \otimes_R P_E$ [4, VI, Proposition 5.2]. If ${}_RP$ is a dominant left R-module, there exists an exact sequence $0 \to U \otimes_R P_E \to P_E$. This gives an exact sequence, $0 \to \mathrm{Hom}_E(P, U \otimes_R P)_R \to \mathrm{Hom}_E(P, P)_R$. By [24, Lemma 2.2] the canonical map of U_R into $\mathrm{Hom}_E(P, U \otimes_R P)_R$ is a monomorphism. Since ${}_RP$ is faithful, $\mathrm{Hom}_E(P, P)_R$ is an essential extension of R_R [13, Theorem 1]. Thus U_R is isomorphic to a minimal right ideal of R.

Conversely, suppose each U_R is isomorphic to a right ideal of R. Then $U \otimes_R P_E$ is isomorphic to a submodule of $R \otimes_R P_E \cong P_E$. Thus ${}_RP$ is a dominant module since each simple right E-module is isomorphic to some

$U \otimes_R P_E$ [24, Lemma 2.4].

A ring R will be called a *left QF-3* ring if it has a minimal faithful left module, that is, as faithful module which is isomorphic to a direct summand of every faithful module. A QF-3 ring is a ring which is both left and right QF-3.

COROLLARY 1.2. *If R is a left QF-3 ring, the minimal faithful left R-module is a dominant module.*

PROOF. By [23, Theorem 1] the minimal faithful left R-module M is isomorphic to $Re_1 + \cdots + Re_n$, where $e_1, \cdots, e_n$ are primitive orthogonal idempotents such that each Re_i is injective and contains a unique minimal left ideal M_i. Since Re_i is an indecomposable injective module, its endomorphism ring e_iRe_i is a local ring. Thus [30, Theorem 4.2], e_iR contains a unique maximal submodule e_iJ and hence (up to isomorphism) a unique simple homomorphic image e_iR/e_iJ, where J is the Jacobson radical of R. Thus (up to isomorphism) the simple homomorphic images of $M_R^* \cong e_1R + \cdots + e_nR$ are $e_1R/e_1J, \cdots, e_nR/e_nJ$. If $0 \neq a_i \in M_i$, a_iR is a minimal right ideal [11, Lemma 1]. Since $a_ie_i \neq 0$, $a_iR \cong e_iR/e_iJ$. Thus Proposition 1.1 implies that M is a dominant module.

This result has also been observed by Kato [14, p. 482, Example] who has studied the role of dominant modules in the structure theory of QF-3 rings. We note, however, that the definition of left QF-3 ring in [14] differs from that used here.

COROLLARY 1.3. *If R satisfies the ascending chain condition on annihilator left ideals and has a faithful projective, injective left module, then R contains a dominant injective left ideal.*

PROOF. By [25, Lemmas 1 and 2] R contains a faithful injective left ideal $Re = Re_1 + \cdots + Re_k$ where $e = e_1 + \cdots + e_k$ and $e_1, \cdots, e_k$ are primitive orthogonal idempotents. Furthermore, R has an essential right socle and a finite number of isomorphism classes of minimal right ideals [25, Lemma 3]. Let $M_1, \cdots, M_n$ be distinct representatives for the isomorphism classes of minimal right ideals. Since Re is faithful, each $M_iRe \neq 0$ and so $M_iRe_{j_1} \neq 0$ for some $1 \leqq j_i \leqq k$. Let $L = Re_{j_1} + \cdots + Re_{j_n}$. Since $M_1, \cdots, M_n$ exhaust the isomorphism classes of minimal right ideals of R, the left annihilator of L intersects the right socle of R in zero. Thus L is faithful. Repeating a portion of the proof of the preceding corollary, shows that $e_{j_i}R$ has (up to isomorphism) a unique simple homomorphic image which is isomorphic to M_i because $M_ie_{j_i} \neq 0$. Thus L is a dominant module by Proposition 1.1.

A right ideal K of R is *dense* if $\mathrm{Hom}_R(R/K, E(R_R)) = 0$ where $E(R_R)$ is the injective hull of R_R. If K is an ideal, K is dense as a right ideal if and only if the left annihilator of K in R is zero [16, Corollary, p. 96]. The dense ideals of R form a filter and the ring of right quotients of R with respect to this filter is the maximal (or complete) ring of right quotients of R. (See [16, Chapter 4, § 4.3] and [28, § 3 and § 7].)

THEOREM 1.4. *If P is a finitely generated projective left R-module, P is a dominant module if and only if the trace ideal T of P is a minimal dense right ideal of R.*

PROOF. Assume P is a dominant left module. Since P is faithful, T has zero left annihilator and is, therefore, dense as a right ideal. If K is a dense right ideal, $\mathrm{Hom}_R(P^*, R/K) = 0$. For if $f\colon P^* \to R/K$ is a non-zero R-homomorphism, $f(P^*)_R$, being finitely generated, has a simple homomorphic image U. Since U is also a homomorphic image of P^*, U is isomorphic to a minimal right ideal of R by Proposition 1.1. This implies that there exists a non-zero homomorphism of $f(P^*)_R$ into R_R. Such a homomorphism could be extended to a non-zero homomorphism of R/K into $E(R_R)$ an obvious contradiction. Since T is the trace ideal of P^*, the above observation implies that $(R/K)T = 0$. Hence $K \supset T$.

Suppose T is a minimal dense right ideal of R. Since T is an ideal, its left annihilator is zero and so ${}_RP$ is faithful. Let U_R be a simple homomorphic image of P^*_R. If U_R is not isomorphic to a minimal right ideal, then $\mathrm{Hom}_R(U, E(R_R)) = 0$. But $U_R \cong R/K$ where K is a right ideal of R. Furthermore, the above observation implies that K is a dense right ideal of R. Thus $K \supset T$ and so $UT = 0$. However, $P^*T = P^*$ and so $UT = U$. This contradiction shows that U is isomorphic to a minimal right ideal of R. Hence ${}_RP$ is a dominant module by Proposition 1.1.

Since the trace ideal of any faithful projective left R-module is a dense right ideal of R, the above theorem shows that when a dominant left R-module exists it is a minimal faithful projective left R-module in the sense that it is isomorphic to a direct summand of a finite direct sum of copies of any faithful projective left R-module.

COROLLARY 1.5. *If R has a dominant left R-module P, then R contains a minimal dense right ideal which is equal to the trace ideal of P.*

Since a left perfect ring has a dominant left R-module [13, Example 4], this corollary is a generalization of the fact that a left perfect ring contains a minimal dense right ideal [10, Theorem 3.1]. Combined with

Proposition 1.1 it also generalizes the description of the minimal dense right ideal of a left perfect ring given in [29, Theorem 2.5].

Two R-modules are called *similar* if each is isomorphic to a direct summand of a finite direct sum of copies of the other.

COROLLARY 1.6. *If P is a dominant left R-module, then a left R-module X is a dominant left R-module if and only if X is similar to P.*

PROOF. Since P is finitely generated and projective, it is readily verified that X is similar to P if and only if X is finitely generated and projective with the same trace ideal as P. Thus this corollary is immediate from Theorem 1.4.

The next corollary is due originally to Kato [15, Corollary 5].

COROLLARY 1.7. *If R has a dominant left R-module P, the bicommutator of P is the maximal ring of right quotients of R.*

PROOF. By [8, Theorem 2.1] the bicommutator of P is the ring of right quotients of R with respect to the filter of right ideals of R which consists of those right ideals that contain the trace ideal of P. By Theorem 1.4, this is precisely the filter of dense right ideals of R.

A ring R is a *right S-ring* if every simple right R-module is isomorphic to a minimal right ideal of R. (Morita [19] calls these left S-rings and also assumes that R is artinian. Following Stenstrom [28] we have reversed Morita's terminology.) Right S-rings are also characterized by the property that they contain no proper dense right ideals [28, Proposition 18.1].

A left R-module M is a *generator* (also called completely faithful) if ${}_R R$ is a homomorphic image of a direct sum of copies of M.

Since a left perfect ring has a dominant left R-module, the following corollary generalizes [3, Theorem 3.3].

COROLLARY 1.8. *If R has a dominant left R-module, the following conditions are equivalent:*

(1) *R is a right S-ring.*

(2) *Every faithful projective left R-module is a generator.*

(3) *Every finitely generated faithful projective left R-module is a generator.*

PROOF. A projective R-module is a generator if and only if its trace ideal equals R. The trace ideal of a faithful projective left R-module is dense as a right ideal of R. Thus (1) implies (2). The implication (2) implies (3) is trivial. If the dominant left R-module is a generator, it is immediate from Theorem 1.4 that R contains no proper dense right ideals. Thus (3) implies (1).

A ring R is *right non-singular* if the right annihilator of each nonzero element of R is not an essential right ideal, i.e., the right singular ideal of R is zero.

PROPOSITION 1.9. *If R is a right non-singular ring, then R has a dominant left R-module if and only if the right socle of R is an essential right ideal and R has only a finite number of isomorphism classes of minimal right ideals.*

PROOF. Assume that R has a dominant left R-module P. The trace ideal T of P is a minimal dense right ideal of R by Theorem 1.4. Thus T_R contains no proper essential submodules. For if K_R is an essential submodule of T_R, it is an essential right ideal. Since R has zero singular ideal, K is a dense right ideal [28, Proposition 3.10] and so $T \subset K$. It, therefore, follows from Zorn's lemma, that every submodule of T_R is a direct summand of T_R. Hence T_R is completely reducible and, therefore, equals the right socle of R. Since $P^*T = P^*$, P^* is a homomorphic image of a direct sum of copies of T_R. Thus P^*_R is completely reducible and, being finitely generated, has (up to isomorphism) only a finite number of simple homomorphic images. If U is a minimal right ideal of R, $UT \neq 0$ and so $UT = U$. Thus U_R is a homomorphic image of P^*_R.

We now prove the converse. Since the right singular ideal of R is zero, the essential right ideals and the dense right ideals of R coincide [28, Proposition 3.10]. Since the right socle of R is always contained in every essential right ideal and is by assumption an essential right ideal, it is a minimal dense right ideal of R. Let $U_1, \cdots, U_n$ be distinct representatives for the minimal right ideals of R. Then $\bigoplus_{i=1}^n U_i$ is projective (See [5, Lemma 1.6].) and its trace ideal is clearly equal to the right socle of R. Thus Theorem 1.4 implies that ${}_RP = (\bigoplus_{i=1}^n U_i)^*$ is a dominant left R-module.

2. Finite localizations. If R is a subring of a ring A, then A is a *finite right localization* of R provided the inclusion map of R into A is an epimorphism in the category of rings and ${}_RA$ is finitely generated and projective as a left R-module. This concept was introduced by Silver [27]. However, we have restricted his original definition to fit the situation with which we will be concerned. Namely, when is the maximal ring of right quotients of R a finite right localization of R? Although it does not discuss finite localizations per se, much information related to the material of this section and many references to other related material can be found in [28].

THEOREM 2.1. *Let R be a subring of a ring A. The following conditions are equivalent:*

(1) *A is a finite right localization of R.*

(2) *R contains a left ideal L of A such that $LA = A$.*

(3) *R contains an ideal T such that $TA = A$ and $AT \subset T$.*

Moreover, in these circumstances T is the trace ideal of ${}_RA$ and T_R is finitely generated and projective.

PROOF. The equivalence of (2) and (3) is obvious.

(1) implies (3). Let T be the trace ideal of ${}_RA$. Since ${}_RA$ is projective, $TA = A$. By assumption the inclusion map of R into A is an epimorphism, so $\mathrm{Hom}_R({}_RA, {}_RA) = \mathrm{Hom}_A({}_AA, {}_AA)$ [27, Corollary 1.3]. Therefore, if $f: {}_RA \to {}_RR$ is an R-homomorphism, then for any $a, b \in A$, $af(b) = f(ab)$. Hence $AT \subset T$.

(3) implies (1). Since $TA = A$, $1 = \sum_{i=1}^n t_i a_i$ with $t_i \in T$ and $a_i \in A$. Furthermore, $AT \subset T$ implies at_i and $a_i t \in T$ for each $i = 1, \cdots, n$ and all $a \in A$ and $t \in T$. Thus it follows from [28, Theorem 13.10] that the inclusion map of R into A is an epimorphism and from the "dual basis lemma" [4, VII, Proposition 3.1] that ${}_RA$ and T_R are finitely generated and projective.

We now prove the rest of the final assertion. Let I be the trace ideal of ${}_RA$. As we have seen, I is a left ideal of A. Thus $TA = A$ implies $I = AI = TAI = TI \subset T$. Similarly, $IA = A$ implies $T = AT = IAT = IT \subset I$. Thus $T = I$.

COROLLARY 2.2. *Let A be a finite right localization of R and T be the trace ideal of ${}_RA$. There exist inverse one-to-one correspondences between the right ideals I of A and the right ideals K of R such that $K = KT$ given by:*

$$I \to IT \quad \text{and} \quad K \to KA .$$

In particular, there exists a one-to-one correspondence between the minimal right ideals of R and A.

PROOF. The first assertion is immediate since $TA = A$ and $AT = T$. The second follows from the first and the observation that for any right ideal V of R, $VT \neq (0)$ since $TA = A$.

Throughout the remainder of this paper Q denotes the maximal ring of right quotients of R. It is uniquely determined up to isomorphism over R by the following properties:

(1) For each $q \in Q$, there exists a dense right ideal K of R such that $qK \subset R$.

(2) If $q \in Q$ and K is a dense right ideal of R, then $qK = 0$ implies $q = 0$.

(3) If K is a dense right ideal of R and $f: K_R \to Q_R$ is an R-homomorphism, there exists $q \in Q$ such that $f(k) = qk$ for all $k \in K$.

THEOREM 2.3. *The following conditions are equivalent:*

(1) *Q is a right S-ring and a finite right localization of R.*

(2) *Q is a right S-ring and ${}_RQ$ is projective.*

(3) *R is a subring of a right S-ring A such that R contains a faithful left ideal L of A.*

(4) *R contains a minimal dense right ideal D with D_R finitely generated and projective.*

(5) *R has a dominant left R-module ${}_RP$ such that P_E is finitely generated and projective.*

Moreover, in these circumstances $A = Q$.

PROOF. (1) implies (2) is clear.

(2) implies (3). Let $A = Q$ and L be the trace ideal of ${}_RQ$. If K is a dense right ideal of R, then KQ is a dense right ideal of Q [31, Lemma 16]. Thus $KQ = Q$ and so $K \supset L$. This implies $qL \subset R$ for all $q \in Q$ and since $L^2 = L$, $qL \subset L$. Finally, $LQ = Q$ implies L is a faithful left ideal of Q.

(3) implies (4). Let $D = LR$. Since DA is an ideal of A whose left annihilator in A is zero, DA is a dense right ideal of A. Hence $DA = A$. Thus Theorem 2.1 implies A is a finite localization of R, D is the trace ideal of ${}_RA$ and D_R is finitely generated and projective. By [27, Corollary 1.3], $\mathrm{Hom}_R({}_RA, {}_RA) = \mathrm{Hom}_A({}_AA, {}_AA) = A$. Since A is a right S-ring, this implies ${}_RA$ is a dominant left R-module. Thus D is the minimal dense right of R by Corollary 1.5 and $A = Q$ by Corollary 1.7.

(4) implies (5). It is readily verified that the product of dense right ideals is again a dense right ideal. Thus $D^2 = D$. Hence D is the trace ideal of D_R and also of ${}_RP = (D_R)^*$. Theorem 1.4 implies that ${}_RP$ is a dominant module. The exact sequence of right R-modules $0 \to D_R \to R_R \to R/D \to 0$ gives an exact sequence of right E-modules,

$$0 \to D \otimes_R P_E \to R \otimes_R P_E \to R/D \otimes_R P_E \,.$$

But $DP_E = P_E$ implies, $R/D \otimes_R P_E = 0$. Thus $D \otimes_R P_E \cong R \otimes_R P_E \cong P_E$. However, ${}_RP = (D_R)^*$ implies $D \otimes_R P_E \cong E_E$ [2, Proposition 4.4, p. 68]. Thus P_E is finitely generated and projective.

(5) implies (1). By Corollary 1.7, Q is the bi-commutator of ${}_RP$. Since

${}_{R}P$ is finitely generated and projective P_E is a generator [18, Lemma 3.3] and by hypothesis it is finitely generated and projective. Thus P_E and hence also ${}_{Q}P$ is a progenerator [18, Lemma 3.3]. If D is the trace ideal of ${}_{R}P$, then $DQP = DP = P$. Thus DQ contains the trace ideal of ${}_{Q}P$ and so $DQ = Q$. By Theorem 1.4, D is the minimal dense right ideal of R. If I is a dense right ideal of Q, $I \cap R \supset D$ [31, Lemma 16] and so $IQ = Q$. Thus Q is a right S-ring. Finally, $qD \subset R$ for any $q \in Q$ and since $D^2 = D$, $qD \subset D$. Thus Q is a finite localization of R by Theorem 2.1.

REMARK 2.4. If Q is a right S-ring, it is shown in [28, Proposition 19.5] or [17, Proposition 3] that ${}_{R}Q$ is flat and the inclusion map of R into Q is an epimorphism in the category of rings. (In [28, Proposition 19.5], it is asserted that these conditions are equivalent to Q being a right S-ring, but this is clearly in error as is shown by any right self injective ring which is not a right S-ring.) Thus if R is left perfect and Q is a right S-ring ${}_{R}Q$ is projective. Therefore, the preceding theorem and most of the subsequent results of this section generalize and in several instances sharpen corresponding results of Storrer [29] concerning the maximal right quotient ring of a left perfect ring.

COROLLARY 2.5. *The following statements are equivalent:*

(1) *Q is semi-simple and is a finite right localization of R.*

(2) *Q is semi-simple and ${}_{R}Q$ is projective.*

(3) *The right socle of R is an essential, finitely generated, projective right ideal of R.*

(4) *R is a subring of a semi-simple ring A and R contains a faithful left ideal L of R.*

Moreover, in these circumstances $A = Q$.

PROOF. The equivalence of (1), (2) and (4) and the validity of the final assertion are immediate from the preceding theorem.

(1) implies (3). Let D be the trace ideal of ${}_{R}Q$. Then D is a dense right ideal and it follows from Theorem 2.1 that D_R is finitely generated and projective. Since Q is semi-simple, Q_Q is a finite direct sum of minimal right ideal of Q. Thus Corollary 2.2 implies $QD = D$ is a finite direct sum of minimal right ideals of R.

(3) implies (1). Let D be the right socle of R. Clearly D is the trace ideal of D_R and so $D^2 = D$. Each minimal right ideal M of R is a homomorphic image of D_R, so $MD = D$. Since D_R is an essential right ideal, each right ideal of R contains a minimal right ideal and so the left annihilator of D in R is zero. Thus D is a dense right ideal and hence the minimal dense right ideal of R. It follows from the preceding theorem

that Q is a right S-ring and a finite right localization of R. By Corollary 2.2, DQ is a sum of minimal right ideals of Q. Moreover, $DQ = Q$ since DQ is a dense right ideal of Q [31, Lemma 16].

An R-module M is *quasi-injective* if for every submodule N of M and every R-homomorphism $f: N \to M$, f can be extended to an R-endomorphism of M.

PROPOSITION 2.6. *The following statements are equivalent*:

(1) *R contains a dense right ideal which is quasi-injective as a right R-module.*

(2) *R is a subring of a right self injective ring A containing a faithful left ideal L of A.*

Moreover, in these circumstance $A = Q$ and LR is a quasi-injective, dense right ideal of R.

PROOF. (1) implies (2). Let $A = Q$ and K be a quasi-injective dense right ideal of R. Since K_R is an essential submodule of R_R and Q_R is an essential extension of R_R, $E(K_R) = E(R_R) = E(Q_R)$. For each $q \in Q$, the R-homomorphism $\psi_q: Q_R \to Q_R$ defined by $\psi_q(x) = qx$ can be extended to an R-endomorphism $\bar{\psi}_q$ of $E(R_R)$. Since K_R is quasi-injective $\bar{\psi}_q(k) = qk \in K$ for each $k \in K$ [9, § 3, Proposition 1]. Thus K is a left ideal of Q. Furthermore, since $qK \neq 0$ for each $0 \neq q \in Q$, K is a faithful left ideal of Q.

Suppose I is a right ideal of R and $f: I_R \to Q_R$ is an R-homomorphism. Since $f(IK) = f(I)K \subset K$ and $IK \subset K \cap I$, the restriction f' of f to IK, can be extended to an R-endomorphism of $\bar{f}$ of K_R. Since K is a dense right ideal of R, there exists $q \in Q$ such that $\bar{f}(k) = qk$ for all $k \in K$. Hence $f(x)k = f'(xk) = qxk$ for all $x \in I$ and $k \in K$. Since K is a faithful left ideal of Q, $f(x) = qx$ for all $x \in I$. Thus Q_R is injective [9, § 1, Theorem 6] and hence Q_Q is injective [16, § 4.3, Proposition 3].

(2) implies (1). If R is a subring of A and $L \subset R$ is a faithful left ideal of A, then $K = LR$ is dense as a right ideal of R. For each $0 \neq a \in A$, $0 \neq aK \subset R$. Thus the identity map of R can be extended to a ring monomorphism ϕ of A into Q [16, § 4.3, Proposition 8]. Since A_A is injective and Q_R an essential extension of R_R, ϕ is onto. We may, therefore, identify A and Q both as rings and as R-modules. Thus $A_R = E(R_R) = E(K_R)$ [16, § 4.3, Proposition 3]. Since $\mathrm{Hom}_R(A_R, A_R) = \mathrm{Hom}_A(A_A, A_A) = A$ [26, Corollary 1.3], and $aK \subset K$ for all $a \in A$, K_R is quasi-injective [9, § 3, Proposition 1].

THEOREM 2.7. *The following statements are equivalent*:

(1) *Q is a right self injective and a finite right localization of R.*

(2) *R contains a dense right ideal K such that K_R is a finitely generated, projective, quasi-injective right R-module.*

(3) *R is a subring of a right self injective ring A containing a left ideal L of A such that $LA = A$.*

Moreover, in these circumstances $A = Q$.

PROOF. (1) implies (2). This implication is immediate from Theorem 2.1 and Proposition 2.6.

(2) implies (3). If we let $A = Q$, it was shown in proving (1) implies (2) of Proposition 2.6, that K is a faithful left ideal of Q. Since K_R is finitely generated and projective, there exists a dual basis $\{f_i\}_{i=1}^n$ and $\{k_i\}_{i=1}^n$ for K_R where $f_i \in K_R^*$ and $k_i \in K$ for each $i = 1, \cdots, n$. Since K is a dense right ideal, there exist $q_i \in Q$ such that $f_i(k) = q_i k$ for each $1 \leqq i \leqq n$ and all $k \in K$. Thus $k = \sum_{i=1}^n k_i q_i k$ for all $k \in K$. Since K is a faithful left ideal of Q, $1 = \sum_{i=1}^n k_i q_i$ and so $KQ = Q$.

(3) implies (1). This implication is immediate from Theorem 2.1.

An R-module M is a *cogenerator* if every R-module (of the same hand) is isomorphic to a submodule of a direct product of copies of M. R is a *right cogenerator* ring of R_R is a cogenerator. A *cogenerator* ring is a ring which is both a left and right cogenerator ring. R is a *right self injective cogenerator* ring (often called a *PF* ring) if R_R is an injective cogenerator. A structure theorem for right self injective cogenerator rings has been given by Osofsky [20]. A right cogenerator ring need not be right self injective [20, Example 2]. However, a cogenerator ring is both left and right self injective [12]. Because of the fact that except for chain conditions cogenerator rings have properties very much analogous to quasi-Frobenius rings, they are sometimes called generalized quasi-Frobenius rings.

COROLLARY 2.8. *The following statements are equivalent:*

(1) *Q is a right self injective cogenerator ring and a finite right localization of R.*

(2) *Q is a right self injective cogenerator ring and $_RQ$ is projective.*

(3) *R contains a minimal dense right ideal D such that D_R is a finitely generated, projective, quasi-injective right R-module.*

(4) *R is a subring of a right self injective cogenerator ring A containing a faithful left ideal of A.*

Moreover, in these circumstances $A = Q$.

PROOF. Since a ring is a right self injective cogenerator ring if and only if it is a right self injective, right S-ring, this corollary is immediate from Theorems 2.3 and 2.7 and Proposition 2.6.

THEOREM 2.9. *The following statements are equivalent*:

(1) *R is a QF-3 ring whose minimal dense right ideal is projective as a right R-module.*

(2) *R is a QF-3 ring with minimal faithful left module Re such that Re_{eRe} is projective.*

(3) *R contains a minimal dense left ideal and a minimal dense right ideal and has a two-sided maximal quotient ring which is a cogenerator ring.*

(4) *Q is a cogenerator ring and a finite left and a finite right localization of R.*

(5) *Q is a cogenerator ring and both ${}_RQ$ and Q_R are projective.*

(6) *R is a subring of a cogenerator ring A containing a faithful left ideal and a faithful right ideal of A.*

(7) *R is a subring of a cogenerator ring A which is both a finite right and a finite left localization of R.*

Moreover, in these circumstances $A = Q$.

PROOF. If Q is an S-ring, it is a two-sided quotient ring of R if and only if Q_R is flat [17, Theorem 2]. Therefore, the equivalence of statements (3)-(7) follows from Theorem 2.1 and Corollary 2.8.

(1) implies (2). Re is a dominant left R-module by Corollary 1.2 so the minimal dense right ideal of R is the trace ideal of Re by Corollary 1.5. Therefore, Re_{eRe} is projective [1, Theorem 3.2].

(2) implies (3). By Corollaries 1.2 and 1.5, R contains a minimal dense left ideal and a minimal dense right ideal. It is known that if R is a QF-3 ring, then R has a two-sided maximal quotient ring Q which is also a QF-3 ring. (See [21, § 1].) Since Re_{eRe} is projective, it follows from [22, Theorem 3.6 and Proposition 5.1] that Re_{eRe} is finitely generated. Re is a dominant left R-module and so Theorem 2.3 implies that Q is a right S-ring and hence a right self injective cogenerator ring [23, Theorem 2]. Thus Q is also a left S-ring [12, Theorem 1] and hence a left self injective cogenerator ring [23, Theorem 2].

(4) implies (1). By [20, Theorem 1], $Q_Q = V_1 \oplus \cdots \oplus V_n$ where each V_i is an indecomposable injective right ideal of Q containing a minimal right ideal of Q. Since Q is a finite right localization of R, each V_i is an indecomposable injective right R-module [27, Corollary 1.8] which contains a minimal right ideal of R by Corollary 2.2. Furthermore, each V_i is a projective right R-module since Q_R is projective. Thus R is right QF-3 [23, Theorem 1]. R is left QF-3 by symmetry. Finally, Theorem

2.3 implies that the minimal dense right ideal of R is projective as a right R-module (and in fact also finitely generated).

COROLLARY 2.10. *The following statements are equivalent*:

(1) *R is a QF-3 ring with the ascending chain condition on annihilator right (or left) ideals whose minimal dense right ideal is projective.*

(2) *R contains a minimal dense left and a minimal dense right ideal and has a two-sided quasi-Frobenius maximal quotient ring.*

(3) *Q is a quasi-Frobenius ring and $_RQ$ and Q_R are projective.*

PROOF. Since the ascending chain condition on annihilators is inherited by subrings, it suffices to prove (1) implies (2). Assume (1). Since Q_R is projective, R satisfies the ascending chain condition on annihilators of subsets of Q_R. Combining this with the fact that Q is a finite right localization of R, we see that Q satisfies the ascending chain condition on annihilator right ideals [28, Proposition 14.3]. Hence Q_Q injective, implies Q is a quasi-Frobenius ring [28, Theorem 18.9].

The parenthetical case is similar.

REMARK 2.11. It follows from the main theorem of [25] that we can replace the assumption that R is QF-3 in (1) of the preceding corollary by a number of other conditions. For example, one may assume instead that R contains a faithful injective left ideal and a faithful injective right ideal.

The following corollary sharpens [29, Theorem 6.4] where it is proved under the hypothesis that R is left and right artinian.

COROLLARY 2.12. *If R is a perfect ring the following statements are equivalent*:

(1) *Q is a quasi-Frobenius ring and a two-sided maximal quotient ring of R.*

(2) *R is a QF-3 ring whose minimal dense right ideal is projective.*

PROOF. (1) implies (2). As noted in Remark 2.4, $_RQ$ and Q_R are projective. Thus this implication is immediate from Theorem 2.9.

(2) implies (1). R satisfies the ascending chain condition on left annihilators by [7, Theorem 1.3] and [28, Propositions 11.7 and 11.8]. Thus this implication follows from Corollary 2.10.

It is well known and readily verified that a right non-singular ring contains a minimal dense right ideal if and only if the right socle of R is an essential right ideal. (See the proof of Proposition 1.9.) Moreover, in these circumstances the right socle of R is projective as a right R-module (See [5, Lemma 1.6].) Recall, also, that R is right non-singular if and only if Q is a regular ring [28, Proposition 20.1]. Combining these observations and their left hand analogs with the fact that a regular right S-ring is semi-simple [11, Lemma 3], we have the following corollary due originally to Colby and Rutter [6, Theorem 5].

COROLLARY 2.13. *The following statements are equivalent:*

(1) *R is a QF-3 ring with zero right singular ideal.*

(2) *The left socle and the right socle of R are essential left, respectively, right ideals of R and R has a two-sided semi-simple maximal quotient ring.*

REMARK 2.14. One can add to the list of equivalent statements in Corollary 2.10 the analogs of statements (2), (4), (6) and (7) of Theorem 2.9. The statements of Theorem 2.9 would of course have to be suitably modified. Likewise, suitable modifications of statements (2) and (4)-(7) of Theorem 2.9 can be added to Corollary 2.13. We have elected not to do so in the interest of avoiding unnecessary repetition as the necessary changes are quite obvious.

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