Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 542040, 10 pages
doi:10.1155/2009/542040

*Research Article*

# Dominating Sets and Domination Polynomials of Paths

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Received 13 November 2008; Accepted 12 March 2009

Recommended by Alexander Rosa

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of $G$, if every vertex in $V \backslash S$ is adjacent to at least one vertex in $S$. Let $\mathcal{P}_n^i$ be the family of all dominating sets of a path $P_n$ with cardinality $i$, and let $d(P_n, j) = |\mathcal{P}_n^j|$. In this paper, we construct $\mathcal{P}_n^i$, and obtain a recursive formula for $d(P_n, i)$. Using this recursive formula, we consider the polynomial $D(P_n, x) = \sum_{i=\lceil n/3 \rceil}^{n} d(P_n, i)x^i$, which we call domination polynomial of paths and obtain some properties of this polynomial.

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## 1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of $G$, if $N[S] = V$, or equivalently, every vertex in $V \backslash S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma$-set, and the family of $\gamma$-sets is denoted by $\Gamma(G)$. For a detailed treatment of this parameter, the reader is referred to [1]. It is well known and generally accepted that the problem of determining the dominating sets of an arbitrary graph is a difficult one (see [2]). A path is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2. Let $P_n$ be the path with $n$ vertices. Let $\mathcal{P}_n^i$ be the family of dominating sets of a path $P_n$ with cardinality $i$ and let $d(P_n, i) = |\mathcal{P}_n^i|$. We call the polynomial $D(P_n, x) = \sum_{i=\lceil n/3 \rceil}^{n} d(P_n, i)x^i$, the domination polynomial of the path $P_n$. For a detailed treatment of the domination polynomial of a graph, the reader is referred to [3].

In the next section we construct the families of the dominating sets of paths by a recursive method. In Section 3, we use the results obtained in Section 2 to study the domination polynomial of paths.

As usual we use $\lceil x \rceil$, for the smallest integer greater than or equal to $x$. In this article we denote the set $\{1, 2, \dots, n\}$ simply by $[n]$.

## 2. Dominating sets of paths

Let $\mathcal{P}_n^i$ be the family of dominating sets of $P_n$ with cardinality $i$. We will investigate dominating sets of path. We need the following lemmas to prove our main results in this article.

**Lemma 2.1** (see [4, page 371]). $\gamma(P_n) = \lceil n/3 \rceil$.

By Lemma 2.1 and the definition of domination number, one has the following lemma.

**Lemma 2.2.** *$\mathcal{P}_j^i = \emptyset$, if and only if $i > j$ or $i < \lceil j/3 \rceil$.*

A *simple path* is a path in which all its internal vertices have degree two. The following lemma follows from observation.

**Lemma 2.3.** *If a graph $G$ contains a simple path of length $3k - 1$, then every dominating set of $G$ must contain at least $k$ vertices of the path.*

To find a dominating set of $P_n$ with cardinality $i$, we do not need to consider dominating sets of $P_{n-4}$ with cardinality $i-1$. We show this in Lemma 2.4. Therefore, we only need to consider $\mathcal{P}_{n-1}^{i-1}$, $\mathcal{P}_{n-2}^{i-1}$, and $\mathcal{P}_{n-3}^{i-1}$. The families of these dominating sets can be empty or otherwise. Thus, we have eight combinations of whether these three families are empty or not. Two of these combinations are not possible (see Lemma 2.5(i) and (ii)). Also, the combination that $\mathcal{P}_{n-1}^{i-1} = \mathcal{P}_{n-2}^{i-1} = \mathcal{P}_{n-3}^{i-1} = \emptyset$ does not need to be considered because it implies $\mathcal{P}_n^i = \emptyset$ (see Lemma 2.5(iii)). Thus we only need to consider five combinations or cases. We consider these cases in Theorem 2.7.

**Lemma 2.4.** *If $Y \in \mathcal{P}_{n-4}^{i-1}$, and there exists $x \in [n]$ such that $Y \cup \{x\} \in \mathcal{P}_n^i$, then $Y \in \mathcal{P}_{n-3}^{i-1}$.*

*Proof.* Suppose that $Y \notin \mathcal{P}_{n-3}^{i-1}$. Since $Y \in \mathcal{P}_{n-4}^{i-1}$, $Y$ contains at least one vertex labeled $n-5$ or $n-4$. If $n-4 \in Y$, then $Y \in \mathcal{P}_{n-3}^{i-1}$, a contradiction. Hence, $n-5 \in Y$, but then in this case, $Y \cup \{x\} \notin \mathcal{P}_n^i$, for any $x \in [n]$, also a contradiction. □

**Lemma 2.5.** (i) *If $\mathcal{P}_{n-1}^{i-1} = \mathcal{P}_{n-3}^{i-1} = \emptyset$, then $\mathcal{P}_{n-2}^{i-1} = \emptyset$.*
(ii) *If $\mathcal{P}_{n-1}^{i-1} \neq \emptyset$ and $\mathcal{P}_{n-3}^{i-1} \neq \emptyset$, then $\mathcal{P}_{n-2}^{i-1} \neq \emptyset$.*
(iii) *If $\mathcal{P}_{n-1}^{i-1} = \mathcal{P}_{n-2}^{i-1} = \mathcal{P}_{n-3}^{i-1} = \emptyset$, then $\mathcal{P}_n^i = \emptyset$.*

*Proof.* (i) Since $\mathcal{P}_{n-1}^{i-1} = \mathcal{P}_{n-3}^{i-1} = \emptyset$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \lceil (n-3)/3 \rceil$. In either case we have $\mathcal{P}_{n-2}^{i-1} = \emptyset$.

(ii) Suppose that $\mathcal{P}_{n-2}^{i-1} = \emptyset$, so by Lemma 2.2, we have $i-1 > n-2$ or $i-1 < \lceil (n-2)/3 \rceil$. If $i-1 > n-2$, then $i-1 > n-3$, and hence, $\mathcal{P}_{n-3}^{i-1} = \emptyset$, a contradiction. Hence $i-1 < \lceil (n-2)/3 \rceil$. So $i-1 < \lceil (n-1)/3 \rceil$, and hence, $\mathcal{P}_{n-1}^{i-1} = \emptyset$, also a contradiction.

(iii) Suppose that $\mathcal{P}_n^i \neq \emptyset$. Let $Y \in \mathcal{P}_n^i$. Then at least one vertex labeled $n$ or $n-1$ is in $Y$. If $n \in Y$, then by Lemma 2.3, at least one vertex labeled $n-1$, $n-2$ or $n-3$ is in $Y$. If

$n-1 \in Y$ or $n-2 \in Y$, then $Y - \{n\} \in \mathcal{D}_{n-1}^{i-1}$, a contradiction. If $n-3 \in Y$, then $Y - \{n\} \in \mathcal{D}_{n-2}^{i-1}$, a contradiction. Now suppose that $n-1 \in Y$. Then by Lemma 2.3, at least one vertex labeled $n-2$, $n-3$ or $n-4$ is in $Y$. If $n-2 \in Y$ or $n-3 \in Y$, then $Y - \{n-1\} \in \mathcal{D}_{n-2}^{i-1}$, a contradiction. If $n-4 \in Y$, then $Y - \{n-1\} \in \mathcal{D}_{n-3}^{i-1}$, a contradiction. Therefore $\mathcal{D}_n^i = \emptyset$. □

**Lemma 2.6.** *If $\mathcal{D}_n^i \neq \emptyset$, then*

(i) $\mathcal{D}_{n-1}^{i-1} = \mathcal{D}_{n-2}^{i-1} = \emptyset$, and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$ if and only if $n = 3k$ and $i = k$ for some $k \in \mathbb{N}$;

(ii) $\mathcal{D}_{n-2}^{i-1} = \mathcal{D}_{n-3}^{i-1} = \emptyset$ and $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$ if and only if $i = n$;

(iii) $\mathcal{D}_{n-1}^{i-1} = \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$ if and only if $n = 3k+2$ and $i = \lceil (3k+2)/3 \rceil$ for some $k \in \mathbb{N}$;

(iv) $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-3}^{i-1} = \emptyset$ if and only if $i = n-1$;

(v) $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$ if and only if $\lceil (n-1)/3 \rceil + 1 \leq i \leq n-2$.

*Proof.* (i) ($\Rightarrow$) Since $\mathcal{D}_{n-1}^{i-1} = \mathcal{D}_{n-2}^{i-1} = \emptyset$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \lceil (n-2)/3 \rceil$. If $i-1 > n-1$, then $i > n$, and by Lemma 2.2, $\mathcal{D}_n^i = \emptyset$, a contradiction. So $i < \lceil (n-2)/3 \rceil + 1$, and since $\mathcal{D}_n^i \neq \emptyset$, together $\lceil n/3 \rceil \leq i < \lceil (n-2)/3 \rceil + 1$, which give us $n = 3k$ and $i = k$ for some $k \in \mathbb{N}$.

($\Leftarrow$) If $n = 3k$ and $i = k$ for some $k \in \mathbb{N}$, then by Lemma 2.2, $\mathcal{D}_{n-1}^{i-1} = \mathcal{D}_{n-2}^{i-1} = \emptyset$, and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$.

(ii) ($\Rightarrow$) Since $\mathcal{D}_{n-2}^{i-1} = \mathcal{D}_{n-3}^{i-1} = \emptyset$, by Lemma 2.2, $i-1 > n-2$ or $i-1 < \lceil (n-3)/3 \rceil$. If $i-1 < \lceil (n-3)/3 \rceil$, then $i-1 < \lceil (n-1)/3 \rceil$, and hence $\mathcal{D}_{n-1}^{i-1} = \emptyset$, a contradiction. So $i > n-1$. Also since $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $i-1 \leq n-1$. Therefore $i = n$.

($\Leftarrow$) If $i = n$, then by Lemma 2.2, $\mathcal{D}_{n-2}^{i-1} = \mathcal{D}_{n-3}^{i-1} = \emptyset$ and $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$.

(iii) ($\Rightarrow$) Since $\mathcal{D}_{n-1}^{i-1} = \emptyset$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \lceil (n-1)/3 \rceil$. If $i-1 > n-1$, then $i-1 > n-2$ and by Lemma 2.2, $\mathcal{D}_{n-2}^{i-1} = \mathcal{D}_{n-3}^{i-1} = \emptyset$, a contradiction. So $i < \lceil (n-1)/3 \rceil + 1$, but $i-1 \geq \lceil (n-2)/3 \rceil$ because $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$. Hence, $\lceil (n-2)/3 \rceil + 1 \leq i < \lceil (n-1)/3 \rceil + 1$. Therefore $n = 3k+2$ and $i = k+1 = \lceil (3k+2)/3 \rceil$ for some $k \in \mathbb{N}$.

($\Leftarrow$) If $n = 3k+2$ and $i = \lceil (3k+2)/3 \rceil$ for some $k \in \mathbb{N}$, then by Lemma 2.2, $\mathcal{D}_{n-1}^{i-1} = \mathcal{D}_{3k+1}^{k} = \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$, and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$.

(iv) ($\Rightarrow$) Since $\mathcal{D}_{n-3}^{i-1} = \emptyset$, by Lemma 2.2, $i-1 > n-3$ or $i-1 < \lceil (n-3)/3 \rceil$. Since $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$, by Lemma 2.2, $\lceil (n-2)/3 \rceil + 1 \leq i \leq n-1$. Therefore $i-1 < \lceil (n-3)/3 \rceil$ is not possible. Hence $i-1 > n-3$. Thus $i = n-1$ or $n$, but $i \neq n$ because $\mathcal{D}_{n-3}^{i-1} = \emptyset$. So $i = n-1$.

($\Leftarrow$) If $i = n-1$, then by Lemma 2.2, $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$, and $\mathcal{D}_{n-3}^{i-1} = \emptyset$.

(v) ($\Rightarrow$) Since $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$, and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$, then by applying Lemma 2.2, $\lceil (n-1)/3 \rceil \leq i-1 \leq n-1$, $\lceil (n-2)/3 \rceil \leq i-1 \leq n-2$, and $\lceil (n-3)/3 \rceil \leq i-1 \leq n-3$. So $\lceil (n-1)/3 \rceil \leq i-1 \leq n-3$ and hence $\lceil (n-1)/3 \rceil + 1 \leq i \leq n-2$.

($\Leftarrow$) If $\lceil (n-1)/3 \rceil + 1 \leq i \leq n-2$,then the result follows from Lemma 2.2. □

By Lemma 2.4, for the construction of $\mathcal{D}_n^i$, it's sufficient to consider $\mathcal{D}_{n-1}^{i-1}$, $\mathcal{D}_{n-2}^{i-1}$, and $\mathcal{D}_{n-3}^{i-1}$. By Lemma 2.5, we need only to consider the following five cases.

**Theorem 2.7.** *For every $n \geq 4$ and $i \geq \lceil n/3 \rceil$.*

(i) If $\mathcal{D}_{n-1}^{i-1} = \mathcal{D}_{n-2}^{i-1} = \emptyset$ and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$, then $\mathcal{D}_n^i = \{\{2, 5, \ldots, n-4, n-1\}\}$.

(ii) If $\mathcal{D}_{n-2}^{i-1} = \mathcal{D}_{n-3}^{i-1} = \emptyset$, and $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, then $\mathcal{D}_n^i = \{[n]\}$.

(iii) If $\mathcal{D}_{n-1}^{i-1} = \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$, then

$$\mathcal{D}_n^i = \left\{\{2, 5, \ldots, n-3, n\}\right\} \cup \left\{X \cup \{n-1\} \mid X \in \mathcal{D}_{n-3}^{i-1}\right\}. \tag{2.1}$$

(iv) If $\mathcal{D}_{n-3}^{i-1} = \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, then $\mathcal{D}_n^i = \{[n] - \{x\} \mid x \in [n]\}$.

(v) If $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$, then

$$\begin{aligned} \mathcal{D}_n^i = &\left\{\{n\} \cup X_1, \{n-1\} \cup X_2 \mid X_1 \in \mathcal{D}_{n-1}^{i-1}, X_2 \in \mathcal{D}_{n-2}^{i-1}\right\} \\ &\cup \left\{\{n-1\} \cup X \mid X \in \mathcal{D}_{n-3}^{i-1} \setminus \mathcal{D}_{n-2}^{i-1}\right\} \\ &\cup \left\{\{n\} \cup X \mid X \in \mathcal{D}_{n-3}^{i-1} \cap \mathcal{D}_{n-2}^{i-1}\right\}. \end{aligned} \tag{2.2}$$

*Proof.* (i) $\mathcal{D}_{n-1}^{i-1} = \mathcal{D}_{n-2}^{i-1} = \emptyset$, and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$. By Lemma 2.6(i), $n = 3k$ and $i = k$ for some $k \in \mathbb{N}$. Therefore $\mathcal{D}_n^i = \mathcal{D}_n^{n/3} = \{\{2, 5, \ldots, n-4, n-1\}\}$.

(ii) $\mathcal{D}_{n-2}^{i-1} = \mathcal{D}_{n-3}^{i-1} = \emptyset$, and $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$. By Lemma 2.6(ii), we have $i = n$. So $\mathcal{D}_n^i = \mathcal{D}_n^n = \{[n]\}$.

(iii) $\mathcal{D}_{n-1}^{i-1} = \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$ and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$. By Lemma 2.6(iii), $n = 3k+2$ and $i = \lceil (3k+2)/3 \rceil = k+1$ for some $k \in \mathbb{N}$. Since $X = \{2, 5, \ldots, 3k-1\} \in \mathcal{D}_{3k}^k$, $X \cup \{3k+2\} \in \mathcal{D}_{3k+2}^{k+1}$. Also, if $X \in \mathcal{D}_{3k-1}^k$, then $X \cup \{3k+1\} \in \mathcal{D}_{3k+2}^{k+1}$. Therefore we have

$$\left\{\{2, 5, \ldots, 3k-1, 3k+2\}\right\} \cup \left\{X \cup \{3k+1\} \mid X \in \mathcal{D}_{3k-1}^k\right\} \subseteq \mathcal{D}_{3k+2}^{k+1}. \tag{2.3}$$

Now let $Y \in \mathcal{D}_{3k+2}^{k+1}$. Then $3k+2$ or $3k+1$ is in $Y$. If $3k+2 \in Y$, then by Lemma 3, at least one vertex labeled $3k+1, 3k$ or $3k-1$ is in $Y$. If $3k+1$ or $3k$ is in $Y$, then $Y - \{3k+2\} \in \mathcal{D}_{3k+1}^k$, a contradiction because $\mathcal{D}_{3k+1}^k = \emptyset$. Hence, $3k-1 \in Y$, $3k \notin Y$, and $3k+1 \notin Y$. Therefore $Y = X \cup \{3k+2\}$ for some $X \in \mathcal{D}_{3k}^k$, that is $Y = \{2, 5, \ldots, 3k-1, 3k+2\}$. Now suppose that $3k+1 \in Y$ and $3k+2 \notin Y$. By Lemma 2.3, at least one vertex labeled $3k, 3k-1$, or $3k-2$ is in Y. If $3k \in Y$, then $Y - \{3k+1\} \in \mathcal{D}_{3k}^k = \{\{2, 5, \ldots, 3k-1\}\}$, a contradiction because $3k \notin X$ for all $X \in \mathcal{D}_{3k}^k$. Therefore $3k-1$ or $3k-2$ is in $Y$, but $3k \notin Y$. Thus $Y = X \cup \{3k+1\}$ for some $X \in \mathcal{D}_{3k-1}^k$. So

$$\mathcal{D}_{3k+2}^{k+1} \subseteq \left\{\{2, 5, \ldots, 3k-1, 3k+2\}\right\} \cup \left\{\{3k+1\} \cup X \mid X \in \mathcal{D}_{3k-1}^k\right\}. \tag{2.4}$$

(iv) $\mathcal{D}_{n-3}^{i-1} = \emptyset$, $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, and $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$. By Lemma 2.6(iv), $i = n-1$. Therefore $\mathcal{D}_n^i = \mathcal{D}_n^{n-1} = \{[n] - \{x\} \mid x \in [n]\}$.

(v) $\mathcal{D}_{n-1}^{i-1} \neq \emptyset$, $\mathcal{D}_{n-2}^{i-1} \neq \emptyset$, and $\mathcal{D}_{n-3}^{i-1} \neq \emptyset$. Let $X_1 \in \mathcal{D}_{n-1}^{i-1}$, so at least one vertex labeled $n-1$ or $n-2$ is in $X_1$. If $n-1$ or $n-2 \in X_1$, then $X_1 \cup \{n\} \in \mathcal{D}_n^i$.

Let $X_2 \in \mathcal{D}_{n-2}^{i-1}$, then $n-2$ or $n-3$ is in $X_2$. If $n-2$ or $n-3 \in X_2$, then $X_2 \cup \{n-1\} \in \mathcal{D}_n^i$.

Now let $X_3 \in \mathcal{D}_{n-3}^{i-1}$, then $n-3$ or $n-4$ is in $X_3$. If $n-3 \in X_3$, then $X_3 \cup \{x\} \in \mathcal{D}_n^i$, for $x \in \{n, n-1\}$. If $n-4 \in X_3$, then $X_3 \cup \{n-1\} \in \mathcal{D}_n^i$. Therefore we have

$$\begin{aligned}&\left\{\{n\} \cup X_1, \{n-1\} \cup X_2 \mid X_1 \in \mathcal{D}_{n-1}^{i-1}, X_2 \in \mathcal{D}_{n-2}^{i-1}\right\}\\ &\quad \cup \left\{\{n-1\} \cup X \mid X \in \mathcal{D}_{n-3}^{i-1} \setminus \mathcal{D}_{n-2}^{i-1}\right\}\\ &\quad \cup \left\{\{n\} \cup X \mid X \in \mathcal{D}_{n-3}^{i-1} \cap \mathcal{D}_{n-2}^{i-1}\right\} \subseteq \mathcal{D}_n^i.\end{aligned} \tag{2.5}$$

Now, let $Y \in \mathcal{D}_n^i$, then $n \in Y$ or $n-1 \in Y$. If $n \in Y$, then by Lemma 2.3, at least one vertex labeled $n-1$, $n-2$, or $n-3$ is in $Y$. If $n-1 \in Y$ or $n-2 \in Y$, then $Y = X \cup \{n\}$ for some $X \in \mathcal{D}_{n-1}^{i-1}$. If $n-3 \in Y$, $n-2 \notin Y$, and $n-1 \notin Y$, then $Y = X \cup \{n\}$ for some $X \in \mathcal{D}_{i-1}^{n-2} \cap \mathcal{D}_{i-1}^{n-3}$. Now suppose that $n-1 \in Y$ and $n \notin Y$, then by Lemma 3, at least one vertex labeled $n-2, n-3$ or $n-4$ is in $Y$. If $n-2 \in Y$ or $n-3 \in Y$, then $Y = X \cup \{n-1\}$ for some $X \in \mathcal{D}_{n-2}^{i-1}$. If $n-4 \in Y, n-3 \notin Y$ and $n-2 \notin Y$, then $Y = X \cup \{n-1\}$ for some $X \in \mathcal{D}_{n-3}^{i-1} \setminus \mathcal{D}_{n-2}^{i-1}$. So

$$\begin{aligned}\mathcal{D}_n^i \subseteq &\left\{\{n\} \cup X_1, \{n-1\} \cup X_2 \mid X_1 \in \mathcal{D}_{n-1}^{i-1}, X_2 \in \mathcal{D}_{n-2}^{i-1}\right\}\\ &\cup \left\{\{n-1\} \cup X \mid X \in \mathcal{D}_{n-3}^{i-1} \setminus \mathcal{D}_{n-2}^{i-1}\right\}\\ &\cup \left\{\{n\} \cup X \mid X \in \mathcal{D}_{n-3}^{i-1} \cap \mathcal{D}_{n-2}^{i-1}\right\}.\end{aligned} \tag{2.6}$$

□

*Example 2.8.* Consider $P_6$ with $V(P_6) = [6]$. We use Theorem 2.7 to construct $\mathcal{D}_6^i$ for $2 \le i \le 6$. Since $\mathcal{D}_5^1 = \mathcal{D}_4^1 = \varnothing$ and $\mathcal{D}_3^1 = \{\{2\}\}$, by Theorem 2.7, $\mathcal{D}_6^2 = \{\{2,5\}\}$.

Since $\mathcal{D}_5^5 = \{[5]\}, \mathcal{D}_4^5 = \varnothing$, and $\mathcal{D}_3^5 = \varnothing$, we get $\mathcal{D}_6^6 = \{[6]\}$.

Since $\mathcal{D}_5^4 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,3,4,5\},\{2,3,4,5\},\{1,2,4,5\}\}$, $\mathcal{D}_4^4 = \{[4]\}$, and $\mathcal{D}_3^4 = \varnothing$, then by Theorem 2.7,

$$\begin{aligned}\mathcal{D}_6^5 &= \{[6] - \{x\} \mid x \in [6]\}\\ &= \{\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,3,4,5,6\},\{2,3,4,5,6\},\{1,2,4,5,6\},\{1,2,3,4,5\}\},\end{aligned} \tag{2.7}$$

and, for the construction of $\mathcal{D}_6^3$, by Theorem 2.7,

$$\begin{aligned}\mathcal{D}_6^3 &= \left\{X_1 \cup \{6\}, X_2 \cup \{5\} \mid X_1 \in \mathcal{D}_5^2, X_2 \in \mathcal{D}_4^2\right\} \cup \left\{\{1,2\} \cup \{5\}, \{1,3\} \cup \{6\}, \{2,3\} \cup \{6\}\right\}\\ &= \left\{\{1,3,5\},\{1,3,6\},\{2,3,6\},\{2,3,5\},\{1,4,6\},\{1,4,5\},\{2,5,6\},\{2,4,6\},\{2,4,5\},\{1,2,5\}\right\}.\end{aligned} \tag{2.8}$$

Finally, since $\mathcal{D}_5^3 = \{\{1,3,5\},\{1,2,4\},\{2,4,5\},\{2,3,4\},\{2,3,5\},\{1,4,5\},\{1,3,4\},\{1,2,5\}\}$, $\mathcal{D}_4^3 = \{\{1,2,3\},\{1,2,4\},\{2,3,4\},\{1,3,4\}\}$, and $\mathcal{D}_3^3 = \{[3]\}$, then

$$\begin{aligned}\mathcal{D}_6^4 &= \left\{X_1 \cup \{6\}, X_2 \cup \{5\} \mid X_1 \in \mathcal{D}_5^3, X_2 \in \mathcal{D}_4^3\right\} \cup \left\{X \cup \{6\} \mid X \in \mathcal{D}_3^3\right\}\\ &= \{\{1,2,3,5\},\{1,2,3,6\},\{1,2,4,6\},\{1,3,5,6\},\{1,3,4,6\},\{1,3,4,5\},\{1,2,5,6\},\\ &\qquad \{1,2,4,5\},\{2,3,4,6\},\{1,4,5,6\},\{2,3,4,5\},\{2,4,5,6\},\{2,3,5,6\}\}.\end{aligned} \tag{2.9}$$

**Table 1:** $d(P_n, j)$, the number of dominating set of $P_n$ with cardinality $j$.

| $j$ / $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | |
| 2 | 2 | 1 | | | | | | | | | | | | | |
| 3 | 1 | 3 | 1 | | | | | | | | | | | | |
| 4 | 0 | 4 | 4 | 1 | | | | | | | | | | | |
| 5 | 0 | 3 | 8 | 5 | 1 | | | | | | | | | | |
| 6 | 0 | 1 | 10 | 13 | 6 | 1 | | | | | | | | | |
| 7 | 0 | 0 | 8 | 22 | 19 | 7 | 1 | | | | | | | | |
| 8 | 0 | 0 | 4 | 26 | 40 | 26 | 8 | 1 | | | | | | | |
| 9 | 0 | 0 | 1 | 22 | 61 | 65 | 34 | 9 | 1 | | | | | | |
| 10 | 0 | 0 | 0 | 13 | 70 | 120 | 98 | 43 | 10 | 1 | | | | | |
| 11 | 0 | 0 | 0 | 5 | 61 | 171 | 211 | 140 | 53 | 11 | 1 | | | | |
| 12 | 0 | 0 | 0 | 1 | 40 | 192 | 356 | 343 | 192 | 64 | 12 | 1 | | | |
| 13 | 0 | 0 | 0 | 0 | 19 | 171 | 483 | 665 | 526 | 255 | 76 | 13 | 1 | | |
| 14 | 0 | 0 | 0 | 0 | 6 | 120 | 534 | 1050 | 1148 | 771 | 330 | 89 | 14 | 1 | |
| 15 | 0 | 0 | 0 | 0 | 1 | 65 | 483 | 1373 | 2058 | 1866 | 1090 | 418 | 103 | 15 | 1 |

## 3. Domination Polynomial of a path

Let $D(P_n, x) = \sum_{i=\lceil n/3 \rceil}^{n} d(P_n, i)x^i$ be the domination polynomial of a path $P_n$. In this section we study this polynomial.

**Theorem 3.1.** (i) *If $\mathcal{P}_n^i$ is the family of dominating set with cardinality* **i** *of $P_n$, then*

$$|\mathcal{P}_n^i| = |\mathcal{P}_{n-1}^{i-1}| + |\mathcal{P}_{n-2}^{i-1}| + |\mathcal{P}_{n-3}^{i-1}|. \tag{3.1}$$

(ii) *For every $n \geq 4$,*

$$D(P_n, x) = x\left[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)\right], \tag{3.2}$$

*with the initial values $D(P_1, x) = x, D(P_2, x) = x^2 + 2x$, and $D(P_3, x) = x^3 + 3x^2 + x$.*

*Proof.* (i) It follows from Theorem 2.7.

(ii) It follows from Part (i) and the definition of the domination polynomial. □

Using Theorem 3.1, we obtain $d(P_n, j)$ for $1 \leq n \leq 15$ as shown in Table 1. There are interesting relationships between numbers in this table. In the following theorem, we obtain some properties of $d(P_n, j)$.

**Theorem 3.2.** *The following properties hold for the coefficients of $D(P_n, x)$:*

(i) $d(P_{3n}, n) = 1$, *for every $n \in \mathbb{N}$.*

(ii) $d(P_{3n+2}, n+1) = n+2$, *for every $n \in \mathbb{N}$.*

(iii) $d(P_{3n+1}, n+1) = ((n+2)(n+3))/2 - 2$, *for every $n \in \mathbb{N}$.*

(iv) $d(P_{3n}, n+1) = n(n+1)(n+8)/6$, *for every $n \in \mathbb{N}$.*

(v) $d(P_n, n) = 1$, *for every* $n \in \mathbb{N}$.

(vi) $d(P_n, n-1) = n$, *for every* $n \geq 2$.

(vii) $d(P_n, n-2) = \binom{n}{2} - 2 = (n(n-1)/2) - 2$, *for every* $n \geq 3$.

(viii) $d(P_n, n-3) = \binom{n}{3} - (3n-8) = (n-4)(n-3)(n+4)/6$, *for every* $n \geq 4$.

(ix) $d(P_n, n-4) = (n-5)(n^3 - n^2 - 42n + 96)/24$, *for every* $n \geq 5$.

(x) $1 = d(P_n, n) < d(P_{n+1}, n) < d(P_{n+2}, n) < \cdots < d(P_{2n-1}, n) < d(P_{2n}, n) > d(P_{2n+1}, n) > \cdots > d(P_{3n-1}, n) > d(P_{3n}, n) = 1$.

(xi) $\sum_{i=j}^{3j} d(P_i, j) = 3\sum_{i=j-1}^{3j-3} d(P_i, j-1)$.

(xii) *If* $s_n = \sum_{j=\lceil n/3 \rceil}^{n} d(P_n, j)$, *then for every* $n \geq 4$, $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ *with initial values* $s_1 = 1, s_2 = 3$ *and* $s_3 = 5$.

(xiii) *For every* $n \in \mathbb{N}$, *and* $k = 0, 1, 2, \ldots, 2n-1$, $d(P_{2n-k}, n) = d(P_{2n+k}, n)$.

(xiv) *For every* $j \geq \lceil n/3 \rceil$, $d(P_{n+1}, j+1) - d(P_n, j+1) = d(P_n, j) - d(P_{n-3}, j)$.

*Proof.* (i) Since $\mathcal{D}_{3n}^{n} = \{\{2, 5, \ldots, 3n-1\}\}$, we have $d(P_{3n}, n) = 1$.

(ii) Proof by induction on $n$. Since $\mathcal{D}_5^2 = \{\{1, 4\}, \{2, 4\}, \{2, 5\}\}$, so $d(P_5, 2) = 3$. Therefore the result is true for $n = 1$. Now suppose that the result is true for all natural numbers less than $n$, and we prove it for $n$. By part (i), Theorem 3.1, and the induction hypothesis we have $d(P_{3n+2}, n+1) = d(P_{3n+1}, n) + d(P_{3n}, n) + d(P_{3(n-1)+2}, n) = n+2$.

(iii) Proof by induction on $n$. The result is true for $n = 2$, because $d(P_7, 3) = 8 = 4 + 4$. Now suppose that the result is true for all natural numbers less than $n$, and we prove it for $n$. By part (i), (ii), Theorem 3.1, and the induction hypothesis we have

$$\begin{aligned} d(P_{3n+1}, n+1) &= d(P_{3n}, n) + d(P_{3n-1}, n) + d(P_{3n-2}, n) \\ &= 1 + d(P_{3(n-1)+2}, n) + d(P_{3(n-1)+1}, n) \\ &= 1 + (n+1) + \frac{(n+1)(n+2)}{2} - 2 \\ &= \frac{(n+2)(n+3)}{2} - 2. \end{aligned} \tag{3.3}$$

(iv) Proof by induction on $n$. Since $d(P_3, 2) = 3$, the result is true for $n = 1$. Now suppose that the result is true for all natural numbers less than $n$, and we prove it for $n$. By Theorem 3.1, parts (ii), (iii), and the induction hypothesis we have

$$\begin{aligned} d(P_{3n}, n+1) &= d(P_{3n-1}, n) + d(P_{3n-2}, n) + d(P_{3n-3}, n) \\ &= n + 1 + \frac{(n+1)(n+2)}{2} - 2 + \frac{n(n-1)(n+7)}{6} \\ &= \frac{n(n+1)(n+8)}{6}. \end{aligned} \tag{3.4}$$

(v) Since $\mathcal{D}_n^n = \{[n]\}$, we have the result.

(vi) Since $\mathcal{D}_n^{n-1} = \{[n] - \{x\} \mid x \in [n]\}$, we have $d(P_n, n-1) = n$.

(vii) By induction on $n$. The result is true for $n = 3$, because $d(P_3, 1) = 1$. Now suppose that the result is true for all numbers less that $n$, and we prove it for $n$. By Theorem 3.1, induction hypothesis, part (v) and part (vi) we have

$$\begin{aligned} d(P_n, n-2) &= d(P_{n-1}, n-3) + d(P_{n-2}, n-3) + d(P_{n-3}, n-3) \\ &= \frac{(n-1)(n-2)}{2} + n - 3 \\ &= \frac{n(n-1)}{2} - 2. \end{aligned} \tag{3.5}$$

(viii) By induction on $n$. The result is true for $n = 4$, since $d(P_4, 1) = 0$. Now suppose that the result is true for all natural numbers less than or equal $n$ and we prove it for $n+1$. By Theorem 3.1, induction hypothesis, parts (vii) and (vi) we have

$$\begin{aligned} d(P_{n+1}, n-2) &= d(P_n, n-3) + d(P_{n-1}, n-3) + d(P_{n-2}, n-3) \\ &= \frac{(n-4)(n-3)(n+4)}{6} + \frac{(n-1)(n-2)}{2} - 2 + n - 2 \\ &= \frac{(n-3)(n-2)(n+5)}{6}. \end{aligned} \tag{3.6}$$

(ix) By induction on $n$. Since $d(P_5, 1) = 0$, the result is true for $n = 5$. Now suppose that the result is true for all natural numbers less than $n$, and we prove it for $n$. By Theorem 3.1, induction hypothesis, parts (viii) and (vii), we have

$$\begin{aligned} d(P_n, n-4) &= d(P_{n-1}, n-5) + d(P_{n-2}, n-5) + d(P_{n-3}, n-5) \\ &= \frac{(n-6)\left((n-1)^3 - (n-1)^2 - 42n + 138\right)}{24} \\ &\quad + \frac{(n-6)(n-5)(n+2)}{6} + \frac{(n-3)(n-4)}{2} - 2 \\ &= \frac{(n-5)\left(n^3 - n^2 - 42n + 96\right)}{24}. \end{aligned} \tag{3.7}$$

(x) We will prove that for every $n$, $d(P_i, n) < d(P_{i+1}, n)$ for $n \le i \le 2n-1$, and $d(P_i, n) > d(P_{i+1}, n)$ for $2n \le i \le 3n-1$. We prove the first inequality by induction on $n$. The result holds for $n = 1$. Suppose that result is true for all $n \le k$. Now we prove it for $n = k+1$, that is $d(P_i, k+1) < d(P_{i+1}, k+1)$ for $k+1 \le i \le 2k+1$. By Theorem 3.1 and the induction hypothesis we have

$$\begin{aligned} d(P_i, k+1) &= d(P_{i-1}, k) + d(P_{i-2}, k) + d(P_{i-3}, k) \\ &< d(P_i, k) + d(P_{i-1}, k) + d(P_{i-2}, k) \\ &= d(P_{i+1}, k+1). \end{aligned} \tag{3.8}$$

Similarly, we have the other inequality.

(xi) Proof by induction on $j$. First, suppose that $j = 2$. Then $\sum_{i=2}^{6} d(P_i, 2) = 12 = 3\sum_{i=1}^{3} d(P_i, 1)$. Now suppose that the result is true for every $j < k$, and we prove for $j = k$:

$$\begin{aligned}\sum_{i=k}^{3k} d(P_i, k) &= \sum_{i=k}^{3k} d(P_{i-1}, k-1) + \sum_{i=k}^{3k} d(P_{i-2}, k-1) + \sum_{i=k}^{3k} d(P_{i-3}, k-1) \\ &= 3\sum_{i=k-1}^{3(k-1)} d(P_{i-1}, k-2) + 3\sum_{i=k-1}^{3(k-1)} d(P_{i-2}, k-2) + 3\sum_{i=k-1}^{3(k-1)} d(P_{i-3}, k-2) \\ &= 3\sum_{i=k-1}^{3k-3} d(P_i, k-1).\end{aligned} \tag{3.9}$$

(xii) By Theorem 3.1, we have

$$\begin{aligned} s_n &= \sum_{j=\lceil n/3\rceil}^{n} d(P_n, j) \\ &= \sum_{j=\lceil n/3\rceil}^{n} \left(d(P_{n-1}, j-1) + d(P_{n-2}, j-1) + d(P_{n-3}, j-1)\right) \\ &= \sum_{j=\lceil n/3\rceil-1}^{n-1} d(P_{n-1}, j) + \sum_{j=\lceil n/3\rceil-1}^{n-2} d(P_{n-2}, j) + \sum_{j=\lceil n/3\rceil-1}^{n-3} d(P_{n-3}, j-1) \\ &= s_{n-1} + s_{n-2} + s_{n-3}.\end{aligned} \tag{3.10}$$

(xiii) Proof by induction on $n$. Since $d(P_1, 1) = d(P_3, 1)$, the theorem is true for $n = 1$. Now, suppose that the theorem is true for all numbers less than $n$, and we will prove it for $n$. By Theorem 3.1 and the induction hypothesis, we can write

$$\begin{aligned} d(P_{2n-k}, n) &= d(P_{2n-k-1}, n-1) + d(P_{2n-k-2}, n-1) + d(P_{2n-k-3}, n-1) \\ &= d(P_{2(n-1)+1-k}, n-1) + d(P_{2(n-1)-k}, n-1) + d(P_{2(n-1)-1-k}, n-1) \\ &= d(P_{2(n-1)+k-1}, n-1) + d(P_{2(n-1)+k}, n-1) + d(P_{2(n-1)+1+k}, n-1) \\ &= d(P_{2n+k-3}, n-1) + d(P_{2n+k-2}, n-1) + d(P_{2n+k-1}, n-1) \\ &= d(P_{2n+k}, n).\end{aligned} \tag{3.11}$$

(xiv) By Theorem 3.1, we have

$$\begin{aligned} d(P_{n+1}, j+1) - d(P_n, j+1) &= \left(d(P_n, j) + d(P_{n-1}, j) + d(P_{n-2}, j)\right) \\ &\quad - \left(d(P_{n-1}, j) + d(P_{n-2}, j) + d(P_{n-3}, j)\right) \\ &= d(P_n, j) - d(P_{n-3}, j).\end{aligned} \tag{3.12}$$

Therefore we have the result. □

In the following theorem we use the generating function technique to find $|\mathcal{D}_n^i|$.

**Theorem 3.3.** *For every* $n \in \mathbb{N}$ *and* $\lceil n/3 \rceil \leq i \leq n$, $|\mathcal{D}_n^i|$ *is the coefficient of* $u^n v^i$ *in the expansion of the function*

$$f(u,v) = \frac{u^3 v(1 + 3v + v^2 + 3uv + uv^2 + 2u^2 v + u^2 v^2)}{1 - uv - u^2 v - u^3 v}. \tag{3.13}$$

*Proof.* Set $f(u,v) = \sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|\mathcal{D}_n^i|u^n v^i$. By recursive formula for $|\mathcal{D}_n^i|$ in Theorem 3.1 we can write $f(u,v)$ in the following form:

$$\begin{aligned}
f(u,v) &= \sum_{n=3}^{\infty}\sum_{i=1}^{\infty}\left(\left|\mathcal{D}_{n-1}^{i-1}\right| + \left|\mathcal{D}_{n-2}^{i-1}\right| + \left|\mathcal{D}_{n-3}^{i-1}\right|\right)u^n v^i \\
&= uv\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}\left|\mathcal{D}_{n-1}^{i-1}\right|u^{n-1}v^{i-1} + u^2 v\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}\left|\mathcal{D}_{n-2}^{i-1}\right|u^{n-2}v^{i-1} + u^3 v\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}\left|\mathcal{D}_{n-3}^{i-1}\right|u^{n-3}v^{i-1} \\
&= uv\left(\left|\mathcal{D}_2^0\right|u^2 + \left|\mathcal{D}_2^1\right|u^2 v + \left|\mathcal{D}_2^2\right|u^2 v^2\right) + uvf(u,v) \\
&\quad + u^2 v\left(\left|\mathcal{D}_1^0\right|u + \left|\mathcal{D}_1^1\right|uv + \left|\mathcal{D}_2^0\right|u^2 + \left|\mathcal{D}_2^1\right|u^2 v + \left|\mathcal{D}_2^2\right|u^2 v^2\right) \\
&\quad + u^2 vf(u,v) + u^3 v\left(\left|\mathcal{D}_0^0\right| + \left|\mathcal{D}_1^0\right|u + \left|\mathcal{D}_1^1\right|uv + \left|\mathcal{D}_2^0\right|u^2 + \left|\mathcal{D}_2^1\right|u^2 v + \left|\mathcal{D}_2^2\right|u^2 v^2\right) \\
&\quad + u^3 vf(u,v).
\end{aligned} \tag{3.14}$$

By substituting the values from Table 1 (note that $|\mathcal{D}_n^0| = 0$ for all natural numbers $n$ and $|\mathcal{D}_0^0| = 1$), we have

$$f(u,v)\left(1 - uv - u^2 v - u^3 v\right) = u^3 v\left(1 + 3v + v^2 + 3uv + uv^2 + 2u^2 v + u^2 v^2\right). \tag{3.15}$$

Therefore we have the result. □

## Acknowledgments

Authors wish to thank the referee for his valuable comments and the research management center (RMC) of University Putra Malaysia for their partial financial support.

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