Dominating Sets in k-Majority Tournaments

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Abstract

A *k-majority* tournament T on a finite vertex set V is defined by a set of $2k-1$ linear orderings of V, with $u \to v$ if and only if u lies above v in at least k of the orders. Motivated in part by the phenomenon of "non-transitive dice", we let $F(k)$ be the maximum over all k-majority tournaments T of the size of a minimum dominating set of T.

We show that $F(k)$ exists for all $k > 0$, that $F(2) = 3$ and that in general $C_1 k/\log k \leq F(k) \leq C_2 k \log k$ for suitable positive constants C_1 and C_2.

1 Introduction

Let $T = (V, E)$ be a tournament. For vertices $x, y \in V$ we say that x *dominates* y and write $x \to y$ if $xy \in E$ or $x = y$. Similarly, for vertex sets $X, Y \subseteq V$ we say that X *dominates* Y and write $X \to Y$ if for every $y \in Y$ there exists $x \in X$ such that $x \to y$. In the case that X or Y is a singleton we may replace the set by its unique element in this notation. A *dominating set* in T is a set $X \subseteq V$ such that $X \to V$.

For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Let $P_1, \ldots, P_{2k-1}$ be linear orders on a finite set V. The tournament T *realized* by the orders $P_1, \ldots, P_{2k-1}$ is the tournament on V in which an ordered pair uv is a directed edge if and only if u

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lies above v in at least k of the orders. We say that a tournament T is a *k-majority tournament* if it is realized by some set of $2k-1$ linear orders. This concept arises in social choice theory, where each linear order represents the preferences of a voter among candidates, and the tournament represents the results of head-to-head contests between pairs of candidates.

Let $k_0(n)$ be the minimum k such that every n-vertex tournament is a k-majority tournament. McGarvey [14] showed that $k_0(n)$ is defined for each n, i.e., that every tournament is a k-majority tournament for some k. Sterns [17] showed that $k_0(n) = \Omega(n/\log n)$, and that $k_0(n) \leq n+1$. Erdős and Moser [10] improved the upper bound to $k_0(n) = O(n/\log n)$. Later, Alon [2] showed that, for every n-vertex tournament, there is a set of $2k = O(n \log n)$ orders, so that for every edge uv, u lies above v in at least $k + \Omega(k/\sqrt{n})$ of the orders.

This paper is concerned with the size of a minimum dominating set in a k-majority tournament. Of course, as first shown in [9], there are tournaments whose minimum dominating set is arbitrarily large but, as conjectured by H.A. Kierstead and W.T. Trotter, this turns out not to be the case if we confine ourselves to k-majority tournaments for some fixed k.

Accordingly, let $F(k)$ be the supremum of the size of a minimum dominating set in a k-majority tournament (where the supremum is taken over all k-majority tournaments, with no restriction on their size). Trivially $F(1) = 1$. We show below that $F(2) = 3$, and that $F(3) \geq 4$, but we have not found a way to generalize these arguments. We do, however, show that $F(k)$ is finite for each k, proving the conjecture of Kierstead and Trotter; in fact, we give two proofs of this conjecture. The first proof demonstrates the result to be a simple consequence of a geometric result of Bárány and Lehel [5], but the upper bound obtained on $F(k)$ from this proof is rather large. Our second proof yields that $F(k) = O(k \log k)$; we also show that $F(k) = \Omega(k/\log k)$, so our bounds are reasonably close. A similar upper bound applies to tournaments defined by orders in a more general way. The technique in our second proof has several additional applications, including an improvement on the result of Bárány and Lehel mentioned above.

A tournament has no dominating set of size t if and only if it satisfies the property S_t: every set U of t vertices is dominated by some vertex not in U. The question of existence of such tournaments was raised by Schütte, and first studied by Erdős in [9], where he proves existence by a (by now) simple probabilistic argument. To demonstrate lower bounds on $F(k)$, we have to construct families of linear orders realizing tournaments with property S_t.

The concept of k-majority tournaments is strongly related to that of *dice tournaments.* It is well known that the faces of three standard dice can be assigned distinct numbers so that the first is more likely to beat the second (has a higher number with probability exceeding a half when they are rolled against each other), the second is more likely to beat the third, and the third is more likely to beat the first. In other words, the tournament determined by such a set of three "non-transitive" dice satisfies property S_1.

Here is an example of a set of three non-transitive dice: one die is labeled with five 3's and a 6, one with five 4's and a 1, and one with three 2's and three 5's. The owner of such a set can offer to roll against a victim, "generously" allowing the victim to choose first which of the three dice they prefer. (A set of dice with these labels was provided to each participant at the Fourth Gathering for

Gardner—one of a series of meetings in honor of Martin Gardner.)

If the goal is to handle t victims at the same time—much as a chess master puts on a simultaneous exhibition—one would need a set of dice with the property that, for any subset of t dice, there is a die in the set that beats them all. This is always possible if $t < F(k)$ and at least $2k-1$ faces are available on each die. Indeed, let T be a k-majority tournament on n vertices (dice) satisfying property S_t. The ith face of each die is assigned a number in the range $\{ni+1, ni+2, \ldots, n(i+1)\}$ according to its rank in the ith order of a realizer of T. The same construction shows that any k-majority tournament can be realized by dice with $2k-1$ faces. Let $G(k)$ denote the supremum of the minimum size of a dominating set in a tournament realized by dice with k faces. (Note that because of possible ties, not all collections of dice realize tournaments.) By the discussion above, $G(2k-1) \geq F(k)$ for all k, and hence our lower bound for $F(k)$ together with the obvious monotonicity of $G(k)$ implies that $G(k) \geq \Omega(k/\log k)$. On the other hand, our methods here enable us to show that $G(k) \leq O(k \log k)$.

Some of the questions addressed in this article arose during a summer class on challenge problems for in-service high school teachers taught by the third author.

2 2-Majority tournaments

In this section we prove that $F(2) = 3$.

Theorem 1. *Every* 2*-majority tournament has a dominating set of size at most* 3*. Moreover, if T does not have a dominating set of size one, then it has a dominating set of size* 3 *that induces a directed cycle.*

Proof. Consider a 2-majority tournament $T = (V, A)$ defined by the three linear orders $P_i = (V, >_i)$, $i \in [3]$. Choose the least vertex c in P_3 such that there exists a vertex $d \leq_3 c$ dominating the set $U = \{x \in V : x >_3 c\}$ of vertices strictly above c in P_3. If U is empty, i.e., c is the top element of P_3, then c is the only vertex dominating the set $\{c\}$, and so $c \to V$. Thus we may assume that U is nonempty.

Let $D = \{x \in V \setminus U : x \to U\}$ be the (nonempty) set of vertices not in U that dominate U, and let $R = V \setminus (U \cup D \cup \{c\})$ be the set of remaining vertices. Let u_i be the maximum element of U in P_i, $i \in [2]$, and fix any $d \in D$ (see Figure 1.)

No element of $D \setminus \{c\}$ can dominate c, since otherwise c's immediate predecessor in P_3 would have been preferred in the definition of c; hence $c \to D$. Any element $x \in V \setminus U$ satisfies $x <_3 u_1, u_2$. So, if x dominates both u_1 and u_2 then it satisfies $u_1, u_2 <_i x$ for both $i \in [2]$, and thus dominates all of U. It follows that $D = \{x \in V \setminus U : x >_1 u_1 \text{ and } x >_2 u_2\} = \{x \in V \setminus U : x \to \{u_1, u_2\}\}$, and therefore $\{u_1, u_2\}$ dominates R. Thus $\{c, d, u_1, u_2\}$ dominates V, but we can do better. Let $R_i = \{x \in R : x <_i u_i\}$, $i \in [2]$. Then $R = R_1 \cup R_2$ and u_i dominates R_i, $i \in [2]$. Since $u_1, u_2 <_i d$ for both $i \in [2]$ and c dominates d, there exists $i \in [2]$ such that $u_i <_i c$; we may suppose $u_2 <_2 c$. Then R_2 is also dominated by c, since c is above R in P_3. It follows that $\{c, d, u_1\}$ is a dominating set for T.

Note also that if $c \in D$ then $c >_1 R_1$, and so $c \to V$. Otherwise $c \to d \to u_1 \to c$. □

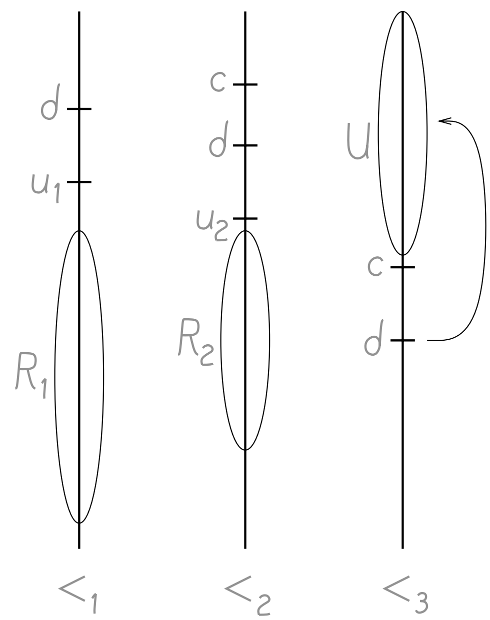

Figure 1: The orders $<_1$, $<_2$ and $<_3$

The following example completes the proof that $F(2) = 3$.

Example 2. *There exists a 2-majority tournament T with property S_2.*

Proof. Let T be the quadratic residue tournament whose vertices are the elements of the finite field $GF(7)$ in which ij is a directed edge if and only if $i - j$ is a quadratic residue, i.e., $(i - j) \bmod 7 \in \{1, 2, 4\}$. It is apparent that T satisfies S_2 since $2 \to \{0, 1\}$, $4 \to \{0, 2\}$, $4 \to \{0, 3\}$ and edges are preserved under translation. Moreover it can be checked that T is realized by the orders

$$P_1 = 0 < 1 < 2 < 3 < 4 < 5 < 6$$
$$P_2 = 4 < 6 < 1 < 3 < 5 < 0 < 2$$
$$P_3 = 5 < 2 < 6 < 3 < 0 < 4 < 1.$$

□

The example shows that, with an appropriate set of (seven) 3-sided dice, we can handle two victims at once; Theorem 1 implies that two is the maximum. In other words, no matter how many 3-sided dice are offered, a group of three "victims" can select one die each so that no choice from the remaining dice beats all three. To see this, let $a_i \geq b_i \geq c_i$ be the numbers labeling die i, where we assume that no number appears on two distinct dice. Let A, B and C be the orderings on the set of dice given by $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ respectively. We claim that if die 1 lies above die 2 in at least two of these orders, then die 1 beats die 2. Indeed, if $a_1 > a_2$ and $b_1 > b_2$, then also $a_1 > b_2$, $a_1 > c_2$ and $b_1 > c_2$, so die 1 beats die 2 in at least 5 cases out of 9. The case where $b_1 > b_2$ and

$c_1 > c_2$ is symmetric. If $a_1 > a_2$ and $c_1 > c_2$, then we also have $a_1 > b_2$, $a_1 > c_2$ and $b_1 > c_2$ so again die 1 wins. It follows that the dice constitute a 2-majority tournament, thus the victims can choose a dominating set of size 3, the elements of which cannot be beaten by any one die in the set.

3 3-Majority tournaments

We have not been able to determine the value of $F(3)$, but the following result shows that $F(3) \geq 4$.

Theorem 3. *There exists a 3-majority tournament with property S_3.*

Proof. Consider the simplex consisting of all vectors $a = (a_1, \ldots, a_5)$ in $\mathbb{R}^5$ with $a_i \geq 0$ for each i, and $\sum_{i=1}^5 a_i = 1$. Now take a finite subset A of this simplex, with the property that, for any vector $(x_1, \ldots, x_5)$ with $x_i \geq 0$ for each i and $\sum_{i=1}^5 x_i = 10/11$, there is some element a of A such that $a_i > x_i$ for each i. The five coordinate orders induce linear orders on this finite set A; we can assume that there are no ties in these orders among elements of A.

These five linear orders on the set A realize a 3-majority tournament. Note that any element $a = (a_1, \ldots, a_5)$ of A is dominated by some other element of A. To see this, define $x = (x_1, \ldots, x_5)$ by $x_i = a_i$ if a_i is one of the three smallest coordinates of a and $a_i = 0$ otherwise. Then $\sum_{i=1}^5 x_i \leq 3/5 \leq 10/11$ and so some element of A dominates a. Perhaps this tournament already has property S_3, but it is easier to work with an augmentation of A. We define the set $\langle A \rangle = \bigcup_{a \in A} \langle a \rangle$ by taking, for each element a of A, a 3-set $\langle a \rangle = \{a^1, a^2, a^3\}$. We call $\langle a \rangle$ the set of "clones" of $a \in A$. We obtain five linear orders on $\langle A \rangle$ from the corresponding orders on A by replacing each element a with an interval formed from its clones and ordered so that each set of clones forms a directed cycle. So we replace a by the interval I_i in the ith order, where

$$I_1 = (a^1, a^2, a^3), I_2 = (a^2, a^3, a^1), I_3 = (a^3, a^1, a^2), I_4 = (a^1, a^2, a^3), I_5 = (a^3, a^2, a^1).$$

We claim that the 3-majority tournament realized by these linear orders on $\langle A \rangle$ has property S_3, i.e., for every triple $U = \{a^i, b^j, c^k\}$ of elements of $\langle A \rangle$, there is some other element of $\langle A \rangle$ that dominates all three.

Let us first deal with some easy cases. Suppose there exist distinct $d, e \in A$ such that $U \subseteq \langle d \rangle \cup \langle e \rangle$, with $e \to d$. Then there exists $f \in A$ such that $f \to e$. If $f \to d$ then any clone of f dominates U. Otherwise any element that dominates d^1, e^1, f^1 also dominates U. Thus it suffices to consider the case that a, b and c are distinct elements of A. If, say, a dominates both b and c in A, then some clone of a dominates U. This is the sole reason for introducing the clones. Therefore it suffices to show that for all $a \to b \to c \to a$ in A there is some other element of A dominating $\{a, b, c\}$.

Consider how the coordinates of a, b and c can be arranged. If there are three coordinates i where a_i is the highest among $\{a_i, b_i, c_i\}$, then $a \to c$ which is a contradiction. And if there are three coordinates i where a_i is the lowest among $\{a_i, b_i, c_i\}$, then $b \to a$ which again is a contradiction. So two of the elements, say a and b, are lowest in two of the five linear orders each, while c is lowest in the other. Suppose next that a is in the middle in two of the five linear orders: to get $a \to b \to c$ we need b bottom in those two orders, and ahead of c in all the others. That gives us, possibly after a renumbering of the orders:

$$\begin{array}{ccccc} c_1 & < & b_1 & < & a_1 \\ b_2 & < & a_2 & < & c_2 \\ b_3 & < & a_3 & < & c_3 \\ a_4 & < & c_4 & < & b_4 \\ a_5 & < & c_5 & < & b_5 \end{array} \qquad \text{Case 1.}$$

The case where b is in the middle twice is impossible, as we can't then have $b \to c$. So the other possibility is that a and b are in the middle just once each, and c is in the middle three times. Then we clearly must have:

$$\begin{array}{ccccc} c_1 & < & b_1 & < & a_1 \\ b_2 & < & a_2 & < & c_2 \\ b_3 & < & c_3 & < & a_3 \\ a_4 & < & c_4 & < & b_4 \\ a_5 & < & c_5 & < & b_5 \end{array} \qquad \text{Case 2.}$$

We will show that we can always find a non-negative vector $x = (x_1, \dots, x_5)$ so that $\sum_{i=1}^5 x_i \leq 10/11$ and, for $d = a, b, c$, we have $x_i \geq d_i$ in at least three coordinates i. By the construction of A, there is then some element of A that is above x (strictly) in all five coordinates, and so dominates all of a, b, c.

We start with Case 2, as it is slightly cleaner. The ten vectors below are candidates for x, in that they are non-negative vectors that at least match each of a, b, c in at least three coordinates each.

$$\begin{array}{cccc} (a_1, c_2, c_3, a_4, 0) & (a_1, b_2, a_3, 0, c_5) & (b_1, 0, a_3, b_4, a_5) & (c_1, a_2, 0, b_4, b_5) \\ (0, c_2, b_3, c_4, b_5) & (c_1, c_2, b_3, a_4, b_5) & (c_1, b_2, a_3, b_4, a_5) & (a_1, b_2, b_3, c_4, c_5) \\ (b_1, a_2, c_3, c_4, a_5) & (b_1, a_2, c_3, a_4, c_5) & & \end{array}$$

Consider the sum S of the coordinates of all ten of these vectors. Since each coordinate of a, b, c appears exactly three times in this sum we have

$$S = 3 \sum_{i=1}^{5} (a_i + b_i + c_i) = 9.$$

Thus one of these vectors satisfies the required inequality

$$\sum_{i=1}^{5} x_i \leq 9/10 \leq 10/11.$$

For Case 1, we present 11 candidates for x (two of them are identical, but that doesn't matter):

$$\begin{array}{cccc} (c_1, b_2, a_3, b_4, c_5) & (c_1, a_2, b_3, c_4, b_5) & (b_1, c_2, c_3, a_4, 0) & (b_1, c_2, c_3, 0, a_5) \\ (b_1, c_2, c_3, 0, a_5) & (b_1, b_2, c_3, a_4, c_5) & (c_1, a_2, 0, b_4, b_5) & (c_1, 0, a_3, b_4, b_5) \\ (a_1, b_2, b_3, c_4, c_5) & (a_1, b_2, 0, c_4, b_5) & (0, c_2, b_3, b_4, c_5) & \end{array}$$

The sum of the fifty-five entries is

$$2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 4b_1 + 4b_2 + 3b_3 + 4b_4 + 4b_5 + 4c_1 + 4c_2 + 4c_3 + 3c_4 + 4c_4$$
$$\leq 2\sum a_i + 4\sum b_i + 4\sum c_i = 10.$$

So again one of them has coordinate sum at most 10/11. □

4 A geometric proof that $F(k)$ is finite

In this section we offer a simple but crude proof that $F(k)$ is finite, that is, that k-majority tournaments cannot have arbitrarily large minimum dominating sets. This proof relies on a geometric result of Bárány and Lehel. We start by describing this result.

For two points x, y in $\mathbb{R}^d$, let $\text{box}(x, y)$ denote the smallest closed box, with faces parallel to the coordinate hyperplanes, that contains both x and y. We say that this box is the box *generated by* x and y, which form two of its corners. Note that this box consists of those points $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ such that for every i, $1 \leq i \leq d$, z_i lies between x_i and y_i. Bárány and Lehel proved in [5] that, for every dimension d, there is a constant $c = c(d)$ (depending only on the dimension), such that every compact subset $V \subset \mathbb{R}^d$ contains a subset S of cardinality at most c, satisfying

$$V \subset \cup_{x,y \in S} \text{ box}(x, y). \tag{1}$$

Their proof shows that $c(d) \leq (2d^{2^d} + 1)^{d2^d}$, but by plugging into it the result of [12] or [1], this can be improved to

$$c(d) \leq (2d^2 + 1)^{d2^d}.$$

Using the main result of [8], Pach [15] has improved this bound to $2^{2^{d+2}}$. In Section 7 we use a similar approach to obtain a still better bound, based on our techniques here.

Let $T = (V, E)$ be a k-majority tournament, and let $P_1, \ldots, P_{2k-1}$ be linear orders that realize it. Put $d = 2k - 1$ and identify each $v \in V$ with the point $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ whose ith coordinate is the rank of v in P_i. (This identification will be useful later as well.) Thus V is a finite set in $\mathbb{R}^d$, and by the theorem of [5] this set contains a subset S of at most $c(2k-1)$ points such that (1) holds. Note that for each box $\text{box}(x, y)$, with $x, y \in S$, one of the vertices x or y beats every point z of V which is inside the box in at least k of our linear orders (as z lies strictly between x and y in each order). It follows that the set S dominates the whole tournament, completing the (first) proof.

5 Improving the upper bound

The previous proof supplies a huge upper bound for $F(k)$. Here we prove the following nearly linear bound. (We make no attempt to optimize the absolute constant 80 in the statement below.)

Theorem 4. *$F(k) \leq (80 + o(1))k \log k$, where the $o(1)$ term tends to zero as k tends to infinity.*

Proof. For a vertex v in a tournament T, let $D(v)$ denote the set consisting of all vertices (including v) that dominate v. We start with the following simple lemma.

Lemma 5. *For every tournament $T = (V, E)$ there is a probability distribution $p : V \mapsto [0, 1]$ such that for every vertex w, the total weight of vertices in $D(w)$ is at least $1/2$.*

Proof. Consider the following two-person zero-sum game played on T. Each of the two players, Alice and Bob, simultaneously picks a vertex of T, and the owner of the dominant vertex collects \$1 from the other player. Suppose Alice is deemed to be the winner if the players pick the same vertex; then the game cannot be in Bob's favor, hence by the Minimax Theorem (see, e.g., [6], Theorem 15.1) there is a mixed strategy of Alice with non-negative expectation against any strategy—in particular any pure strategy—of Bob.

This mixed strategy of Alice is a probability distribution on the vertices that satisfies the assertion of the lemma. □

Given a tournament $T = (V, E)$, let $H = H(T)$ be the hypergraph whose vertices are all the vertices of T, and whose edges are all the sets $D(v)$, $v \in V$. The *transversal number* $\tau(H)$ of H is the minimum cardinality of a set of vertices that intersects every edge; since such a set is exactly a dominating set of T, $\tau(H(T))$ is just another way to describe the size of a minimum dominating set of T.

The *fractional transversal number* $\tau^*(H)$ of H is the minimum possible value of $\sum_{v\in V} f(v)$, where the minimum is taken over all functions $f : V \mapsto [0, 1]$ such that, for every edge of the hypergraph, the total weight of vertices in the edge is at least 1. The following is an immediate consequence of Lemma 5.

Corollary 6. *For every tournament $T = (V, E)$, $\tau^*(H(T)) \le 2$.*

Proof. Let $p : V \mapsto [0, 1]$ be as in Lemma 5, and define $f(v) = \min\{1, 2p(v)\}$ for every v. □

The *Vapnik-Chervonenkis dimension* or *VC-dimension* $\mathrm{VC}(H)$ of a hypergraph $H = (V, E)$ is the maximum cardinality of a set of vertices $A \subset V$ such that for every $B \subset A$ there is an edge $e \in E$ so that $e \cap A = B$. We need the following result, which follows from the work of Vapnik and Chervonenkis and of Haussler and Welzl (see [13], Corollary 10.2.7).

Lemma 7. *For every hypergraph H with $VC(H) \le h$*

$$\tau(H) < 20h\tau^*(H)\log(\tau^*(H)).$$

Returning to our original problem, let $T = (V, E)$ be a k-majority tournament, let $P_1, \ldots, P_{2k-1}$ be linear orders realizing T, and let $H = H(T)$ be the hypergraph defined above.

Lemma 8. *If $h = VC(H)$ is the VC-dimension of H, then $(h + 1)^{2k-1} + h \ge 2^h$. Therefore $h \le (1 + o(1))2k \log_2 k$, where the $o(1)$ term tends to zero as k tends to infinity.*

Proof. As before, we identify each vertex u of T with the vector $(u_1, \dots, u_{2k-1})$ of length $2k-1$ whose ith coordinate is the rank of u in P_i. Let $A = \{u(1), u(2), \dots, u(h)\}$ be a set of h vertices of T and suppose that for each $B \subset A$ there is an edge $D(v)$ of H such that $A \cap D(v) = B$. For each $i \in [2k-1]$, the ith coordinates of the $u(j)$'s split the range of possible ranks in P_i of vertices in $T - A$ into $h+1$ open intervals. Thus, if we know for each i the interval in which the ith coordinate of a particular vertex $v \in T - A$ is found, we can determine precisely which $u(j)$ lie in $D(v)$ and which do not. It follows that the total number of possibilities of the intersection $A \cap D(v)$, including $v \in A$, cannot exceed $(h+1)^{2k-1} + h$, implying the desired result. □

Combining Corollary 6, Lemma 7 and Lemma 8 we conclude that every k-majority tournament contains a dominating set of size at most $20(2+o(1))k \log_2 k (2 \log 2) \leq (80 + o(1))k \log k$, as claimed. □

6 A lower bound on $F(k)$

The following theorem shows that $F(k) \geq (\frac{1}{5} + o(1))k/\log k$.

Theorem 9. *For every integer $t \geq 2$ there exists a $3t\lceil \log_2 t \rceil$-majority tournament that satisfies property S_t.*

Proof. Example 2 proves the case $t = 2$. Let $t \geq 3$. Let $a = t\lceil \log_2 t \rceil$ and $k = 3a$. We shall construct a k-majority tournament $T = (V, E)$ with property S_t, that is, in which every dominating set has size greater than t. Let V be the set of pairs (A, B) where A is a subset of $[k]$ of size a and B a subset of $[k] \setminus A$ of size $b = \lceil \log_2 t \rceil$.

Fix any linear order $L = (V, \succ)$ on V. For all $j \in [k]$ define the linear order $P_j = (V, >_j)$ to agree with L except that all (A, B) with $j \in A$ are placed on top, and all (A, B) with $j \in B$ are placed next. In other words, $(A, B) >_j (A', B')$ if and only if either

$$j \in (A \cap A') \cup (B \cap B') \cup \big(([k] \setminus A \setminus B) \cap ([k] \setminus A' \setminus B')\big) \text{ and } (A, B) \succ (A', B')$$

or

$$j \in (A \setminus A') \cup (B \setminus A' \setminus B').$$

We also define companion orders $Q_j = (V, >^j)$ which are consistent with the *dual* of L, except that (A, B) with $j \in B$ are placed on the bottom and (A, B) with $j \in A$ are placed next going up. Thus, $(A, B) >^j (A', B')$ if and only if either

$$j \in (A \cap A') \cup (B \cap B') \cup \big(([k] \setminus A \setminus B) \cap ([k] \setminus A' \setminus B')\big) \text{ and } (A', B') \succ (A, B)$$

or

$$j \in (B' \setminus B) \cup \big(A' \setminus A \setminus B\big).$$

The orders are pictured in Fig. 2.

Let $\Omega = \{P_j : j \in [k]\} \cup \{Q_j : j \in [k]\}$. Then the number of orders in Ω in which (A, B) beats (A', B'), minus the number in which (A', B') beats (A, B), is exactly twice $|A \cap B'| - |A' \cap B|$. Let

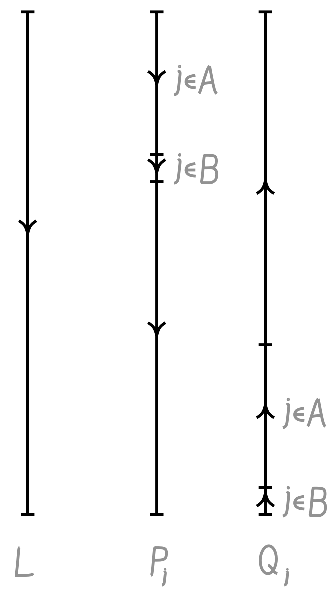

Figure 2: The orders L, P_j and Q_j

T be the k-majority tournament generated by the $2k-1$ orders in Ω other than P_1; then we have $(A,B) \to (A',B')$ whenever $|A \cap B'| > |A' \cap B|$ (and in some cases when the two quantities are equal).

We claim that T satisfies property S_t. To see this, consider a t-set $\mathcal{S} = \{(A_i, B_i) : i \in [t]\}$ of elements of V: we need to find an element (A,B) of V that dominates $\mathcal{S}$. Choose A of size a so that $\bigcup_{i\in[t]} B_i \subseteq A$. We now wish to choose $B \subseteq [k] \setminus A$ so that $B \not\subseteq A_i$ for all $i \in [t]$. Since $|[k] \setminus A| = 2a$, we have $\binom{2a}{b}$ possibilities for the choice of B, and each A_i 'spoils' at most $\binom{a}{b}$ of them. Note that $\binom{2a}{b} > 2^b \binom{a}{b}$ for $b \geq 2$, and this is the case since $t \geq 3$. As $2^b \geq t$, this implies that we can choose $B \not\subseteq A_i$ for all $i \in [t]$. Now indeed (A,B) dominates $\mathcal{S}$, since $|A \cap B_i| = |B_i| = b = |B| > |A_i \cap B|$ for each $i \in [t]$. □

7 Improving the Bárány-Lehel estimate

The technique used to prove Theorem 4 enables us to give a new, simple proof of the main result of Bárány and Lehel [5], stated in Section 4. The resulting bound is better than any previously obtained.

Theorem 10. *Every set V of n points in $\mathbb{R}^d$ is contained in the union of at most*

$$2^{2^d + d + \log d + \log\log d + O(1)}$$

boxes of the form box(p,q) with $p, q \in V$.

Proof. We first claim that there is an $\varepsilon = \varepsilon(d) > 0$ such that every set V of n points in $\mathbb{R}^d$ contains two points p, q so that $|V \cap \text{box}(p, q)| \geq \varepsilon n$. Indeed, by a well-known (but unpublished) result of N.G. de Bruijn (c.f., e.g., [3]—the result itself follows by iterating the Erdős-Szekeres Theorem, the fact that it is tight requires a construction), every set of $m = m(d) = 2^{2^{d-1}} + 1$ points contains distinct points p, q, r such that $r \in \text{box}(p, q)$. This implies, by the known estimates for the Turán number for 3-uniform hypergraphs (see [7]) and choosing $n \geq 4m$, that there are at least

$$\frac{n^2(n-m)}{3m^2} \geq \frac{n^3}{4m^2}$$

triples as above, since otherwise the hypergraph whose vertices are all points and whose edges are all such triples would contain an independent set of size m, which is impossible. By averaging, the same pair p, q appears as the two corners of the box in at least

$$\frac{n^3/4m^2}{\binom{n}{2}} \geq \frac{n}{2m^2}$$

triples, giving the required claim with $\varepsilon(d) = \frac{1}{2m^2}$. By duplicating some of the points, if needed, we conclude that the claim holds with weights as well; for every set V of n points in $\mathbb{R}^d$, and every probability distribution on the points, there are $p, q \in V$ so that the total measure of $V \cap \text{box}(p, q)$ is at least $\varepsilon = \varepsilon(d)$. Duality now implies that there is a probability distribution on the boxes, so that for every point v, the measure of all boxes that contain v is at least ε.

Consider the hypergraph H whose vertices are all boxes box(p, q) with $p, q \in V$, where the edges are all sets $R_v = \{\text{box}(p, q) : v \in \text{box}(p, q)\}$. Our objective is to bound the transversal number $\tau(H)$ of this hypergraph by a function depending only on d. By the above discussion, $\tau^*(H) \leq 1/\varepsilon(d) \leq 2m^2$, and thus $\tau^*(H)$ is bounded by such a function. Hence, to complete the proof we only have to bound the VC-dimension of this hypergraph. This, however, is easy, as the defining corners of the h boxes split each coordinate axis into at most $4h + 1$ open pieces and $2h$ points, showing that if the VC-dimension is h then $(6h + 1)^d \geq 2^h$ and implying that $h \leq (1/\log 2 + o(1))d \log d$.

Plugging in Lemma 7 we conclude that every set V of n points in $\mathbb{R}^d$ is contained in the union of at most

$$20(1/\log 2 + o(1))d \log d(2m^2) \log(2m^2) \leq 2^{2^d + d + \log d + \log\log d + O(1)}$$

boxes of the form box(p, q) with $p, q \in V$. □

As noted in [5] (and as follows from the fact that de Bruijn's result is tight), $2^{2^{d-1}}$ points is a lower bound for Bárány and Lehel's theorem. This translates to $\frac{1}{2}2^{2^{d-1}}$ pairs of points in Theorem 10 above.

8 Dice tournaments

In this section we observe that the method of Section 5 provides an $O(k \log k)$ upper bound for $G(k)$, the supremum of the size of a minimum dominating set in a tournament realized by dice with k faces.

Theorem 11. *There are two absolute positive constants c_1, c_2 such that $c_1 k/\log k \le G(k) \le c_2 k \log k$ for all $k > 1$.*

Proof. The lower bound follows from the lower bound for $F(k)$ obtained in Section 6, together with the fact that any k-majority tournament is realizable by dice with $2k-1$ faces each. This implies that $G(2k-1) \ge (\frac{1}{5} + o(1))k/\log k$. It is not difficult to see that $G(k)$ is a monotone non-decreasing function of k, and thus the lower bound follows.

To prove the upper bound using the method of Section 5, fix a tournament T realized by dice with k faces, and fix a set of such dice realizing it. Let $H = H(T)$ be the hypergraph corresponding to T, defined as in Section 5. Its vertices are the vertices of T, and its edges are all the sets $D(v)$. By Corollary 6, the fractional transversal number of H is at most 2. By Lemma 7, it suffices to show that its VC-dimension is at most $O(k \log k)$. Given a set A of d vertices of H, consider the kd numbers on the faces of the dice that correspond to them. These numbers partition the real line into at most $kd+1$ open intervals and at most kd points. Knowing the location in this partition of the k numbers on the faces of a die corresponding to any vertex v, determines precisely which members of A lie in $D(v)$ and which do not. As there are $\binom{2kd+k}{k}$ ways to choose the location of k numbers in the partition, it follows that there are at most that many possibilities for the intersection $A \cap D(v)$, and hence if d is the VC-dimension of H, then $\binom{2kd+k}{k} \ge 2^d$, implying the desired estimate. $\square$

9 Additional Remarks

1. The notion of k-majority tournaments can be extended as follows. Put $K = \{1, 2, \ldots, k\}$, and let $\mathcal{F}$ be an arbitrary collection of subsets of K such that for every $A \subset K$, either A or its complement $\overline{A}$ lies in $\mathcal{F}$, but not both.

Given k-orders $P_1, P_2, \ldots, P_k$ on a finite set V, define a tournament on V by letting uv be a directed edge if and only if $\{i \in K : u \text{ precedes } v \text{ in } P_i\} \in \mathcal{F}$. Thus, if k is odd and the set $\mathcal{F}$ consists of all subsets of K of cardinality bigger than $k/2$, we get the previous notion of a $(k+1)/2$-majority tournament. More generally, we can consider any probability distribution on the set K, such that no subset has measure exactly a half, and let $\mathcal{F}$ consist of all subsets of K of measure exceeding a half.

Our proof extends easily to show that for each $\mathcal{F}$ and for every set of k linear orders, the corresponding tournament has a dominating set of size $O(k \log k)$.

2. Let $T = (V, E)$ be a tournament in which the smallest size of a dominating set, t, is large. As mentioned in the introduction, such tournaments have first been studied by Erdős in [9], motivated by a question of Schütte. It is easy to see that for a random tournament on n vertices, t is roughly $\log n$, and there are several explicit constructions of tournaments in which the smallest dominating set is of size $\Theta(\log n)$. Most of these constructions have some pseudo-random properties. In particular, it seems plausible to suspect that if indeed, t is large, then the tournament must contain a large number of pairwise non-isomorphic sub-tournaments on s vertices, for some s that grows with t. This, indeed, follows from Lemma 7. If t is large, then the VC-dimension h of $H(T)$ is at least $t/30$, so there is a set A of $h \geq t/30$ vertices such that, for every subset $B \subseteq A$, there is a vertex x_B dominating all members of B and no member of $A \setminus B$. Let C be the set consisting of all the 2^h vertices x_B, and consider all the subtournaments arising by taking A together with h elements from C: there are $\binom{2^h}{h}$ such subtournaments, and each isomorphism type occurs at most $\binom{2h}{h}$ times, so we have $2^{\Omega(h^2)}$ pairwise non-isomorphic subtournaments on $s = 2h$ vertices. On the other hand, it is easy to see that k-majority tournaments can have only $(s!)^{2k-1} < s^{2ks}$ distinct isomorphism types of induced subtournaments on s vertices, showing (again, but essentially with the same proof) that $F(k) = O(k \log k)$.

3. As mentioned in the Introduction, Alon [2] investigated the problem of finding a family of linear orders realizing a tournament T so that, for every edge uv of T, u is above v in *substantially* more than half of the orders. The same issue arises for representations of a tournament by dice.

We define the *quality* of a representation of a tournament by dice to be the largest $\varepsilon > 0$, such that for every two vertices u, v of the tournament, with $u \to v$, the die of u beats that of v with probability at least $\frac{1}{2} + \varepsilon$. Thus, for example, an old result of Steinhaus and Trybula [18] asserts that the supremum of all ε such that there is a dice representation of quality ε for the cyclic tournament on 3 vertices is $\frac{\sqrt{5}-2}{2}$. See also [16] and its references for some related results. It seems plausible to suspect that if a tournament T satisfies property S_t for large t, then it does not have a high quality representation, namely, for some directed edge $u \to v$, the die of u will be only slightly superior to that of v. It can be shown that this is indeed the case. Here is a sketch.

Let T have property S_t, and fix a dice representation of it of quality ε. By the previous remark, each such tournament contains $2^{\Omega(d^2)}$ pairwise non-isomorphic subtournaments on some $d = \Theta(t)$ vertices. For each such subtournament, if we throw the dice corresponding to its vertices we get a linear order, and every directed edge of the tournament is consistent with the order with probability at least $\frac{1}{2} + \varepsilon$. Thus, if we take randomly some $2k - 1 = C/\varepsilon^2$ such linear orders, for an appropriate large constant C, then most of the edges of the subtournament will be consistent with those of the k-majority tournament obtained. It follows by a simple counting which is omitted, that one can get some $2^{\Omega(d^2)}$ pairwise non-isomorphic tournaments on d vertices, each of which is realizable by some $s = C/\varepsilon^2$ linear orders. As there are obviously only $(d!)^s < d^{sd}$ ways to choose s linear orders on d elements, this implies that $d^{sd} \geq 2^{\Omega(d^2)}$, implying that $s \geq \Omega(d/\log d)$ and hence that $\varepsilon \leq O(\sqrt{\log d/d}) = O(\sqrt{\log t/t})$.

4. The estimate in Lemma 8 is tight, up to a constant factor. To see this, let $D(m)$ be the maximum VC-dimension of a hypergraph H arising from m linear orders. We show below that $D(4k+3) \geq 2D(2k+1) + 2k$ for any non-negative integer k. As we also have $D(4k+5) \geq D(4k+3)$, this implies by induction that $D(2k+1) \geq (k+2)\log_2(k+2) - 3k - 1$ for any k, so $D(m) \geq (\frac{1}{2} + o(1)) m \log_2 m$.

Our aim is to construct a set of $4k+3$ linear orders on a set S of $2D(2k+1) + 2k$ elements, so that, whatever subset S' of S we choose, we can "cut" each order into an upper and a lower segment such that each element of S' is in at least $2k+2$ of the lower segments, and each element of $S - S'$ is in at most $2k + 1$ of the lower segments: we say that these linear orders can *exhibit* every subset S'.

Take two disjoint sets A and B, each with $D(2k + 1)$ elements, and let $(L_1, \ldots, L_{2k+1})$ and $(M_1, \ldots, M_{2k+1})$ be families of linear orders on A and B respectively, so that $(L_1, \ldots, L_{2k+1})$ can exhibit every subset A' of A, and $(M_1, \ldots, M_{2k+1})$ can exhibit every subset B' of B. Also let $C = \{c_1, \ldots, c_{2k}\}$ be a further disjoint set of $2k$ elements. Set $S = A \cup B \cup C$.

For $j = 1, \ldots, k$, form the pair of linear orders on S:

$$\begin{aligned} K_{2j-1} &: M_j < L_j < C, \\ K_{2j} &: (C - c_j) < M_j < c_j < L_j. \end{aligned}$$

For $j = k + 1, \ldots, 2k$, form:

$$\begin{aligned} K_{2j-1} &: L_j < M_j < C, \\ K_{2j} &: (C - c_j) < L_j < c_j < M_j. \end{aligned}$$

Finally take three more linear orders:

$$\begin{aligned} K_{4k+1} &: B < L_{2k+1} < C, \\ K_{4k+2} &: C < A < M_{2k+1}, \\ K_{4k+3} &: C < A < B. \end{aligned}$$

In the above, if no order is specified among a set of elements, then it is immaterial.

Now consider any subset S' of S, with $S' = A' \cup B' \cup C'$, where $A' \subseteq A$, $B' \subseteq B$, $C' \subseteq C$. By the choice of the linear orders, there are places to cut the L_i and M_i to exhibit A' and B'.

For each $j = 1, \ldots, 2k$: (a) if $c_j \in C'$, cut: K_{2j-1} at the appropriate point of M_j, and K_{2j} at the appropriate point of L_j, (b) if $c_j \notin C'$, cut: K_{2j-1} at the appropriate point of L_j, and K_{2j} at the appropriate point of M_j. Cut K_{4k+1} at the appropriate point of L_{2k+1}, K_{4k+2} at the appropriate point of M_{2k+1}, and K_{4k+3} above all of C.

For an element a of A, each of $L_1, \ldots, L_{2k+1}$ is cut at the required point, so a appears in more than half of the lower segments in these linear orders if and only if $a \in A'$. In the other $2k+2$ orders, a appears in the lower segment exactly half the time. Similarly for the elements of B.

An element c_j is in the upper segment in each of $K_1, K_3, \ldots, K_{4k-1}, K_{4k+1}$, and in the lower segment in each of the other K_i except possibly K_{2j}, where c_j is in the lower segment if and only if $c_j \in C'$. Hence c_j is in more than half of the lower segments if and only if it is in C'. Therefore this choice of cuts exhibits S', as required.

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