Domination and Total Domination Contraction Numbers of Graphs *

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Abstract

In this paper we consider the effect of edge contraction on the domination number and total domination number of a graph. We define the (total) domination contraction number of a graph as the minimum number of edges that must be contracted in order to decrease the (total) domination number. We show both of this two numbers are at most three for any graph. In view of this result, we classify graphs by their (total) domination contraction numbers and characterize these classes of graphs.

Keywords: Domination, total domination, domination contraction number.

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1 Introduction

In the domination area, edge addition and deletion have attracted much attention. Much research has been done on the changes of domination-type parameters resulting from edge addition and deletion. Besides addition and deletion the most fundamental operations on the edges of a graph are subdivision and contraction. However, it appears that subdivision and contraction have not received much attention in the domination area until recently. Some people investigated the effect of edge subdivision on the domination number and total domination number, two most fundamental ones among the domination-type parameters that have been studied.

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The domination subdivision number $sd_\gamma(G)$ of a graph is the minimum number of edges that must be subdivided (where an edge can be subdivided at most once) in order to increase the domination number. Arumugam and Paulraj Joseph [1] first defined this number, showed that $sd_\gamma(T) \leqslant 3$ for any tree T on at least three vertices, and conjectured that this upper bound holds for every graph with at least three vertices. However, in 2001, Haynes et al. [5] gave a counterexample to the above conjecture by showing that $sd_\gamma(G) = 4$ for the Cartesian product $G = K_t \times K_t$ where $t \geqslant 4$. Later Swaminathan and Sumathi [9] constructed a graph G with $sd_\gamma(G) = 5$. General bounds for domination subdivision number can be found in [2, 3, 4]. A parallel conception is the total domination subdivision number $sd_{\gamma_t}(G)$. In 2003, Haynes et al. [6] introduced this number and established some upper bounds in terms of vertex degree. They also presented several sufficient conditions which imply that $sd_{\gamma_t}(G) \leqslant 3$, and showed that $sd_{\gamma_t}(T) \leqslant 3$ for any tree T. A constructive characterization of trees T with $sd_{\gamma_t}(T) = 3$ was given in [8]. However, for general graphs, this number can be arbitrarily large (see [7]).

Motivated by recent researches focusing on subdivision, we consider the edge contraction and introduce similar conceptions, namely the domination and total domination contraction numbers. For a graph G with (total) domination number at least (three) two, we define the *(total) domination contraction number* $(ct_{\gamma_t}(G))$ $ct_\gamma(G)$ as the minimum number of edges which must be contracted in order to decrease the (total) domination number. If the (total) domination number is (two) one, then we define $(ct_{\gamma_t}(G))ct_\gamma(G) = 0$ for convenience.

The relevance of (total) domination contraction number can be illustrated as follows. In a facility location problem, one target is to find in the corresponding graph G a minimum (total) dominating set where to locate valuable facilities. If we want to reduce the cost on these facilities, we must decrease the (total) domination number. To this aim, we need to make some changes on edges of G, that is, to add or contract some edges. The former increases the number of edges and hence requires extra cost, while the latter does not (in fact, it decreases the number of edges, by Proposition2.1). Therefore edge contraction may be a better choice than edge addition. Next, one wants to know at least how many edges must be contracted in order to decrease the (total) domination number. We answer this question by showing that $ct_\gamma(G) \leqslant 3$ and $ct_{\gamma_t}(G) \leqslant 3$ hold for every graph G. This result also provides an interesting comparison with that on $sd_\gamma(G)$ and $sd_{\gamma_t}(G)$. Furthermore, according to these two upper bounds of $ct_\gamma(G)$ and $ct_{\gamma_t}(G)$, we give classifications of graphs and characterize them, respectively.

Before we enter the next section, we would like to give some terminology and notation. By the definitions, contraction numbers of a graph are equal to the minimum value of the contraction numbers of all components of the graph. Therefore we only consider the connected graph $G = (V, E)$ with vertex-set $V = V(G)$ and edge-set $E = E(G)$. For a vertex $v \in V(G)$, let $N(v)$ and $N[v] = N(v) \cup \{v\}$ be the open and closed neighborhoods of v, respectively. We use $d(x, y)$ to denote the distance between two vertices x and y. Denote by G/e the graph obtained from G by contracting an edge e. Since edge contraction is a commutative operation, i.e., $(G/e)/f = (G/f)/e$ for any two edges e and f, we can use G/E' to denote the resulting graph of contracting all edges in $E' \subseteq E(G)$ one by one from G. For terminology and notation on graph theory not given here, the reader is referred to [10].

Given two vertices u and v in G, we say u *dominates* v if $v \in N[u]$. A subset $D \subseteq V(G)$ is called a *dominating set* if its vertices dominate all vertices of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets. We call a dominating set consisting of $\gamma(G)$ vertices a *γ-set* for short. A *total dominating set* T of G is a dominating set whose induced subgraph has no isolated vertex, namely, every vertex of G has a neighbor in T. The *total domination number* γ_t and *γ_t-set* of G are defined similarly to $\gamma(G)$ and γ-set. It is easy to see that γ_t is well-defined for any nontrivial connected graph.

The rest of the paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we show $ct_\gamma(G) \leqslant 3$, classify graphs according to $ct_\gamma(G)$, and then characterize these classes. In Section 4 we do the same thing for $ct_{\gamma_t}(G)$.

2 Preliminaries

In this section we give preparations for proving the main results in Section 3 and Section 4.

Proposition 2.1 *If $E' \subseteq E(G)$, then $|V(G/E')| = |V(G)| - |E'|$ and $|E(G/E')| \leqslant |E(G)| - |E'|$.*

Proof. Note that contracting an edge decreases the number of vertices by one and the number of edges by at least one. The results follows by induction on $|E'|$. ∎

Lemma 2.2 *Let G be a connected graph.*

1) If D is a γ-set of G and x, y are two vertices in D, then $ct_\gamma(G) \leqslant d(x, y)$.

2) If T is a γ_t-set of G and x, y, z are three vertices in D such that x, y are adjacent in G, then $ct_{\gamma_t}(G) \leqslant \min\{d(x, z), d(y, z)\}$.

Proof. 1) Let P be a shortest path between x and y and consider $G' = G/E(P)$. We show $\gamma(G') < \gamma(G)$, which implies that $ct_\gamma(G) \leqslant |E(P)| = d(x,y)$.

Suppose that D is a γ-set of G and v is the contracted vertex in G'. Let $D' = (D \setminus V(P)) \cup \{v\}$. We will show that D' is a dominating set of G'. To this aim, consider $u \in V(G')$.

If $u \in N[v]$, then u is dominated by the contracted vertex v in G'.

If $u \notin N[v]$, then u is also a vertex of G and so D contains a vertex w which dominates u in G. It is clear that $w \notin V(P)$(otherwise $u \in N[v]$). It follows that $w \in D'$, and w dominates u in G' by the definition of contraction.

Therefore D' is a dominating set of G' and $\gamma(G') \leqslant |D'| = |D| - 1 = \gamma(G) - 1$.

2) Suppose that T is a γ_t-set of G and $x, y, z \in D$ satisfying $xy \in E(G)$. Assume $d(x,z) \geqslant d(y,z)$, without loss of generality. Let P be a shortest path between y and z. Then P does not contain x (otherwise $d(x,z) < d(y,z)$). We will show that $\gamma_t(G/E(P)) < \gamma_t(G)$, which implies $ct_{\gamma_t}(G) \leqslant |E(P)| = d(y,z) = \min\{d(x,z), d(y,z)\}$.

Denote the contracted vertex in $G/E(P)$ by v and let $T' = (T \setminus V(P)) \cup \{v\}$. In order to prove T' is a total dominating set of G, we need only to consider each vertex $u \notin N[x] \cup N[v]$, since x and v are two adjacent vertex in T'. Note that u is also a vertex of G. Then there exists a vertex $w \in T$ such that $uw \in E(G)$. It is clear that $w \notin V(P)$ (otherwise $w \in N[v]$). Hence $w \in T'$ and $uw \in E(G/E(P))$ by the definition of contraction. Therefore T' is a total dominating set of $G/E(P)$), which implies that $\gamma_t(G) \leqslant |T'| = |T| - 1 = \gamma_t(G) - 1$. ∎

Now we investigate the relationship between the (total) dominating sets in the original graph and the (total) dominating sets in the contracted graph.

Lemma 2.3 (Contraction Lemma) *Let G be a connected graph.*

1) If D is a dominating set of G and E is a subset of $E(G[D])$, then G/E' contains a dominating set D' such that $|D'| = |D| - |E'|$.

2) If T is a total dominating set of G and E' is a subset of $E(G[T])$ such that $G[T]/E'$ contains no isolated vertex, then G/E' contains a total dominating set T' such that $|T'| = |T| - |E'|$.

Proof. 1) We will show that $D' = V(G[D]/E')$ is a dominating set of G/E'. Then $|D'| = |D| - |E'|$ by Proposition 2.1.

Let S be the set of all contracted vertices in G/E'. Since $E' \subseteq E(G[D])$, then $S \subseteq D'$. Hence we need only to consider such vertex v that $v \notin N[s]$ for any $s \in S$. In that case, v is also a vertex of G. Thus D contains a vertex $u \in D$ such that $u = v$ or $uv \in E(G)$. It is easy to observe that u is not incident with any edge of E'; otherwise $u \in S$, a contradiction to the assumption that $v \notin N[s]$ for any $s \in S$. Therefore u lies in D'. If $u = v$ then we are done. Otherwise u remains adjacent to v in G/E'. Finally we conclude that D' is a dominating set of G/E'.

2) Since T is a dominating set of G, then by 1), T' is a dominating set of G/E' and $|T'| = |T| - |E'|$. Moreover, T' is a total dominating set, since $G[T'] = G[T]/E'$ contains no isolated vertex. ∎

Lemma 2.4 (Expansion Lemma) *Let G be a connected graph.*

1) If E' is a subset of $E(G)$ and D' is a dominating set of $G' = G/E'$, then G has a dominating set D such that $G'[D']$ is a spanning subgraph of $G[D]/F$ where $F \subseteq E(G[D])$ and $|F| = |E'|$. As a consequence, $|D| = |D'| + |E'|$ and $|E(G[D])| \geqslant |E(G'[D'])| + |E'|$.

2) If E' is a subset of $E(G)$ and T' is a total dominating set of $G' = G/E'$, then G has a total dominating set T such that $G'[T']$ is a spanning subgraph of $G[T]/F$ where $F \subseteq E(G[T])$ and $|F| = |E'|$. As a consequence, $|T| = |T'| + |E'|$ and $|E(G[T])| \geqslant |E(G'[T'])| + |E'|$.

Proof. 1) By induction on $k = |E'|$. First consider $k = 1$. Suppose that $E' = \{xy\}$ and v is the contracted vertex in $G' = G/xy$. We distinguish two cases.

Case 1. $v \in D'$. Let $D = (D' \setminus \{v\}) \cup \{x, y\}$. Then $G[D]/xy = G'[D']$. In order to show that D is a dominating set of G, we need only to consider such vertex u that $u \notin N[x] \cup N[y]$. Since u is also a vertex of G/xy, then D' contains a vertex w which dominates u. It is easy to see $w \in D$ since $w \neq v$ (otherwise $u \in N[x] \cup N[y]$). If $u = w$ then we are done. Otherwise $uw \in E(G')$, which implies $uw \in E(G)$. Thus D is a dominating set of G.

Case 2. $v \notin D'$. Then D' contains a vertex z such that $vz \in E(G')$. Assume $xz \in E(G)$, without loss of generality. Let $D = D' \cup \{x\}$. Then $G'[D']$ is a spanning subgraph of $G[D]/xz$. To show that D is a dominating set of G, we need only to consider each vertex $u \notin \{x, y\}$ of G. Since u is also a vertex of G', then there exists a vertex $w \in D' \subseteq D$ such that $u = w$ or $uw \in E(G')$. If $u = w$ then we are done. Otherwise $uw \in E(G')$. Since $v \notin D'$, then $w \neq v$, which implies $uw \in E(G)$. Therefore D is a dominating set of G.

Now consider $k \geqslant 2$ and let D' be a dominating set of $G' = G/E'$. By the induction hypothesis, the result holds for $E'' = E' \setminus \{e\}$ where $e \in E'$. Note that $G/E' = (G/e)/E''$. Then $G'' = G/e$ has a dominating set D'' such that

$$G'[D'] \text{ is a spanning subgraph of } G''[D'']/F'',\ |F''| = |E''|. \tag{1}$$

Applying the result of $k = 1$ to $G'' = G/e$ we have that G has a dominating set D such that

$$G''[D''] \text{ is a spanning subgraph of } G[D]/F', \ |F'| = 1. \tag{2}$$

Combining (1) and (2) yields that G has a dominating set D such that $G'[D']$ is a spanning subgraph of $(G[D]/F')/F'' = G[D]/F$ where $F = F' \cup F''$. It is easy to see that $F'' \subseteq E(G''[D'']) \setminus E(G'[D'])$ and $F' \subseteq E(G[D]) \setminus E(G''[D''])$, which implies that $|F| = |F''|+|F'| = |E''|+1 = |E'|$. It follows from Proposition 2.1 that

$$\begin{aligned} |D| = &\ |V(G[D])| = |V(G[D]/F)| + |F| = |V(G'[D'])| + |E'| \\ = &\ |D'| + E'| \end{aligned}$$

and

$$|E(G[D])| \geqslant |E(G[D]/F)| + |F| \geqslant |E(G'[D'])| + |E'|.$$

2) Let T' be a total dominating set of $G' = G/E'$. Then T' is a dominating set. By 1), G has a dominating set T such that $G'[T']$ is a spanning subgraph of $G[T]/F$ where $F \subseteq E(G[T])$ and $|F| = |E'|$. If $G[T]$ has an isolated vertex, then this vertex remains isolated in $G[T]/F$. Hence $G'[T']$ contains an isolated vertex, since $G'[T']$ is a spanning subgraph of $G[T]/F$. That contracts the hypothesis that T' is a total dominating set of G'. ∎

Corollary 2.5 *Let G be a connected graph.*

1) If $ct_\gamma(G) = k \geqslant 1$, then there exists a set $E' \subseteq E(G)$ such that $|E'| = k$ and $\gamma(G/E') = \gamma(G) - 1$.

2) If $ct_{\gamma_t}(G) = k \geqslant 1$, then there exists a set $E' \subseteq E(G)$ such that $|E'| = k$ and $\gamma_t(G/E') = \gamma_t(G) - 1$.

Proof. 1) First we show that contracting an edge e may decrease $\gamma(G)$ by at most one. Suppose to the contrary that $\gamma(G/e) \leqslant \gamma(G) - 2$. It follows from the Expanding Lemma that G has a dominating set consisting of at most $\gamma(G) - 1$ vertices. That is a contradiction.

Now by the definition of $ct_\gamma(G)$, there exists a set $E' \subseteq E(G)$ such that $|E'| = ct_\gamma(G)$ and $\gamma(G/E') < \gamma(G)$. Let $e \in E'$ and $E'' = E' \setminus \{e\}$. Then $\gamma(G/E'') = \gamma(G)$; otherwise $ct_\gamma(G) \leqslant |E''| < |E'| = ct_\gamma(G)$, a contradiction. It follows that

$$\gamma(G/E') = \gamma((G/E'')/e) \geqslant \gamma(G/E'') - 1 = \gamma(G) - 1.$$

2) The proof is similar to that of 1). ∎

We conclude this section with a remark on the Contraction Lemma and Expansion Lemma. Both of them are essential to proving our main results. The proofs in Section 3 and Section 4 mainly use the results of cases $|E'| = 1, 2$ of these two lemmas. Nevertheless, we present general cases for integrality.

3 Domination Contraction Number

In this section we consider domination contraction number of a graph. Let us begin with some simple examples.

Proposition 3.1 *For a path P_n and a cycle C_n on n vertices, $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Proposition 3.2 *For a path P_n and a cycle C_n on $n \geqslant 4$ vertices, $ct_\gamma(P_n) = ct_\gamma(C_n) = i$, where $n = 3k + i$, $1 \leqslant i \leqslant 3$.*

Theorem 3.3 *For a connected graph G, $ct_\gamma(G) \leqslant 3$.*

Proof. Let D be a γ-set of G. If $|D| = 1$ then $ct_\gamma(G) = 0$ by the definition. Assume $|D| \geqslant 2$ below. Choose two vertices x, y in D such that $d(x, y)$ is as small as possible. We claim that $d(x, y) \leqslant 3$, which implies $ct_\gamma(G) \leqslant 3$ by Lemma 2.2.

Suppose to the contrary that $d(x, y) = k \geqslant 4$. Let $P = xv_1v_2 \ldots v_{k-1}y$ be a shortest path between x and y. Then neither x nor y can dominate v_2 since P is a shortest path. Thus there exists a vertex z in $D \setminus \{x, y\}$ such that $v_2 \in N[z]$. It follows that

$$d(x, z) \leqslant d(x, v_2) + d(v_2, z) \leqslant 2 + 1 = 3,$$

a contradiction to the choice of x, y. ∎

Next we determine when a graph has domination contraction number $0, 1, 2$ or 3.

Lemma 3.4 *For a connected graph G, $ct_\gamma(G) = 0$ if and only if G admits a star as its spanning tree.*

Proof. It is clear that $\gamma(G) = 1$ if and only if G has a vertex joined to all other vertices in G, i.e. G admits a star as its spanning tree. ∎

Lemma 3.5 *For a connected graph G, $ct_\gamma(G) = 1$ if and only if there exists a γ-set D which is not independent.*

Proof. If D is a γ-set which is not independent, then there exist two adjacent vertices x, y in D. By Lemma 2.2, $ct_\gamma(G) \leqslant d(x, y) = 1$. Clearly $ct_\gamma(G) \neq 0$, since $\gamma(G) = |D| \geqslant 2$. Hence $ct_\gamma(G) = 1$.

Conversely, assume $ct_\gamma(G) = 1$. Then there exists an edge xy such that $\gamma(G/xy) = \gamma(G) - 1$. Let D' be a γ-set of $G' = G/xy$. By the Expansion Lemma, G has a dominating set D such that $|D| = |D'| + 1$ and $|E(G[D])| \geqslant |E(G'[D'])| + 1 \geqslant 1$. Then D is a γ-set of G since $|D'| = \gamma(G') = \gamma(G) - 1$, and D is not independent. ∎

By Lemma 3.5, a connected graph G has $ct_\gamma(G) \neq 1$ if and only if every γ-set of G is independent. As a consequence, $\gamma(G) = i(G)$, where $i(G)$ is the *independent domination number* of G, defined as the minimum cardinality of all maximal independent sets in G. However, we can not conclude from $\gamma(G) = i(G)$ that $ct_\gamma(G) \neq 1$. (See P_4.)

We go forward to characterize graphs with contraction domination number 2. For convenience, we call a dominating set D of G with $|D| = \gamma(G)+1$ a $(\gamma+1)$-*set*.

Lemma 3.6 *Let G be a connected graph. Then $ct_\gamma(G) = 2$ if and only every γ-set of G is independent and there exists a $(\gamma+1)$-set D such that $G[D]$ contains at least two edges.*

Proof. Suppose that every γ-set of G is independent. Let D be a $(\gamma+1)$-set such that $G[D]$ contains at least two edges. Then $ct_\gamma(G) \geqslant 2$, By Lemma 3.4 and Lemma 3.5. Let $E' \subseteq E(G[D])$ with $|E'| = 2$. It follows from the Contraction Lemma that G/E' contains a dominating set D' such that $|D'| = |D| - |E'| = \gamma(G) + 1 - 2 = \gamma(G) - 1$. Thus $ct_\gamma(G) \leqslant 2$.

Conversely, assume $ct_\gamma(G) = 2$. Then every γ-set of G is independent, and there exists $E' \subseteq E(G)$ with $|E'| = 2$ such that $\gamma(G/E') = \gamma(G) - 1$. Let D' be a γ-set of $G' = G/E'$. By the Expansion Lemma, G has a dominating set D such that $|D| = |D'| + |E'| = \gamma(G') + 2 = \gamma(G) + 1$ and $|E(G[D])| \geqslant |E(G'[D'])| + |E'| \geqslant 2$. ∎

In view of Theorem 3.3, all graphs can be classified into four categories according to their domination contraction numbers. We can obtain characterizations of them from Lemma 3.4, 3.5 and 3.6. Denoted by $\mathscr{C}_\gamma^i$ the graphs with domination contraction number i for $i = 0, 1, 2, 3$. Also denote by $\mathscr{P}_\gamma^j$ the set of all connected graphs satisfying Property j below. If $\mathscr{A}$ is a family of connected graphs, then $\overline{\mathscr{A}}$ means the family of all connected graphs not in $\mathscr{A}$.

Property 1 G admits a star as its spanning tree.

Property 2 G has a γ-set which is not independent.

Property 3 G has a $(\gamma+1)$-set D such that $G[D]$ contains at least two edges.

Theorem 3.7 $\mathscr{C}_\gamma^0 = \mathscr{P}_\gamma^1$, $\mathscr{C}_\gamma^1 = \mathscr{P}_\gamma^2$, $\mathscr{C}_\gamma^2 = \overline{\mathscr{P}_\gamma^2} \cap \mathscr{P}_\gamma^3$, *and* $\mathscr{C}_\gamma^3 = \overline{\mathscr{P}_\gamma^1} \cap \overline{\mathscr{P}_\gamma^3}$.

Proof. $\mathscr{C}_\gamma^0 = \mathscr{P}_\gamma^1$, $\mathscr{C}_\gamma^1 = \mathscr{P}_\gamma^2$ and $\mathscr{C}_\gamma^2 = \overline{\mathscr{P}_\gamma^2} \cap \mathscr{P}_\gamma^3$ follow immediately from Lemma 3.4, 3.5 and 3.6. We need only to prove $\mathscr{C}_\gamma^3 = \overline{\mathscr{P}_\gamma^1} \cap \overline{\mathscr{P}_\gamma^3}$.

First we show $\mathscr{P}_\gamma^2 \subseteq \mathscr{P}_\gamma^3$. Suppose that $G \in \mathscr{P}_\gamma^2$, i.e., G has a γ-set D such that $G[D]$ contains an edge xy. Then $V(G) \setminus D \neq \emptyset$. (Otherwise

$D \setminus \{x\}$ is a dominating set smaller than D.) Hence we can choose a vertex z outside D. Clearly, z has a neighbor in D. Thus $D' = D \cup \{z\}$ is a $(\gamma+1)$-set and $|E(G[D'])| \geqslant |E(G[D])| + 1 \geqslant 2$. Therefore $G \in \mathscr{P}^3_\gamma$.

Now we can compute that

$$\begin{aligned}
\mathscr{C}_3 &= \overline{\mathscr{C}^0_\gamma \cup \mathscr{C}^1_\gamma \cup \mathscr{C}^2_\gamma} = \overline{\mathscr{P}^1_\gamma} \cap \overline{\mathscr{P}^2_\gamma} \cap \overline{\overline{\mathscr{P}^2_\gamma} \cap \mathscr{P}^3_\gamma} \\
&= \overline{\mathscr{P}^1_\gamma} \cap \overline{\mathscr{P}^2_\gamma} \cap (\mathscr{P}^2_\gamma \cup \overline{\mathscr{P}^3_\gamma}) \\
&= (\overline{\mathscr{P}^1_\gamma} \cap \overline{\mathscr{P}^2_\gamma} \cap \mathscr{P}^2_\gamma) \cup (\overline{\mathscr{P}^1_\gamma} \cap \overline{\mathscr{P}^2_\gamma} \cap \overline{\mathscr{P}^3_\gamma}) \\
&= \overline{\mathscr{P}^1_\gamma} \cap \overline{\mathscr{P}^3_\gamma}
\end{aligned}$$

since $\overline{\mathscr{P}^3_\gamma} \subseteq \overline{\mathscr{P}^2_\gamma}$. ∎

4 Total Domination Contraction Number

First we compute $ct_\gamma(G)$ for paths and cycles.

Proposition 4.1 [8] *For the path P_n and cycle C_n on $n \geqslant 3$ vertices,*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} n/2 & \text{if } n \equiv 0 \ (mod\ 4); \\ \lfloor n/2 \rfloor + 1 & \text{otherwise.} \end{cases}$$

Proposition 4.2 *For the path P_n and cycle C_n on $n \geqslant 5$ vertices,*

$$ct_{\gamma_t}(C_n) = ct_{\gamma_t}(P_n) = \begin{cases} 1 & \text{if } n \equiv 1, 2 \ (mod\ 4); \\ 2 & \text{if } n \equiv 3 \ (mod\ 4); \\ 3 & \text{if } n \equiv 0 \ (mod\ 4). \end{cases}$$

Next we show $ct_{\gamma_t}(G) \leqslant 3$ for any graph G.

Theorem 4.3 *For any nontrivial connected graph G, $ct_{\gamma_t}(G) \leqslant 3$.*

Proof. Let T be a γ_t-set of G. If $|T| = 2$ then $ct_{\gamma_t}(G) = 0$. Assume $|T| \geqslant 3$ below. We need only to show that there exist three vertices x, y, z in T such that $d(x, y) = 1$, $d(x, z) \geqslant d(y, z)$ and $d(y, z) \leqslant 3$. Then it follows from Lemma 2.2 that $ct_{\gamma_t}(G) \leqslant d(y, z) \leqslant 3$.

Since $G[T]$ contains no isolated vertex, we can choose a vertex x and its neighbor y such that $x, y \in T$. Since $|T| \geqslant 3$, then there exists a vertex z in $V(G) \setminus (N(x) \cup N(y))$. Assume $d(x, z) \geqslant d(y, z)$, without loss of generality. Let $P = yuv \dots z$ be a shortest path between y and z. Then x is not in P. (Otherwise $d(x, z) < d(y, z)$.) If $d(y, z) \leqslant 3$, then we are done. Assume $d(y, z) \geqslant 4$ below.

Clearly, $v \notin N(y)$. If $v \in N(x)$ then $d(x, z) < d(y, z)$, which contradicts the assumption. Thus $v \notin N(x)$. Then T must contain a vertex z' such that

$z \neq x$, $z \neq y$ and $v \in N(z')$. It follows that $d(y, z') \leqslant d(y, v) + d(v, z') = 2 + 1 = 3$. ∎

Finally we characterize graphs with total domination contraction numbers equal to $0, 1, 2, 3$. By the definition, $ct_{\gamma_t}(G) = 0$ if and only if $\gamma_t(G) = 2$. Such G can be characterized as a graph containing a spanning tree isomorphic to the *double-star*, which is the graph formed by joining the center vertices of two stars.

Lemma 4.4 *Let G be a connected graph. Then $ct_{\gamma_t}(G) = 0$ if and only if G admits a double-star as its spanning tree.*

Lemma 4.5 *Let G be a connected graph. Then $ct_{\gamma_t}(G) = 1$ if and only if there exists a γ_t-set T such that $G[T]$ contains a 3-path.*

Proof. If G has a γ_t-set T such that $G[T]$ contains a 3-path $P = xyz$, then $G[T]/xy$ contains no isolated vertex since z is adjacent to y. By the Contraction Lemma, G/xy has a γ_t-set T' such that $|T'| = |T| - 1$. It follows that $\gamma_t(G/xy) \leqslant |T'| < \gamma_t(G)$ and $ct_{\gamma_t}(G) \leqslant 1$. But $ct_{\gamma_t}(G) \geqslant 1$ since $\gamma_t(G) = |T| \geqslant 3$. Thus $ct_{\gamma_t}(G) = 1$.

Conversely, assume $ct_{\gamma_t}(G) = 1$. Then there exists an edge e of G such that $\gamma_t(G/e) = \gamma_t(G) - 1$. Let T' be a γ_t-set of $G' = G/e$. It follows from the Expansion Lemma that G has a total dominating set T such that $G'[T']$ is a spanning subgraph of $G[T]/f$ for some $f = uv \in E(G)$. Since $|T| = |T'| + 1 = \gamma_t(G)$, then T is a γ_t-set of G. Furthermore, the contracted vertex corresponding to f has a neighbor, say w, in $G'[T']$, since $G'[T']$ has no isolated vertex. Hence uvw is a 3-path in $G[T]$. ∎

The contrapositive of Lemma 4.5 says that, for a nontrivial connected graph G, $ct_{\gamma_t}(G) \neq 1$ if and only if $G[T]$ contains no 3-path for every γ_t-set T of G. In other words, $G[T]$ is an induced 1-factor, since $G[T]$ contains no isolated vertex. As a consequence, $\gamma_t(G) = \gamma_{pr}(G)$, where $\gamma_{pr}(G)$, called the *paired domination number*, is the minimum cardinality of all paired dominating sets of G. (A dominating set whose induced subgraph contains a perfect matching is called a *paired dominating set*. The paired dominating set of cardinality equal to $\gamma_{pr}(G)$ is called a γ_{pr}-set. Clearly, $\gamma_t(G) \leqslant \gamma_{pr}(G)$.) However, $\gamma_t(G) = \gamma_{pr}(G)$ does not generally yields $ct_{\gamma_t}(G) \neq 1$. (For example, see P_6.)

Next we consider when $ct_{\gamma_t}(G) = 2$. Call a total dominating set of cardinality $\gamma_t(G)+1$ a *(γ_t+1)-set*. Denote by $2H$ the disjoint union of two copies of a graph H.

Lemma 4.6 *Let G be a connected graph. Then $ct_{\gamma_t}(G) = 2$ if and only if every γ_t-set of G is a γ_{pr}-set and there exists a $(\gamma_t + 1)$-set T such that $G[T]$ contains a subgraph isomorphic to $P_4, K_{1,3}$ or $2P_3$.*

Proof. If every γ_t-set is a γ_{pr}-set, then $ct_{\gamma_t}(G) \neq 1$ by Lemma 4.5. Suppose that G has a (γ_t+1)-set T such that $G[T]$ contains a subgraph isomorphic to $P_4, K_{1,3}$ or $2P_3$. Then $|T| \geqslant 4$, which implies $ct_{\gamma_t}(G) \neq 0$. We will show $ct_{\gamma_t}(G) \leqslant 2$. Then $ct_{\gamma_t}(G) = 2$ follows.

It is easy to choose two edges in $P_4, K_{1,3}$ or $2P_3$ such that no isolated vertex appears after contracting these two edges. Let E' be the set of these two edges. Then $G[T]/E'$ contains no isolated vertex. By the Contraction Lemma, G/E' has a total dominating set T' such that $|T'| = |T| - |E'| = \gamma_t(G) + 1 - 2 = \gamma_t(G) - 1$. Hence $\gamma_t(G/E') < \gamma_t(G)$ and so $ct_{\gamma_t}(G) \leqslant 2$.

Conversely, Assume $ct_{\gamma_t}(G) = 2$. It follows from Lemma 4.5 that every γ_t-set is a γ_{pr}-set. By Corollary 2.5, there exists a set $E' \subseteq E(G)$ such that $|E'| = 2$ and $\gamma_t(G/E') = \gamma_t(G) - 1$. Let T' be a γ_t-set of $G' = G/E'$. The Expansion Lemma yields that G has a total dominating set T such that $G'[T']$ is a spanning subgraph of $G[T]/F$ where $F = \{e, f\} \subseteq E(G)$. Then $|T| = |T'| + |E'| = \gamma_t(G) - 1 + 2 = \gamma_t(G) + 1$. To show $G[T]$ contains $P_4, K_{1,3}$ or $2P_3$, we distinguish two cases.

Case 1. $e = xy, f = yz$. The contracted vertex v in G' must has a neighbor u in $G'[T']$, since T' is a total dominating set. Then u is adjacent to one of x, y, z in $G[T]$. If u is adjacent to x or z, then $uxyz$ or $uzyz$ is a 4-path. If u is adjacent to y, then $G[\{u, x, y, z\}]$ contains a $K_{1,3}$.

Case 2. $e = ab$ and $f = cd$ are not incident. Denote the neighbors of the contracted vertices corresponding to e, f by u, v respectively. Similarly to Case 1, we have that, if $u = v$ then $baucd$ is a 5-path, and if $u \neq v$ then uab and vcd are two disjoint 3-paths. ∎

In view of Theorem 4.3, we can also classify graphs into four categories according to their total domination contraction numbers. Characterizations of these categories follows from Lemma 4.4, 4.5 and 4.6. Denoted by $\mathscr{C}^i_{\gamma_t}$ the graphs with total domination contraction number i for $i = 0, 1, 2, 3$. Also denote by $\mathscr{P}^j_{\gamma_t}$ the set of all connected graphs satisfying the j-th one of the following properties. We would like to remark that $\overline{\mathscr{A}}$ means the family of all *nontrivial* connected graphs not in $\mathscr{A}$ whenever we consider $ct_{\gamma_t}(G)$.

Property 1' G admits a double-star as its spanning tree.

Property 2' G has a γ_t-set T such that $G[T]$ contains a 3-path.

Property 3' G has a $(\gamma_t + 1)$-set T such that $G[T]$ contains $P_4, K_{1,3}$ or $2P_3$.

Theorem 4.7 *$\mathscr{C}^0_{\gamma_t} = \mathscr{P}^1_{\gamma_t}$, $\mathscr{C}^1_{\gamma_t} = \mathscr{P}^2_{\gamma_t}$, $\mathscr{C}^2_{\gamma_t} = \overline{\mathscr{P}^2_{\gamma_t}} \cap \mathscr{P}^3_{\gamma_t}$, and $\mathscr{C}^3_{\gamma_t} = \overline{\mathscr{P}^1_{\gamma_t}} \cap \overline{\mathscr{P}^3_{\gamma_t}}$.*

Proof. $\mathscr{C}^0_{\gamma_t} = \mathscr{P}^1_{\gamma_t}$, $\mathscr{C}^1_{\gamma_t} = \mathscr{P}^2_{\gamma_t}$ and $\mathscr{C}^2_{\gamma_t} = \overline{\mathscr{P}^2_{\gamma_t}} \cap \mathscr{P}^3_{\gamma_t}$ follow immediately from Lemma 3.4, 3.5 and 3.6. In order to show $\mathscr{C}^3_{\gamma_t} = \overline{\mathscr{P}^1_{\gamma_t}} \cap \overline{\mathscr{P}^3_{\gamma_t}}$, we first prove $\mathscr{P}^2_{\gamma_t} \subseteq \mathscr{P}^3_{\gamma_t}$. Suppose that $G \in \mathscr{P}^2_{\gamma_t}$, i.e., G has a γ_t-set T such that $G[T]$ contains a 3-path xyz. Then $V(G) \setminus T \neq \emptyset$. (Otherwise $T \setminus \{x\}$ is a dominating set smaller than T.) Hence we can choose a vertex u outside T. Clearly, u has a neighbor, say v in T. Let $T' = T \cup \{u\}$. Then T' is a $(\gamma_t + 1)-$set.

If $v \in \{x, y, z\}$, then $G[\{x, y, z, u\}]$ contains either a P_4 or a $K_{1,3}$.

If $v \notin \{x, y, z\}$, then v has a neighbor w in $G[T]$. If $w \in \{x, y, z\}$ then $G[\{x, y, z, v\}]$ contains either a P_4 or a $K_{1,3}$. Otherwise, xyz and uvw are two disjoint 3-paths.

Therefore $G[T']$ contains one of $P_4, K_{1,3}, 2P_3$ as its subgraph. That proves $\mathscr{P}^2_{\gamma_t} \subseteq \mathscr{P}^3_{\gamma_t}$. Then $\mathscr{C}^3_{\gamma_t} = \overline{\mathscr{P}^1_{\gamma_t}} \cap \overline{\mathscr{P}^3_{\gamma_t}}$ follows from a computation similar to what has appeared in the proof of Theorem 3.7. ∎

5 Conclusion

The general bounds on the domination and total domination contraction numbers present an interesting comparison with those on subdivision numbers. According to these bounds, graphs can be explicitly classified and characterized, as what we have done in Section 3 and Section 4.

It still remains for us to focus on particular classes of graphs, such as trees, and give constructive characterizations depending on their contraction numbers. It is also attractive to consider operations of graphs, such as graph products. On the other hand, the relationship between contraction numbers and subdivision numbers is worthy of further research.

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