

Domination in the Corona and Join of Graphs

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Abstract

In this paper we characterized the dominating sets, total dominating sets, and secure total dominating sets in the corona of two graphs. The secure total dominating sets in the join of two graphs were also investigated. As direct consequences, the domination, total domination, and secure total domination numbers of these graphs were obtained.

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1 Introduction

Several results about domination number $\gamma(G)$ and total domination number $\gamma_t(G)$ are upper bounds in terms of the *order* n , *minimum degree* δ , *maximum*

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degree Δ , diameter $\text{diam}(G)$, and girth $g(G)$ of the graph G . Some of these results on domination number were obtained by Berge in [3], DelaViña in [7], Flach and Volkmann in [8], Löwenstein and Rautenbach in [13], McCuaig and Shepherd in [14] and Volkmann in [15]. Also, some of the results on total domination number were obtained by Atapour and Soltankhah in [1], Brigham, et. al. in [4], Cockayne, et. al. in [5], Lam and Wei in [12], Haynes, Henning and Yeo in [9], [10], and [11].

There are also other types of domination in graphs which are being studied such as secure domination and independent domination. One of the most recent types of dominating set in a graph is the so-called *secure total dominating set*. This concept was introduced by Benecke et. al. [2] in 2007.

In this paper, we characterized the dominating, total dominating, and secure total dominating sets in the corona of two connected graphs. The secure total dominating sets in the join of two graphs were also investigated. As quick consequences, we determined the domination, total domination, and secure total domination numbers of these graphs. To facilitate the results obtained, we need the following definitions.

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $X \subseteq V(G)$, then the *open neighborhood* of X is the set $N_G(X) = N(X) = \cup_{v \in X} N_G(v)$. The *closed neighborhood* of X is $N_G[X] = N[X] = X \cup N(X)$.

A subset X of $V(G)$ is a *dominating set* of G if for every $v \in (V(G) \setminus X)$, there exists $x \in X$ such that $xv \in E(G)$, i.e., $N[X] = V(G)$. It is a *total dominating set* if $N(X) = V(G)$. A total dominating set X is a *secure total dominating set* if for every $u \in V(G) \setminus X$, there exists $v \in X$ such that $uv \in E(G)$ and $[X \setminus \{v\}] \cup \{u\}$ is a total dominating set. The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$, or *secure total domination number* $\gamma_{st}(G)$) of G is the cardinality of a minimum dominating (resp., total dominating or secure total dominating) set of G .

The *join* $G + H$ of two graphs G and H is the graph with vertex set

$$V(G + H) = V(G) \cup V(H)$$

and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H , and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H . For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v . Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$.

2 Domination in the Corona of Graphs

Theorem 2.1 *Let G be a connected graph of order m and let H be any graph of order n . Then $C \subseteq V(G \circ H)$ is a dominating set in $G \circ H$ if and only if $V(v + H^v) \cap C$ is a dominating set of $v + H^v$ for every $v \in V(G)$.*

Proof. Let C be a dominating set in $G \circ H$ and let $v \in V(G)$. If $v \in C$, then $\{v\}$ is a dominating set of $v + H^v$. It follows that $V(v + H^v) \cap C$ is a dominating set of $v + H^v$. Suppose that $v \notin C$ and let $x \in V(v + H^v) \setminus C$ with $x \neq v$. Since C is a dominating set of $G \circ H$, there exists $y \in C$ such that $xy \in E(G \circ H)$. Then $y \in V(H^v) \cap C$ and $xy \in E(v + H^v)$. This proves that $V(v + H^v) \cap C$ is a dominating set of $v + H^v$.

For the converse, suppose that $V(v + H^v) \cap C$ is a dominating set of $v + H^v$ for every $v \in V(G)$. Then, clearly, C is a dominating set of $G \circ H$. ■

Corollary 2.2 *Let G be a connected graph of order m and let H be any graph of order n . Then $\gamma(G \circ H) = m$.*

Proof. Let $C = V(G)$. Then $V(v + H^v) \cap C = \{v\}$ is a dominating set of $v + H^v$ for every $v \in V(G)$. By Theorem 2.1, C is a dominating set of $G \circ H$; hence, $\gamma(G \circ H) \leq |C| = m$.

Next, let C^* be a minimum dominating set of $G \circ H$. Then, by Theorem 2.1, $V(v + H^v) \cap C^*$ is a dominating set of $v + H^v$ for every $v \in V(G)$. It follows that $\gamma(G \circ H) = |C^*| \geq m$. Therefore, $\gamma(G \circ H) = m$. ■

3 Total Domination in the Corona of Graphs

Theorem 3.1 *Let G be a connected graph of order m and let H be any graph of order n . Then $C \subseteq V(G \circ H)$ is a total dominating set in $G \circ H$ if and only if for every $v \in V(G)$, either*

- (i) $V(v + H^v) \cap C$ is a total dominating set of $v + H^v$ or
- (ii) $v \in C$ and $N_G(v) \cap C \neq \emptyset$.

Proof. Let C be a total dominating set in $G \circ H$ and let $v \in V(G)$. If $V(v + H^v) \cap C$ is a total dominating set of $v + H^v$, then we are done. So, suppose that $V(v + H^v) \cap C$ is not a total dominating set of $v + H^v$. Suppose further that $v \notin C$. Since C is a dominating set of $G \circ H$, $V(H^v) \cap C$ must be a dominating set of $v + H^v$. Now, since $V(v + H^v) \cap C = V(H^v) \cap C$ is not a total dominating set of $v + H^v$, there exists $u \in V(H^v) \setminus C$ such that $N_{G \circ H}(u) \cap C = \emptyset$. This contradicts the fact that C is a total dominating set of $G \circ H$. Thus, $v \in C$. By assumption, $V(v + H^v) \cap C = \{v\}$ (otherwise,

the set is total dominating set). Since C is a total dominating set of $G \circ H$, it follows that $N_G(v) \cap C \neq \emptyset$.

For the converse, suppose that the condition holds for C . Let $x \in V(G \circ H)$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Consider the following cases:

Case 1. $x = v$

If $x \in C$, then there exists $u \in V(G) \cap (C \setminus \{x\})$ such that $xu \in E(G \circ H)$ by (ii). If $x \notin C$, then $V(H^v) \cap C$ is a total dominating set of $v + H^v$ by (i). Hence, there exists $y \in V(H^v) \cap C$ such that $xy \in E(G \circ H)$.

Case 2. $x \neq v$

If $v \in C$, then $xv \in E(G \circ H)$. If $v \notin C$, then there exists $w \in V(H^v) \cap C$ such that $xw \in E(G \circ H)$ by (i).

In both cases, we have $N_{G \circ H}(x) \cap C \neq \emptyset$. Therefore, C is a total dominating set of $G \circ H$. ■

Corollary 3.2 *Let G be a connected graph of order m and let H be any graph of order n . Then $\gamma_t(G \circ H) = m$.*

Proof. Let $C = V(G)$. Then C is a total dominating set of $G \circ H$ by Theorem 3.1. Thus, $\gamma_t(G \circ H) \leq |C| = m$.

Next, let C^* be a minimum total dominating set of $G \circ H$. Then, by Theorem 3.1, $|V(v + H^v) \cap C^*| \geq 1$ for every $v \in V(G)$. It follows that $\gamma_t(G \circ H) = |C^*| \geq m$. Therefore, $\gamma_t(G \circ H) = m$. ■

4 Secure Total Domination in the Corona of Graphs

Lemma 4.1 *Let G be a connected graph and let S be a secure total dominating set of G . Then the set $S \setminus \{v\}$ is a dominating set of G for every $v \in S$. In particular, $1 + \gamma(G) \leq \gamma_{st}(G)$.*

Proof. Let $v \in S$ and let $S^* = S \setminus \{v\}$. Suppose S^* is not a dominating set of G . Then there exists $z \in V(G) \setminus S^*$ such that $zw \notin E(G)$ for all $w \in S^*$. Then $z \neq v$ and v is the only element of S with $zv \in E(G)$. However, the set $(C \setminus \{v\}) \cup \{z\}$ cannot be a total dominating set because $zw \notin E(G)$ for all $w \in S^*$. This contradicts the fact that S is a secure total dominating set of G . Therefore, $S \setminus \{v\}$ is a dominating set of G . Moreover, if S is a minimum secure total dominating set of G , then the result implies that $\gamma(G) \leq \gamma_{st}(G) - 1$. ■

Theorem 4.2 *Let G be a connected graph of order m and let H be any graph of order n . Then $C \subseteq V(G \circ H)$ is a secure total dominating set of $G \circ H$ if and only if for every $v \in V(G)$, either*

- (i) $V(H^v) \cap C$ is a secure total dominating set of H^v or
- (ii) $v \in C$ and $V(H^v) \cap C$ is a dominating set of H^v .

Proof. Let C be a secure total dominating set $G \circ H$ and let $v \in V(G)$. If $V(H^v) \cap C$ is a secure total dominating set of H^v , then we are done. So suppose that $V(H^v) \cap C$ is not a secure total dominating set of H^v . Suppose further that $v \notin C$. Since C is a total dominating set of $G \circ H$, $V(H^v) \cap C$ must be a total dominating set of H^v . By assumption, there exists $x \in V(H^v) \setminus C$ such that $[(V(H^v) \cap C) \setminus \{y\}] \cup \{x\}$ is not a total dominating set for every $y \in V(H^v) \cap C$ with $xy \in E(H^v)$. This implies that $(C \setminus \{y\}) \cup \{x\}$ is not a total dominating set of $G \circ H$ for every $y \in C$ with $xy \in E(G \circ H)$, contrary to our assumption of the set C . Therefore, $v \in C$. If $V(H^v) \cap C = \emptyset$ and $w \in V(H^v)$, then $(C \setminus \{v\}) \cup \{w\}$ is not a total dominating set of $G \circ H$, contrary to our assumption. Thus $V(H^v) \cap C \neq \emptyset$. Using a similar argument, it can be shown that $V(H^v) \cap C$ is a dominating set of H^v .

Suppose the condition holds for C . Then C is clearly a total dominating set of $G \circ H$. Let $x \in V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Consider the following cases:

Case 1. $x = v$

By assumption $V(H^v) \cap C$ is a secure total dominating of H^v . Pick $y \in V(H^v) \cap C$. Then $(C \setminus \{y\}) \cup \{x\}$ is a total dominating set of $G \circ H$.

Case 2. $x \neq v$

Then $x \in V(H^v)$. If $v \notin C$, then $V(H^v) \cap C$ is a secure total dominating of H^v , and so there exists $u \in V(H^v) \cap C$ such that $[(V(H^v) \cap C) \setminus \{u\}] \cup \{x\}$ is a total dominating set of H^v . It follows that $(C \setminus \{u\}) \cup \{x\}$ is a total dominating set of $G \circ H$. If $v \in C$, then $V(H^v) \cap C$ is a dominating set of H^v . Pick $w \in V(H^v) \cap C$ such that $wx \in E(H^v)$. Then $(C \setminus \{w\}) \cup \{x\}$ is a total dominating set of $G \circ H$.

Therefore, C is a secure total dominating set of $G \circ H$. ■

Corollary 4.3 *Let G be a connected graph of order m and let H be any graph of order n . Then $\gamma_{st}(G \circ H) = m(1 + \gamma(H))$.*

Proof. For each $v \in V(G)$, let S_v be a minimum dominating set of H^v and set $C = \cup_{v \in V(G)} (S_v \cup \{v\})$. Then C is a secure total dominating set of $G \circ H$ by Theorem 4.2. Thus, $\gamma_{st}(G \circ H) \leq |C| = m(1 + \gamma(H))$.

Next, let C^* be a minimum secure total dominating set of $G \circ H$. Then, by Theorem 4.2, $|V(H^v) \cap C^*| \geq \gamma_{st}(H)$ or $|(V(H^v) \cup \{v\}) \cap C^*| \geq \gamma(H) + 1$ for every $v \in V(G)$. With Lemma 4.1, it follows that $\gamma_{st}(G \circ H) = |C^*| \geq m(1 + \gamma(H))$. Therefore, $\gamma_{st}(G \circ H) = |C^*| = m(1 + \gamma(H))$. ■

5 Secure Total Domination in the Join of Graphs

It is worth noting that if G is a connected non-trivial graph, then $\gamma_{st}(G) \geq 2$. Our next result characterizes all connected graphs G with $\gamma_{st}(G) = 2$.

Theorem 5.1 *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{st}(G) = 2$ if and only if there exist $x, y \in V(G)$ such that $xy \in E(G)$ and $N(x) = V(G) \setminus \{x\}$ and $N(y) = V(G) \setminus \{y\}$.*

Proof. Suppose $\gamma_{st}(G) = 2$ and let $S = \{x, y\}$ be a secure total dominating set. Since S is a total dominating set, $xy \in E(G)$. Let $z \in V(G) \setminus S$ and suppose $xz \notin E(G)$. Since S is a dominating set, $zy \in E(G)$. Hence, $(S \setminus \{y\}) \cup \{z\} = \{x, z\}$ is a total dominating set by assumption. This is not possible because $xz \notin E(G)$. Therefore, $N(x) = V(G) \setminus \{x\}$. Similarly, $N(y) = V(G) \setminus \{y\}$.

For the converse, suppose there exist $x, y \in V(G)$ such that $xy \in E(G)$ and $N(x) = V(G) \setminus \{x\}$ and $N(y) = V(G) \setminus \{y\}$. Then $X = \{x, y\}$ is a total dominating set. Moreover, if $z \in V(G) \setminus X$, then $xz \in E(G)$ and $(X \setminus \{x\}) \cup \{z\} = \{y, z\}$ is a total dominating set. Therefore, X is a secure total dominating set in G . Accordingly, $\gamma_{st}(G) = 2$. ■

Corollary 5.2 *Let G be a graph and K_n the complete graph of order $n \geq 2$. Then $\gamma_{st}(G + K_n) = 2$.*

Before we give our next result, we state the following

Remark 5.3 *Let G and H be non-complete graphs. Then*

$$2 \leq \gamma_{st}(G + H) \leq 4.$$

To see this, let $a, b \in V(G)$ and $x, y \in V(H)$. Then, clearly, $X = \{a, b, x, y\}$ is a total dominating set in $G + H$. Thus, $2 \leq \gamma_{st}(G) \leq 4$.

Theorem 5.4 *Let G and H be connected non-complete graphs of orders m and n , respectively. Then $\gamma_{st}(G + H) = 2$ if and only if at least one of the following holds:*

- (i) $\gamma_{st}(G) = 2$
- (ii) $\gamma_{st}(H) = 2$
- (iii) $\Delta(G) = m - 1$ and $\Delta(H) = n - 1$.

Proof. Suppose $\gamma_{st}(G+H) = 2$. By Theorem 5.1, there exist $x, y \in V(G+H)$ such that $xy \in E(G+H)$ and $N_{G+H}(x) = V(G+H) \setminus \{x\}$ and $N_{G+H}(y) = V(G+H) \setminus \{y\}$. In particular, $S = \{x, y\}$ is a secure total dominating set in $G+H$. Consider the following cases:

Case 1. Suppose $x, y \in V(G)$.

Then $xy \in E(G)$, $N_G(x) = V(G) \setminus \{x\}$ and $N_G(y) = V(G) \setminus \{y\}$. It follows from Theorem 5.1 that $\gamma_{st}(G) = 2$.

Case 2. Suppose $x, y \in V(H)$.

Then, as in Case 1, $\gamma_{st}(H) = 2$.

Case 3. Suppose $x \in V(G)$ and $y \in V(H)$.

Suppose $\Delta(G) \neq m - 1$. Then there exists $z \in V(G)$ such that $xz \notin E(G)$. It follows that $(S \setminus \{y\}) \cup \{z\} = \{x, z\}$ is not a total dominating set in $G+H$. This contradicts the fact that S is a secure total dominating set in $G+H$. Hence, $\Delta(G) = m - 1$. Similarly, $\Delta(H) = n - 1$.

The converse is obvious. ■

Theorem 5.5 *Let G and H be connected non-complete graphs of orders m and n , respectively, and suppose $\gamma_{st}(G+H) \neq 2$. Then $\gamma_{st}(G+H) = 3$ if and only if at least one of the following holds:*

- (i) $\gamma(G) = 2$
- (ii) $\gamma(H) = 2$
- (iii) $\Delta(G) = m - 1$ or $\Delta(H) = n - 1$.

Proof. Suppose $\gamma_{st}(G+H) = 3$, say $S = \{x, y, z\}$ is an stds in $G+H$. Suppose that condition (iii) is not true, i.e., $\Delta(G) \neq m - 1$ and $\Delta(H) \neq n - 1$. Consider the following cases:

Case 1. Suppose $S \subseteq V(G)$ or $S \subseteq V(H)$.

Assume that $S \subseteq V(G)$ and let $X = \{x, y\}$. Since S is a tds in G , $z \in N_G(X)$. Let $w \in V(G) \setminus X$. Suppose that $w \notin N_G(X)$. Then $(S \setminus \{z\}) \cup \{w\} = \{x, y, w\}$ cannot be a tds in $G+H$, contrary to our assumption about S . Therefore, $N_G(X) = V(G)$, i.e., X is a dominating set in G . Since $\Delta(G) \neq m - 1$, it follows that $\gamma(G) = 2$. Similarly, $\gamma(H) = 2$ whenever $S \subseteq V(H)$. Therefore, (i) or (ii) holds.

Case 2. Suppose $|V(G) \cap S| = 2$ or $|V(H) \cap S| = 2$.

Assume that $|V(G) \cap S| = 2$, say $x, y \in V(G)$. Then $z \in V(H)$. Let $X = \{x, y\}$. Let $a \in V(G) \setminus X$. If $a \notin N_G(X)$, then $(S \setminus \{z\}) \cup \{a\} = \{x, y, a\}$ cannot be a tds in $G+H$. This is contrary to our assumption about S . Therefore, $a \in N_G(X)$, showing that X is a dominating set in G . Since $\Delta(G) \neq m - 1$, it follows that $\gamma(G) = 2$. Using a similar argument, it can be shown that $\gamma(H) = 2$ whenever $|V(H) \cap S| = 2$.

Therefore, (i) or (ii) holds.

For the converse, assume first that condition (i) holds, say $\Delta(G) = m - 1$ (of course, $\Delta(H) \neq n - 1$ by assumption). Let $v \in V(G)$ with $\deg_G(v) = m - 1$, and pick $u \in V(G) \setminus \{v\}$ and $w \in V(H)$. Then $S = \{u, v, w\}$ is an stds in $G + H$. Since $\gamma_{st}(G + H) \neq 2$, it follows that $\gamma_{st}(G + H) = 3$. We obtain the same conclusion if we assume that $\Delta(H) = n - 1$.

Next, suppose that $\gamma(G) = 2$, say $S = \{a, b\}$ is a dominating set of G . Pick $c \in V(H)$ and set $X = \{a, b, c\}$. Then X is an stds in $G + H$. Thus, $\gamma_{st}(G + H) = 3$. A similar result is obtained if $\gamma(H) = 2$. ■

References

- [1] M. Atapour and N. Soltankhah, On Total Dominating Sets in Graphs. *Int. J. Contemp. Math Sciences*, Vol. 4, 2009, no. 6, 253-257.
- [2] S. Benecke, E.J. Cockayne, C.M. Mynhardt, Secure total domination in graphs, *Util. Math.*, 74(2007), 247-259.
- [3] C. Berge, *Graphs and Hypergraphs*, North-Holland, London (1973).
- [4] R.C. Brigham, J.R. Carrington, and R.P. Vitray, Connected graphs with maximum total domination number. *J. Combin. Comput. Math.* 34 (2000), 81-95.
- [5] E.J. Cockayne, R.M. Dawes, and S.T. Hedetnieme, Total domination in graphs. *Networks* 10 (1980), 211-219.
- [6] E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh, J. Munganga and J.H. van Vuuren, Protection of a graph, *Utilitas Math.*, to appear.
- [7] E. DelaViña, R. Pepper, and B. Waller, A Note on Dominating Sets and Average Distance. (1991) *Mathematics Subject Classification*, 05C35.
- [8] P. Flach and L. Volkmann, Estimations for the domination number of a graph. *Discrete Math.* 80 (1990), 145-151.
- [9] T.W. Haynes and M.A. Henning, Upper bounds on the Total Domination Number. *AMS subject classification*:05C69.
- [10] M.A. Henning, Graphs with large total domination number. *J. Graph Theory* 35 (1) (2000), 21-45.
- [11] M.A. Henning and A. Yeo, A new upper bound on the total domination number of a graph. *The Electronic Journal of Combinatorics* 14 (2007), #R65.

- [12] P.C.B. Lam and B. Wei, On the total domination number of graphs. *Utilitas Math.* 72 (2007), 223-240.
- [13] C. Löwenstein and D. Rautenbach, Domination in Graphs of Minimum Degree at least Two and large Girth. *Discrete Math.*, to appear.
- [14] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two, *J. Graph Theory* 13(1989), 749-762.
- [15] L. Volkmann, Upper bounds on the domination number of a graph in terms of diameter and girth. *J. Combin. Comput.. Combin. Math.* 52 (2005), 131-141.

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