Domination numbers of cardinal products $P_6 \times P_n$

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Abstract. *Here we determine the domination numbers of the cardinal product of path graphs $P_6 \times P_n$. For $P_7 \times P_n$ and $P_8 \times P_n$ we give some bounds.*

Key words: *graph, dominating set, cardinal product*

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1. Terminology and introduction

For a graph G a subset D of the vertex-set of G is called a dominating set if every vertex x not in D, is adjacent to at least one vertex of D. The domination number $\gamma(G)$ is the cardinality of the smallest dominating set.

For any graph G we denote by V(G) and E(G) the vertex-set and the edge-set of G, respectively. The cardinal product $G \times H$ of two graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$ and $\{(g_1, h_1), (g_2, h_2)\} \in E(G \times H)$ if and only if $\{g_1, g_2\} \in E(G)$ and $\{h_1, h_2\} \in E(H)$.

(This product is also known as the Kronecker product, cross product, direct product or tensor product.)

The problem of determining the domination numbers of graphs first occurs in the paper of de Jaenisch [3]. He wanted to find the minimal number of queens on a chessboard, such that every square is either occupied by a queen or can be reached by a queen with a single move.

A variety of applications of domination theory can be discussed: the problem of keeping all points in a network under surveillance by a set of radar stations [1], or application of domination to communications in a network, where a dominating set represents a set of cities, which acting as transmitting stations can transmit messages to every city in the network [16]. Some other applications are listed in [2] and [11].

Starting in the eighties domination numbers of cartesian products were intensively investigated (see e. g. [4], [5], [6], [8], [9], [12]). In the meantime, some papers on domination numbers of cardinal products of graphs were also published. We refer the interested reader to [7], [10], [11], [13], [14], [15].

In [14] the domination numbers of $P_2 \times P_n$, $P_3 \times P_n$, $P_4 \times P_n$ and $P_5 \times P_n$ are determined. Here are the minimal dominating sets for these cases:

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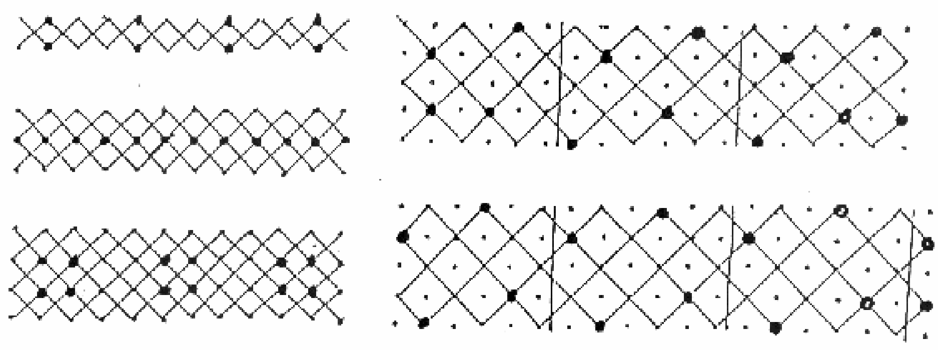

Figure 1.

For completeness we first recall the following (obvious) result:

Proposition 1. *If P_n is the path of order n, then*

$$\gamma(P_n) = \lceil \frac{n}{3} \rceil.$$

To fix the terminology for the proofs of our results we need some more definitions.

Observation 1 *Let $1, ..., k$ and $1, ..., n$ be the vertices of P_k and P_n, respectively. Then the vertices of $P_k \times P_n$ are denoted by (i, j) where $i = 1, ..., k$ and $j = 1, ..., n$.*

Definition 1. *The cardinal product $P_k \times P_n$, $k, n \geq 3$, consists of two components. The component containing the vertex (1,1) is denoted by C_1, and the other component by C_2 .*

Remark 1. *If both, k and n are odd, these components are not isomorphic. If at least one of these two numbers is even, the components are isomorphic.*

Definition 2. *For a fixed m, $1 \leq m \leq n$, the set $(P_k)_m := P_k \times m$ is called a column of $P_k \times P_n$; the set ${}_r(P_n) := r \times P_n$ is called a row of $P_k \times P_n$. $P_k \times P_n$ always consists of two components. A column (row) of one of those components then only consists of those vertices which are contained in the respective component. Any set $B = \{(P_k)_m, (P_k)_{m+1}, ..., (P_k)_{m+l}, |l \geq 0, m \geq 1, m + l \leq n\}$, of consecutive columns is called a block of size $k \times (l + 1)$ of $P_k \times P_n$. If another block $B^{'}$ ends with the column $(P_k)_{m-1}$ or begins with the column $(P_k)_{m+l+1}$, then we say that $B^{'}$ is adjacent to B. A block B is called internal, if it is adjacent to two other blocks. It is called external, if it is adjacent only to one block.*

2. The domination number of $P_6 \times P_n$

The domination number of the cardinal product $P_6 \times P_n$ of paths P_6 and P_n is given by the following theorem:

Theorem 1. *For* $n \geq 6$

$$\gamma(P_6 \times P_n) = 2(n - \lfloor \frac{n}{5} \rfloor).$$

Proof. Let $n \geq 6$. Recall that $P_6 \times P_n$ has two isomorphic components C_1 and C_2. So, it is sufficient to consider only one component (C_2) (which does not contain (1,1)). We give a dominating set S of C_2 as follows: Let $n \geq 5$. If $n = 5q$, then we can partition (split) the set of columns of $P_6 \times P_n$ into q 6-by-5 blocks Q_i, $i = 1, \cdots, q$ and dominate each such block by a set isomorphic to set $P = \{(1,2),(2,5),(4,3),(5,2)\}$. See *Figure 2.*

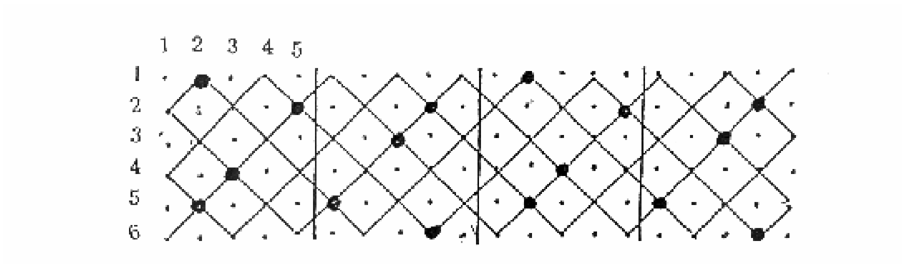

Figure 2.

If $n = 5q + l$, $2 \leq l \leq 4$, then in addition to blocks Q_i $i = 1, ..., q$ we dominate the last $6 \times l$ block $Q_{q+1}^{(l)}$ by a set isomorphic to R_l $(2 \leq l \leq 4)$. See *Figure 3.*

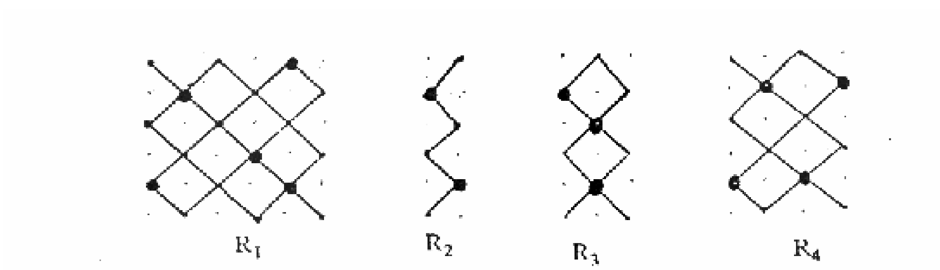

Figure 3.

If $n = 5q+1$, then we dominate Q_i $i = 1, ..., q-1$ by a copy of P, and $Q_q \cup Q_{q+1}^{(1)}$ by a copy of R_1.

Then

$$|S \cap C_2| = 4\lfloor \frac{n}{5} \rfloor + n \ \ mod\ 5.$$

The set S chosen in this way is dominating. In the sequel we prove the minimality of S i.e.

$$\gamma(P_6 \times P_n) \geq |S|.$$

We partition the vertex set of $(P_6 \times P_n)$ into 6×5 blocks. If a block is external, we denote it by R. If it is internal, it is denoted by M. The whole proof is done for C_2. If n is not divisible by 5, there exists one block (R'), which is not a 6×5 block. Let R' be the last block (it contains the column $(P_6)_n$).

Lemma 1. $|D \cap R| \geq 4$, *for every dominating set* D.

Proof. Without loss of generality, we may assume that $R = \{(P_6)_1, ..., (P_6)_5\}$. Even if the column $(P_6)_5$ is dominated by vertices from the adjacent column, we need at least 4 vertices to dominate all vertices of the first four columns. □

Lemma 2. $|D \cap M| \geq 3$, *for every dominating set* D.

Proof. Only the vertices in the first and the last column in M can be dominated by vertices from the adjacent block. To dominate the remaining 6×3 block we obviously need at least 3 vertices which must lie in M. □

Lemma 3. *Let* $n \geq 15$. *If* $|D \cap B_k| = 3$ *for some internal* 6×5 *block* B_k, *then* $|D \cap B_{k-1}| \geq 4$ *and* $|D \cap B_{k+1}| \geq 4$. *But if* B_{k-1} *(*B_{k+1}*) is external, then* $|D \cap B_{k-1}| \geq 5$ *(*$|D \cap B_{k+1}| \geq 5$*).*

Proof. Let $B_k = \{(P_6)_j, (P_6)_{j+1}, ..., (P_6)_{j+4}\}$, $j = 5(k-1)+1$, $k = 2, ..., \lfloor \frac{n}{5} \rfloor - 1$. Without loss of generality, let $(P_6)_j$ consists of the vertices (1,j),(3,j),(5,j). To dominate all the vertices of B_k we need at least 5 vertices which are either contained in B_k, or in adjacent blocks. There are three possibilities to dominate all vertices of the columns $(P_6)_{j+1}$, $(P_6)_{j+2}$, $(P_6)_{j+3}$ by three vertices:
1) (2,j+1),(2,j+3),(5,j+2) $\in D$
2) (2,j+1),(3,j+4),(5,j+2) $\in D$
3) (3,j),(2,j+3),(5,j+2) $\in D$

Case 1. The vertices (5,j) and (5,j+4) are not dominated by the three vertices in 1). The vertex (5,j) must be dominated by vertices of B_{k-1} and (5,j+4) by vertices of B_{k+1}. Without loss of generality, we consider only the case when (5,j) is dominated from B_{k-1}. (See *Figure 4*.)

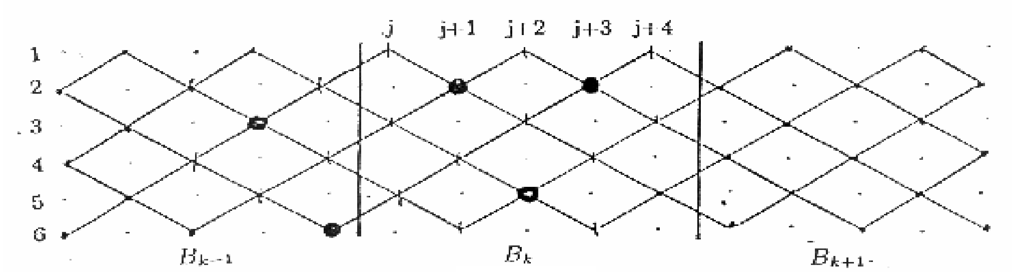

Figure 4.

To dominate (5,j) we need at least 1 vertex from $(P_6)_{j-1}$. If we have only 1 dominating vertex in $(P_6)_{j-1}$, two vertices in this column are not dominated. To dominate them, we need at least one vertex of $(P_6)_{j-2}$. These 2 vertices dominate at most the column $(P_6)_{j-1}$ and two vertices from $(P_6)_{j-2}$ and $(P_6)_{j-3}$. If B_{k-1} is internal, we can dominate at most the column $(P_6)_{j-5}$ by vertices of B_{k-2}. Then still 3 vertices in $(P_6)_{j-4}$ and 1 vertex of $(P_6)_{j-3}$ and 1 vertex of $(P_6)_{j-2}$ are undominated. To dominate them, we need at least 2 vertices. So, $|D \cap B_{k-1}| \geq 4$. If B_{k-1} is external, all vertices in $(P_6)_{j-5}$, $(P_6)_{j-4}$ and one vertex from $(P_6)_{j-3}$

and $(P_6)_{j-2}$ must be dominated by vertices of B_{k-1}. To dominate these vertices we need at least 3 vertices. Then $|D \cap B_{k-1}| \geq 5$. The assertion about B_{k+1} can be proved analogously.

Case 2. The assertion about B_{k-1} can be shown as above.

So, we only have to consider B_{k+1}. To dominate the vertices (1,j+4) and (5,j+4) we need at least 2 vertices from $(P_6)_{j+5}$. These vertices together with (3,j+4) dominate at most the columns $(P_6)_{j+5}$ and $(P_6)_{j+6}$. Even if all vertices of $(P_6)_{j+9}$ are dominated by vertices of B_{k+2}, we need at least two more vertices of B_{k+1} to dominate $(P_6)_{j+7}$ and $(P_6)_{j+8}$. If B_{k+1} is external, we need at least three more vertices of B_{k+1}. Hence, our assertion also holds in this case.

Case 3. As Case 2, only the roles of B_{k-1} and B_{k+1} are interchangeable. □

Lemma 4. *Let $n \geq 20$. If $|D \cap B_k| = 3$ and $|D \cap B_{k-1}| = 4$, then there exists at least one block B_i, $i \in \{1, ..., k-2\}$, such that $|D \cap B_i| \geq 5$. For all blocks B_j, $i < j \leq k-1$, $|D \cap B_j| \geq 4$ holds.*

Proof. If $|D \cap B_k| = 3$ and $|D \cap B_{k-1}| = 4$, then it follows from the proof of *Lemma 3.* that at least one vertex of B_{k-1} is not dominated by vertices of $B_{k-1} \cup B_k$. By the same arguments as above we can conclude that B_{k-2} contains at least four vertices of D, etc. If $i = 1$, the block B_i must contain at least five vertices of D. □

Lemma 5. *Let $n \geq 15$. If $|D \cap B_k| = 5$ for some internal 6×5 block B_k, then $|D \cap B_{k-1}| \geq 3$ and $|D \cap B_{k+1}| \geq 3$. If B_{k-1} (B_{k+1}) is external, then $|D \cap B_{k-1}| \geq 4$ ($|D \cap B_{k+1}| \geq 4$).*

Proof. Only the vertices in the first and the last column of a 6×5 block can be dominated by vertices from adjacent blocks. To dominate the vertices of the three remaining columns we need at least three vertices which are contained in this block.

The assertion about external blocks also follows immediately since in this case at most the vertices of one column can be dominated by vertices of an adjacent block. □

Case 1. $\mathbf{n \equiv 0 \pmod{5}}$

Lemma 6. *Let $n \geq 15$. If $|D \cap B_k| = 3$ holds for any internal 6×5 block B_k, D is not minimal.*

Proof. Similarly as in *Lemma 4*, it can be shown that there exists at least one 6×5 block B_j such that $|D \cap B_j| \geq 5$ holds, where $k < j$. For all B_m $k < m < j$ there holds $|D \cap B_k| \geq 4$.

Let $n \geq 15$, and let D be any dominating set. $|D \cap B_k| \geq 3$ holds for each block B_k, $1 \leq k \leq \frac{n}{5}$, by *Lemma 2.*. Assume that there are s 6×5 blocks each of which contains only three vertices of D. By *Lemma 1.* these blocks are internal. From *Lemma 4.*, there are at least $s+1$ 6×5 blocks which contain at least five vertices of D. Let B_{i_j}, $1 \leq j \leq 2s+1$, denote these blocks which contain either three or five vertices. Then $\mathbf{B} = \cup_{j=1}^{2s+1} B_{i_j}$ contains at least $8s+5$ vertices of D. By the above description of S, the set $\mathbf{B}$ contains at most $8s+4$ vertices of S. Hence D is not minimal. □

For $n \geq 15$, the result follows from *Lemma 6.* For n=10, the statement follows from *Lemma 1.* and for $n = 5$ the result is obvious.

Case 2. $\mathbf{n \equiv 1 \pmod 5}$

We partition the graph $P_6 \times P_n$ into $\lfloor \frac{n}{5} \rfloor$ 6×5 blocks and one block $R' = (P_6)_n$ of size 6×1. By B_m we denote the 6×5 block adjacent to R'.

Lemma 7. *If $D \cap R' = \emptyset$, then $|D \cap (B_m \cup R')| \geq 5$ for any dominating set D.*

Proof. $B_m \cup R' = \{(P_6)_{n-5}, ..., (P_6)_n\}$. Even if the column $(P_6)_{n-5}$ is dominated by vertices of the adjacent block, we need at least 5 vertices to dominate all vertices of the last five columns. □

Let D be any dominating set. We now assume that *Lemma 7.* holds and that there are s blocks B_{j_i}, $1 \leq s, j_i \leq m-1$, containing only three vertices of D. Then *Lemma 4.* implies that there are s blocks B_{k_i} , $k_i \neq m$, which contain at least 5 vertices of D. Then again $|D| \geq |S|$.

If there is no block B_i with $|B_i \cap D| \geq 3$, then *Lemma 9* implies that $|D| \geq |S|$.

If R' contains at least one vertex and each 6×5 block contains at least 4 vertices, then our result clearly holds again.

If R' contains at least one vertex and there is at least one internal 6×5 block containing only 3 vertices, then our result is an immediate consequence of *Lemma 4* again.

For $n = 6$, the statement follows from the proof of *Lemma 7.*. Let $n = 11$. From *Lemma 7.* it follows $|D \cap (B_m \cup R')| \geq 5$. From *Lemma 1.* it follows $|D \cap B_{m-1}| \geq 4$. Then $|D| \geq 9 = |S|$.

Case 3. $\mathbf{n \equiv 2 \pmod 5}$

We partition the graph $P_6 \times P_n$ into $\lfloor \frac{n}{5} \rfloor$ 6×5 blocks and one block $R' = \{(P_6)_{n-1}, (P_6)_n\}$ of size 6×2.

Lemma 8. *For every dominating set D, $|D \cap R'| \geq 2$.*

Proof. By vertices from the adjacent block, we can only dominate vertices of $(P_6)_{n-1}$. □

Lemma 9. *If $|D \cap R'| \geq 3$, then D is not minimal.*

Proof. By *Lemma 2* we have $|D \cap B_m| \geq 3$ where $m = \lfloor \frac{n}{5} \rfloor$. By *Lemma 3* and *Lemma 4* we again know that there are at least s blocks which contain at least 5 vertices of D if there are s blocks which contain only three vertices of D. Hence $|D| > |S|$ if $|D \cap R'| \geq 3$. □

Let D be any dominating set. By the same methods as for $n \equiv 0, 1 \ (mod 5)$ it can be seen that $|D| \geq |S|$ also holds for $n \equiv 2 \ (mod 5)$.

Case 4. $\mathbf{n \equiv 3 \pmod 5}$

We partition $P_6 \times P_n$ into 6×5 blocks and one block $R' = \{(P_6)_{n-2}, (P_6)_{n-3}, (P_6)_{n-3}\}$.

Lemma 10. *For every dominating set D, $|D \cap R'| \geq 2$.*

Proof. At most the first column of R' can be dominated by vertices from the previous block. To dominate the vertices of the last two columns we need at least two vertices. □

Lemma 11. *If $|D \cap R'| = 2$, then $|D \cap B_m| \geq 4$ where $m = \lfloor \frac{n}{5} \rfloor$.*

Proof. R' is a 6×3 block. To dominate all vertices of such a block we need at least 3 vertices. If $|D \cap R'| = 2$, then the vertices (2,n-1), (5,n) must be contained in D if m is odd. If m is even, then (2,n), (5,n-1) must be in D. Here we only consider the case when m is odd, the other case can be done similarly.

The vertices (2,n-1) and (5,n) dominate all vertices of R' with the exception of (5,n-2). To dominate it, we need at least 1 vertex from $(P_6)_{n-3}$. If we have only one dominating vertex in $(P_6)_{n-3}$, then two vertices in $(P_6)_{n-3}$ are not dominated. To dominate them, we need at least one vertex from $(P_6)_{n-4}$. These 2 vertices can only dominate $(P_6)_{n-3}$, $(P_6)_{n-4}$ and two vertices on $(P_6)_{n-5}$. By vertices not in B_m we can dominate only the first column in B_m, i.e. $(P_6)_{n-7}$. Then $(P_6)_{n-6}$ and one vertex in $(P_6)_{n-5}$ remain undominated. To dominate them, we need at least 2 vertices. So, $|D \cap B_m| \geq 4$. □

Remark 2. *If $|D \cap B_m| = 4$, we have the same case as in* Lemma 4.. *At least one vertex on B_m is not dominated and then there exists at least one block B_i $i \in \{1, ..., m-1\}$ such that $|D \cap B_i| \geq 5$, and for all blocks B_j $i < j \leq m-1$ $|D \cap B_j| = 4$ holds. (See* Figure 5.*)*

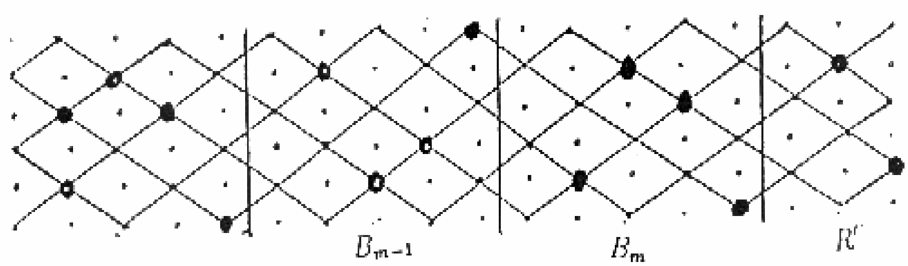

Figure 5.

Lemma 12. *If $|D \cap R'| \geq 4$, then D is not minimal.*

Proof. If $|D \cap R'| \geq 4$, these 4 (or more) vertices can at most dominate all vertices on R' and the column $(P_6)_{n-3}$ on the 6×5 block B_m where $m = \lfloor \frac{n}{5} \rfloor$. Of course it may happen that $|B_m \cap D| = 3$ holds, but *Lemma 4* again implies that there are at least s blocks containing at least 5 vertices of D if there are s blocks containing only three vertices of D. Hence $|D| \geq 4m + 4 > |S|$. □

By the same methods as for $n \equiv 0, 1$, from *Lemmas 3, 11* and *12* it follows that $|D| \geq |S|$ holds for any dominating set D.

Case 5. n ≡ 4 (mod5)

We partition the graph $P_6 \times P_n$ into $\lfloor \frac{n}{5} \rfloor$ 6×5 blocks and one block $R' = \{(P_6)_{n-3}, (P_6)_{n-2}, (P_6)_{n-1}, (P_6)_n\}$.

Lemma 13. *For every dominating set D $|D \cap R'| \geq 3$.*

Proof. At most the first column in R' can be dominated by vertices from the adjacent block. To dominate the remaining 6×3 block, we clearly need at least 3 dominating vertices which are contained in R'. □

Lemma 14. *If* $|D \cap R'| = 3$*, then* $|D \cap B_m| \geq 4$*, where* $m = \lfloor \frac{n}{5} \rfloor$*.*

Proof. Here we only consider the case when m is odd, and the case when m is even can be done similarly. If m is odd, in C_2 in the column $(P_6)_n$ there are vertices (2,n),(4,n) and (6,n). There are three possibilities in case $|D \cap R'| = 3$.
1) (1,n-1),(3,n-1),(5,n-1) are in D. They dominate $(P_6)_n$, $(P_6)_{n-1}$ and $(P_6)_{n-2}$.
2) (1,n-3),(2,n),(5,n-1) are in D. They dominate on R' the columns $(P_6)_n$, $(P_6)_{n-1}$, $(P_6)_{n-2}$ and vertex (1,n-3) on R', and on B_m vertex (2,n-4).
3) (3,n-3),(2,n),(5,n-1) are in D. They dominate on R' the columns $(P_6)_n$,$(P_6)_{n-1}$ $(P_6)_{n-2}$ and the vertex (3,n-3), and on B_m they dominate the vertices (2,n-4), (4,n-4).

We will consider case 3), because most vertices are dominated.

In case 3) on R' only the vertices (1,n-3) and (5,n-3) are not dominated. The distance between them is 4. So, to dominate them, we need at least 2 vertices from the column $(P_6)_{n-4}$ on B_m. These vertices dominate the column $(P_6)_{n-3}$ too. By vertices not in B_m we can dominate at most the first column in B_m (if B_m is an internal block). Then one 6×2 block remains undominated on B_m. To dominate it, we need at least 2 vertices from B_m. So, $|D \cap B_m| \geq 4$. If B_m is external $|D \cap B_m| \geq 5$. □

Remark 3. *If* $|D \cap B_m| = 4$*, there also exists at least one block* B_i $i \in \{1, ..., m-1\}$ *such that* $|D \cap B_i| \geq 5$*, and for all blocks* B_j $i < j \leq m-1$ $|D \cap B_j| = 4$ *holds.*

Lemma 15. *If* $|D \cap R'| \geq 5$*, then* D *is not minimal.*

Proof. If $|D \cap R'| \geq 5$, these 5 (or more) vertices can at most dominate all vertices in R' and the column $(P_6)_{n-4}$ of the next block B_m, where $m = \lfloor \frac{n}{5} \rfloor$. Of course it may happen that $|B_m \cap D| = 3$ holds, but *Lemma 4* again implies that there are at least s blocks containing at least 5 vertices of D if there are s blocks containing only three vertices of D. Hence $|D| \geq 4m + 5 > |S|$. □

From *Lemmas 1,2,3,13,14* and *15*, by the same arguments as for cases $n \equiv 0, 1 \pmod 5$, there holds $|D| \geq |S|$. □

3. The bounds for $\gamma(P_7 \times P_n)$ and $\gamma(P_8 \times P_n)$

Proposition 2.

$$\gamma(P_7 \times P_n) \leq \begin{cases} 2n, & n \geq 6 \quad and\ n = 4 \\ 6, & n = 2 \\ 7, & n = 3 \\ 11, & n = 5 \end{cases}$$

Proof. For even n we have two isomorphic connectivity components. We consider only component C_1 (which contains (1,1)). A dominating set S' of C_1 is given as follows

$$(2, 2 + 4m) \;\; for \;\; m = 0, 1, ..., \lceil \frac{n-1}{4} \rceil - 1$$

$$(3, 3 + 4m) \;\; for \;\; m = 0, 1, ..., \lceil \frac{n-2}{4} \rceil - 1$$

$$(6, 2m) \;\; for \;\; m = 1, ..., \frac{n}{2} - 1 \;\; and \;\; (5, n-1)$$

Then $|S'| = n$ or on both components

$$|S| = 2n.$$

For odd n, $n > 5$: The dominating set on the component C_1 (S_1) contains vertices

$$(2, 2 + 4m) \;\; for \;\; m = 0, 1, ..., \lceil \frac{n-1}{4} \rceil - 1$$

$$(3, 3 + 4m) \;\; for \;\; m = 0, 1, ..., \lceil \frac{n-2}{4} \rceil - 2$$

$$(6, 2m) \;\; for \;\; m = 1, ..., \lfloor \frac{n}{2} \rfloor - 2 \;\; and \;\; (2, n-1), (5, n-2), (6, n-1).$$

Then

$$|S_1| = n - 1.$$

The dominating set S_2 on the component C_2 contains vertices

$$(2, 1 + 2m) \;\; for \;\; m = 0, 1, ... \lfloor \frac{n}{2} \rfloor$$

$$(5, 2 + 4m) \;\; for \;\; m = 0, 1, ..., \lfloor \frac{n-2}{4} \rfloor$$

$$(6, 3 + 4m) \;\; for \;\; m = 0, 1, ..., \lfloor \frac{n-3}{4} \rfloor \;\; and \;\; (5, n-1).$$

Then

$$|S_2| = n + 1.$$

On both components $|S| = |S_1 \cup S_2| = 2n$. Obviously, S is a dominating set. It follows that $\gamma(P_7 \times P_n) \leq 2n$.

Proposition 3.

$$\gamma(P_8 \times P_n) \leq \begin{cases} 2n, & n \equiv 0 (mod 4) \\ 2(n+1), & otherwise \end{cases}$$

Proof. There are 2 isomorphic connectivity components. We will study only C_1 and multiply the results by 2. We partition the graph $(P_8 \times P_n)$ into 8×4 blocks. A dominating set S_1 of C_1 is given as follows

$$(2, 2 + 4m), (3, 3 + 4m), (6, 2 + 4m), (7, 3 + 4m), m = 0, 1, ..., \lfloor \frac{n}{4} \rfloor - 1.$$

S_1 dominates all vertices if n is divisible by 4. If $n = 4m + 1$, then we add (2,n-1) and (6,n-1) to S_1, if n = 4m + 2, we add (2,n), (5,n-1) and (6,n), and if n = 4m + 3, we add (2,n-1), (3,n), (6,n-1), (7,n). Therefore, it holds:

$$\gamma(P_8 \times P_n) \leq 2|S_1| = \begin{cases} 2n, & n \equiv 0 (mod\, 4) \\ 2(n+1), & otherwise \end{cases}$$

□

References

[1] C. Berge, *Some theorems on coverings,* Studia Sci. Math. Hungar. **5**(1970), 303–315.

[2] E. J. Cockayne, S. T. Hedetniemi, *Towards a theory of domination in graphs*, Netwoks **7**(1977), 247–261.

[3] C. F. de Jaenisch, *Applications de l'Analyse Mathematique an Jenudes Echecs*, Petrograd 1862.

[4] M. El-Zahar, C. M. Pareek, *Domination number of products of graphs*, Ars Combin. **31**(1991), 223–227.

[5] R. J. Faudree, R. H. Schelp, *The domination number for the product of graphs*, Congr. Numer. **79**(1990), 29–33.

[6] D. C. Fisher, *The domination number of complete grid graphs,* J. Graph Theory, to appear.

[7] S. Gravier, A. Khelladi, *On the dominating number of cross product of graphs*, Discrete Math. **145**(1995),273–277.

[8] M. S. Jacobson, L. F. Kinch, *On the domination number of products of graphs I*, Ars Combin. **18**(1983), 33–44.

[9] M. S. Jacobson, L. F. Kinch, *On the domination number of the products of graphs II : Trees*, J. Graph Theory **10**(1986), 97–106.

[10] P. K. Jha, S. Klavžar, *Independance and matching in direct-product graphs*, Ars combinatoria **50**(1998), 53–63.

[11] P. K. Jha, S. Klavžar, B. Zmazek, *Isomorphic components of Kronecker product of bipartite graphs*, Discuss. Math. Graph Theory **17**(1997), 301–309.

[12] S. Klavžar, N. Seifter, *Dominating Cartesian products of cycles,* Discrete Appl. Math. **59**(1995), 129–136.

[13] S. Klavžar, B. Zmazek, *On a Vizing-like conjecture for direct product graphs*, Discrete Math. **156**(1996), 243–246.

[14] A. Klobučar, *Domination numbers of cardinal products of graphs*, Mathematica Slovaca, to appear.

[15] A. Klobučar, N. Seifter, *K-dominating sets of the cardinal products of paths*, Ars Combinatoria, to appear.

[16] C. L. Liu, Introduction to Combionatorial Mathematics, McGraw-Hill, New York, 1968.

[17] V. G. Vizing, *The cartesian product of graphs,* Vychisl. Sistemy **9**(1963), 30–43.