Technical Report
FLLL–TR–0207
SCCH–TR–0237

FUZZY.LOGIC.LAB.LINZ

Domination of Aggregation Operators and Preservation of Transitivity

Susanne Saminger
Fuzzy Logic Laboratorium Linz-Hagenberg
saminger@flll.uni-linz.ac.at

Radko Mesiar
Slovak University of Technology
mesiar@cvt.stuba.sk

Ulrich Bodenhofer
Software Competence Center Hagenberg
ulrich.bodenhofer@scch.at

Abstract — Aggregation processes are fundamental in any discipline where the fusion of information is of vital interest. For aggregating binary fuzzy relations such as equivalence relations or fuzzy orderings, the question arises which aggregation operators preserve specific properties of the underlying relations, e.g. T-transitivity. It will be shown that preservation of T-transitivity is closely related to the domination of the applied aggregation operator over the corresponding t-norm T. Furthermore, basic properties for dominating aggregation operators, not only in the case of dominating some t-norm T, but dominating some arbitrary aggregation operator, will be presented. Domination of isomorphic t-norms and ordinal sums of t-norms will be treated. Special attention is paid to the four basic t-norms (minimum t-norm, product t-norm, Łukasiewicz t-norm, and the drastic product).

Key words — *aggregation operators, domination, fuzzy relations, T-transitivity.*

accepted for publication in *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*

Johannes Kepler Universität Linz | Fuzzy Logic Laboratorium Linz-Hagenberg

Institut für Algebra, Stochastik und
wissensbasierte mathematische Systeme
Johannes Kepler Universität Linz
A-4040 Linz
Austria

FLLL
Softwarepark Hagenberg
Hauptstrasse 99
A-4232 Hagenberg
Austria

1 Introduction

Aggregation is a fundamental process in decision making and in any other discipline where the fusion of different pieces of information is of vital interest. Consider, for example, some process of comparing different objects which is based on some of their characteristic properties, where we are interested in an overall comparison of objects.

For instance, think of flexible (fuzzy) querying systems. Such systems are usually designed not just to give results that match a query exactly, but to give a list of possible answers ranked by their closeness to the query—which is particularly beneficial if no record in the database matches the query in an exact way ([1]). The closeness of a single value of a record to the respective value in the query is usually measured using a fuzzy equivalence relation, that is, a reflexive, symmetric, and T-transitive fuzzy relation. Recently, a generalization has been proposed ([2]) which also allows flexible interpretation of ordinal queries (such as "at least" and "at most") by using fuzzy orderings ([3]). In any case, if a query consists of at least two expressions that are to be interpreted vaguely, it is necessary to combine the degrees of matching with respect to the different fields in order to obtain an overall degree of matching. Assume that we have a query $(q_1, \ldots, q_n)$, where each $q_i \in X_i$ is a value referring to the i-th field of the query. Given a data record $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for all $i = 1, \ldots, n$, the overall degree of matching is computed as

$$\tilde{R}\big((q_1, \ldots, q_n), (x_1, \ldots, x_n)\big) = \mathbf{A}\big(R_1(q_1, x_1), \ldots, R_n(q_n, x_n)\big),$$

where every R_i is a T-transitive binary fuzzy relation on X_i which measures the degree to which the value x_i matches the query value q_i.

It is natural to require that $\tilde{R}$ is fuzzy relation on the Cartesian product of all X_i and, therefore, the range of $\mathbf{A}$ should be the unit interval, i.e. $\mathbf{A} : [0,1]^n \to [0,1]$. Furthermore, it is desirable that if a data record matches one of the criteria of the query better than a second one, then the overall degree of matching for the first should be higher or at least the same as the overall degree of matching for second one. Clearly, if some data record matches all criteria, i.e. all $R_i(x_i, q_i) = 1$, then the overall degree of matching should also be 1. On the other hand, if a data record fulfills none of the criteria to any level, i.e. all $R_i(x_i, q_i) = 0$, then the overall degree should vanish to 0. Aggregation operators are functions which guarantee all these properties ([4, 5, 6, 7]).

Moreover, it would be desirable that, if all relations R_i on X_i are T-transitive, also $\tilde{R}$ is still T-transitive in order to have a clear interpretation of the aggregated fuzzy relation $\tilde{R}$. It is, therefore, necessary to investigate which aggregation operators are able to guarantee that $\tilde{R}$ maintains T-transitivity.

It turns out that the preservation of T-transitivity in aggregating fuzzy relations is closely related to the domination of an aggregation operator over the corresponding t-norm T. Therefore, a concept of domination for aggregation operators will be introduced and the relationship to the preservation of T-transitivity will be proved. Some construction methods for dominating aggregation operators will be proposed as well as a characterization of aggregation operators dominating the four basic t-norms (minimum t-norm $T_{\mathbf{M}}$, product t-norm $T_{\mathbf{P}}$, Łukasiewicz t-norm $T_{\mathbf{L}}$, and the drastic product $T_{\mathbf{D}}$).

2 Basic Definitions and Preliminaries

2.1 Aggregation operators and t-norms

Definition 1. A function

$$\mathbf{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$$

is called an *aggregation operator* if it fulfills the following properties ([5, 7]):

(AO1) $\mathbf{A}(x_1, \ldots, x_n) \leq \mathbf{A}(y_1, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$

(AO2) $\mathbf{A}(x) = x$ for all $x \in [0,1]$

(AO3) $\mathbf{A}(0, \ldots, 0) = 0$ and $\mathbf{A}(1, \ldots, 1) = 1$

Each aggregation operator $\mathbf{A}$ can be represented by a family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, i.e. functions $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ given by

$$\mathbf{A}_{(n)}(x_1, \ldots, x_n) = \mathbf{A}(x_1, \ldots, x_n).$$

In that case, $\mathbf{A}_{(1)} = \mathrm{id}_{[0,1]}$ and, for $n \geq 2$, each $\mathbf{A}_{(n)}$ is non-decreasing and satisfies $\mathbf{A}_{(n)}(0, \ldots, 0) = 0$ and $\mathbf{A}_{(n)}(1, \ldots, 1) = 1$. Usually, the aggregation operator $\mathbf{A}$ and the corresponding family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations are identified with each other.

Note that, for $n \geq 2$, n-ary operations $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ which fulfill properties **(AO1)** and **(AO3)** are referred to as *n-ary aggregation operators.*

Unless explicitly mentioned otherwise, we will restrict ourselves to aggregation operators acting on the unit interval (according to Definition 1). With only simple and obvious modifications, aggregation operators can be defined to act on any closed interval $I = [a,b] \subseteq [-\infty, \infty]$. While **(AO1)** and **(AO2)** basically remain the same, only **(AO3)** has to be modified accordingly:

(AO3') $\mathbf{A}(a, \ldots, a) = a$ and $\mathbf{A}(b, \ldots, b) = b$

Consequently, we will speak of an *aggregation operator acting on I*.

Definition 2. Consider some aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$.

(i) $\mathbf{A}$ is called *symmetric*, if

$$\forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in [0,1] : \mathbf{A}(x_1, \ldots, x_n) = \mathbf{A}(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$$

for all permutations $\alpha = (\alpha(1), \ldots, \alpha(n))$ of $\{1, \ldots, n\}$.

(ii) $\mathbf{A}$ is called *associative* if

$$\begin{aligned}&\forall n, m \in \mathbb{N}, \forall x_1, \ldots, x_n, y_1, \ldots, y_m \in [0,1] :\\ &\qquad \mathbf{A}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \mathbf{A}(\mathbf{A}(x_1, \ldots, x_n), \mathbf{A}(y_1, \ldots, y_m)).\end{aligned}$$

(iii) An element $e \in [0,1]$ is called a neutral element of $\mathbf{A}$ if

$$\forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in [0,1] \text{ if } x_i = e \text{ for some } i \in \{1, \ldots, n\} \text{ then}$$
$$\mathbf{A}(x_1, \ldots, x_n) = \mathbf{A}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Triangular norms were originally introduced in the context of probabilistic metric spaces ([8, 9, 10]). A *triangular norm* (t-norm for short) is a binary operation T on the unit interval which is commutative, associative, non-decreasing in each component, and has 1 as a neutral element. In fact, triangular norms are nothing else than binary, associative, and symmetric aggregation operators with 1 as neutral element.

Example 3. The following are the four basic t-norms:

Minimum t-norm: $T_{\mathbf{M}}(x,y) = \min(x,y)$,
Product t-norm: $T_{\mathbf{P}}(x,y) = x \cdot y$,
Łukasiewicz t-norm: $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$,
Drastic product: $T_{\mathbf{D}}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$

Observe that, for a given aggregation operator $\mathbf{A}$, the operators $\mathbf{A}_{(n)}$ and $\mathbf{A}_{(m)}$ need not be related in general, if $n \neq m$. However, if $\mathbf{A}$ is an associative aggregation operator, for $n \geq 3$, all n-ary operators $\mathbf{A}_{(n)}$ can be derived from the binary operator $\mathbf{A}_{(2)}$. Therefore, in the case of associative aggregation operators, the distinction between $\mathbf{A}_{(2)}$ and $\mathbf{A}$ itself is often omitted. This justifies to speak about t-norms as general aggregation operators, although only the binary operations have been defined.

2.2 Transitivity and preservation of transitivity

We have already mentioned that binary fuzzy relations R_i on the subspaces X_i can be used for the comparison of two objects on the subspaces' level. For details on fuzzy relations, especially fuzzy equivalence relations ([11, 12, 13, 14, 15]) and fuzzy orderings ([3, 11, 16, 17, 18]) and their properties, we refer to the relevant literature. We only recall the definition of T-transitivity, since we are interested in its preservation during the aggregation process.

Definition 4. Consider a binary fuzzy relation R on some universe X and an arbitrary t-norm T. R is called T*-transitive* if and only if, for all $x, y, z \in X$ the following property holds:

$$T\big(R(x,y), R(y,z)\big) \leq R(x,z)$$

Definition 5. An aggregation operator $\mathbf{A}$ *preserves* T*-transitivity* if, for all $n \in \mathbb{N}$ and for all binary T-transitive fuzzy relations R_i on X_i with $i \in \{1, \ldots, n\}$, the aggregated relation $\tilde{R} = \mathbf{A}(R_1, \ldots, R_n)$ on the Cartesian product of all X_i, i.e.

$$\tilde{R}(A,B) = \tilde{R}((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \mathbf{A}\big(R_1(a_1,b_1), \ldots, R_n(a_n,b_n)\big),$$

is also T-transitive, that means, for all $A, B, C \in \prod_{i=1}^{n} X_i$,

$$T\big(\tilde{R}(A,B), \tilde{R}(B,C)\big) \leq \tilde{R}(A,C).$$

Without loss of generality, we will restrict our considerations in the sequel to fuzzy relations on the same universe $X_i = X$.

2.3 Domination

Similar to t-norms, the concept of domination has been introduced in the framework of probabilistic metric spaces ([19, 20]) when constructing the Cartesian products of such spaces. In the framework of t-norms, domination is also needed when constructing fuzzy equivalence relations and fuzzy orderings ([14, 15, 17, 18]). We will now extend the concept of domination for the framework of aggregation operators.

Definition 6. Consider an n-ary aggregation operator $\mathbf{A}_{(n)}$ and an m-ary aggregation operator $\mathbf{B}_{(m)}$. We say that $\mathbf{A}_{(n)}$ *dominates* $\mathbf{B}_{(m)}$ ($\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$) if, for all $x_{i,j} \in [0,1]$ with $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, the following property holds:

$$\begin{aligned} \mathbf{B}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1},\dots,x_{1,n}),\dots,\mathbf{A}_{(n)}(x_{m,1},\dots,x_{m,n})\big) \\ \leq \mathbf{A}_{(n)}\big(\mathbf{B}_{(m)}(x_{1,1},\dots,x_{m,1}),\dots,\mathbf{B}_{(m)}(x_{1,n},\dots,x_{m,n})\big). \end{aligned} \tag{1}$$

Note that if either n or m or both are equal to 1, because of the boundary condition **(AO2)**, $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$ is trivially fulfilled for any two aggregation operators $\mathbf{A}, \mathbf{B}$.

Definition 7. Let $\mathbf{A}$ and $\mathbf{B}$ be aggregation operators. We say that $\mathbf{A}$ *dominates* $\mathbf{B}$ ($\mathbf{A} \gg \mathbf{B}$), if $\mathbf{A}_{(n)}$ dominates $\mathbf{B}_{(m)}$ for all $n, m \in \mathbb{N}$.

Note that, if two aggregation operators $\mathbf{A}$ and $\mathbf{B}$ are both acting on some closed interval $I = [a,b] \subseteq [-\infty,\infty]$, then the property of domination can be easily adapted by requiring that the Ineq. (1) must hold for all arguments $x_{i,j}$ from the interval I and for all $n, m \in \mathbb{N}$.

Further on, we will denote the class of all aggregation operators $\mathbf{A}$ which dominate an aggregation operator $\mathbf{B}$ by

$$\mathcal{D}_{\mathbf{B}} = \{\mathbf{A} \mid \mathbf{A} \gg \mathbf{B}\}.$$

Since t-norms are special kinds of associative aggregation operators, the following proposition will be helpful for considering the domination of an aggregation operator over a t-norm T.

Proposition 8. *Let $\mathbf{A}, \mathbf{B}$ be two aggregation operators. Then the following holds:*

(i) If $\mathbf{B}$ is associative and $\mathbf{A}_{(n)} \gg \mathbf{B}_{(2)}$ for all $n \in \mathbb{N}$, then $\mathbf{A} \gg \mathbf{B}$.

(ii) If $\mathbf{A}$ is associative and $\mathbf{A}_{(2)} \gg \mathbf{B}_{(m)}$ for all $m \in \mathbb{N}$, then $\mathbf{A} \gg \mathbf{B}$.

Proof. Consider two aggregation operators $\mathbf{A}, \mathbf{B}$. Further, let $\mathbf{B}$ be associative and $\mathbf{A}_{(n)} \gg \mathbf{B}_{(2)}$ for all $n \in \mathbb{N}$. Consider arbitrary $n, m \in \mathbb{N}$ and arbitrary $x_{i,j} \in [0,1]$ with $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Note that for better readability we introduce the notation $(x_{i,\bullet}) = (x_{i,1},\dots,x_{i,n})$

for all $i \in \{1, \ldots, m\}$. Then we get

$$\begin{aligned}
&\mathbf{B}\big(\mathbf{A}(x_{1,1}, \ldots, x_{1,n}), \ldots, \mathbf{A}(x_{m,1}, \ldots, x_{m,n})\big)\\
&\quad = \mathbf{B}\big(\mathbf{A}(x_{1,\cdot}), \ldots, \mathbf{A}(x_{m,\cdot})\big)\\
&\quad = \mathbf{B}\big(\mathbf{B}(\mathbf{A}(x_{1,\cdot}), \mathbf{A}(x_{2,\cdot})), \mathbf{A}(x_{3,\cdot}), \ldots, \mathbf{A}(x_{m,\cdot})\big)\\
&\quad \leq \mathbf{B}\big(\mathbf{A}(\mathbf{B}(x_{1,1}, x_{2,1}), \ldots, \mathbf{B}(x_{1,n}, x_{2,n})), \mathbf{A}(x_{3,\cdot}), \ldots, \mathbf{A}(x_{m,\cdot})\big)\\
&\quad = \mathbf{B}\big(\mathbf{B}[\mathbf{A}(\mathbf{B}(x_{1,1}, x_{2,1}), , \ldots, \mathbf{B}(x_{1,n}, x_{2,n})), \mathbf{A}(x_{3,1}, \ldots, x_{3n})], \ldots, \mathbf{A}(x_{m,\cdot})\big)\\
&\quad \leq \mathbf{B}\big(\mathbf{A}[\mathbf{B}(\mathbf{B}(x_{1,1}, x_{2,1}), x_{3,1}), \ldots, \mathbf{B}(\mathbf{B}(x_{1,n}, x_{2,n}), x_{3n})], \ldots, \mathbf{A}(x_{m,\cdot})\big)\\
&\quad = \mathbf{B}\big(\mathbf{A}(\mathbf{B}(x_{1,1}, x_{2,1}, x_{3,1}), \ldots, \mathbf{B}(x_{1,n}, x_{2,n}, x_{3n})), \ldots, \mathbf{A}(x_{m,\cdot})\big)\\
&\quad \leq \mathbf{B}\big(\mathbf{A}(\mathbf{B}(x_{1,1}, \ldots, x_{(m-1),1}), \ldots, \mathbf{B}(x_{1,n}, \ldots, x_{(m-1),n})), \mathbf{A}(x_{m,\cdot})\big)\\
&\quad \leq \mathbf{A}\big(\mathbf{B}(x_{1,1}, \ldots, x_{m,1}), \ldots, \mathbf{B}(x_{1,n}, \ldots, x_{m,n})\big).
\end{aligned}$$

It can be shown analogously that, if $\mathbf{A}$ is associative and $\mathbf{A}_{(2)} \gg \mathbf{B}_{(m)}$ for all $m \in \mathbb{N}$, $\mathbf{A} \gg \mathbf{B}$ holds, i.e. $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$ for arbitrary $n \in \mathbb{N}$. □

Consequently, if two aggregation operators $\mathbf{A}$ and $\mathbf{B}$ are both associative, as it would be in the case of two t-norms, it is sufficient to show that $\mathbf{A}_{(2)} \gg \mathbf{B}_{(2)}$ for proving that $\mathbf{A} \gg \mathbf{B}$.

We summarize a few well-known, basic results on domination in the framework of t-norms ([7, 14]):

(i) For any t-norm T, it holds that T itself and $T_{\mathbf{M}}$ dominate T.

(ii) Furthermore, for any two t-norms T_1, T_2, $T_1 \gg T_2$ implies $T_1 \geq T_2$ and, therefore, we know that $T_{\mathbf{D}} \gg T$ if and only if $T = T_{\mathbf{D}}$ and $T \gg T_{\mathbf{M}}$ if and only if $T = T_{\mathbf{M}}$.

Note that $T_{\mathbf{M}}$ dominates not only all t-norms, but also any aggregation operator $\mathbf{A}$, because of its monotonicity property

$$\mathbf{A}(\min(x_1, y_1), \ldots, \min(x_n, y_n)) \leq \min(\mathbf{A}(x_1, \ldots, x_n), \mathbf{A}(y_1, \ldots, y_n)).$$

Further note, that the property of selfdomination of an aggregation operator, i.e. $\mathbf{A} \gg \mathbf{A}$, is nothing else than the property of bisymmetry in the sense of Aczél ([21]), i.e. for all $n, m \in \mathbb{N}$ and all $x_{i,j} \in [0, 1]$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$

$$\begin{aligned}
&\mathbf{A}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1}, \ldots, x_{1,n}), \ldots, \mathbf{A}_{(n)}(x_{m,1}, \ldots, x_{m,n})\big)\\
&\qquad = \mathbf{A}_{(n)}\big(\mathbf{A}_{(m)}(x_{1,1}, \ldots, x_{m,1}), \ldots, \mathbf{A}_{(m)}(x_{1,n}, \ldots, x_{m,n})\big).
\end{aligned}$$

3 T-Transitivity and Domination

Standard aggregation of fuzzy equivalence relations (fuzzy orderings) preserving the T-transitivity is done either by means of T itself or $T_{\mathbf{M}}$, but in fact, any t-norm $\tilde{T}$ dominating T can be applied, i.e., if R_1, R_2 are two T-transitive binary relations on a universe X and $\tilde{T} \gg T$, then also $\tilde{T}(R_1, R_2)$ is T-transitive ([14, 17, 18]).

As already mentioned above, in several applications, other types of aggregation processes preserving T-transitivity are required ([2]). Especially the introduction of different weights (degrees of importance) for input fuzzy equivalences (orderings) cannot be properly done by aggregation with t-norms, because of the commutativity. Therefore, we are investigating aggregation operators preserving the T-transitivity of the aggregated fuzzy relations. The following theorem generalizes the result known for triangular norms ([14]).

Theorem 9. *Let $|X| \geq 3$ and let T be an arbitrary t-norm. An aggregation operator $\mathbf{A}$ preserves the T-transitivity of fuzzy relations on X if and only if $\mathbf{A} \in \mathcal{D}_T$.*

Proof. First we show that if $\mathbf{A}$ dominates T, then it also preserves T-transitivity. Therefore we have to show that $\tilde{R}$ is T-transitive for some binary, T-transitive relations R_i on X with $i \in \{1, \ldots n\}$ and some $n \in \mathbb{N}$.
Consider arbitrary $A, B, C \in X^n$, then we get

$$\begin{aligned}
&T\big(\tilde{R}(A,B), \tilde{R}(B,C)\big) \\
&\quad = T\left(\mathbf{A}(R_1(a_1,b_1), \ldots, R_n(a_n,b_n)), \mathbf{A}(R_1(b_1,c_1), \ldots, R_n(b_n,c_n))\right) \\
&\quad \leq \mathbf{A}\left(T(R_1(a_1,b_1), R_1(b_1,c_1)), \ldots, T(R_n(a_n,b_n), R_n(b_n,c_n))\right) \\
&\quad \leq \mathbf{A}\left(R_1(a_1,c_1), \ldots R_n(a_n,c_n)\right) = \tilde{R}(A,C).
\end{aligned}$$

On the other hand, we have to show that an aggregation operator $\mathbf{A}$ which preserves T-transitivity also dominates the corresponding t-norm T. Due to Proposition 8, it is sufficient to show that

$$T\big(\mathbf{A}(x_1, \ldots, x_n), \mathbf{A}(y_1, \ldots, y_n)\big) \leq \mathbf{A}\big(T(x_1,y_1), \ldots, T(x_n,y_n)\big)$$

holds for all $x_i, y_i \in [0,1]$ with $i \in \{1, \ldots, n\}$ and arbitrary $n \in \mathbb{N}$.

Since the universe X contains at least three elements a_i, b_i, c_i, there exists a binary fuzzy relation R_i on X, which is T-transitive and fulfills the following equations $x_i = R_i(a_i,b_i), y_i = R_i(b_i,c_i)$ and $T(x_i,y_i) = R_i(a_i,c_i)$ for some $x_i, y_i \in [0,1]$, e.g. consider the following binary fuzzy relation $\tilde{R}_i$ on X defined by

- $\tilde{R}_i(x,x) = 1$ for all $x \in X$,
- $\tilde{R}_i(a_i,b_i) = \tilde{R}_i(b_i,a_i) = x_i$,
- $\tilde{R}_i(b_i,c_i) = \tilde{R}_i(c_i,b_i) = y_i$,
- $\tilde{R}_i(a_i,c_i) = \tilde{R}_i(c_i,a_i) = T(x_i,y_i)$,
- $\tilde{R}_i(x,y) = 0$ for all $x \neq y$ and at least one argument from $X \setminus \{a_i, b_i, c_i\}$.

For proving the T-transitivity of $\tilde{R}_i$, we have to show that the following inequality holds for all $x, y, z \in X$:

$$T(\tilde{R}_i(x,y), \tilde{R}_i(y,z)) \leq \tilde{R}_i(x,z).$$

If at least one of the arguments is from $X \setminus \{a_i, b_i, c_i\}$ the inequality is trivially fulfilled. For arguments $x, y, z \in \{a_i, b_i, c_i\}$ we get the following situations proving the T-transitivity of $\tilde{R}_i$

$$\begin{aligned}
&T(\tilde{R}_i(a_i,b_i), \tilde{R}_i(b_i,c_i)) = T(x_i,y_i) = \tilde{R}_i(a_i,c_i), \\
&T(\tilde{R}_i(b_i,c_i), \tilde{R}_i(c_i,a_i)) = T(y_i, T(x_i,y_i)) \leq \min(x_i,y_i) \leq x_i = \tilde{R}_i(b_i,a_i), \\
&T(\tilde{R}_i(c_i,a_i), \tilde{R}_i(a_i,b_i)) = T(T(x_i,y_i), x_i) \leq \min(x_i,y_i) \leq y_i = \tilde{R}_i(c_i,b_i).
\end{aligned}$$

Consequently, for arbitrary $x_i, y_i \in [0,1]$, we can find a T-transitive binary fuzzy relation R_i on X which fulfills $x_i = R_i(a_i, b_i)$, $y_i = R_i(b_i, c_i)$, and $T(x_i, y_i) = R_i(a_i, c_i)$. Therefore, we conclude

$$\begin{aligned}
&T\big(\mathbf{A}(x_1,\ldots,x_n), \mathbf{A}(y_1,\ldots,y_n)\big)\\
&\quad = T\big(\mathbf{A}(R_1(a_1,b_1),\ldots,R_n(a_n,b_n)), \mathbf{A}(R_1(b_1,c_1),\ldots,R_n(b_n,c_n))\big)\\
&\quad = T\big(\tilde{R}(A,B), \tilde{R}(B,C)\big) \leq \tilde{R}(A,C) = \mathbf{A}\big(R_1(a_1,c_1),\ldots,R_n(a_n,c_n)\big)\\
&\quad = \mathbf{A}\big(T(x_1,y_1),\ldots T(x_n,y_n)\big),
\end{aligned}$$

showing the domination of $\mathbf{A}$ over T. □

4 Construction of Dominating Aggregation Operators

We have shown the close relationship between the preservation of T-transitivity and the domination of the involved aggregation operator $\mathbf{A}$ over T. Therefore, we are interested in the characterization of the class $\mathcal{D}_T$ with respect to some t-norm T but also in construction and transformation methods for such dominating aggregation operators. Clearly, some of the following results are not only restricted to the domination of an aggregation operator $\mathbf{A}$ over some fixed t-norm T, but are also valid with respect to any fixed aggregation operator $\tilde{\mathbf{A}}$.

4.1 Combination of dominating aggregation operators

Proposition 10. *Consider an aggregation operator $\tilde{\mathbf{A}}$ and the corresponding class of dominating aggregation operators $\mathcal{D}_{\tilde{\mathbf{A}}}$. Then the following properties hold:*

(i) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}_{\tilde{\mathbf{A}}}$ where $\mathbf{A}_{(2)}$ is idempotent, also $\mathbf{D} = \mathbf{A}(\mathbf{B}, \mathbf{C}) \in \mathcal{D}_{\tilde{\mathbf{A}}}$, with

$$\mathbf{D}(x_1,\ldots,x_n) = \mathbf{A}(\mathbf{B}(x_1,\ldots,x_n), \mathbf{C}(x_1,\ldots,x_n)).$$

(ii) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}_{\tilde{\mathbf{A}}}$, also $\mathbf{D}^{(k)} = \mathbf{A}(\mathbf{B}, \mathbf{C}) \in \mathcal{D}_{\tilde{\mathbf{A}}}$ for all $k \in \{1,\ldots,n-1\}$ with

$$\mathbf{D}^{(k)}(x_1,\ldots,x_n) = \mathbf{A}(\mathbf{B}(x_1,\ldots,x_k), \mathbf{C}(x_{k+1},\ldots,x_n)).$$

Note that the idempotency of $\mathbf{A}_{(2)}$, i.e. $\mathbf{A}(x,x) = x$ for all $x \in [0,1]$, ensures that $\mathbf{D}$ is an aggregation operator fulfilling $\mathbf{D}(x) = x$. However, the idempotency of $\mathbf{A}_{(2)}$ can be omitted, whenever we put $\mathbf{D}_{(1)} = \mathrm{id}_{[0,1]}$ by convention and apply $\mathbf{D} = \mathbf{A}(\mathbf{B}, \mathbf{C})$ for $n \geq 2$.

Proof. Consider some $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}_{\tilde{\mathbf{A}}}$, arbitrary $x_{1,1},\ldots,x_{m,n} \in [0,1]$ for some $n, m \in \mathbb{N}$. Once again, we introduce for better readability the following notations:

- $(x_{i,\bullet}) = (x_{i,1},\ldots,x_{i,n})$ for all $i \in \{1,\ldots,m\}$,
- $(x_{\bullet,j}) = (x_{1,j},\ldots,x_{m,j})$ for all $j \in \{1,\ldots,n\}$,
- $(x_{i,\overline{k\bullet l}}) = (x_{i,k},\ldots,x_{i,l})$ for all $i \in \{1,\ldots,m\}$ and $k, l \in \{1,\ldots,n\}$ with $k \leq l$.

Then the following holds:

(i) $\mathbf{D} \in \mathcal{D}_{\tilde{\mathbf{A}}}$, i.e. $\mathbf{D} \gg \tilde{\mathbf{A}}$:

$$\begin{aligned}
&\tilde{\mathbf{A}}\big(\mathbf{D}(x_{1,1},\ldots,x_{1,n}),\ldots,\mathbf{D}(x_{m,1},\ldots,x_{m,n})\big)\\
&\quad= \tilde{\mathbf{A}}\big(\mathbf{A}(\mathbf{B}(x_{1,\bullet}),\mathbf{C}(x_{1,\bullet})),\ldots,\mathbf{A}(\mathbf{B}(x_{m,\bullet}),\mathbf{C}(x_{m,\bullet}))\big)\\
&\quad\le \mathbf{A}\big(\tilde{\mathbf{A}}(\mathbf{B}(x_{1,\bullet}),\ldots,\mathbf{B}(x_{m,\bullet})),\tilde{\mathbf{A}}(\mathbf{C}(x_{1,\bullet}),\ldots,\mathbf{C}(x_{m,\bullet}))\big)\\
&\quad\le \mathbf{A}\big(\mathbf{B}(\tilde{\mathbf{A}}(x_{\bullet,1}),\ldots,\tilde{\mathbf{A}}(x_{\bullet,n})),\mathbf{C}(\tilde{\mathbf{A}}(x_{\bullet,1}),\ldots,\tilde{\mathbf{A}}(x_{\bullet,n}))\big)\\
&\quad= \mathbf{D}\big(\tilde{\mathbf{A}}(x_{\bullet,1}),\ldots,\tilde{\mathbf{A}}(x_{\bullet,n})\big)\\
&\quad= \mathbf{D}\big(\tilde{\mathbf{A}}(x_{1,1},\ldots,x_{m,1}),\ldots,\tilde{\mathbf{A}}(x_{1,n},\ldots,x_{m,n})\big).
\end{aligned}$$

(ii) $\mathbf{D}^{(k)} \in \mathcal{D}_{\tilde{\mathbf{A}}}$ for all $k \in \{1,\ldots,n-1\}$, i.e. $\mathbf{D}^{(k)} \gg \tilde{\mathbf{A}}$:

$$\begin{aligned}
&\tilde{\mathbf{A}}\big(\mathbf{D}(x_{1,1},\ldots,x_{1,n}),\ldots,\mathbf{D}^{(k)}(x_{m,1},\ldots,x_{m,n})\big)\\
&\quad= \tilde{\mathbf{A}}\big(\mathbf{A}(\mathbf{B}(x_{1,\overline{1\bullet k}}),\mathbf{C}(x_{1,\overline{(k+1)\bullet n}})),\ldots,\mathbf{A}(\mathbf{B}(x_{m,\overline{1\bullet k}}),\mathbf{C}(x_{m,\overline{(k+1)\bullet n}}))\big)\\
&\quad\le \mathbf{A}\big(\tilde{\mathbf{A}}(\mathbf{B}(x_{1,\overline{1\bullet k}}),\ldots,\mathbf{B}(x_{m,\overline{1\bullet k}})),\tilde{\mathbf{A}}(\mathbf{C}(x_{1,\overline{(k+1)\bullet n}}),\ldots,\mathbf{C}(x_{m,\overline{(k+1)\bullet n}}))\big)\\
&\quad\le \mathbf{A}\big(\mathbf{B}(\tilde{\mathbf{A}}(x_{\bullet,1}),\ldots,\tilde{\mathbf{A}}(x_{\bullet,k})),\mathbf{C}(\tilde{\mathbf{A}}(x_{\bullet,(k+1)}),\ldots,\tilde{\mathbf{A}}(x_{\bullet,n}))\big)\\
&\quad= \mathbf{D}^{(k)}\big(\tilde{\mathbf{A}}(x_{\bullet,1}),\ldots,\tilde{\mathbf{A}}(x_{\bullet,n})\big)\\
&\quad= \mathbf{D}^{(k)}\big(\tilde{\mathbf{A}}(x_{1,1},\ldots,x_{m,1}),\ldots,\tilde{\mathbf{A}}(x_{1,n},\ldots,x_{m,n})\big).
\end{aligned}$$

□

Remark 11. Note that for the cases where $k = 1$ and $k = n-1$ we have

$$\begin{aligned}
\mathbf{D}^{(1)}(x_1,\ldots,x_n) &= \mathbf{A}(x_1,\mathbf{C}(x_2,\ldots,x_n)),\\
\mathbf{D}^{(n-1)}(x_1,\ldots,x_n) &= \mathbf{A}(\mathbf{B}(x_1,\ldots,x_{n-1}),x_n)
\end{aligned}$$

and therefore the inductive extensions ([5])

$$\begin{aligned}
\mathbf{A}^{(ext)}(x_1,\ldots,x_n) &= \mathbf{A}_{(2)}(x_1,\mathbf{A}_{(2)}(\ldots,\mathbf{A}_{(2)}(x_{n-1},x_n)\ldots)),\\
\mathbf{A}_{(ext)}(x_1,\ldots,x_n) &= \mathbf{A}_{(2)}(\ldots,\mathbf{A}_{(2)}(\mathbf{A}_{(2)}(x_1,x_2),x_3)\ldots,x_n)
\end{aligned}$$

of a binary aggregation operator $\mathbf{A}_{(2)}$ also dominate the corresponding aggregation operator $\tilde{\mathbf{A}}$, if $\mathbf{A}_{(2)} \gg \tilde{\mathbf{A}}$.

Further note, that Proposition 8 (ii) is an immediate consequence of Remark 11 since, in the case of associative aggregation operators, the inductive extensions coincide. Therefore, it is sufficient to show that $\mathbf{A}_{(2)} \gg T_{(2)}$, if $\mathbf{A}$ is an associative aggregation operator.

Proposition 10 has shown how new dominating aggregation operators can be constructed from already known dominating aggregation operators. In case of continuous Archimedean t-norms, other construction methods based on their additive generators can be formulated.

4.2 Generated and weighted t-norms

Aggregation operators preserving T-transitivity, where T is some continuous Archimedean t-norm with additive generator f, are closely related to pseudo-metric-preserving transformations ([22, 23]).

Definition 12. A function $F : (\mathbb{R}^+)^n \to \mathbb{R}^+$ is a *pseudo-metric-preserving* function if it fulfills the following properties:

(i) $F(0,,\ldots,0) = 0$,

(ii) for any family of pseudo-metrics $d_i : X_i \times X_i \to \mathbb{R}^+$ and any $x_i, y_i, z_i \in X_i$, with $i = 1,\ldots,n$,

$$F(d_1(x_1,z_1),\ldots,d_n(x_n,z_n)) \\ \leq F(d_1(x_1,y_1),\ldots,d_n(x_n,y_n)) + F(d_1(y_1,z_1),\ldots,d_n(y_n,z_n)).$$

A sufficient condition for a function to be pseudo-metric-preserving can be adapted from results for metric-preserving functions ([22, 23, 24]): If a function $F : (\mathbb{R}^+)^n \to \mathbb{R}^+$ is non-decreasing and subadditive, fulfilling $F(0,\ldots,0) = 0$, then it is pseudo-metric-preserving.

Definition 13. A function $F : [0,c]^n \to [0,c]$ is *subadditive* on $[0,c]$, if the following inequality holds for all $x_i, y_i \in [0,c]$ with $x_i + y_i \in [0,c]$:

$$F(x_1+y_1,\ldots,x_n+y_n) \leq F(x_1,\ldots,x_n) + F(y_1,\ldots,y_n).$$

An aggregation operator $\mathbf{H} : \bigcup_{n\in\mathbb{N}} [0,c]^n \to [0,c]$ acting on $[0,c]$ is *subadditive*, if all n-ary operations $\mathbf{H}_{(n)} : [0,c]^n \to [0,c]$ are subadditive on $[0,c]$.

Before turning to aggregation operators dominating a continuous, Archimedean t-norm T, we recall that a function $f : [0,1] \to [0,\infty]$ is an additive generator of such a t-norm T if and only if f is continuous, strictly decreasing, fulfilling $f(1) = 0$, and for all $x, y \in [0,1]$:

$$T(x,y) = f^{-1}\big(\min(f(0), f(x) + f(y))\big).$$

Then we also have that $T(x_1,\ldots,x_n) = f^{-1}\big(\min(f(0), \sum_{i=1}^n f(x_i))\big)$.

Theorem 14. *Consider some continuous, Archimedean t-norm T with an additive generator $f : [0,1] \to [0,\infty]$, $f(0) = c$, and let $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be an aggregation operator. Then $\mathbf{A} \in \mathcal{D}_T$ if and only if there exists a subadditive aggregation operator $\mathbf{H} : \bigcup_{n\in\mathbb{N}} [0,c]^n \to [0,c]$, such that for all $n \in \mathbb{N}$ and for all $x_i \in [0,1]$ with $i \in \{1,\ldots,n\}$*

$$f(\mathbf{A}(x_1,\ldots,x_n)) = \mathbf{H}(f(x_1),\ldots,f(x_n)). \tag{2}$$

Proof. Let $\mathbf{A} \in \mathcal{D}_T$, i.e., for all $n \in \mathbb{N}$ and for all $x_i, y_i \in [0,1], i \in \{1,\ldots,n\}$, the following inequality holds

$$T(\mathbf{A}(x_1,\ldots,x_n),\mathbf{A}(y_1,\ldots,y_n)) \leq \mathbf{A}((T(x_1,y_1),\ldots,T(x_n,y_n))$$

and can be rewritten by

$$f^{-1}\big(\min[c, f(\mathbf{A}(x_1,\dots,x_n)) + f(\mathbf{A}(y_1,\dots,y_n))]\big) \le$$
$$\le \mathbf{A}\big(f^{-1}(\min[c, f(x_1)+f(y_1)]),\dots,f^{-1}(\min[c, f(x_n)+f(y_n)])\big). \quad (3)$$

Consider some $n \in \mathbb{N}$. Note that, for arbitrary $u_i, v_i \in [0,c]$ with $u_i + v_i \in [0,c]$ and $i \in \{1,\dots,n\}$, there exist unique $x_i, y_i \in [0,1]$ such that $u_i = f(x_i)$ and $v_i = f(y_i)$ for all $i \in \{1,\dots,n\}$. Moreover, applying f two both sides of Ineq. (3), we get

$$\min[c, f(\mathbf{A}(x_1,\dots,x_n)) + f(\mathbf{A}(y_1,\dots,y_n))] \ge f\big(\mathbf{A}(f^{-1}(u_1+v_1),\dots,f^{-1}(u_n+v_n))\big)$$

Define $\mathbf{H}_{(n)} : [0,c]^n \to [0,c]$ by

$$\mathbf{H}_{(n)}(u_1,\dots,u_n) = f\big(\mathbf{A}(f^{-1}(u_1),\dots,f^{-1}(u_n))\big), \quad (4)$$

then $\mathbf{H}_{(n)}$ is a non-decreasing mapping fulfilling

$$\begin{aligned}
\mathbf{H}_{(n)}(0,\dots,0) &= f(\mathbf{A}(1,\dots,1)) = f(1) = 0,\\
\mathbf{H}_{(n)}(c,\dots,c) &= f(\mathbf{A}(0,\dots,0)) = f(0) = c,\\
\mathbf{H}_{(n)}(u_1+v_1,\dots,u_n+v_n) &\le \min[c, \mathbf{H}_{(n)}(u_1,\dots,u_n) + \mathbf{H}_{(n)}(v_1,\dots,v_n)]\\
&\le \mathbf{H}_{(n)}(u_1,\dots,u_n) + \mathbf{H}_{(n)}(v_1,\dots,v_n),
\end{aligned}$$

i.e., for arbitrary n, $\mathbf{H}_{(n)}$ is an n-ary aggregation operator, which is subadditive on $[0,c]$ and satisfies for all $x_i \in [0,1]$ and $i \in \{1,\dots,n\}$

$$\mathbf{H}_{(n)}(f(x_1),\dots,f(x_n)) = f(\mathbf{A}(x_1,\dots,x_n)).$$

Define an aggregation operator $\mathbf{H} : \bigcup_{n\in\mathbb{N}} [0,c]^n \to [0,c]$ acting on $[0,c]$ by

$$\mathbf{H}(x_1,\dots,x_n) = \mathbf{H}_{(n)}(x_1,\dots,x_n),$$

for all $n \in \mathbb{N}$ and with $\mathbf{H}_{(n)}$ defined by Equation (4), then $\mathbf{H}$ is a subadditive aggregation operator acting on $[0,c]$ and fulfilling Equation (2).

On the other hand, for a given subadditive aggregation operator $\mathbf{H} : \bigcup_{n\in\mathbb{N}} [0,c]^n \to [0,c]$, define $\mathbf{A} : \bigcup_{n\in\mathbb{N}} : [0,1]^n \to [0,1]$ by

$$\mathbf{A}(x_1,\dots,x_n) = f^{-1}\big(\mathbf{H}(f(x_1),\dots,f(x_n))\big).$$

Evidently, $\mathbf{A}$ is an aggregation operator. Due to the subadditivity of $\mathbf{H}$, the domination inequality (3) holds for all $x_i, y_i \in [0,1]$ such that $f(x_i) + f(y_i) \le c$ with $i \in \{1,\dots,n\}$. In general, we can introduce for any given $x_i, y_i \in [0,1]$ and for all $i \in \{1,\dots,n\}$ the value z_i defined by

$$z_i = \begin{cases} y_i & \text{if } f(x_i) + f(y_i) \le c,\\ f^{-1}(c - f(x_i)) & \text{otherwise.} \end{cases}$$

It is easy to see that $z_i \ge y_i$ for all $i \in \{1,\dots,n\}$ and therefore we get

$$\begin{aligned}
T(\mathbf{A}(x_1,\dots,x_n), \mathbf{A}(y_1,\dots,y_n)) &\le T(\mathbf{A}(x_1,\dots,x_n), \mathbf{A}(z_1,\dots,z_n))\\
\le \mathbf{A}(T(x_1,z_1),\dots,T(x_n,z_n)) &= \mathbf{A}(T(x_1,y_1)\dots,T(x_n,y_n)),
\end{aligned}$$

where the first inequality is a consequence of the monotonicities of T and $\mathbf{A}$ and the second inequality follows from the subadditivity of the aggregation operator $\mathbf{H}$, proving that $\mathbf{A} \in \mathcal{D}_T$. □

One of the main purposes for investigating aggregation operators dominating t-norms was the request for introducing weights into the aggregation process. Hence, considering continuous Archimedean t-norms, we have to find subadditive aggregation operators, which provide this possibility.

Example 15. Consider some some weights $p_1, \ldots, p_n \in [0, \infty]$, $n \geq 2$, and some $c \in [0, 1]$, then $\mathbf{H}_{(n)}\ [0, c]^n \to [0, c]$ given by

$$\mathbf{H}_{(n)}(x_1, \ldots, x_n) = \min(c, \sum_{i=1}^{n} p_i x_i)$$

is an n-ary, subadditive aggregation operator on $[0, c]$, fulfilling $\mathbf{H}_{(n)}(c, \ldots, c) = c$, whenever $c \leq c \cdot \sum_{i=1}^{n} p_i$. This means, with convention $0 \cdot \infty = 0$, if $c = \infty$, the sum must fulfill $\sum_{i=1}^{n} p_i > 0$ and ,if $c < \infty$, then also $\sum_{i=1}^{n} p_i \geq 1$.

If we combine such an aggregation operator with an additive generator of a continuous Archimedean t-norm by applying the construction method as proposed in Theorem 14 we can introduce weights into the aggregation process without losing T-transitivity.

Corollary 16. *Consider a continuous Archimedean t-norm T with additive generator f, $f(0) = c$, and a weighting vector $\overrightarrow{p} = (p_1, \ldots, p_n)$, $n \geq 2$, with weights $p_i \in [0, \infty]$ fulfilling $c \leq c \cdot \sum_{i=1}^{n} p_i$. Further, let $\mathbf{A}_{(n)} : [0, 1]^n \to [0, 1]$ be an n-ary aggregation operator defined by Eq.* (2) *from the aggregation operator $\mathbf{H}_{(n)}$ introduced in Example 15. Then the n-ary aggregation operator can be rewritten by*

$$\mathbf{A}_{(n)}(x_1, \ldots, x_n) = f^{-1}\big(\min(f(0), \sum_{i=1}^{n} p_i \cdot f(x_i))\big) \tag{5}$$

and it dominates the t-norm T, i.e. $\mathbf{A}_{(n)} \gg T$.

Remark 17.

(i) The n-ary aggregation operator defined by Equation (5) is also called weighted t-norm $T_{\overrightarrow{p}}$ ([6, 7]).

(ii) Note that, for any strict t-norm T, it holds, that not only $T_{\overrightarrow{p}} \gg T$, but also $T \gg T_{\overrightarrow{p}}$. In case of some nilpotent t-norm T it is clear, that $T_{\overrightarrow{p}} \gg T$, but $T \gg T_{\overrightarrow{p}}$ only if all weights $p_i \notin]0, 1[$.

Example 18. The strongest subadditive aggregation operator acting on $[0, c]$ is given by $\mathbf{H} : \bigcup_{n \in \mathbb{N}} [0, c]^n \to [0, c]$ with

$$\mathbf{H}(u_1, \ldots, u_n) = \begin{cases} 0 & \text{if } u_1 = \ldots = u_n = 0, \\ c & \text{otherwise.} \end{cases}$$

Then, for any additive generator $f : [0, 1] \to [0, \infty]$ with $f(0) = c$, we have

$$f(\mathbf{A}(x_1, \ldots, x_n)) = \mathbf{H}\big(f(x_1), \ldots, f(x_n)\big),$$

for all $x_i \in [0,1]$ with $i \in \{1,\dots,n\}$ and some $n \in \mathbb{N}$, if and only if

$$\mathbf{A}(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } x_1 = \ldots = x_n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. $\mathbf{A} = \mathbf{A}_w$ is the weakest aggregation. Observe that $\mathbf{A}_w$ dominates all t-norms, but not all aggregation operators, e.g. $\mathbf{A}_w$ does not dominate the arithmetic mean.

4.3 Isomorphic t-norms

Another interesting aspect is the relationship or invariance of domination with respect to transformations — transformation of the dominating aggregation operator as well as of the dominated aggregation operator or of both of them. These transformations will be necessary when thinking about ordinal sums of t-norms and about isomorphic t-norms. First, we recall the transformation of an aggregation operators and the property of invariance.

Consider an aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}} [a,b]^n \to [a,b]$ on $[a,b]$ and a monotone bijection $\varphi : [c,d] \to [a,b]$. The operator $\mathbf{A}_\varphi : \bigcup_{n\in\mathbb{N}} [c,d]^n \to [c,d]$ defined by

$$\mathbf{A}_\varphi(x_1,\dots,x_n) = \varphi^{-1}\big(\mathbf{A}(\varphi(x_1),\dots,\varphi(x_n))\big)$$

is an aggregation operator on $[c,d]$, which is isomorphic to $\mathbf{A}$.

Definition 19. An aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is called *invariant with respect to a monotone bijective transformation* $\varphi : [0,1] \to [0,1]$ if $\mathbf{A} = \mathbf{A}_\varphi$. An aggregation operator $\mathbf{A}$ is called *invariant* if it is invariant with respect to all monotone bijective transformations ([5, 25, 26]).

It is trivial to see that, if $\mathbf{A} \gg \mathbf{B}$ and $\mathbf{A}$ (resp. $\mathbf{B}$) is an invariant aggregation operator, then $\mathbf{A}_\varphi \gg \mathbf{B}$ (resp. $\mathbf{A} \gg \mathbf{B}_\varphi$) for all monotone, bijective transformations $\varphi : [0,1] \to [0,1]$.

The following proposition summarizes the results for transformations of both involved aggregation operators.

Proposition 20. *Consider two aggregation operators* $\mathbf{A}$ *and* $\mathbf{B}$ *on* $[a,b]$.

(i) $\mathbf{A} \gg \mathbf{B}$ *if and only if* $\mathbf{A}_\varphi \gg \mathbf{B}_\varphi$ *for all non-decreasing bijections* $\varphi : [c,d] \to [a,b]$.

(ii) $\mathbf{A} \gg \mathbf{B}$ *if and only if* $\mathbf{B}_\varphi \gg \mathbf{A}_\varphi$ *for all non-increasing bijections* $\varphi : [c,d] \to [a,b]$.

Proof. First we show that if $\mathbf{A} \gg \mathbf{B}$ then $\mathbf{A}_\varphi \gg \mathbf{B}_\varphi$ for all non-decreasing bijections $\varphi : [c,d] \to [a,b]$. Therefore, consider some arbitrary non-decreasing bijection $\varphi : [c,d] \to [a,b]$, some $n,m \in \mathbb{N}$ and some $x_{i,j} \in [c,d]$ with $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Once again, we will use the notations $(x_{i,\bullet}) = (x_{i,1},\dots,x_{i,n})$ for all $i \in \{1,\dots,m\}$ and $(x_{\bullet,j}) = (x_{1,j},\dots,x_{m,j})$ for all $j \in \{1,\dots,n\}$, note that parentheses $[,]$ are used for indicating arguments of aggregation operators.

$$\begin{aligned}
&\mathbf{B}_\varphi\big[\mathbf{A}_\varphi[x_{1,1},\ldots,x_{1,n}],\ldots,\mathbf{A}_\varphi[x_{m,1},\ldots,x_{m,n}]\big]\\
&\quad=\varphi^{-1}\big(\mathbf{B}\big[\varphi(\mathbf{A}_\varphi[x_{1,\bullet}]),\ldots,\varphi(\mathbf{A}_\varphi[x_{m,\bullet}])\big]\big)\\
&\quad=\varphi^{-1}\big(\mathbf{B}\big[\varphi(\varphi^{-1}(\mathbf{A}[\varphi(x_{1,\bullet})])),\ldots,\varphi(\varphi^{-1}(\mathbf{A}[\varphi(x_{m,\bullet})]))\big]\big)\\
&\quad=\varphi^{-1}\big(\mathbf{B}\big[\mathbf{A}[\varphi(x_{1,\bullet})],\ldots,\mathbf{A}[\varphi(x_{m,\bullet})]\big]\big)\\
&\quad\le\varphi^{-1}(\mathbf{A}\big[\mathbf{B}[\varphi(x_{\bullet,1})],\ldots,\mathbf{B}[\varphi(x_{\bullet,n})]\big])\\
&\quad=\varphi^{-1}\big(\mathbf{A}\big[\varphi(\varphi^{-1}(\mathbf{B}[\varphi(x_{\bullet,1})])),\ldots,\varphi(\varphi^{-1}(\mathbf{B}[\varphi(x_{\bullet,n})]))\big]\big)\\
&\quad=\varphi^{-1}\big(\mathbf{A}\big[\varphi(\mathbf{B}_\varphi[x_{\bullet,1}]),\ldots,\varphi(\mathbf{B}_\varphi[x_{\bullet,n}])\big]\big)\\
&\quad=\mathbf{A}_\varphi\big[\mathbf{B}_\varphi[x_{1,1},\ldots,x_{m,1}],\ldots,\mathbf{B}_\varphi[x_{1,n},\ldots,x_{m,n}]\big]
\end{aligned}$$

If $\mathbf{A}_\varphi \gg \mathbf{B}_\varphi$ then also $(\mathbf{A}_\varphi)_\psi \gg (\mathbf{B}_\varphi)_\psi$ for all non-decreasing bijections $\psi:[a,b]\to[c,d]$, also especially for $\varphi^{-1}:[a,b]\to[c,d]$, and therefore

$$(\mathbf{A}_\varphi)_{\varphi^{-1}}=\mathbf{A}\gg\mathbf{B}=(\mathbf{B}_\varphi)_{\varphi^{-1}}.$$

The property for non-increasing bijections can be shown analogously. □

Since we are especially interested in aggregation operators dominating some t-norm T, we recall some basic properties of t-norms and their transformations. If we consider some t-norm T and demand a function T_φ defined by

$$T_\varphi(x_1,\ldots,x_n)=\varphi^{-1}\big(T(\varphi(x_1),\ldots,\varphi(x_n))\big)$$

also to be a t-norm, then $\varphi:[0,1]\to[0,1]$ has to be a strictly increasing bijection. Then the t-norms T and T_φ are called *isomorphic* t-norms ([7]). As a direct consequence of Proposition 20 we can formulate the following corollary.

Corollary 21. *Consider some aggregation operator $\mathbf{A}$ and some t-norm T. Then $\mathbf{A}\in\mathcal{D}_T$ if and only if $\mathbf{A}_\varphi\in\mathcal{D}_{T_\varphi}$ for all strictly increasing bijections $\varphi:[0,1]\to[0,1]$.*

Note that the only t-norms invariant with respect to all strictly increasing bijections are the minimum t-norm $T_{\mathbf{M}}$ and the drastic product $T_{\mathbf{D}}$.

Corollary 22. *Consider some t-norm T and some aggregation operator $\mathbf{A}\in\mathcal{D}_T$. If $\mathbf{A}$ is an invariant aggregation operator, then it dominates all isomorphic t-norms T_φ, i.e. $\mathbf{A}\in\mathcal{D}_{T_\varphi}$.*

As already mentioned, transformations and scaling of t-norms are important in constructing new t-norms from a family of given t-norms. Aggregation operators dominating such t-norms will be investigated in the next section.

4.4 Ordinal sums

Definition 23. Let $(T_\alpha)_{\alpha\in I}$ be a family of t-norms and let $(]a_\alpha,e_\alpha[)_{\alpha\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T:[0,1]^2\to[0,1]$ is a t-norm ([7]):

$$T(x,y)=\begin{cases}T^*_\alpha(x,y)=a_\alpha+(e_\alpha-a_\alpha)\cdot T\big(\frac{x-a_\alpha}{e_\alpha-a_\alpha},\frac{y-a_\alpha}{e_\alpha-a_\alpha}\big) & \text{if } (x,y)\in[a_\alpha,e_\alpha]^2\,,\\ \min(x,y) & \text{otherwise.}\end{cases}$$

The t-norm T is called the *ordinal sum* of the *summands* $\langle a_\alpha, e_\alpha, T_\alpha\rangle, \alpha \in I$, and we shall write $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in I}$.

Corresponding to t-norms, aggregation operators can also be constructed from several aggregation operators acting on non-overlapping domains. We will use the lower ordinal sum of aggregation operators ([5, 27]). Observe that this ordinal sum was originally proposed only for finitely many summands, however, we generalize this concept to an arbitrary (countable) number of summands.

Definition 24. Consider a family of aggregation operators

$$\big(\mathbf{A}_i : \bigcup_{n\in\mathbb{N}} [a_i, e_i]^n \to [a_i, e_i]\,\big)_{i\in\{1,\ldots,k\}}$$

acting on non-overlapping domains $[a_i, e_i]$ with $i \in \{1, \ldots, k\}$ and

$$0 \le a_1 < e_1 \le a_2 < e_2 \le \ldots \le e_k \le 1.$$

The aggregation operator $\mathbf{A}^{(w)}$ defined by ([5])

$$\mathbf{A}^{(w)}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } u < a_1, \\ \mathbf{A}_i\big(\min(x_1, e_i), \ldots, \min(x_n, e_i)\big) & \text{if } a_i \le u < a_{i+1}, \\ 1 & \text{if } u = 1. \end{cases}$$

with $u = \min(x_1, \ldots, x_n)$ is called the *lower ordinal sum* (of aggregation operators $\mathbf{A}_i$) and it is the weakest aggregation operator that coincides with $\mathbf{A}_i$ at inputs from $[a_i, e_i]$.

If $(\mathbf{A}_\alpha)_{\alpha\in I}$ is a family of aggregation operators all acting on $[0,1]$ and $(]a_\alpha, e_\alpha[)_{\alpha\in I}$ a (countable) family of non-empty, pairwise disjoint open subintervals of $[0,1]$, then the lower ordinal sum of this family $\mathbf{A}^{(w)} = (\langle a_\alpha, e_\alpha, \mathbf{A}_\alpha\rangle)_{\alpha\in I}$ can be constructed in the following way:

$$\mathbf{A}^{(w)}(x_1, \ldots, x_n) = \begin{cases} \sup\limits_{\alpha\in I}\{\mathbf{A}^*_\alpha\big(\min(x_1, e_\alpha), \ldots, \min(x_n, e_\alpha)\big) \mid a_\alpha \le u\} & \text{if } u < 1, \\ 1 & \text{otherwise,} \end{cases}$$

with $\sup\emptyset = 0$ and $u = \min(x_1, \ldots, x_n)$. $\mathbf{A}^*_\alpha$ denotes the aggregation operator $\mathbf{A}_\alpha$, scaled for acting on $[a_\alpha, e_\alpha]$ by

$$\mathbf{A}^*_\alpha(x_1, \ldots, x_n) = a_\alpha + (e_\alpha - a_\alpha)\cdot \mathbf{A}_\alpha\big(\frac{x_1 - a_\alpha}{e_\alpha - a_\alpha}, \ldots, \frac{x_n - a_\alpha}{e_\alpha - a_\alpha}\big).$$

Proposition 25. *Let $(T_\alpha)_{\alpha\in I}$ be a family of t-norms, $(\mathbf{A}_\alpha)_{\alpha\in I}$ a family of aggregation operators, and $(]a_\alpha, e_\alpha[)_{\alpha\in I}$ a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. If for all $\alpha \in I : \mathbf{A}_\alpha \in \mathcal{D}_{T_\alpha}$, then the lower ordinal sum $A^{(w)} = (\langle a_\alpha, e_\alpha, \mathbf{A}_\alpha\rangle)_{\alpha\in I}$ dominates the ordinal sum $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in I}$, i.e. $A^{(w)} \in \mathcal{D}_T$.*

Proof. We have to show that for all $n \in \mathbb{N}$ and for all $x_i, y_i \in [0,1], i \in \{1, \ldots, n\}$ the following inequality holds:

$$T(\mathbf{A}^{(w)}(x_1, \ldots, x_n), \mathbf{A}^{(w)}(y_1, \ldots, y_n)) \le \mathbf{A}^{(w)}(T(x_1, y_1), \ldots, T(x_n, y_n)). \tag{6}$$

Consider arbitrary $x_i, y_i \in [0,1]$ with $i, j \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and let $u = \min\{x_i, y_i \mid i \in \{1, \ldots, n\}\}$ be the smallest of these arguments, i.e. there exists some $j \in \{1, \ldots, n\}$ such that $u = x_j$ or $u = y_j$. Without loss of generality, we will suppose that $u = x_j$ for the rest of the proof. If $u = 1$ then Inequality (6) is trivially fulfilled. Therefore, we have to consider the following two cases:

Case 1. There exists some $\alpha \in I$ such that $u \in [a_\alpha, e_\alpha[$, i.e. $x_j \in [a_\alpha, e_\alpha[$, and thus also $T(x_j, y_j) \in [a_\alpha, e_\alpha[$. Therefore, $T(x_i, y_i) \geq a_\alpha$ for all arguments x_i, y_i with $i \in \{1, \ldots, n\}$ and we see from the construction of $\mathbf{A}^{(w)}$ that, with

$$x_i' = \min(x_i, e_\alpha) \text{ and } y_i' = \min(y_i, e_\alpha) \text{ for all } i \in \{1, \ldots, n\},$$

the following equality is fulfilled

$$\mathbf{A}^{(w)}\big(T(x_1, y_1), \ldots, T(x_n, y_n)\big) = \mathbf{A}^{(w)}\big(T(x_1', y_1'), \ldots, T(x_n', y_n')\big) \in [a_\alpha, e_\alpha]\,.$$

On the other hand, applying $\mathbf{A}^{(w)}$ to arguments x_i and y_i, we get

$$\mathbf{A}^{(w)}(x_1, \ldots, x_n) = \mathbf{A}^{(w)}(x_1', \ldots, x_n') \in [a_\alpha, e_\alpha] \text{ and } \mathbf{A}^{(w)}(y_1, \ldots, y_n) \geq a_\alpha.$$

If $\min\{y_i \mid i \in \{1, \ldots, n\}\} < e_\alpha$ then

$$\mathbf{A}^{(w)}(y_1, \ldots, y_n) = \mathbf{A}^{(w)}(y_1', \ldots, y_n') \in [a_\alpha, e_\alpha]\,.$$

Since $\mathbf{A}_\alpha \gg T_\alpha$, we obtain in that case

$$\begin{aligned}
&T\big(\mathbf{A}^{(w)}(x_1, \ldots, x_n), \mathbf{A}^{(w)}(y_1, \ldots, y_n)\big) \\
&\quad = T\big(\mathbf{A}^{(w)}(x_1', \ldots, x_n'), \mathbf{A}^{(w)}(y_1', \ldots, y_n')\big) \\
&\quad \leq \mathbf{A}^{(w)}\big(T(x_1', y_1'), \ldots, T(x_n', y_n')\big) \\
&\quad = \mathbf{A}^{(w)}\big(T(x_1, y_1), \ldots, T(x_n, y_n)\big).
\end{aligned}$$

If $\min\{y_i \mid i \in \{1, \ldots, n\}\} \geq e_\alpha$, then $\mathbf{A}^{(w)}(y_1, \ldots, y_n) \geq e_\alpha$ and therefore

$$\begin{aligned}
&T\big(\mathbf{A}^{(w)}(x_1, \ldots, x_n), \mathbf{A}^{(w)}(y_1, \ldots, y_n)\big) \\
&\quad = \mathbf{A}^{(w)}(x_1, \ldots, x_n) = \mathbf{A}^{(w)}(x_1', \ldots, x_n') \\
&\quad = \mathbf{A}^{(w)}(T(x_1', y_1), \ldots, T(x_n', y_n)) \\
&\quad \leq \mathbf{A}^{(w)}(T(x_1, y_1), \ldots, T(x_n, y_n)).
\end{aligned}$$

Case 2. If $u \notin \bigcup\limits_{\alpha \in I} [a_\alpha, e_\alpha[$, we know that

$$\mathbf{A}^{(w)}(x_1, \ldots, x_n) = \sup_{\alpha \in I}\{e_\alpha \mid e_\alpha \leq x_j\} = v.$$

Since $y_i \geq v$ for all $i \in \{1, \ldots, n\}$ and, therefore, $\mathbf{A}^{(w)}(y_1, \ldots, y_n) \geq v$, we obtain

$$T(\mathbf{A}^{(w)}(x_1, \ldots, x_n), \mathbf{A}^{(w)}(y_1, \ldots, y_n)) = v.$$

On the other hand, the fact that $T(x_j, y_j) = x_j$ and $T(x_i, y_i) \geq x_j$ for all $i \in \{1, \ldots, n\}$ ensuring that $\mathbf{A}^{(w)}(T(x_1, y_1) \ldots, T(x_n, y_n)) = v$.

□

Note that not all dominating aggregation operators are lower ordinal sums of dominating aggregation operators, e.g. the aggregation operator $\mathbf{A}_w$ introduced in Example 18 dominates all t-norms T, but is not a lower ordinal sum constructed by means of some index set I (in fact it is the empty lower ordinal sum). The following example also shows that weighted t-norms as proposed by Calvo and Mesiar ([6]) dominate the original t-norm but are no lower ordinal sums as proposed here. As a consequence we can conclude that $\big(\langle a_\alpha, e_\alpha, \mathcal{D}_{T_\alpha}\rangle\big)_{\alpha\in I} \subset \mathcal{D}_T$, whenever $T = (\langle a_\alpha, e_\alpha, T_\alpha)_{\alpha\in I}$.

Let $(]a_\alpha, e_\alpha[)_{\alpha\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $t_\alpha : [a_\alpha, e_\alpha] \to [0,\infty]$ be continuous, strictly decreasing mappings fulfilling $t_\alpha(e_\alpha) = 0$. Then (and only then) the following function $T : [0,1]^2 \to [0,1]$ is a continuous t-norm ([6]):

$$T(x,y) = \begin{cases} t_\alpha^{-1}\big(\min(t_\alpha(0), t_\alpha(x) + t_\alpha(y)\big) & \text{if } (x,y) \in [a_\alpha, e_\alpha]\,, \\ \min(x,x) & \text{otherwise.} \end{cases}$$

The corresponding weighted t-norm $T_{\overrightarrow{p}}$ in the sense of Calvo and Mesiar ([6]) is defined by

$$T_{\overrightarrow{p}}(x_1,\ldots,x_n) = \begin{cases} t_\alpha^{-1}(\min(t_\alpha(a_\alpha), \sum_{i=1}^n p_i \cdot t_\alpha(\min(x_i, e_\alpha)))) & \text{if } u \in [a_\alpha, e_\alpha[\,, \\ \min(x_i \mid p_i > 0) & \text{otherwise,} \end{cases}$$

with $u = \min(x_i \mid p_i > 0)$ and some weighting vector $\overrightarrow{p} = (p_1,\ldots,p_n) \neq (0,\ldots,0)$ such that, if $a_\alpha = 0$ for some $\alpha \in I$ and the corresponding $t_\alpha(a_\alpha)$ is finite, then $\sum_{i=1}^n p_i \geq 1$.

Example 26. Consider the t-norm $T = (\langle 0, \frac{1}{2}, T_{\mathbf{P}}\rangle)$, i.e.

$$T(x,y) = \begin{cases} 2xy & \text{if } (x,y) \in \left[0,\tfrac{1}{2}\right]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

We know that the geometric mean $G(x,y) = \sqrt{x\cdot y} = T_{\mathbf{P}(\frac{1}{2},\frac{1}{2})}$ dominates $T_{\mathbf{P}}$. Therefore we can construct

- the lower ordinal sum $\mathbf{A}^{(w)} = (\langle 0, \frac{1}{2}, G\rangle)$ with

$$\mathbf{A}^{(w)}(x,y) = \begin{cases} 1 & \text{if} (x,y) = (1,1), \\ \sqrt{\min(x,\frac{1}{2})\cdot\min(y,\frac{1}{2})} & \text{otherwise} \end{cases}$$

- and the weighted t-norm $T_{\overrightarrow{p}} = T_{(\frac{1}{2},\frac{1}{2})}$ by

$$T_{(\frac{1}{2},\frac{1}{2})}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in \left]\tfrac{1}{2},1\right]^2, \\ \sqrt{\min(x,\frac{1}{2})\cdot\min(y,\frac{1}{2})} & \text{otherwise.} \end{cases}$$

Both aggregation operators—$\mathbf{A}^{(w)}$ as well as $T_{\overrightarrow{p}}$—dominate the t-norm T. Note that they coincide in any values except for arguments $(x,y) \in \left]\frac{1}{2},1\right]^2 \setminus \{(1,1)\}$.

5 Domination of Basic t-Norms

Finally we will discuss the classes of aggregation operators dominating one of the basic t-norms as introduced in Example 3.

5.1 Domination of the minimum t-norm

As already observed, $T_{\mathbf{M}}$ dominates any t-norm T and any aggregation operator $\mathbf{A}$, but no t-norm T, except $T_{\mathbf{M}}$ itself, dominates $T_{\mathbf{M}}$. The class of all aggregation operators dominating $T_{\mathbf{M}}$ is described in the following proposition.

Proposition 27. *For any $n \in \mathbb{N}$, the class of all n-ary aggregation operators $\mathbf{A}_{(n)}$ dominating the strongest t-norm $T_{\mathbf{M}}$ is given by*

$$\begin{aligned}\mathcal{D}^{(n)}_{\min} = \{\min_{\mathcal{F}} \mid \mathcal{F} = (f_1, \ldots, f_n),&\\ &f_i : [0,1] \to [0,1],\ \textit{non-decreasing},\ \textit{with}\\ &f_i(1) = 1 \textit{ for all } i \in \{1, \ldots, n\},\\ &f_i(0) = 0 \textit{ for at least one } i \in \{1, \ldots, n\}\},\end{aligned}$$

where $\min_{\mathcal{F}}(x_1, \ldots, x_n) = \min(f_1(x_1), \ldots, f_n(x_n))$.

Proof. If $\mathbf{A}_{(n)} \gg T_{\mathbf{M}}$, we know that

$$\begin{aligned}\mathbf{A}_{(n)}(x_1, \ldots, x_n) &\leq \min\big(\mathbf{A}_{(n)}(x_1, 1, \ldots, 1), \ldots, \mathbf{A}_{(n)}(1, \ldots, 1, x_n)\big)\\ &\leq \mathbf{A}_{(n)}(x_1, \ldots, x_n),\end{aligned}$$

where the first inequality holds due to the monotonicity of $\mathbf{A}_{(n)}$ and the second one due to domination. Consequently,

$$\mathbf{A}_{(n)}(x_1, \ldots, x_n) = \min\big(\mathbf{A}_{(n)}(x_1, 1, \ldots, 1), \ldots, \mathbf{A}_{(n)}(1, \ldots, 1, x_n)\big).$$

Define functions $f_i : [0,1] \to [0,1]$ for all $i \in \{1, \ldots, n\}$ by

$$f_i(x_i) = \mathbf{A}_{(n)}(1, \ldots, 1, x_i, 1 \ldots, 1).$$

Since $\mathbf{A}_{(n)}(1, \ldots, 1) = 1$ and $\mathbf{A}_{(n)}(0, \ldots, 0) = 0$, we know that $f_i(1) = 1$ for all $i \in \{1, \ldots, n\}$ and $f_i(0) = 0$ for at least one $i \in \{1, \ldots, n\}$. The monotonicity of $\mathbf{A}_{(n)}$ assures that all f_i are non-decreasing and therefore $\mathbf{A}_{(n)} = \min_{\mathcal{F}}$, where $\mathcal{F} = (f_1, \ldots, f_n)$.

On the other side, if $\mathbf{A}_{(n)} = \min_{\mathcal{F}}$, we can deduce from the non-decreasingness of all f_i that

$$\begin{aligned}&T_{\mathbf{M}}\big(\mathbf{A}_{(n)}(x_1, \ldots, x_n), \mathbf{A}_{(n)}(y_1, \ldots, y_n)\big)\\ &\quad= \min\big(\min_{\mathcal{F}}(x_1, \ldots, x_n), \min_{\mathcal{F}}(y_1, \ldots, y_n)\big)\\ &\quad= \min(f_1(x_1), f_1(y_1), \ldots, f_n(x_n), f_n(y_n))\\ &\quad\leq \min(f_1(\min(x_1, y_1)), \ldots, f_n(\min(x_n, y_n)))\\ &\quad= \mathbf{A}_{(n)}(T_{\mathbf{M}}(x_1, y_1), \ldots, T_{\mathbf{M}}(x_n, y_n)),\end{aligned}$$

concluding that $\mathbf{A}_{(n)} \gg T_{\mathbf{M}}$. $\square$

Evidently, $\mathbf{A}_{(n)} \in \mathcal{D}_{\min}^{(n)}$ is symmetric if and only if

$$\mathbf{A}_{(n)}(x_1, \dots, x_n) = f\big(\min(x_1, \dots, x_n)\big)$$

for some non-decreasing function $f : [0,1] \to [0,1]$ fulfilling $f(0) = 0$ and $f(1) = 1$.

Example 28. As already observed in Example 18, the weakest aggregation operator $\mathbf{A}_w$ dominates all t-norms T. Since this aggregation operator is symmetric, it can be described by $\mathbf{A}_w(x_1, \dots, x_n) = f\big(\min(x_1, \dots, x_n)\big)$ with $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 29. Any aggregation operator $\mathbf{A}$ dominating $T_{\mathbf{M}}$ is also dominated by $T_{\mathbf{M}}$, i.e. for arbitrary $n, m \in \mathbb{N}$ and for all $x_{i,j} \in [0,1]$ with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ the following equality holds

$$\begin{aligned} &\mathbf{A}\big(\min(x_{1,1}, \dots, x_{1,n}), \dots, \min(x_{m,1}, \dots, x_{m,n})\big) \\ &\quad = \min\big(\mathbf{A}(x_{1,1}, \dots, x_{m,1}), \dots, \mathbf{A}(x_{1,n}, \dots, x_{m,n})\big). \end{aligned}$$

Consequently, for any necessity measures ([28, 29, 30]) $\mathrm{Nec}_1, \dots, \mathrm{Nec}_n$ on some measurable space $(X, \mathcal{A})$, also $\mathrm{Nec} = \mathbf{A}(\mathrm{Nec}_1, \dots, \mathrm{Nec}_n)$ is a necessity measure. Note that only operators $\mathbf{A} \in \mathcal{D}_{T_{\mathbf{M}}}$ fulfill this property. By duality, a similar result for the aggregation of possibility measures ([28, 29, 30]) can be obtained.

5.2 Domination of the drastic product

Oppositely to the case of $T_{\mathbf{M}}$, the weakest t-norm $T_{\mathbf{D}} : [0,1]^2 \to [0,1]$, i.e.

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is dominated by any t-norm T. This can also be seen from the characterization of all aggregation operators dominating $T_{\mathbf{D}}$ as given in the next proposition.

Proposition 30. *Consider an arbitrary $n \in \mathbb{N}$ and an n-ary aggregation operator $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$. Then $\mathbf{A}_{(n)} \gg T_{\mathbf{D}}$ if and only if there exists a non-empty subset $I = \{k_1, \dots, k_m\} \subseteq \{1, \dots, n\}, k_1 < \dots < k_m$, and a non-decreasing mapping $B : [0,1]^m \to [0,1]$ satisfying the following conditions*

(i) $B(0, \dots, 0) = 0$,

(ii) $B(u_1, \dots, u_m) = 1$ *if and only if* $u_1 = \dots = u_m = 1$,

such that $\mathbf{A}(x_1, \dots, x_n) = B(x_{k_1}, \dots, x_{k_m})$.

Proof. We have to show that the inequality

$$T_{\mathbf{D}}\big(\mathbf{A}_{(n)}(x_1,\ldots,x_n),\mathbf{A}_{(n)}(y_1,\ldots,y_n)\big) \le \mathbf{A}_{(n)}\big(T_{\mathbf{D}}(x_1,y_1,\ldots,T_{\mathbf{D}}(x_n,y_n)\big) \tag{7}$$

holds for all $x_i, y_i \in [0,1], i \in \{1,\ldots,n\}$, if and only if $\mathbf{A}_{(n)}$ can be described by a non-decreasing mapping B as introduced above.

To see the sufficiency, it is enough to observe that for all $x_i, y_i \in [0,1]$ with $i \in \{1,\ldots,n\}$ the expression

$$T_{\mathbf{D}}\big(\mathbf{A}_{(n)}(x_1,\ldots,x_n),\mathbf{A}_{(n)}(y_1,\ldots,y_n)\big) = T_{\mathbf{D}}\big(B(x_{k_1},\ldots,x_{k_m}),B(y_{k_1},\ldots,y_{k_m})\big)$$

is positive only if either $(x_{k_1},\ldots,x_{k_m}) = (1,\ldots,1)$ or $(y_{k_1},\ldots,y_{k_m}) = (1,\ldots,1)$. Without loss of generality, we suppose that $(x_{k_1},\ldots,x_{k_m}) = (1,\ldots,1)$.
As a consequence, $T_{\mathbf{D}}(x_{k_j},y_{k_j}) = y_{k_j}$ for all $j \in \{1,\ldots,m\}$ and therefore

$$\begin{aligned}\mathbf{A}_{(n)}(T_{\mathbf{D}}(x_1,y_1),\ldots,T_{\mathbf{D}}(x_n,y_n)) &= B(y_{k_1},\ldots,y_{k_m})\\ &= T_{\mathbf{D}}(\mathbf{A}_{(n)}(x_1,\ldots,x_n),\mathbf{A}(y_1,\ldots,y_n)),\end{aligned}$$

i.e. $\mathbf{A} \gg T_{\mathbf{D}}$.

Concerning necessity, suppose $\mathbf{A}_{(n)} \gg T_{\mathbf{D}}$, i.e. Inequality (7) is fulfilled for all $x_i, y_i \in [0,1]$ with $i, j \in \{1,\ldots,n\}$.

We have for all $i \in \{1,\ldots,n\}$ that if $x_i \in [0,1[$, then $T_{\mathbf{D}}(x_i,x_i) = 0$ implying $\mathbf{A}_{(n)}(x_1,\ldots,x_n) < 1$. Hence, if $\mathbf{A}(v_1,\ldots,v_n) = 1$ for some $v_i \in [0,1], i \in \{1,\ldots,n\}$, there exists necessarily some index set $J = \{k \in \{1,\ldots,n\} \mid v_k = 1\} \neq \emptyset$. Moreover,

$$T_{\mathbf{D}}(\mathbf{A}_{(n)}(v_1,\ldots,v_n),\mathbf{A}_{(n)}(v_1,\ldots,v_n)) = 1 \le \mathbf{A}_{(n)}(T_{\mathbf{D}}(v_1,v_1),\ldots,T_{\mathbf{D}}(v_n,v_n)), \text{ i.e.}$$

$$\mathbf{A}_{(n)}(z_1^{(J)},\ldots,z_n^{(J)}) = 1, \text{ where } z_i^{(J)} = \begin{cases}1 & \text{if } i \in J,\\ 0 & \text{otherwise.}\end{cases}$$

If it holds for two subsets $J, K \subseteq \{1,\ldots,n\}$ that

$$\mathbf{A}_{(n)}(z_1^{(J)},\ldots,z_n^{(J)}) = \mathbf{A}_{(n)}(z_1^{(K)},\ldots,z_n^{(K)}) = 1,$$

we can conclude that

$$\begin{aligned}&T_{\mathbf{D}}\big(\mathbf{A}_{(n)}(z_1^{(J)},\ldots,z_n^{(J)}),\mathbf{A}_{(n)}(z_1^{(K)},\ldots,z_n^{(K)})\big) = 1\\ &\quad\le \mathbf{A}_{(n)}\big(T_{\mathbf{D}}(z_1^{(J)},z_1^{(K)}),\ldots,T_{\mathbf{D}}(z_n^{(J)},z_n^{(K)})\big) = \mathbf{A}_{(n)}\big(z_1^{(J\cap K)},\ldots,z_n^{(J\cap K)}\big),\end{aligned}$$

showing that also $\mathbf{A}_{(n)}\big(z_1^{(J\cap K)},\ldots,z_1^{(J\cap K)}\big) = 1$. This fact ensures the existence of a unique, minimal non-empty subset $I \subseteq \{1,\ldots,n\}$, for which

$$\mathbf{A}_{(n)}\big(z_1^{(I)},\ldots,z_n^{(I)}\big) = 1.$$

The monotonicity of $\mathbf{A}_{(n)}$ ensures that, for arbitrary $x_i \in [0,1], i \in \{1,\ldots,n\}$, the following inequalities hold

$$\mathbf{A}_{(n)}(x_1^{'},\ldots,x_n^{'}) \le \mathbf{A}_{(n)}(x_1,\ldots,x_n) \le \mathbf{A}_{(n)}(x_1^{''},\ldots,x_n^{''}) \tag{8}$$

$$\text{where} \qquad x_i^{'} = \begin{cases}x_i & \text{if } i \in I,\\ 0 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad x_i^{''} = \begin{cases}x_i & \text{if } i \in I,\\ 1 & \text{otherwise.}\end{cases}$$

Therefore, we know that

$$\mathbf{A}_{(n)}(x_1^{''},\dots,x_n^{''}) = T_{\mathbf{D}}\big(\mathbf{A}_{(n)}(z_1^{(I)},\dots,z_n^{(I)}),\mathbf{A}_{(n)}(x_1^{''},\dots,x_n^{''})\big)$$
$$\leq \mathbf{A}_{(n)}\big(T_{\mathbf{D}}(z_1^{(I)},x_1^{''}),\dots,T_{\mathbf{D}}(z_n^{(I)},x_n^{''})\big) = \mathbf{A}_{(n)}(x_1^{'},\dots,x_n^{'}),$$

concluding that the inequality signs of Eq. (8) can be replaced by equality signs, i.e.

$$\mathbf{A}_{(n)}(x_1^{'},\dots,x_n^{'}) = \mathbf{A}(x_1,\dots,x_n) = \mathbf{A}(x_1^{''},\dots,x_n^{''}). \tag{9}$$

Now we define a mapping $B : [0,1]^m \to [0,1]$ by

$$B(t_1\dots,t_m) = \mathbf{A}_{(n)}(s_1,\dots,s_n),\ \text{where } s_i = \begin{cases} t_j & \text{if } i = k_j, \\ 0 & \text{otherwise.}\end{cases}$$

Then B is a non-decreasing mapping, fulfilling $B(0,\dots,0) = \mathbf{A}_{(n)}(0,\dots,0) = 0$ and $B(1,\dots,1) = \mathbf{A}(z_1^{(I)},\dots,z_n^{(I)}) = 1$ and due to the minimality of I, $B(t_1,\dots,t_m) < 1$ whenever $(t_1,\dots,t_m) \neq (1,\dots,1)$. Moreover, it holds that

$$\mathbf{A}(x_1,\dots,x_n) = B(x_{k_1},\dots,x_{k_m})$$

because of Eq. (9). □

Observe that mapping B in the above proposition is an m-ary aggregation operator whenever $m \geq 2$. However, if $m = 1$, i.e. $I = \{k\}$, then $B : [0,1] \to [0,1]$ is a non-decreasing mapping with strict maximum $B(1) = 1$ and $B(0) = 0$ as well as $A(x_1,\dots,x_n) = B(x_k)$ and is therefore a distortion of the k-th projection.

Concerning t-norms, for any t-norm T, we have $T(x_1,\dots,x_n) = 1$ if and only if $x_i = 1$ for all $i \in \{1,\dots,n\}$ and thus $I = \{1,\dots,n\}$. Therefore $B = T$ and $T \in \mathcal{D}_{T_{\mathbf{D}}}$.

5.3 Domination of product t-norm and Łukasiewicz t-norm

Each strict t-norm T is isomorphic to the product t-norm ([7]) and, therefore, know-ledge about aggregation operators dominating the product t-norm gives immediately knowledge about aggregation operators dominating any strict t-norm T. Analogously, the characterization of aggregation operators dominating the Łukasiewicz t-norm $T_{\mathbf{L}}$ leads directly to aggregation operators dominating some given nilpotent t-norm, since any nilpotent t-norm is isomorphic to the Łukasiewicz t-norm ([7]).

Though the classes $\mathcal{D}_{T_{\mathbf{P}}}$ and $\mathcal{D}_{T_{\mathbf{L}}}$ are completely characterized either by Theorem 9 or by Theorem 14, there is no counterpart of Proposition 27 in these cases. However, it is possible to give examples of members of these classes and, of course, apply Proposition 10, Proposition 20 or Theorem 14 to obtain new members.

Example 31. For any $n \geq 2$ and any $\overrightarrow{p} = (p_1,\dots,p_n)$ with $\sum_{i=1}^n p_i > 0$ and $p_i \in [0,\infty]$, the function $\mathbf{H} : [0,\infty]^n \to [0,\infty]$ defined by

$$\mathbf{H}(x_1,\dots,x_n) = \sum_{i=1}^n p_i \cdot x_i$$

is an n-ary, subadditive aggregation operator acting on $[0, \infty]$. Therefore, any n-ary aggregation operator

$$\mathbf{A}_{\overrightarrow{p}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{p_i}$$

dominates the product t-norm $T_{\mathbf{P}}$. Particularly, if we consider a binary aggregation operator $\mathbf{A}_{p,q}$ and let $p, q \in]0, \infty[$ with $p + q = 1$, then $\mathbf{A}_{p,q}$ is a weighted geometric mean dominating the product t-norm ([22, 23]).

However, observing that for all $\lambda \geq 1$, the function

$$\mathbf{H}_\lambda : [0, \infty]^2 \to [0, \infty], \mathbf{H}_\lambda(x, y) = (x^\lambda + y^\lambda)^{\frac{1}{\lambda}},$$

is also a binary, subadditive aggregation operator acting on $[0, \infty]$, also any member of the Aczél-Alsina family of t-norms ([7]) $(T_\lambda^{\mathbf{AA}})_{\lambda \in [1,\infty]}$, is contained in $\mathcal{D}_{T_{\mathbf{P}}}$ because of Theorem 14.

Example 32. Further, for any $n \geq 2$ and any $\overrightarrow{p} = (p_1, \ldots, p_n)$ with $\sum_{i=1}^{n} p_i \geq 1$ and $p_i \in [0, \infty]$, the function $\mathbf{H} : [0, 1]^n \to [0, 1]$ defined by

$$\mathbf{H}(x_1, \ldots, x_n) = \min(1, \sum_{i=1}^{n} p_i \cdot x_i)$$

is again an n-ary, subadditive aggregation operator acting on $[0, 1]$. Therefore, any n-ary aggregation operator

$$\mathbf{A}_{\overrightarrow{p}}(x_1, \ldots, x_n) = \max(0, \sum_{i=1}^{n} p_i \cdot x_i + 1 - \sum_{i=1}^{n} p_i)$$

dominates the Łukasiewicz t-norm $T_{\mathbf{L}}$. Particularly, if we just consider a binary operator and let $p, q \in]0, \infty[$ with $p + q = 1$, then any $\mathbf{A}_{p,q} = px + qy$, i.e. any weighted mean dominates $T_{\mathbf{L}}$ ([22, 23]).

Based on $\mathbf{H}_\lambda : [0, 1]^2 \to [0, 1], \mathbf{H}_\lambda(x, y) = (x^\lambda + y^\lambda)^{\frac{1}{\lambda}}$, any Yager t-norm $T_\lambda^{\mathbf{Y}}$ dominates $T_{\mathbf{L}}$, whenever $\lambda \geq 1$.

6 Summary

An aggregation operator $\mathbf{A}$ preserves T-transitivity of fuzzy relations if and only if it dominates the corresponding t-norm T ($\mathbf{A} \in \mathcal{D}_T$). Several methods for constructing aggregation operators within a certain class $\mathcal{D}_{\mathbf{A}}$ with $\mathbf{A}$ some t-norm or some aggregation operator have been mentioned. An explicit description of $\mathcal{D}_T$ could be presented for the minimum t-norm $T_{\mathbf{M}}$ and the drastic product $T_{\mathbf{D}}$.

Acknowledgements

This work was partly supported by network CEEPUS SK-42 and COST Action 274 “TARSKI”. Radko Mesiar was also supported by the grant VEGA 1/8331/01. Ulrich Bodenhofer acknowledges support of the K*plus* Competence Center Program which is funded by the Austrian Government, the Province of Upper Austria, and the Chamber of Commerce of Upper Austria.

References

[1] F. E. Petry and P. Bosc. *Fuzzy Databases: Principles and Applications*. International Series in Intelligent Technologies. Kluwer Academic Publishers, Boston, 1996.

[2] U. Bodenhofer and J. Küng. Enriching vague queries by fuzzy orderings. In *Proc. 2nd Int. Conf. in Fuzzy Logic and Technology (EUSFLAT 2001)*, pages 360–364, Leicester, UK, September 2001.

[3] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000.

[4] B. Bouchon-Meunier, editor. *Aggregation and Fusion of Imperfect Information*, volume 12 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 1998.

[5] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar. Aggregation operators: Properties, classes and construction methods. In T. Calvo, G. Mayor, and R. Mesiar, editors, *Aggregation Operators*, volume 97 of *Studies in Fuzziness and Soft Computing*, pages 3 – 104. Physica-Verlag, Heidelberg, 2002.

[6] T. Calvo and R. Mesiar. Weighted means based on triangular conorms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 9(2):183–196, 2001.

[7] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 2000.

[8] K. Menger. Statistical metrics. *Proc. Nat. Acad. Sci. U.S.A.*, 8:535–537, 1942.

[9] B. Schweizer and A. Sklar. Statistical metric spaces. *Pacific J. Math.*, 10:313–334, 1960.

[10] B. Schweizer and A. Sklar. Associative functions and statistical triangle inequalities. *Publ. Math. Debrecen*, 8:169–186, 1961.

[11] L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.

[12] J. C. Bezdek and J. D. Harris. Fuzzy partitions and relations: An axiomatic basis for clustering. *Fuzzy Sets and Systems*, 1:111–127, 1978.

[13] U. Höhle. Fuzzy equalities and indistinguishability. In *Proc. EUFIT'93*, volume 1, pages 358–363, 1993.

[14] B. De Baets and R. Mesiar. Pseudo-metrics and T-equivalences. *J. Fuzzy Math.*, 5:471–481, 1997.

[15] B. De Baets and R. Mesiar. $\mathcal{T}$-partitions. *Fuzzy Sets and Systems*, 97:211–223, 1998.

[16] U. Höhle and N. Blanchard. Partial ordering in L-underdeterminate sets. *Inform. Sci.*, 35:133–144, 1985.

[17] U. Bodenhofer. *A Similarity-Based Generalization of Fuzzy Orderings*, volume C 26 of *Schriftenreihe der Johannes-Kepler-Universität Linz*. Universitätsverlag Rudolf Trauner, 1999.

[18] U. Bodenhofer. Representations and constructions of similarity-based fuzzy orderings. Fuzzy Sets and Systems. (to appear).

[19] R. M. Tardiff. Topologies for probabilistic metric spaces. *Pacific J. Math.*, 65:233–251, 1976.

[20] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North-Holland, Amsterdam, 1983.

[21] J. Aczél. *Lectures on Functional Equations and their Applications*. Academic Press, New York, 1966.

[22] A. Pradera, E. Trillas, and E. Castiñeira. On the aggregation of some classes of fuzzy relations. In B. Bouchon-Meunier, J. Guitiérrez-Ríoz, L. Magdalena, and R. R. Yager, editors, *Technologies for Constructing Intelligent Systems 1: Tasks*. Springer, 2002. (to appear).

[23] A. Pradera and E. Trillas. A note on pseudometric aggregation. submitted to Internat. J. Gen. Systems.

[24] J. Dobos. *Metric Preserving Functions*. Stroffek, 1998.

[25] T. Rückschlossová. Invariant aggregation operators. *Journal of Electrical Engineering*, 52(10/s):78–81, 2001.

[26] S. Ovchinnikov and A. Dukhovny. Integral representation of invariant functionals. *J. Math. Anal. Appl.*, 244:228–232, 2000.

[27] R. Mesiar and B. De Baets. New construction methods for aggregation operators. In *Proc. 8th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2000)*, volume 2, pages 701–706, Madrid, 2000.

[28] D. Dubois and H. Prade. *Possibility Theory. An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York, 1988.

[29] D. Dubois and H. Prade. Aggregation of possibility measures. In J. Kacprzyk and M. Fedrizzi, editors, *Multipperson Decision Making Using Fuzzy Sets and Possibility Theory*, pages 55 –63. Kluwer Academic Publishers, Dordrecht, 1990.

[30] D. Dubois, J. Fodor, H. Prade, and M. Roubens. Aggregation of decomposbabe measures with application to utility theory. *Theory and Decision*, 41:59 – 95, 1996.