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Domination of multilinear singular integrals by positive sparse forms

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Abstract

We establish a uniform domination of the family of trilinear multiplier forms with singularity over a one-dimensional subspace by positive sparse forms involving L^p-averages. This class includes the adjoint forms to the bilinear Hilbert transforms. Our result strengthens the L^p-boundedness proved by Muscalu, Tao and Thiele, and entails as a corollary a novel rich multilinear weighted theory. A particular case of this theory is the $L^q(v_1) \times L^q(v_2)$-boundedness of the bilinear Hilbert transform when the weight v_j belong to the class $A_{\frac{q+1}{2}} \cap RH_2$. Our proof relies on a stopping time construction based on newly developed localized outer-L^p embedding theorems for the wave packet transform. In an Appendix, we show how our domination principle can be applied to recover the vector-valued bounds for the bilinear Hilbert transforms recently proved by Benea and Muscalu.

1. *Introduction and main results*

The L^p-boundedness theory of Calderón-Zygmund operators, whose prototype is the Hilbert transform, plays a central role in harmonic analysis and in its applications to elliptic partial differential equations, geometric measure theory and related fields.

A recent remarkable discovery is that the action of a singular integral operator T on a function f can be dominated in a pointwise sense by the averages of f over a *sparse*, i.e. essentially disjoint, collection of cubes in $\mathbb{R}^n$. This control is much stronger than L^p-norm bounds and carries significantly more information on the operator itself. As of now, the most striking consequence is that sharp weighted norm inequalities for T follow from the corresponding, rather immediate estimates for the averaging operators. Such a pointwise domination principle, albeit in a slightly weaker sense, appears explicitly for the first time in the proof of the A_2 theorem by Lerner [**24**]. We also point out the recent improvements by Lacey [**21**] and Lerner [**25**], and the analogue for multilinear Calderón-Zygmund operators obtained independently by Conde-Alonso and Rey [**7**] and by Lerner and Nazarov [**23**]. Most recently, Bernicot, Frey and Petermichl [**4**] extend this approach to non-integral singular operators associated with a second-order elliptic operator, lying outside the scope of classical Calderón-Zygmund theory.

The main focus of the present article is to formulate a similar principle for the class of multilinear multiplier operators, invariant under simultaneous modulations of the input functions, which includes the bilinear Hilbert transforms. Besides their intrinsic interest, our results yield a rich, and sharp in a suitable sense, family of multilinear weighted bounds for this class of operators. In fact, Theorem 1.6 below is the first result of this kind. Weighted estimates for the bilinear Hilbert transforms have been mentioned as an open problem in several related works [**11**, **12**, **15**].

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Let $\Gamma = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 + \xi_2 + \xi_3 = 0\}$ and $\beta \in \Gamma$ be a fixed unit vector, nondegenerate in the sense that

$$\Delta_\beta = \min_{k \neq j} |\beta_k - \beta_j| > 0.$$

We are concerned with the trilinear forms

$$\Lambda_m(f_1, f_2, f_2) = \int_\Gamma m(\xi) \prod_{j=1}^{3} \widehat{f}_j(\xi_j)\, \mathrm{d}\xi \tag{1.1}$$

acting on triples of Schwartz functions on $\mathbb{R}$, where $m : \Gamma \to \mathbb{C}$ is a Fourier multiplier satisfying, in multi-index notation,

$$\sup_{|\alpha| \leq N} \sup_{\xi \in \Gamma} \big(\mathrm{dist}(\xi, \beta^\perp)\big)^\alpha \big|\partial_\alpha m(\xi)\big| \leq C_N. \tag{1.2}$$

The one-parameter family (with respect to β) of trilinear forms adjoint to the bilinear Hilbert transforms is obtained by choosing

$$m(\xi) = \mathrm{sign}(\xi \cdot \beta).$$

In [**31**], substantially elaborating on the seminal work by Lacey and Thiele [**18**, **19**], Muscalu, Tao and Thiele prove the following result.

Theorem 1.1. *[**31**, Theorem 1.1] Let m be a multiplier satisfying (1.2). Then the adjoint bilinear operators T_m to the forms Λ_m of (1.1) have the mapping properties*

$$T_m : L^{q_1}(\mathbb{R}) \times L^{q_2}(\mathbb{R}) \to L^{\frac{q_1 q_2}{q_1+q_2}}(\mathbb{R}) \tag{1.3}$$

for all exponent pairs (q_1, q_2) satisfying $1 < \inf\{q_1, q_2\} < \infty$ and

$$\tfrac{1}{q_1} + \tfrac{1}{q_2} < \tfrac{3}{2}. \tag{1.4}$$

Not unexpectedly, a pointwise domination principle for this class of bilinear operators is not allowed to hold, as we elaborate in Remark 1.5 below. This obstruction is overcome by introducing the closely related notion of *domination by sparse positive forms* of the adjoint trilinear form, which we turn to in what follows.

We say that $\mathcal{S}$ is a *η-sparse collection* of intervals $I \subset \mathbb{R}$ if for every $I \in \mathcal{S}$ there exists a measurable $E_I \subset I$ with $|E_I| \geq \eta |I|$ such that $\{E_I : I \in \mathcal{S}\}$ are pairwise disjoint. The positive sparse trilinear form of type $\vec{p} = (p_1, p_2, p_3)$ associated to the sparse collection $\mathcal{S}$ is defined by

$$\mathsf{PSF}^{\vec{p}}_{\mathcal{S}}(f_1, f_2, f_3)(x) = \sum_{I \in \mathcal{S}} |I| \prod_{j=1}^{3} \langle f_j \rangle_{I, p_j}, \qquad \langle f \rangle_{I,p} := \left(\frac{1}{|I|} \int_I |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}; \tag{1.5}$$

we omit the subscript and write $\langle f \rangle_I$ when $p = 1$. A rather immediate consequence of the Hardy-Littlewood maximal theorem is the following proposition. We omit the proof, which is a simplified version of the proof of Corollary A.1 given in the appendix.

Proposition 1.2. *Let T be a bilinear operator. Suppose that for all tuples $(f_1, f_2, f_3) \in \mathcal{C}_0^\infty(\mathbb{R})^3$ there holds*

$$|\langle T(f_1, f_2), f_3 \rangle| \leq K \sup_{\mathcal{S}\ \eta-\mathrm{sparse}} \mathsf{PSF}^{\vec{p}}_{\mathcal{S}}(f_1, f_2, f_3)$$

Then for all $(f_1, f_2) \in \mathcal{C}_0^\infty(\mathbb{R})^2$ there holds

$$\|T(f_1, f_2)\|_{\frac{q_1 q_2}{q_1+q_2}} \leq K C_{q_1, q_2, \eta} \prod_{j=1}^{2} \|f_j\|_{q_j} \tag{1.6}$$

provided that $p_j < q_j \le \infty$ *for* $j = 1, 2$ *and* $\inf\{q_1, q_2\} < \infty$.

Our main result is a strengthening of Theorem 1.1 to a domination by positive sparse forms. To formulate it, we need one more notion. We say that $\vec{p} = (p_1, p_2, p_3)$ is an *admissible tuple* if

$$1 \le p_1, p_2, p_3 < \infty, \qquad \varepsilon(\vec{p}) := 2 - \sum_{j=1}^{3} \tfrac{1}{\min\{p_j, 2\}} \ge 0 \tag{1.7}$$

If all the constraints hold with strict inequality, we say that $\vec{p}$ is an *open admissible tuple*.

THEOREM 1.3. *Let* $\vec{p}$ *be an open admissible tuple. There exists* $K = K(\vec{p}), N = N(\vec{p})$ *such that the following holds. For any tuple* $(f_1, f_2, f_3) \in \mathcal{C}_0^\infty(\mathbb{R})^3$ *there exists a* $\frac{1}{6}$*-sparse collection* $\mathcal{S}$ *such that*

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \le K C_N \mathsf{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3), \tag{1.8}$$

where the supremum is being taken over the family of multipliers m *satisfying* (1.2).

We stress that the constants K and N depend only on the exponent tuple $\vec{p}$, and the choice of the sparse collection $\mathcal{S}$ depends only on f_1, f_2, f_3 and $\vec{p}$ and is, in particular, independent of the multiplier m.

REMARK 1.4 Sharpness of Theorem 1.3. Let (q_1, q_2) be an exponent pair with

$$1 < \inf\{q_1, q_2\} < \infty.$$

Then there exists an open admissible tuple $\vec{p} = (p_1, p_2, p_3)$ with $p_1 < q_1, p_2 < q_2$ if and only if (1.4) holds for (q_1, q_2). This observation, coupled with Proposition 1.2, yields Theorem 1.1 as a corollary of Theorem 1.3.

On the other hand, let ϕ be an even Schwartz function with $\mathbf{1}_{[-2^{-4}, 2^4]} \le \widehat{\phi} \le \mathbf{1}_{[-2^{-3}, 2^{-3}]}$, $\{\beta, \gamma\}$ be an orthonormal basis of Γ. Define the family of multipliers on Γ

$$m_{\vec{\sigma}, M}(\xi) = \sum_{n=0}^{M-1} \sigma_n \widehat{\phi}\left(2^8(\xi_1 - (\eta^n)_1)\right) \widehat{\phi}\left(2^8(\xi_2 - (\eta^n)_2)\right) \widehat{\phi}\left(\xi_3 - (\eta^n)_3\right) \tag{1.9}$$

where $\eta^n = n\gamma + \beta$, $n \in \mathbb{N}$. The same argument as in [**20**, Section 2.2] yields

$$\sup_{\vec{\sigma} \in \{-1,1\}^M} \left\| T_{m_{\vec{\sigma}, M}} \right\|_{L^{q_1} \times L^{q_2} \to L^{\frac{q_1 q_2}{q_1 + q_2}}} \ge C M^{\frac{1}{q_1} + \frac{1}{q_2} - \frac{3}{2}}$$

while the family $\{m_{\vec{\sigma}, M} : M \in \mathbb{N}, \vec{\sigma} \in \{-1, 1\}^M\}$ satisfies (1.2) uniformly. This implies that the range (1.4) of Theorem 1.1 is sharp up to equality holding in (1.4) and, in turn, that (1.8) cannot hold for any tuple violating (1.7). Hence, Theorem 1.3 is sharp up to possibly replacing the assumption *open admissible* with the stronger *admissible*. The behavior of the forms Λ_m for tuples at the boundary of the admissible region is studied in detail in [**10**].

REMARK 1.5 No uniform control by a bilinear positive sparse operator. For bilinear Calderón-Zygmund operators T, there holds a pointwise domination by sparse operators of the type

$$|T(f_1, f_2)(x)| \le C \sum_{I \in \mathcal{S}(f_1, f_2)} \langle f_1 \rangle_{I, p_1} \langle f_2 \rangle_{I, p_2} \mathbf{1}_I(x).$$

One can take $p_1 = p_2 = 1$: see [**23**]. Essentially self-adjoint operators T enjoying such pointwise domination inherit the boundedness property

$$T : L^1 \times L^{p_j} \to L^{\frac{p_j}{1+p_j},\infty}$$

which, as described in the previous Remark 1.4, fails for the generic T_m of the class (1.2) when $\inf\{p_1, p_2\} < 2$. In fact, no L^1-boundedness properties are expected to hold even for the bilinear Hilbert transforms. Summarizing, no such pointwise domination principle can be obtained for T_m when $\inf\{p_1, p_2\} < 2$ and, most likely, neither for the case when $\inf\{p_1, p_2\} \geq 2$. Our formulation in terms of positive sparse forms overcomes this obstacle: a similar idea, albeit not explicit, appears in the linear setting in [**4**]. After the first version of this article was made public, several works based on sparse form domination have appeared within and beyond Calderón-Zygmund theory, see for example [**2**, **6**, **22**, **27**] and references therein.

Theorem 1.3 implies multilinear weighted bounds for the forms Λ_m. Our main weighted theorem will involve multilinear $A^{\vec p}_{\vec q}$ Muckenhoupt constants. Given any tuple $\vec p$, a Hölder tuple $\vec q$ and a weight vector $\vec v = (v_1, v_2, v_3)$ satisfying

$$\prod_{j=1}^{3} v_j^{\frac{1}{q_j}} = 1, \tag{1.10}$$

these are defined as

$$[\vec v]_{A^{\vec p}_{\vec q}} := \sup_{I\subset\mathbb{R}} \prod_{j=1}^{3} \langle v_j^{\frac{p_j}{p_j-q_j}} \rangle_I^{\frac{1}{p_j}-\frac{1}{q_j}}. \tag{1.11}$$

For $\vec p = (1, 1, 1)$, these weight classes have been introduced in [**26**], to which we send for an exhaustive discussion of their properties. A particular case of (1.11) (where $p_1 = 1$) can be found in [**16**] as a necessary and sufficient condition for weighted L^q-boundedness of the bilinear fractional integrals. Furthermore, the classes (1.11) appear in ongoing work on multilinear Calderón-Zygmund operators satisfying Hörmander type conditions [**5**].

Theorem 1.6. *Let $\vec q$ be a Hölder tuple with $1 < q_1, q_2, q_3 < \infty$ and $\vec v$ be a weight vector satisfying* (1.10)*. Then there holds*

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \leq \left(\inf_{\vec p} C(\vec p, \vec q) [\vec v]_{A^{\vec p}_{\vec q}}^{\max\left\{\frac{q_j}{q_j-p_j}\right\}} \right) \prod_{j=1}^{3} \|f_j\|_{L^{q_j}(v_j)}$$

where the supremum is being taken over the family of multipliers m satisfying (1.2)*, the infimum is taken over open admissible tuples $\vec p$ with $p_j < q_j$, and*

$$C(\vec p, \vec q) = K(\vec p) C_{N(\vec p)} \left(\prod_{j=1}^{3} \frac{q_j}{q_j-p_j} \right) 2^{3\left(\sum_{j=1}^{3} \frac{1}{p_j} - 1\right) \max\left\{\frac{p_j}{q_j-p_j}\right\}}. \tag{1.12}$$

One is usually interested in weighted estimates involving Muckenhoupt and reverse Hölder constants of each single weight. Recall that the A_q and RH_α constant of a weight v on $\mathbb{R}$ are defined as

$$[v]_{A_q} := \sup_{I\subset\mathbb{R}} \langle v\rangle_I \langle v^{\frac{1}{1-q}}\rangle_I^{q-1}, \qquad [v]_{RH_\alpha} := \sup_{I\subset\mathbb{R}} \langle v^\alpha\rangle_I^{\frac{1}{\alpha}} \langle v\rangle_I^{-1},$$

A suitable choice of admissible tuple $\vec{p}$ in Theorem 1.6 yields the following corollary.[†]

COROLLARY 1.7. *Let*

$$1 < q_1, q_2, r = \tfrac{q_1 q_2}{q_1+q_2} < \infty.$$

and v_1, v_2 *be given weights with* $v_1^2 \in A_{q_1}, v_2^2 \in A_{q_2}$. *Then the operator norms*

$$T_m : L^{q_1}(v_1) \times L^{q_2}(v_2) \to L^r(u_3), \qquad u_3 := \prod_{j=1}^{2} v_j^{\frac{r}{q_j}}$$

of the family of multipliers satisfying (1.2) *with uniform constants* C_N *are uniformly bounded above by a positive constant depending on* $\{q_j,\, [v_j^2]_{A_{q_j}} : j = 1, 2\}$ *only.*

We refer to the recent monograph [**8**] for details on the A_q and RH_α classes. Here we remark that if $q > 1$ then [**8**, Section 3.8] $v \in A_{\frac{q+1}{2}} \cap RH_2$ if and only if $v^2 \in A_q$. We mention that a theory of *linear* extrapolation for weights in the $A_q \cap RH_\alpha$ classes has been introduced in [**1**]; see also the already mentioned monograph [**8**].

As a further application of Corollary 1.7, weighted, vector-valued estimates for multipliers T_m satisfying condition (1.2), extending the results of [**3**, **32**] can be obtained by a multilinear version of the extrapolation theory of [**1**]. These extensions are the object of an upcoming companion article by the same authors. However, Theorem 1.3 can be employed to recover the unweighted vector-valued estimates of [**3**, **32**] in a rather direct fashion. In order to keep our outline as simple as possible, we postpone the complete statement and proof of the vector-valued estimates to Appendix A.

Structure of the article and proof techniques

The class of multipliers (1.2), in addition to the familiar invariances under isotropic dilations and translations proper of Coifman-Meyer type multipliers, enjoys a one-parameter invariance under simultaneous modulation of the three input functions along the line $\mathbb{R}\gamma = \{\beta, (1, 1, 1)\}^\perp$. The invariance properties of the class (1.2) are essentially shared by a family of discretized trilinear forms involving the maximal wave packet coefficients of the input functions parametrized by rank 1 collection of tritiles, which we call *tritile form*.

The first step in the proof of Theorem 1.3, carried out in Section 2, is to establish that for any multiplier m satisfying (1.2), the form Λ_m lies in the convex hull of finitely many tritile forms. This discretization procedure is largely the same as the one employed in [**31**]. Theorem 1.3 then reduces to the analogous result for tritile forms, Theorem 2.2. It is of paramount importance here that the sparse collection $\mathcal{S}$ constructed in Theorem 2.2 is independent of the particular tritile form.

The explicit construction of the collection $\mathcal{S}$, and in fact the proof of Theorem 2.2, is performed in Section 5 by means of an inductive argument. The intervals of $\mathcal{S}$ are, roughly speaking, the stopping intervals of the p_j-Hardy-Littlewood maximal function of the j-th input. At each stage of the argument, the contribution of those wave packets localized within one of the stopping intervals will be estimated at the next step of the induction, after a careful removal of the tail terms. The main term, which is the contribution of the wave packets whose spatial localization is not contained in the union of the stopping interval is estimated by means of a localized outer L^{p_j} embedding Theorem for the wave packet transform.

[†]We have come to know that Xiaochun Li [**28**] has some unpublished results about weighted estimates for the bilinear Hilbert transforms.

This outer L^p embedding, which is the concern of Proposition 4.1, is a close relative of the main result of [**9**] by two of us, namely, a localized embedding theorem for the *continuous* wave packet transform. In fact, while Proposition 4.1 is proved here via a transference argument based upon [**9**, Theorem 1], a direct proof can be given by repeating the arguments of [**9**] in the discrete setting. The construction of the outer L^p spaces on rank 1 collections, which parallels the outer L^p theory introduced by Do and Thiele in [**13**], is performed in Section 3.

Section 6 contains the proof of the weighted estimates of Theorem 1.6 and 1.7, and the concluding Section A is dedicated to vector-valued extensions.

Notation

Let $\chi(x) = (1+|x|^2)^{-1}$. For an interval I centered at $c(I)$ and of length $\ell(I) = |I|$, we write

$$\chi_I(x) := \chi\left(\tfrac{x-c(I))}{\ell(I)}\right). \tag{1.13}$$

We will make use of the weighted L^p spaces

$$\|f\|_{L^p(\chi_I^N)} := \left(\frac{1}{|I|}\int_{\mathbb{R}} |f(x)|^p(\chi_I(x))^N\right)^{1/p}, \ 1 \le p < \infty, \qquad \|f\|_{L^\infty(\chi_I^N)} = \|f\chi_I^N\|_\infty.$$

with N positive integer. We write

$$\mathrm{M}_p(f)(x) = \sup_{I\subset\mathbb{R}} \langle f\rangle_{I,p}\mathbf{1}_I(x)$$

for the p-Hardy Littlewood maximal functions. Finally, the positive constants denoted by C or those implied by almost inequality sign $\lesssim$ and the comparability sign $\sim$ are meant to be absolute throughout the article and may vary from line to line without explicit mention.

2. *Tritile maps*

In this section, we reduce Theorem 1.3 to the corresponding statement for a class of multilinear forms which we call *tritile maps*. Throughout, we assume that the nondegenerate unit vector $\beta \in \Gamma$ is fixed and let $\gamma \in \Gamma$ be a unit vector perpendicular to β, spanning the singular line of the multipliers m from (1.2).

2.1. *Rank 1 collection of tri-tiles*

A *tile* $T = I_T \times \omega_T$ is the cartesian product of a *time interval* I_T and a *frequency interval* ω_T obeying the uncertainty principle $|I_T||\omega_T| \sim 1$. Heuristically, a tile specifies the time-frequency localization of the corresponding associated wave packet *adapted family* $\mathbf{\Phi}(T)$, see below. A tri-tile $P = (P_1, P_2, P_3)$ is an ordered triple of tiles $P_j, j = 1, 2, 3$ with the property that

$$I_{P_1} = I_{P_2} = I_{P_3} =: I_P.$$

The spatial interval I_P and the *frequency cube* $\omega_{P_1} \times \omega_{P_2} \times \omega_{P_3}$ corresponding to P specify the time-frequency localization of the trilinear multiplier forms associated to each tri-tile appearing in the sum (2.4) below. In the following definitions, it is convenient to denote by $\overline{\omega_P}$ the convex hull of the intervals $3\omega_{P_j}$, $j = 1, 2, 3$.

Let $\mathsf{g} > 10$ be a large parameter. We say that the collection of tri-tiles $\mathbb{P}$ is of rank 1 if

a. $\mathcal{I} = \{I_P : P \in \mathbb{P}\}$ and $\Omega_j = \{\omega_P : P \in \mathbb{P}\}$, $j = 1, 2, 3$ are $\log \mathsf{g}$ scale-separated dyadic grids;
b. if $P \ne P' \in \mathbb{P}$ are such that $I_P = I_{P'}$ then $\omega_{P_j} \cap \omega_{P'_j} = \emptyset$ for each $j \in \{1, 2, 3\}$;
c. if $P, Q \in \mathbb{P}$ are such that $\omega_{P_j} \subset \omega_{Q_j}$ for some $j \in \{1, 2, 3\}$ then $\mathsf{g}\overline{\omega_P} \subset \mathsf{g}\overline{\omega_Q}$;
d. if $P, Q \in \mathbb{P}$ are such that $\omega_{P_j} \subset \omega_{Q_j}$ for some $j \in \{1, 2, 3\}$ then $3\omega_{P_k} \cap 3\omega_{Q_k} = \emptyset$ for $k \in \{1, 2, 3\} \setminus \{j\}$.

In the remainder of the paper, we can and will use the value $\mathsf{g} \sim (\Delta_\beta)^{-1}$. If $\mathbb{P}$ is a rank 1 collection of tri-tiles, we will also make use of the notations

$$\mathbb{P}_{\leq}(I) := \{P \in \mathbb{P} : I_P \subset I\}, \qquad \mathbb{P}_{=}(I) := \{P \in \mathbb{P} : I_P = I\} \tag{2.1}$$

for the localizations if $\mathbb{P}$ to the interval $I \subset \mathbb{R}$.

2.2. *Tritile forms*

Let A_N be a fixed increasing sequence of positive constants. For each tile T we define the *adapted family* $\mathbf{\Phi}(T)$ to be the collection of Schwartz functions ϕ_T satisfying

$$\sup_{n\leq N} \sup_{x\in\mathbb{R}} |I_T|^{n+1}\chi_I(x)^{-N} \left| \left(\mathrm{e}^{-ic(\omega_T)\cdot}\phi_T(\cdot)\right)(x)\right| \leq A_N, \qquad \operatorname{supp}\widehat{\phi_T} \subset \omega_T. \tag{2.2}$$

Let $\mathbb{P}$ be a rank 1 collection of tritiles and $f_j \in L^1_{\mathrm{loc}}(\mathbb{R})$. We define the tritile maps $F_j : \mathbb{P} \to \mathbb{C}$ by

$$F_j(f)(P) = \sup_{\phi_{P_j}\in\mathbf{\Phi}(P_j)} |\langle f_j, \phi_{P_j}\rangle|, \qquad j = 1, 2, 3 \tag{2.3}$$

and the trisublinear tritile form associated to $\mathbb{P}$ by

$$\Lambda_{\mathbb{P}}(f_1, f_2, f_3) = \sum_{P\in\mathbb{P}} |I_P| \prod_{j=1}^{3} F_j(f_j)(P). \tag{2.4}$$

2.3. *Reduction to uniform bounds for tritile forms*

The following lemma is a reformulation of the well-known discretization procedure from [**31**]. Several versions of this procedure have since appeared, see for instance the monographs [**30**, **33**]. We omit the standard (by now) proof.

Lemma 2.1. *There exists a finite collection $\{\mathbb{P}^1, \ldots, \mathbb{P}^{\mathsf{G}}\}$ of rank 1 collections of tritiles such that, for any multiplier m satisfying* (1.2) *and any tuple of Schwartz functions f_1, f_2, f_3, there holds*

$$|\Lambda_m(f_1, f_2, f_3)| \leq \sum_{j=1}^{\mathsf{G}} \Lambda_{\mathbb{P}^j}(f_1, f_2, f_3)$$

and the adaptation constants $\{A_N\}$ of the adapted families defining $\Lambda_{\mathbb{P}^j}$ depend on $\{C_N\}$ only. Furthermore, the character G depends only on the nondegeneracy constant of β.

Theorem 1.3 is then an immediate consequence of Lemma 2.1 and of the following discretized version, whose proof is given in Section 5.

Theorem 2.2. *Let $\vec{p}$ be an open admissible tuple. There exists $K = K(\vec{p}), N = N(\vec{p})$ such that the following holds. For any tuple (f_1, f_2, f_3) with $f_j \in L^{p_j}(\mathbb{R})$ and compactly supported there exists a $\frac{1}{6}$-sparse collection $\mathcal{S}$ such that*

$$\sup_{\mathbb{P}} \Lambda_{\mathbb{P}}(f_1, f_2, f_3) \leq K(A_N)^3 \mathsf{PSF}^{\vec{p}}_{\mathcal{S}}(f_1, f_2, f_3),$$

where the supremum is being taken over all rank 1 collections of tritiles $\mathbb{P}$ of finite cardinality and adaptation sequence $\{A_N\}$. In particular, $\mathcal{S}$ depends only on f_1, f_2, f_3 and the tuple $\vec{p}$.

3. *Outer L^p spaces of tritiles*

In this section, we formulate the outer measure space that is needed for our proof, which is based on a finite rank 1 collection of tritiles $\mathbb{P}$. Recall that $\omega_P = \mathrm{co}(3\omega_{P_1}, 3\omega_{P_2}, 3\omega_{P_3})$. The generating collection is the set of trees $\mathcal{T} \subset \mathcal{P}(\mathbb{P})$. The set $\mathsf{T} \subset \mathbb{P}$ is a tree with top data $(I_{\mathsf{T}}, \xi_{\mathsf{T}})$, where I_{T} is an interval on the real line and $\xi_{\mathsf{T}} \in \mathbb{R}$ if

$$I_P \subset I_{\mathsf{T}}, \qquad \xi_{\mathsf{T}} \in \omega_P \qquad \forall P \in \mathbb{P}.$$

By property d. of the rank 1 collections, we have that each tree T can be written as the union

$$\mathsf{T} = \bigcup_{1\leq j<k\leq 3} \mathsf{T} \setminus (\mathsf{T}_j \cup \mathsf{T}_k) \tag{3.1}$$

where each T_j is a tree with the same top data as T and has the additional property

$$\{3\omega_{P_k} : P \in \mathsf{T}_j\} \text{ are a pairwise disjoint collection for } k \neq j.$$

We consider the outer measure on $\mathcal{P}(\mathbb{P})$

$$\mu(E) = \inf \left\{ \sum_{j=1}^{\infty} |I_{\mathsf{T}_j}| : \{\mathsf{T}_j : j \in \mathbb{N}\} \subset \mathcal{T} \text{ is a cover of } E \right\}$$

In this article, with *size* we indicate a sublinear map $\mathsf{s} : \mathbb{C}^{\mathbb{P}} \to [0, +\infty]$: for all $\lambda \in \mathbb{R}$, and $F, G : \mathbb{P} \to \mathbb{C}$

$$\mathsf{s}_j(\lambda F + G) \leq |\lambda| \mathsf{s}_j(F) + \mathsf{s}_j(G).$$

For $F : \mathbb{P} \to \mathbb{C}$, the *super level measure* $\mu(\mathsf{s}(F) > \lambda)$ is defined to be the infimum of all values $\mu(E)$ over all $E \subset \mathbb{P}$ such that

$$\sup_{\mathsf{T}\in\mathcal{T}} \mathsf{s}(f \mathbf{1}_{E^c})(\mathsf{T}) \leq \lambda.$$

For $0 < p \leq \infty$ we make use of the super level measures to define the strong and weak outer L^p norms induced by the size s as

$$\begin{aligned}
\|F\|_{L^\infty(\mathbb{P},\mathsf{s})} &:= \sup_{\mathsf{T}\in\mathcal{T}} \mathsf{s}(F)(\mathsf{T}), \\
\|F\|_{L^p(\mathbb{P},\mathsf{s})} &:= \left(\int_0^\infty p\lambda^{p-1} \mu(\mathsf{s}(F) > \lambda) \,\mathrm{d}\lambda \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \\
\|F\|_{L^{p,\infty}(\mathbb{P},\mathsf{s})} &:= \sup_{\lambda>0} \lambda \left[\mu(\mathsf{s}(F) > \lambda)\right]^{\frac{1}{p}}, \quad 0 < p < \infty.
\end{aligned} \tag{3.2}$$

In particular, we will be concerned with the sizes

$$\mathsf{s}_j(F)(\mathsf{T}) := \left(\frac{1}{|I_{\mathsf{T}}|} \sum_{P \in \mathsf{T}\setminus\mathsf{T}_j} |I_P| |F(P)|^2 \right)^{\frac{1}{2}} + \sup_{P\in\mathsf{T}} |F(P)|, \qquad j = 1, 2, 3. \tag{3.3}$$

We will make use of the natural outer Hölder's inequality involving the sizes s_j, which is stated and proved below.

Lemma 3.1. *Let*

$$\vec{q} = (q_1, q_2, q_3), \qquad 1 \leq q_j \leq \infty, \qquad \sum_{j=1}^{3} \frac{1}{q_j} = 1$$

be a Hölder tuple. Let $G_j : \mathbb{P} \to \mathbb{C}$, $j = 1, 2, 3$. Then

$$\sum_{P\in\mathbb{P}} |I_P| \prod_{j=1}^{3} |G_j(P)| \lesssim \prod_{j=1}^{3} \|G_j\|_{L^{q_j}(\mathbb{P},\mathsf{s}_j)}.$$

Proof. Define another size

$$\mathsf{s}^1(F)(\mathsf{T}) := \frac{1}{|I_\mathsf{T}|} \sum_{P\in\mathsf{T}} |I_P||F(P)|.$$

Then it is obvious that for any T there holds

$$\sum_{P\in\mathsf{T}} |I_P| \prod_{j=1}^{3} |G_j(P)| \le |I_\mathsf{T}|\mathsf{s}^1\left(\prod_{j=1}^{3} G_j\right)(\mathsf{T}),$$

which by the Radon-Nikodym proposition [**13**, Proposition 3.6] implies that

$$\sum_{P\in\mathbb{P}} |I_P| \prod_{j=1}^{3} |G_j(P)| \lesssim \Big\| \prod_{j=1}^{3} G_j \Big\|_{L^1(\mathbb{P},\mathsf{s}^1)}.$$

Furthermore, according to (3.1) and the classical Hölder's inequality, one can easily check that for any fixed T,

$$\begin{aligned}
&\mathsf{s}^1\left(\prod_{j=1}^{3} G_j\right)(\mathsf{T}) \\
\le &\frac{1}{|I_\mathsf{T}|} \sum_{1\le j<k\le 3} \sum_{P\in\mathsf{T}\setminus(\mathsf{T}_j\cup\mathsf{T}_k)} |I_P|| \prod_{j=1}^{3} G_j(P)| \\
\le &\sum_{1\le j<k\le 3} \left[\left(\sum_{P\in\mathsf{T}\setminus\mathsf{T}_j} |I_P||G_j(P)|^2 \right)^{1/2} \left(\sum_{P\in\mathsf{T}\setminus\mathsf{T}_k} |I_P||G_k(P)|^2 \right)^{1/2} \prod_{i\ne j,k} \sup_{P\in\mathsf{T}} |G_i(P)| \right] \\
\lesssim &\prod_{j=1}^{3} \mathsf{s}_j(G_j)(\mathsf{T}).
\end{aligned}$$

Hence the outer Hölder inequality in [**13**] yields that

$$\Big\| \prod_{j=1}^{3} G_j \Big\|_{L^1(\mathbb{P},\mathsf{s}^1)} \lesssim \prod_{j=1}^{3} \|G_j\|_{L^{q_j}(\mathbb{P},\mathsf{s}_j)},$$

which completes the proof. □

4. *Localized Carleson embeddings*

In this section, when we write *dyadic interval*, we mean intervals $I \in \mathcal{D}$, where $\mathcal{D}$ is a fixed dyadic grid on $\mathbb{R}$. Fix a dyadic interval $Q \subset \mathbb{R}$ and $f \in L^p(\mathbb{R})$. We define the p-stopping intervals of f on Q by

$$\mathcal{I}_{f,p,Q} = \text{maximal dyadic } I \subset Q \text{ s.t. } I \subset \{x \in \mathbb{R} : \mathrm{M}_p(f\mathbf{1}_{3Q})(x) > C\langle f\rangle_{3Q,p}\} \tag{4.1}$$

Notice that $\mathcal{I}_{f,p,Q}$ is a pairwise disjoint collection of dyadic intervals and that the maximal theorem guarantees the sparsity condition

$$\sum_{I\in\mathcal{I}_{f,p,Q}} |I| \leq |\{x\in\mathbb{R} : \mathrm{M}_p(f\mathbf{1}_{3Q})(x) > C\langle f\rangle_{3Q,p}\}| \leq \frac{|Q|}{6} \tag{4.2}$$

provided the constant C in (4.1) is chosen large enough. Furthermore, from the very definition of $\mathcal{I}_{f,p,Q}$, there holds

$$\inf_{x\in 3I} \mathrm{M}_p(f\mathbf{1}_{3Q})(x) \lesssim \langle f\rangle_{3Q,p} \qquad \forall I\in\mathcal{I}_{f,p,Q}. \tag{4.3}$$

In what follows, we fix a finite collection of rank 1 tritiles $\mathbb{P}$ whose intervals $\{I_P : P\in\mathbb{P}\}$ are dyadic. Referring to the notation (2.1) for $\mathbb{P}_{\leq}(I)$ we define the set of *good tritiles*

$$\mathsf{G}_{f,p,Q} = \mathbb{P}\setminus\left(\bigcup_{I\in\mathcal{I}_{f,p,Q}} \mathbb{P}_{\leq}(I)\right). \tag{4.4}$$

Recalling the definition of the tritile maps from (2.3), we have the following proposition, which is used to control the main term of the tritile forms (2.4) localized to $3Q$.

Proposition 4.1. *Let $j=1,2,3$, $Q\subset\mathbb{R}$ be a dyadic interval and f be a Schwartz function. For any $1<p<2$, $q>p'$ there exists a positive integer $N=N(p,q)$ and a positive constant $\Theta=\Theta(p,q)$ such that*

$$\left\|F_j(f\mathbf{1}_{3Q})\mathbf{1}_{\mathsf{G}_{f,p,Q}}\right\|_{L^q(\mathbb{P},\mathsf{s}_j)} \leq \Theta A_N |Q|^{\frac{1}{q}}\langle f\rangle_{3Q,p}. \tag{LC$_{q,p}$}$$

4.1. *Proof of Proposition 4.1*

The Proposition will be proved by a transference argument using the main result of [**9**], which is the continuous parameter version recalled below. However, Proposition 4.1 may also be obtained directly, by repeating the arguments of [**9**] in the (simpler, in fact) discrete parameter setting. We leave the details to the interested reader.

4.1.1. *A continuous parameters version of Proposition 4.1* We need to define the continuous outer measure space on the base set

$$\mathbb{P}^\circ = [-R,R]\times(0,R]\times[-R,R], \qquad R = 10\max\{c(I_P)+|I_P|+c(\omega_P)+|\omega_P| : P\in\mathbb{P}\};$$

we are using that $\mathbb{P}$ is a finite set. Let $I\subset\mathbb{R}$ be an interval and $\xi\in\mathbb{R}$. The corresponding *generalized tent* and its *lacunary part*, with fixed geometric parameters g,b, are defined by

$$\mathsf{T}^\circ(I,\xi) = \{(u,t,\eta)\in\mathbb{P}^\circ : 0<t<|I|, |u-c(I)|<|I|-t, |\eta-\xi|\leq \mathsf{g}t^{-1}\},$$
$$\mathsf{T}^\circ_\ell(I,\xi) = \{(u,t,\eta)\in\mathsf{T}^\circ(I,\xi) : t|\xi-\eta|>\mathsf{b}\}.$$

We use the superscript $^\circ$ to distinguish discrete *trees* T with top data $(I_\mathsf{T},\xi_\mathsf{T})$ from continuous *tents* T° with same top data $(I_\mathsf{T},\xi_\mathsf{T})$. It will also be convenient to use the notation

$$\mathsf{T}^\circ(I) = \{(u,t,\eta)\in\mathbb{P}^\circ : 0<t<|I|, |u-c(I)|<|I|-t,\}$$

for the projection of $\mathsf{T}^\circ(I,\xi)$ on the first two components. An outer measure μ° on $\mathbb{P}^\circ$, with

$$\mathcal{T}^\circ = \{\mathsf{T}^\circ(I,\xi) : I\subset[-R,R], \xi\in[-R,R]\}$$

as generating collection is then defined for $E\subset\mathbb{P}^\circ$ as

$$\mu^\circ(E) = \inf\left\{\sum_{j=1}^\infty |I_{\mathsf{T}^\circ_j}| : \{\mathsf{T}^\circ_j : j\in\mathbb{N}\}\subset\mathcal{T}^\circ \text{ is a cover of } E\right\}$$

For $F : Z \to \mathbb{C}$ Borel measurable, we define the size

$$\mathsf{s}^\circ(F)(\mathsf{T}^\circ(I,\xi)) := \left(\frac{1}{|I|}\int_{\mathsf{T}^\circ_\ell(I,\xi)} |F(u,t,\eta)|^2 \,\mathrm{d}u\mathrm{d}t\mathrm{d}\eta\right)^{\frac{1}{2}} + \sup_{(u,t,\eta)\in\mathsf{T}^\circ(I,\xi)} |F(u,t,\eta)|. \tag{4.5}$$

Denoting by $L^p(\mathbb{P}^\circ,\mathsf{s}^\circ)$, $L^{p,\infty}(\mathbb{P}^\circ,\mathsf{s}^\circ)$ the corresponding strong and weak outer L^p-spaces defined in a totally analogous way to (3.2), we turn to the reformulation of the main result of [**9**]. A family of Schwartz functions

$$\Phi := \{\phi_{u,t,\eta} : (u,t,\eta)\in\mathbb{P}^\circ\}$$

is said to be an *adapted system* with *adaptation constants* A_N if

$$\sup_{(u,t,\eta)\in\mathbb{P}^\circ}\ \sup_{n\le N}\ \sup_{x\in\mathbb{R}} t^{n+1}\chi\left(\frac{x-u}{t}\right)^{-N}\left|(\mathrm{e}^{-i\eta\cdot}\phi_{u,t,\eta}(\cdot))^{(n)}(x)\right| \le A_N \tag{4.6}$$

for all nonnegative integers N and furthermore

$$t|\zeta-\eta|>1 \implies \widehat{\phi_{t,\eta}}(\zeta)=0.$$

The wave packet transform of a Schwartz function f is then a function on $\mathbb{P}^\circ$ defined by

$$F^\circ(f)(u,t,\eta) = |\langle f,\phi_{u,t,\eta}\rangle|.$$

With the same notation as in (4.1) for $\mathcal{I}_{f,p,Q}$, and introducing the corresponding good set of parameters

$$\mathsf{G}^\circ_{f,p,Q} = \mathbb{P}^\circ \setminus \bigcup_{I\in\mathcal{I}_{f,p,Q}} \mathsf{T}^\circ(3I) \tag{4.7}$$

we have the following continuous parameter version of Proposition 4.1.

Proposition 4.2. *[**9**, Theorem 1] Let $Q\subset\mathbb{R}$ be a dyadic interval and f be a Schwartz function. For any $1<p<2$, $q>p'$ there exists a positive integer $N=N(p,q)$ and a positive constant $\Theta=\Theta(p,q)$ such that*

$$\left\|F^\circ(f\mathbf{1}_{3Q})\mathbf{1}_{\mathsf{G}^\circ_{f,p,Q}}\right\|_{L^q(\mathbb{P}^\circ,\mathsf{s}^\circ)} \le \Theta A_N |Q|^{\frac{1}{q}}\langle f\rangle_{3Q,p}. \tag{4.8}$$

Remark 4.3. The above proposition is obtained by choosing $\lambda = |Q|^{-\frac{1}{p}}$ in [**9**, Theorem 1]. There are, however, two minor discrepancies between the result of [**9**] and the one recalled above. The first one is that, in definition (4.7), the intervals $\mathcal{I}_{\mathrm{M}_1 f,p,Q}$ are used in place of $\mathcal{I}_{f,p,Q}$. This change is necessary in order to perform a reduction argument to compact support in η of $F^\circ(f)$ see [**9**, Section 7.3.1], and can thus be avoided in the setup of Proposition 4.2 since the parameter η is already in a compact interval. The second difference is that the adapted family Φ used in [**9**] to define the wave packet transform is obtained by applying dilation, translation and modulation symmetries to a fixed mother wave packet. However, the arguments of [**9**] adapt naturally to the more general transform obtained from (4.6). We leave the details for the interested reader.

4.1.2. *Transference* For each $P\in\mathbb{P}$ define

$$P^\circ := \{(u,t,\eta)\in\mathbb{P}^\circ : |I_P|\le t\le 2|I_P|,\ u\in I_P,\ \eta\in\omega_P\}.$$

Up to possibly splitting $\mathbb{P}$ into finitely many subcollections the sets $\{P^\circ : P\in\mathbb{P}\}$ are pairwise disjoint subsets of $\mathbb{P}^\circ$. Furthermore, the $\mathrm{d}u\mathrm{d}t\mathrm{d}\eta$-measure of P° is comparable to $|I_P|$ up to a constant factor. Let f be a fixed Schwartz function and $\{\phi_{P_j} : P \in \mathbb{P}\}$ be chosen such that

$$F_j(f)(P) \le 2|\langle f, \phi_{P_j}\rangle| =: \bar{F}_j(f)(P) \qquad \forall P \in \mathbb{P}.$$

Then the family defined by $\phi_{u,t,\eta} = \phi_{P_j}$ for all $(u,t,\eta) \in P^\circ$, $\phi_{u,t,\eta} = 0$ if (u,t,η) does not belong to any P° is an adapted system. As $j = 1,2,3$ is fixed we may write s in place of s_j for simplicity. We claim that, if $F^\circ(f)$ is the corresponding wave packet transform

$$\mu\left(\mathsf{s}\left(\bar{F}_j(f)\mathbf{1}_{\mathsf{G}_{f,p,Q}}\right) > C\lambda\right) \le \mu^\circ\left(\mathsf{s}^\circ\left(F^\circ(f)\mathbf{1}_{\mathsf{G}^\circ_{f,p,Q}}\right) > \lambda\right) \tag{4.9}$$

which, by virtue of the above definitions and of Proposition 4.2, implies the estimate of Proposition 4.1. Let λ be fixed and L denote the right hand side of (4.9). Let $\{\mathsf{T}^\circ_j(I_j,\xi_j)\}$ be a countable collection of tents such that

$$\sum_j |I_j| \le L + \varepsilon, \qquad \sup_{\mathsf{T}^\circ} \mathsf{s}^\circ\left(F^\circ(f)\mathbf{1}_{\mathsf{G}^\circ_{f,p,Q}}\mathbf{1}_{E^{\circ c}}\right)(\mathsf{T}^\circ) \le \lambda, \qquad E^\circ := \bigcup_j \mathsf{T}^\circ_j$$

Now, for each j, let $\mathsf{T}_j = \mathsf{T}_j(I_j,\xi_j)$ be the maximal tree of tritiles with top data (I_j,ξ_j) same as T°_j and set

$$E := \bigcup_j \mathsf{T}_j \implies \mu(E) \le \sum_j |I_j| \le L + \varepsilon.$$

To obtain (4.9) and conclude the proof it then suffices to show that for all $\mathsf{T} \in \mathcal{T}$ we have

$$\mathsf{s}\left(\bar{F}_j(f)\mathbf{1}_{\mathsf{G}_{f,p,Q}}\mathbf{1}_{E^c}\right)(\mathsf{T}) \le C\mathsf{s}^\circ\left(F^\circ(f)\mathbf{1}_{\mathsf{G}^\circ_{f,p,Q}}\mathbf{1}_{E^{\circ c}}\right)(\mathsf{T}^\circ) \tag{4.10}$$

where T° is the tent with same top data as T. Let us verify this for the L^2 portion of the size s. This is a consequence of the following observations

– if $P \in \mathsf{T} \setminus \mathsf{T}_1$ (i.e. P belongs to the lacunary part), then $P^\circ \subset \mathsf{T}^\circ_\ell$

– if $P \in E^c \cap \mathsf{G}_{f,p,Q}$, then $\widetilde{P^\circ} := P^\circ \cap E^{\circ c} \cap \mathsf{G}^\circ_{f,p,Q}$ has dudtdη-measure larger than $C^{-1}|I_P|$

of which we leave the verification to the reader, and of the computation

$$\begin{aligned}&\sum_{\substack{P\in\mathsf{T}\setminus\mathsf{T}_1\\ P\in E^c\cap\mathsf{G}_{f,p,Q}}} |I_P||\bar{F}_j(f)(P)|^2 \\ &= \sum_{\substack{P\in\mathsf{T}\setminus\mathsf{T}_1\\ P\in E^c\cap\mathsf{G}_{f,p,Q}}} \frac{|I_P|}{\nu(\widetilde{P^\circ})}\int_{\widetilde{P^\circ}} |F^\circ(f)(u,t,\eta)|^2\,\mathrm{d}u\mathrm{d}t\mathrm{d}\eta \le C\int_{\mathsf{T}^\circ_\ell\cap E^{\circ c}\cap\mathsf{G}^\circ_{f,p,Q}} |F^\circ(f)(u,t,\eta)|^2\,\mathrm{d}u\mathrm{d}t\mathrm{d}\eta\end{aligned}$$

where we have denoted by ν the dudtdη measure. The proof is complete.

5. *Proof of Theorem 2.2*

Now we are ready to prove Theorem 2.2, to which Theorem 1.3 has been reduced. Since for any open admissible tuple $\vec{r}$ there exists an open admissible tuple $\vec{p}$ with $\max\{p_j\} < 2$ and $p_j \le r_j$, it suffices to prove the case $\max\{p_j\} < 2$. Such a tuple $\vec{p}$ is fixed from now on.

5.1. *Construction of the sparse collection*

Let $f_j \in L^{p_j}(\mathbb{R})$, $j = 1,2,3$, be three compactly supported functions and $\mathcal{D}$ be a dyadic grid. For all $Q \in \mathcal{D}$, referring to the notation (4.1) for $\mathcal{I}_{f,p,Q}$ we may then define

$$\mathcal{I}_{\vec{f},\vec{p},Q} := \text{ maximal elements of } \bigcup_{j=1}^3 \mathcal{I}_{f_j,p_j,Q} \tag{5.1}$$

It is clear that the intervals $I \in \mathcal{I}_{\vec{f},\vec{p},Q}$ are pairwise disjoint and that

$$\sum_{I \in \mathcal{I}_{\vec{f},\vec{p},Q}} |I| \le \frac{|Q|}{2}. \tag{5.2}$$

Furthermore, as a consequence of (4.3) for each $f = f_j, p = p_j$, there holds

$$\inf_{x \in 3I} \mathrm{M}_{p_j} f_j(x) \lesssim \langle f \rangle_{3Q,p} \qquad \forall I \in \mathcal{I}_{\vec{f},\vec{p},Q},\ j = 1,2,3. \tag{5.3}$$

We put together these stopping intervals in a single sparse collection $\mathcal{S} = \mathcal{S}(\mathcal{D}, f_1, f_2, f_3)$ of stopping intervals for the condition (4.1). Let us begin by choosing a partition of $\mathbb{R}$ by intervals $\{Q_k \in \mathcal{D} : k \in \mathbb{N}\}$ with the property that $\operatorname{supp} f_j \subset 3Q_k$ for all $j = 1,2,3$ and $k \in \mathbb{N}$. For each k, let

$$\mathcal{S}(Q_k) = \bigcup_{\ell=0}^{\infty} \mathcal{S}_\ell(Q_k)$$

where $\mathcal{S}_0(Q_k) = \{Q_k\}$ and, proceeding iteratively,

$$\mathcal{S}_\ell(Q_k) = \bigcup_{Q \in \mathcal{S}_{\ell-1}(Q_k)} \mathcal{I}_{\vec{f},\vec{p},Q}, \qquad l = 1,2,\ldots$$

Finally, define

$$\mathcal{S} = \mathcal{S}(\mathcal{D}, f_1, f_2, f_3) = \bigcup_{k=0}^{\infty} \mathcal{S}(Q_k).$$

By construction and by the packing property (5.2), $\mathcal{S}$ is a $\frac{1}{2}$-sparse subcollection of $\mathcal{D}$.

5.2. *Reduction to a single shifted dyadic grid*

It is convenient to reduce to a canonical choice of dyadic grids, as follows. Let

$$\mathcal{D}_j = \left\{2^k[0,1) + \left(n + \tfrac{j}{3}\right) 2^k : k, n \in \mathbb{Z}\right\}, \qquad j = 0,1,2$$

be the three canonical shifted dyadic grids on $\mathbb{R}$. Recall the well known fact that for all intervals $I \subset \mathbb{R}$ there exists a unique $\tilde{I} \in \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ with $3I \subset \tilde{I}$, $|\tilde{I}| \le 6 \cdot |3I|$, and $c(\tilde{I})$ is least possible. We say that I has type $j \in \{0,1,2\}$ if $\tilde{I} \in \mathcal{D}_j$.

Fix a finite rank 1 collection $\mathbb{P}$ and a tuple of functions $\vec{f} = (f_1, f_2, f_3)$ as above. We split $\mathbb{P} = \mathbb{P}_0 \cup \mathbb{P}_1 \cup \mathbb{P}_2$ where $\mathbb{P}_j = \{P \in \mathbb{P} : I_P \text{ has type } j\}$. For each $j \in \{0,1,2\}$ we use the previous construction with $\mathcal{D} = \mathcal{D}_j$ to obtain a $\frac{1}{2}$-sparse collection of intervals $\mathcal{S}_j = \mathcal{S}(\mathcal{D}_j, \vec{f})$ such that

$$\Lambda_{\mathbb{P}_j}(f_1, f_2, f_3) \le K(A_N)^3 \sum_{Q \in \mathcal{S}_j} |3Q| \prod_{\ell=1}^{3} \langle f_\ell \rangle_{3Q,p_\ell} \tag{5.4}$$

for a suitably large $N = N(\vec{p})$ and $K = K(\vec{p})$. Once (5.4) is performed, we achieve the estimate

$$\begin{aligned}\Lambda_{\mathbb{P}}(f_1, f_2, f_3) &= \sum_{j=0}^{2} \Lambda_{\mathbb{P}_j}(f_1, f_2, f_3) \\ &\le \sum_{j=0}^{2} K(A_N)^3 \sum_{Q \in \mathcal{S}_j} |3Q| \prod_{\ell=1}^{3} \langle f_\ell \rangle_{3Q,p_\ell} \lesssim K(A_N)^3 \mathsf{PSF}_{\widetilde{\mathcal{S}}}^{\vec{p}}(f_1, f_2, f_3),\end{aligned}$$

where $\widetilde{\mathcal{S}} = \{3Q : Q \in \mathcal{S}_{j_0}\}$ and $j_0 \in \{0,1,2\}$ is such that the right hand side of (5.4) is maximal. Since $\mathcal{S}_{j_0}$ is $\frac{1}{2}$ sparse it immediately follows that $\widetilde{\mathcal{S}}$ is a $\frac{1}{6}$-sparse collection. This completes the proof of Theorem 2.2, up to (5.4). In the next three subsections, we give the proof of (5.4).

5.3. *Proof of* (5.4)*: main argument*

A first observation is that, since the intervals I_P and $\widetilde{I_P}$ are comparable, and in view of the maximal definition of the tritile maps, there is no loss in generality in what follows to assume $I_P = \widetilde{I_P}$ for all $P \in \mathbb{P}_j$, that is $\{I_P : P \in \mathbb{P}\} \subset \mathcal{D}_j$. In fact, we are free to work with $j = 0$ and accordingly forgo the subscript j till the end of this section.

The main step of the argument for Theorem 1.3 is summarized in the next lemma, whose proof is postponed to the next subsection. Before the statement, it is convenient to recall from (2.1) the notation $\mathbb{P}_{\leq}(Q)$ associated to a generic finite collection of tritiles $\mathbb{P}$ and an interval $Q \subset \mathbb{R}$. Let $\{Q_k : k \in \mathbb{N}\}$ be the intervals employed in the construction of $\mathcal{S}$ in Subsection 5.1. Since $\{Q_k : k \in \mathbb{N}\}$ partition $\mathbb{R}$, we have the splitting

$$\mathbb{P} = \bigcup_{k=0}^{\infty} \mathbb{P}_{\leq}(Q_k);$$

in fact the union is finite, as the collection $\mathbb{P}$ is. Since $\mathcal{S} = \cup_k \mathcal{S}(Q_k)$, (5.4) is a consequence of

$$\Lambda_{\mathbb{P}_{\leq}(Q_k)}(f_1, f_2, f_3) \leq K(A_N)^3 \sum_{Q \in \mathcal{S}(Q_k)} |3Q| \prod_{j=1}^{3} \langle f_j \rangle_{3Q,p_j}. \tag{5.5}$$

Estimate (5.5) is obtained by iteration of the lemma below, starting with $Q = Q_k$, which is legitimate because $\operatorname{supp} f_j \subset 3Q_k$ for any $j = 1, 2, 3$, and following the construction of $\mathcal{S}(Q_k)$.

Lemma 5.1. *Let $\vec{f} = (f_1, f_2, f_3)$ be as above and $Q \in \mathcal{D}$. For any rank 1 collection of tritiles $\mathbb{P}$ such that $\{I_P : P \in \mathbb{P}\} \subset \mathcal{D}$, there holds*

$$\Lambda_{\mathbb{P}_{\leq}(Q)}(f_1\mathbf{1}_{3Q}, f_2\mathbf{1}_{3Q}, f_3\mathbf{1}_{3Q}) \leq K(A_N)^3 |3Q| \prod_{j=1}^{3} \langle f_j \rangle_{3Q,p_j} + \sum_{I \in \mathcal{I}_{\vec{f},\vec{p},Q}} \Lambda_{\mathbb{P}_{\leq}(I)}(f_1\mathbf{1}_{3I}, f_2\mathbf{1}_{3I}, f_3\mathbf{1}_{3I}).$$

Observe that since $\mathbb{P}_{\leq}(Q_k)$ is finite, the collections $\mathbb{P}_{\leq}(I)$ will be empty after a finite number of iterations, at which point the iterative procedure leading to (5.5) is complete. We are left with the task of showing that Lemma 5.1 holds true.

5.4. *Proof of Lemma 5.1*

Throughout this proof only, we make an exception concerning our use of the almost inequality sign: the implied constant is allowed to be of the form $K(A_N)^3$ where K and N depend only on the tuple $\vec{p}$. For the sake of brevity, we assume that all f_j's are supported on $3Q$. With reference to (5.1) for $\mathcal{I}_{\vec{f},\vec{p},Q}$, let

$$\mathsf{G}_{\vec{f},\vec{p},Q} := \mathbb{P}_{\leq}(Q) \setminus \left(\bigcup_{I \in \mathcal{I}_{\vec{f},\vec{p},Q}} \mathbb{P}_{\leq}(I) \right).$$

We decompose

$$\Lambda_{\mathbb{P}_{\leq}(Q)}(f_1, f_2, f_3) \leq \sum_{P \in \mathsf{G}_{\vec{f},\vec{p},Q}} |I_P| \prod_{j=1}^{3} F_j(f_j)(P) + \sum_{I \in \mathcal{I}_{\vec{f},\vec{p},Q}} \Lambda_{\mathbb{P}_{\leq}(I)}(f_1, f_2, f_3).$$

We claim that the first term satisfies the following estimate:

$$\sum_{P \in \mathsf{G}_{\vec{f},\vec{p},Q}} |I_P| \prod_{j=1}^{3} F_j(f_j)(P) \lesssim |Q| \prod_{j=1}^{3} \langle f_j \rangle_{3Q,p_j}. \tag{5.6}$$

Indeed, since $\vec{p}$ is open admissible and $\max\{p_j\} < 2$, using Proposition 4.1 we learn that there exists a Hölder tuple $\vec{q}$ such that F_j has the (LC_{q_j,p_j}) property, $j = 1, 2, 3$, i.e.

$$\left\| F_j(f_j)\mathbf{1}_{\mathsf{G}_{f_j,p_j,Q}} \right\|_{L^{q_j}(\mathbb{P},\mathsf{s}_j)} \leq \Theta(p_j, q_j) A_{N(p_j,q_j)} |Q|^{\frac{1}{q_j}} \langle f_j \rangle_{3Q,p_j}.$$

Let $G_j(P) := F_j(f_j)(P)\mathbf{1}_{\mathsf{G}_{\vec{f},\vec{p},Q}}(P)$. By the Hölder inequality of Lemma 3.1, we have that

$$\sum_{P\in\mathsf{G}_{\vec{f},\vec{p},Q}} |I_P| \prod_{j=1}^{3} F_j(f_j)(P) \lesssim \prod_{j=1}^{3} \|G_j\|_{L^{q_j}(\mathbb{P},\mathsf{s}_j)} \tag{5.7}$$

Notice that the implicit constant is of the form $K(A_N)^3$, where $K = \prod_{j=1}^{3} \Theta(p_j, q_j)$ and $N = \max N(p_j, q_j)$. Now, since $\mathsf{G}_{\vec{f},\vec{p},Q} \subset \mathsf{G}_{f_j,p_j,Q}$ for $j = 1, 2, 3$, we have

$$\|G_j\|_{L^{q_j}(\mathbb{P},\mathsf{s}_j)} \leq \|F_j(f_j)\mathbf{1}_{\mathsf{G}_{f_j,p_j,Q}}\|_{L^{q_j}(\mathbb{P},\mathsf{s}_j)} \lesssim |Q|^{\frac{1}{q_j}} \langle f_j \rangle_{3Q,p_j}.$$

Inserting the above three inequalities into (5.7) yields (5.6).

We are left with estimating the second term

$$\sum_{I\in\mathcal{I}_{\vec{f},\vec{p},Q}} \Lambda_{\mathbb{P}_{\leq}(I)}(f_1, f_2, f_3),$$

for which we claim

$$\sum_{I\in\mathcal{I}_{\vec{f},\vec{p},Q}} \Lambda_{\mathbb{P}_{\leq}(I)}(f_1, f_2, f_3) \leq K(A_N)^3 |Q| \prod_{j=1}^{3} \langle f_j \rangle_{3Q,p_j} + \sum_{I\in\mathcal{I}_{\vec{f},\vec{p},Q}} \Lambda_{\mathbb{P}_{\leq}(I)}(f_1\mathbf{1}_{3I}, f_2\mathbf{1}_{3I}, f_3\mathbf{1}_{3I}). \tag{5.8}$$

To see this, for each $I \in \mathcal{I}_{\vec{f},\vec{p},Q}$, define

$$\Lambda_{\mathbb{P}_{\leq}(I)}{}^{\vec{t}}(f_1, f_2, f_3) := \Lambda_{\mathbb{P}_{\leq}(I)}(f_1\mathbf{1}_{I^{t_1}}, f_2\mathbf{1}_{I^{t_2}}, f_3\mathbf{1}_{I^{t_3}}) = \sum_{P\in\mathbb{P}_{\leq}(I)} |I_P| \prod_{j=1}^{3} F_j(f_j\mathbf{1}_{I^{t_j}})(P),$$

where $\vec{t} = (t_1, t_2, t_3) \in \{\mathsf{in}, \mathsf{out}\}^3$ and

$$I^{\mathsf{in}} := 3I, \qquad I^{\mathsf{out}} := \mathbb{R} \setminus 3I.$$

Therefore, one can split

$$\Lambda_{\mathbb{P}_{\leq}(I)}(f_1, f_2, f_3) \leq \sum_{\vec{t}\in\{\mathsf{in},\mathsf{out}\}^3} \Lambda_{\mathbb{P}_{\leq}(I)}{}^{\vec{t}}(f_1, f_2, f_3).$$

Among the 2^3 forms on the right hand side, the one corresponding with $\vec{t}$ such that $t_j = \mathsf{in}$ for all j appears exactly in the second term on the right hand side of (5.8), hence it suffices for us to bound the rest of the $2^3 - 1$ forms. According to Proposition 5.2, which we state and prove later, for any $\vec{t}$ such that $t_j = \mathsf{out}$ for at least one $j = 1, 2, 3$, there holds

$$\Lambda_{\mathbb{P}_{\leq}(I)}{}^{\vec{t}}(f_1, f_2, f_3) \lesssim |I| \prod_{j=1}^{3} \inf_{x\in 3I} \mathrm{M}_{p_j} f_j(x) \lesssim |I| \prod_{j=1}^{3} \langle f_j \rangle_{3Q,p_j}$$

where the last step follows from (5.3). Therefore, multiplying the three inequalities together and summing over I yields (5.8), which also completes the proof of the lemma.

5.5. *Handling the tail terms*

Now we proceed with the proposition that has been used in the proof of Lemma 5.2 to estimate the tail term. In fact, we are going to derive it in a more general form, which not only includes our tritile maps F_j as a special case, but also applies to more general tritile maps. A tritile map $F : L^1_{\mathrm{loc}}(\mathbb{R}) \to \mathbb{C}^{\mathbb{P}}$ is said to be *almost localized* if it satisfies

$$\begin{aligned} \sup_{P\in\mathbb{P}_=(J)} F(f)(P) &\lesssim \|f\|_{L^1(\chi_J^M)}, \\ \left(\frac{1}{|J|}\sum_{P\in\mathbb{P}_=(J)} |I_P|F(f)(P)^2\right)^{\frac{1}{2}} &\lesssim \|f\|_{L^2(\chi_J^M)}, \end{aligned} \tag{5.9}$$

where M is a fixed large integer (say $M = 10^3$), and the notation $\mathbb{P}_=(J)$ has been introduced in (2.1).

Proposition 5.2. *Assume the type $\vec{t}$ is such that $t_j = \mathsf{out}$ for at least one $j = 1, 2, 3$. Let F_j be almost localized tritile maps for $j = 1, 2, 3$, and $\vec{p}$ be an open admissible tuple. Then,*

$$\Lambda^{\vec{t}}_{\mathbb{P}_\leq(I)}(f_1, f_2, f_3) \lesssim |I| \prod_{j=1}^{3} \inf_{x\in 3I} \mathrm{M}_{p_j} f_j(x). \tag{5.10}$$

The proof of the proposition will rely on the following key lemma.

Lemma 5.3. *Let J be an interval. Assume that* $\operatorname{supp} f_3 \cap AJ = \emptyset$ *for some $A \geq 3$. Let $\vec{p}$ be an open admissible tuple and F_j be an almost localized tritile map for j=1,2,3. Then*

$$\Lambda_{\mathbb{P}_=(J)}(f_1, f_2, f_3) := \sum_{P\in\mathbb{P}_=(J)} |I_P| \prod_{j=1}^{3} F_j(f_j)(P) \lesssim A^{-100}|J| \prod_{j=1}^{3} \inf_{x\in 3J} \mathrm{M}_{p_j} f_j(x).$$

Proof. The almost localized assumptions (5.9) can be rephrased in the form

$$\|F_j(f)(\cdot)\|_{\ell^\infty(\mathbb{P}_=(J))} \lesssim \|f\|_{L^1(\chi_J^M)}, \qquad \|F_j(f)(\cdot)\|_{\ell^2(\mathbb{P}_=(J))} \lesssim \|f\|_{L^2(\chi_J^M)},$$

which by off-diagonal Marcinkiewicz interpolation yields for $1 \leq p \leq 2$

$$\|F_j(f)(\cdot)\|_{\ell^{p'}(\mathbb{P}_=(J))} \lesssim \|f\|_{L^p(\chi_J^M)}. \tag{5.11}$$

Notice that for $j = 1, 2$,

$$\|f_j\|_{L^p(\chi_J^M)} \lesssim \inf_{x\in 3J} \mathrm{M}_p f_j(x) \tag{5.12}$$

while if M is sufficiently large

$$\|f_3\|_{L^p(\chi_J^M)} \lesssim \left(\sup_{x\in \operatorname{supp} f_3} \chi_J^{100}(x)\right) \|f_3\|_{L^p(\chi_J^{M-100})} \lesssim A^{-100} \inf_{x\in 3J} \mathrm{M}_p f_3(x). \tag{5.13}$$

Since $\vec{p}$ is open admissible, there exists a Hölder tuple $\vec{q} = (q_1, q_2, q_3)$ with $(q_j)' \leq p_j$. Therefore, using (5.11) for each $f = f_j$

$$\Lambda_{\mathbb{P}_=(J)}(f_1, f_2, f_3) \leq |J| \prod_{j=1}^{3} \|F_j(f)(\cdot)\|_{\ell^{q_j}(\mathbb{P}_=(J))} \lesssim A^{-100}|J| \prod_{j=1}^{3} \inf_{x\in 3J} \mathrm{M}_{q_j'} f_j(x),$$

which is stronger than the estimate claimed of the lemma. □

Proof Proof of Proposition 5.2. For the sake of definiteness, let us assume that $t_3 = \mathsf{out}$ and Let $\mathcal{J} = \{J : J = I_P \text{ for some } P \in \mathbb{P}_\leq(I)\}$. We partition

$$\mathcal{J}_k = \{J \in \mathcal{J} : 2^k J \subset I, 2^{k+1}J \not\subset I\}, \qquad \mathbb{P}_{\leq,k}(I) = \{P \in \mathbb{P}_\leq(I) : I_P \in \mathcal{J}_k\}$$

Let us observe the following properties of the intervals $J \in \mathcal{J}_k$:

$$\begin{aligned} &\operatorname{dist}(J, \operatorname{supp} f_3 \mathbf{1}_{I^{\text{out}}}) \sim 2^k |J|, \\ &J \in \mathcal{J}_k \text{ have finite overlap and } \sum_{J \in \mathcal{J}_k} |J| \lesssim |I|, \\ &\inf_{x \in 3J} \mathrm{M}_{p_j} f_j(x) \lesssim 2^k \inf_{x \in 3I} \mathrm{M}_{p_j} f_j(x). \end{aligned} \tag{5.14}$$

We then estimate, using Lemma 5.3 and the above properties

$$\Lambda^{\vec{t}}_{\mathbb{P}_{\leq}(I)}(f_1, f_2, f_3) \leq \sum_{k \geq 0} \sum_{J \in \mathcal{J}_k} \Lambda_{\mathbb{P}_=(J)}(f_1 \mathbf{1}_{I^{t_1}}, f_2 \mathbf{1}_{I^{t_2}}, f_3 \mathbf{1}_{I^{\text{out}}})$$

$$\lesssim \sum_{k \geq 0} \sum_{J \in \mathcal{J}_k} 2^{-100k} |J| \prod_{j=1}^{3} \inf_{x \in 3J} \mathrm{M}_{p_j} f_j(x) \lesssim |I| \prod_{j=1}^{3} \inf_{x \in 3I} \mathrm{M}_{p_j} f_j(x).$$

The proof of the proposition is thus completed. □

Now that we have proved Proposition 5.2, in order to complete the proof of Lemma 5.1, it suffices to verify that the tritile maps F_j, $j = 1, 2, 3$ given in (2.3) are indeed almost localized.

Lemma 5.4. *Tritile maps*

$$F_j(f)(P) = \sup_{\phi_{P_j} \in \mathbf{\Phi}(P_j)} |\langle f_j, \phi_{P_j} \rangle|, \qquad j = 1, 2, 3$$

are almost localized. In other words,

$$\sup_{P \in \mathbb{P}_=(J)} F_j(f)(P) \lesssim \|f\|_{L^1(\chi_J^M)}, \tag{5.15}$$

and

$$\left(\frac{1}{|J|} \sum_{P \in \mathbb{P}_=(J)} |I_P| F_j(f)(P)^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2(\chi_J^M)}. \tag{5.16}$$

Proof. To see (5.15), for any $P \in \mathbb{P}_=(J)$ and $\phi_{P_j} \in \mathbf{\Phi}(P_j)$, write

$$|\langle f, \phi_{P_j} \rangle| = |\langle f \chi_J^M |J|^{-1}, \phi_{P_j} \chi_J^{-M} |J| \rangle| \leq \left\| f \chi_J^M \right\|_{L^1(\chi_J^M)} \left\| \phi_{P_j} \chi_J^{-M} |J| \right\|_{L^\infty}.$$

Then according to (2.2), (5.15) follows immediately from $\left\| \phi_{P_j} \chi_J^{-M} |J| \right\|_{L^\infty} \leq A_M$.

Now we verify that (5.16) holds true. Without loss of generality, one can assume that there exists $\{\phi_{P_j}\}$ such that the supremum in the definition of F_j are attained up to an ϵ. This can certainly be done if the collection $\mathbb{P}$ is finite. Since our estimate will not depend on the cardinality of the collection, a limiting argument will pass this to the infinite collection case as well. Hence, we are now trying to show that

$$\left(\frac{1}{|J|} \sum_{P \in \mathbb{P}_=(J)} |I_P| |\langle f, \phi_{P_j} \rangle|^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2(\chi_J^M)}.$$

To see this, write

$$|I_P| |\langle f, \phi_{P_j} \rangle|^2 = |\langle f \chi_J^M, \phi_{P_j} \chi_J^{-M} |I_P|^{1/2} \rangle|^2.$$

Define $\tilde{\phi}_{P_j} := |I_P|^{1/2} \phi_{P_j} \chi_J^{-M}$. We claim that $\{\tilde{\phi}_{P_j}\}$ is an orthogonal system with L^2 normalization, which yields (5.16) immediately.

The L^2 normalization can be easily seen from

$$\int_{\mathbb{R}} |\tilde{\phi}_{P_j}|^2(x)\,dx \le A_{M+1}|J|^{-1}\int_{\mathbb{R}} \chi_J^2(x)\,dx \lesssim A_{M+1}.$$

And the orthogonality follows from the disjoint frequency supports consideration of $\{\phi_{P_j}\}$. More precisely, since $I_P = J$ for all $P \in \mathbb{P}_=(J)$, $\{\operatorname{supp}\widehat{\phi}_{P_j} \subset \omega_{P_j}\}$ are pairwise disjoint. Therefore, since the Fourier transform of $\tilde{\phi}_{P_j}$ is a finite linear combination of derivatives (up to order $2M$) of the Fourier transform of ϕ_{P_j}, $\tilde{\phi}_{P_j}$ and ϕ_{P_j} have the same frequency support, which implies the desired orthogonality. □

6. *Proof of Theorem 1.6 and Corollary 1.7*

6.1. *Proof of Theorem 1.6*

Fixing a tuple $\vec{q} = (q_1, q_2, q_3)$ and weights $\vec{v} = (v_1, v_2, v_3)$ as in the statement of the theorem, and any open admissible tuple $\vec{p}$ with $p_j < q_j$ for $j = 1, 2, 3$, proving the theorem amounts to showing that

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \le K(\vec{p}, \vec{q}, \vec{v}) \prod_{j=1}^{3} \|f_j\|_{L^{q_j}(v_j)} \tag{6.1}$$

where $K(\vec{p}, \vec{q}, \vec{v})$ is the constant appearing in the statement of the theorem, holds for all tuples $\vec{f} = (f_1, f_2, f_3) \in \mathcal{C}^\infty(\mathbb{R})^3$. We define

$$w_j = v_j^{\frac{p_j}{p_j - q_j}}, \qquad j = 1, 2, 3.$$

Note that the finiteness of the $A_{\vec{q}}^{\vec{p}}$ constant of $\vec{v}$ implies $w_j \in L^1_{\mathrm{loc}}(\mathbb{R})$. Setting $f_j = g_j w_j^{\frac{1}{p_j}}$ one notices that $\|f_j\|_{L^{q_j}(v_j)} = \|g_j\|_{L^{q_j}(w_j)}$. Applying the domination result from Theorem 1.3, we bound the left hand side of (6.1) by

$$\sup_{\mathcal{S}\text{ sparse}} \mathsf{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3) = \sup_{\mathcal{S}\text{ sparse}} \mathsf{PSF}_{\mathcal{S}}^{\vec{p}}\big(g_1 w_1^{\frac{1}{p_1}}, g_2 w_2^{\frac{1}{p_2}}, g_3 w_3^{\frac{1}{p_3}}\big).$$

By possibly splitting $\mathcal{S}$ into three subcollections and using the three grid lemma recalled in Subsection 5.2, we can restrict to the case of $\mathcal{S}$ being a sparse subset of the standard dyadic grid $\mathcal{D}_0$. Therefore, (6.1) will follow from the estimate of the lemma below.

Lemma 6.1. *For any $g_j \in L^{q_j}(w_j)$, $j = 1, 2, 3$, there holds*

$$\sup_{\mathcal{S}\subset\mathcal{D}_0\ \frac{1}{6}-\text{sparse}} \mathsf{PSF}_{\mathcal{S}}^{\vec{p}}(g_1 w_1^{\frac{1}{p_1}}, g_2 w_2^{\frac{1}{p_2}}, g_3 w_3^{\frac{1}{p_3}}) \lesssim \mu_{\vec{p},\vec{q}}[\vec{v}]_{A_{\vec{q}}^{\vec{p}}}^{\max\left\{\frac{q_j}{q_j - p_j}\right\}} \prod_{j=1}^{3} \|g_j\|_{L^{q_j}(w_j)}$$

where

$$\mu_{\vec{p},\vec{q}} := \left(\prod_{j=1}^{3} \frac{q_j}{q_j - p_j}\right) 2^{3\left(\sum_{j=1}^{3}\frac{1}{p_j} - 1\right)\max\left\{\frac{p_j}{q_j - p_j}\right\}}.$$

Proof. We largely follow the argument from [**23**]. We may work with $g_j \geq 0$. Let $\mathcal{S}$ be a fixed 1/2-sparse grid. Then

$$\mathsf{PSF}^{\vec{p}}_{\mathcal{S}}(g_1 w_1^{\frac{1}{p_1}}, g_2 w_2^{\frac{1}{p_2}}, g_3 w_3^{\frac{1}{p_3}}) = \sum_{Q\in\mathcal{S}} |Q| \prod_{j=1}^{3} \left(\langle g_j^{p_j} w_j\rangle_Q\right)^{\frac{1}{p_j}}$$

$$= \sum_{Q\in\mathcal{S}} \left(\prod_{j=1}^{3} w_j(E_Q)^{\frac{1}{q_j}} \left(\frac{\langle g_j^{p_j} w_j\rangle_Q}{\langle w_j\rangle_Q}\right)^{\frac{1}{p_j}} \right) \times \left(\prod_{j=1}^{3} \langle w_j\rangle_Q^{\frac{1}{p_j}-\frac{1}{q_j}} \right) \times \left(|Q| \prod_{j=1}^{3} \left(\frac{\langle w_j\rangle_Q}{w_j(E_Q)}\right)^{\frac{1}{q_j}} \right). \tag{6.2}$$

The second product inside the sum of (6.2) is the precursor to $[\vec{v}]_{A^{\vec{p}}_{\vec{q}}}$. Arguing as in [**23**], the rightmost factor in (6.2) is bounded above uniformly in Q by

$$2^{3\left(\sum_{j=1}^{3}\frac{1}{p_j}-1\right)\max\left\{\frac{p_j}{q_j-p_j}\right\}} [\vec{v}]_{A^{\vec{p}}_{\vec{q}}}^{\max\left\{\frac{1}{p_j q_j}\right\}}.$$

Introducing the dyadic weighted maximal functions

$$\mathrm{M}_{p_j,w_j}(f)(x) = \sup_{Q\in\mathcal{D}_0} \left(\frac{\langle |f|^{p_j} w_j\rangle_Q}{\langle w_j\rangle_Q}\right)^{\frac{1}{p_j}} \mathbf{1}_Q(x)$$

and using the disjointness of E_Q and Hölder's inequality, we estimate the remaining part of (6.2) by

$$\sum_{Q\in\mathcal{S}} \left(\prod_{j=1}^{3} w_j(E_Q)^{\frac{1}{q_j}} \left(\frac{\langle g_j^{p_j} w_j\rangle_Q}{\langle w_j\rangle_Q}\right)^{\frac{1}{p_j}} \right) \leq \prod_{j=1}^{3} \|\mathrm{M}_{p_j,w_j} g_j\|_{L^{q_j}(w_j)}.$$

The claimed estimate then follows by bookkeeping the last three observations and by relying upon the sharp $L^{q_j}(w_j)$-boundedness of $\mathrm{M}_{p_j,w_j}(f_j)$ (see [**29**] for a proof). The proof of the lemma is complete. □

6.2. *Proof of Corollary 1.7*

Let

$$\Theta = \max_{j=1,2} [v_j^2]_{A_{q_j}}.$$

The openness of the A_q classes (see [**17**] for a quantified statement) allows us to to find $\varepsilon = \varepsilon(\Theta, q_1, q_2) > 0$ such that

$$[v_j^{\frac{2}{1-\varepsilon}}]_{A_{q_j}} \leq 2\Theta, \qquad j = 1, 2. \tag{6.3}$$

We denote by q_3 the dual exponent of r and by $v_3 = u_3^{1-q_3}$ the dual weight. To prove the corollary, in light of (6.3), it suffices to find an open admissible tuple $\vec{p}$ with $p_j < q_j$ such that

$$[\vec{v}]_{A^{\vec{p}}_{\vec{q}}} \leq \prod_{j=1}^{2} [v_j^{\frac{2}{1-\varepsilon}}]_{A_{q_j}}^{\frac{1-\varepsilon}{2q_j}} \tag{6.4}$$

and subsequently applying Theorem 1.6, which is made possible by (6.3).

Referring to the notation of (1.7), let $\vec{p}$ be an open admissible tuple with $p_j < q_j$ and $\varepsilon = \varepsilon(\vec{p})$. We set $\delta = 1 + \varepsilon$ and reparametrize

$$\tfrac{1}{p_j} = 1 - \tfrac{\delta\theta_j}{r_j}, \qquad r_j = \tfrac{q_j}{q_j - 1}, \qquad \theta_j \geq 0, \qquad \sum_{j=1}^{3} \tfrac{\theta_j}{r_j} = 1. \tag{6.5}$$

This leads to the following lemma.

LEMMA 6.2. *There holds*

$$[\vec{v}]_{A^{\vec{p}}_{\vec{q}}} \leq \prod_{j=1}^{2} \sup_{Q\subset\mathbb{R}} \left(\left\langle v_j^{\frac{1}{(1-\delta\theta_3)}} \right\rangle_Q^{1-\delta\theta_3} \left\langle v_j^{\frac{1}{1-\delta\theta_j}\frac{1}{1-q_j}} \right\rangle_Q^{(q_j-1)(1-\delta\theta_j)} \right)^{\frac{1}{q_j}}$$

Proof. Observe that

$$\tfrac{1}{p_j} - \tfrac{1}{q_j} = \tfrac{1-\delta\theta_j}{r_j}$$

Using the relation $1 = v_1^{\frac{1}{q_1}} v_2^{\frac{1}{q_2}} v_3^{\frac{1}{q_3}}$, the definition of w_j and Hölder, one has

$$\langle w_n \rangle_Q^{\frac{1}{p_3}-\frac{1}{q_3}} = \left\langle \prod_{j=1}^{2} v_j^{\frac{r_3}{q_j(1-\delta\theta_3)}} \right\rangle_Q^{\frac{1-\delta\theta_3}{r_3}} \leq \prod_{j=1}^{2} \left\langle v_j^{\frac{1}{(1-\delta\theta_3)}} \right\rangle_Q^{\frac{1-\delta\theta_3}{q_j}}$$

and for $j = 1, 2$

$$\langle w_j \rangle_Q^{\frac{1}{p_j}-\frac{1}{q_j}} = \left\langle v_j^{\frac{1}{1-\delta\theta_j}\frac{1}{1-q_j}} \right\rangle_Q^{\frac{1-\delta\theta_j}{r_j}}$$

which, rearranging and taking suprema, completes the proof of the lemma. □

Now, comparing with (6.5), we may choose $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}$ in Lemma 6.2. This leads to the estimate

$$[\vec{v}]_{A^{\vec{p}}_{\vec{q}}} \leq \prod_{j=1}^{2} \sup_{Q\subset\mathbb{R}} \left(\left\langle v_j^{\frac{2}{2-\delta}} \right\rangle_Q \left\langle v_j^{\frac{2}{2-\delta}\frac{1}{1-q_j}} \right\rangle_Q^{q_j-1} \right)^{\frac{2-\delta}{2}\frac{1}{q_j}}$$

whose right hand side is the same as that of (6.4). This completes the proof of Corollary 1.7.

Appendix A. *Vector-valued estimates from sparse domination*

In this section, the tuple $\vec{r} = (r_1, r_2, r_3)$ always satisfies

$$1 < r_1, r_2, r_3 \leq \infty, \qquad \sum_{j=1}^{3} \tfrac{1}{r_j} = 1. \tag{A.1}$$

We turn to the study of the trilinear forms

$$\Lambda_{\mathbf{m}}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) := \sum_k \Lambda_{m_k}(f_{1k}, f_{2k}, f_{3k})$$

acting on ℓ^{r_j}-valued sequences $\mathbf{f}_j = \{f_{jk}\}$, where $\mathbf{m} = \{m_k\}$ is a sequence of multipliers satisfying (1.2) uniformly. The adjoints to the above trilinear forms are the sequence-valued bilinear operators

$$T_{\mathbf{m}}(\mathbf{f}_1, \mathbf{f}_2) = \{T_{m_k}(f_{1k}, f_{2k})\}. \tag{A.2}$$

A consequence of Theorem 1.3 and of the classical Fefferman-Stein inequalities [**14**]

$$\begin{aligned} &\|\{\mathrm{M}_p f_k\}\|_{L^q(\mathbb{R};\ell^r)} \leq C(p,q,r)\,\|\{f_k\}\|_{L^q(\mathbb{R};\ell^r)}, \quad 1 \leq p < \min\{q,r\},\ \sup\{q,r\} < \infty \\ &\|\{\mathrm{M}_p f_k\}\|_{L^{p,\infty}(\mathbb{R};\ell^r)} \leq C(p,r)\,\|\{f_k\}\|_{L^p(\mathbb{R};\ell^r)}, \quad 1 \leq p < r < \infty. \end{aligned} \tag{A.3}$$

are the following vector-valued estimates for the operators $T_{\mathbf{m}}$ of (A.2)

COROLLARY A.1. *Let $\vec{r}$ be a fixed tuple as in* (A.1) *and $\mathbf{m} = \{m_k\}$ be a sequence of multipliers satisfying* (1.2) *uniformly. Then the bilinear operator $T_{\mathbf{m}}$ of* (A.2) *has the mapping* *properties*

$$T_{\mathbf{m}} : L^{q_1}(\mathbb{R};\ell^{r_1}) \times L^{q_2}(\mathbb{R};\ell^{r_2}) \to L^{\frac{q_1 q_2}{q_1+q_2}}(\mathbb{R};\ell^{s_3}), \qquad s_3 := \frac{r_3}{r_3-1} \tag{A.4}$$

for all exponent pairs (q_1, q_2) *satisfying*

$$1 < \inf\{q_1, q_2\} < \infty, \qquad \sum_{j=1}^{3} \frac{1}{\min\{q_j, r_j, 2\}} < 2, \qquad \frac{1}{q_3} := \max\left\{1 - \left(\tfrac{1}{q_1} + \tfrac{1}{q_2}\right), 0\right\}. \tag{A.5}$$

For each pair (q_1, q_2), the range of tuples $\vec{r}$ for which T_m admits $L^{q_1} \times L^{q_2}$ a bounded vector-valued extension is the same as the one recently obtained in [**3**, Theorem 7] for the vector valued bilinear Hilbert transforms. Condition (A.5) needs to be imposed in order to ensure that the set

$$\left\{\vec{p} = (p_1, p_2, p_3) \text{ open admissible} : p_j < \min\{r_j, q_j\}, j = 1, 2, 3.\right\} \tag{A.6}$$

is nonempty.

A.1. *Proof of Corollary A.1*

By an approximation argument, there is no loss in generality in working with multipliers $\mathbf{m} = \{m_k\}$ with $m_k = 0$ for all but finitely many k.

Fix a tuple $\vec{r}$ as in (A.1). We assume $\sup r_j < \infty$: the case $r_j = \infty$ for (at most one) j requires only minor modifications. We first prove the case where (q_1, q_2) is an exponent pair satisfying (A.5) with $q_3 < \infty$. In this range $\vec{q} = (q_1, q_2, q_3)$ is a Hölder tuple and the claimed estimate on $T_{\mathbf{m}}$ is equivalent to proving that

$$\left|\Lambda_{\mathbf{m}}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)\right| \le \sum_k |\Lambda_{m_k}(f_{1k}, f_{2k}, f_{3k})| \lesssim \prod_{j=1}^{3} \|\mathbf{f}_j\|_{L^{q_j}(\mathbb{R};\ell^{r_j})} \tag{A.7}$$

Since the set (A.6) is nonempty, we may choose an open admissible tuple $\vec{p} = (p_1, p_2, p_3)$ with $p_j < \min\{q_j, r_j\}$. We apply the domination Theorem 1.3 to each m_k in the above sum, yielding the existence of sparse collections $\mathcal{S}_k$ for which the estimate

$$|\Lambda_{m_k}(f_{1k}, f_{2k}, f_{3k})| \lesssim \mathsf{PSF}^{\vec{p}}_{\mathcal{S}_k}(f_{1k}, f_{2k}, f_{3k})$$

holds true. If $\{E_I : I \in \mathcal{S}_k\}$ are the distinguished pairwise disjoint major subsets of $I \in \mathcal{S}_k$, we have

$$\mathsf{PSF}^{\vec{p}}_{\mathcal{S}_k}(f_{1k}, f_{2k}, f_{3k}) \lesssim \sum_{I \in \mathcal{S}_k} |E_I| \left(\prod_{j=1}^{3} \inf_{x \in E_I} \mathrm{M}_{p_j} f_{jk}(x)\right) \lesssim \int_{\mathbb{R}} \left(\prod_{j=1}^{3} \mathrm{M}_{p_j} f_{jk}(x)\right) \mathrm{d}x \tag{A.8}$$

Summing over k and using Hölder's inequality first for the tuple $\vec{r}$ in the sum, and later for the Hölder tuple $\vec{q}$ in the integral the left-hand side of (A.7) is bounded by

$$\int_{\mathbb{R}} \sum_k \left(\prod_{j=1}^{3} \mathrm{M}_{p_j} f_{jk}\right) \mathrm{d}x \le \int_{\mathbb{R}} \left(\prod_{j=1}^{3} \left\|\{\mathrm{M}_{p_j} f_{jk}\}\right\|_{\ell^{r_j}}\right) \mathrm{d}x$$

$$\le \prod_{j=1}^{3} \left\|\{\mathrm{M}_{p_j} f_{jk}\}\right\|_{L^{q_j}(\ell^{r_j})} \lesssim \prod_{j=1}^{3} \left\|\mathbf{f}_j\right\|_{L^{q_j}(\ell^{r_j})},$$

having employed the Fefferman-Stein inequality (A.3) in the last step. This completes the proof of the case $q_3 < \infty$.

We pass to the case $q_3 = \infty$. In this range, we are able to choose an open admissible tuple with

$$p_1 < \min\{q_1, r_1\}, \quad p_2 < \min\{q_2, r_2\}, \quad p_3 < \min\{2, r_3\}.$$

Also, by virtue of the fact that $1/q_1 + 1/q_2 > 1$, we can find a tuple of exponents $\vec{t} = (t_1, t_2, t_3)$ satisfying

$$t_1 > q_1, \quad t_2 > q_2, \quad t_3 > p_3, \qquad \tfrac{1}{t_1} + \tfrac{1}{t_2} + \tfrac{1}{t_3} = 1. \tag{A.9}$$

Since the claimed range of exponents (q_1, q_2) is open, it suffices to prove the weak-type analogue of (A.4) and then invoke multilinear vector-valued Marcinkiewicz interpolation. Such a weak-type estimate is equivalent to proving that for all $\mathbf{f}_j \in L^{q_j}(\mathbb{R}; \ell^{r_j})$, $j = 1, 2$ of unit norm and for all sets $F_3 \subset \mathbb{R}$ of finite measure, there exists $F_3' \subset F_3$ with $|F_3| \leq 2|F_3'|$ so that

$$\begin{aligned} &\left|\Lambda_{\mathbf{m}}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)\right| \leq \sum_k |\Lambda_{m_k}(f_{1k}, f_{2k}, f_{3k})| \lesssim |F_3|^{1-\left(\frac{1}{q_1}+\frac{1}{q_2}\right)} \\ &\forall \mathbf{f}_3 : \|\mathbf{f}_3(x)\|_{\ell^{r_3}} \leq \mathbf{1}_{F_3'}(x). \end{aligned} \tag{A.10}$$

Fix such $\mathbf{f}_1, \mathbf{f}_2, F_3$. We proceed with the definition of F_3' in two steps. First, set

$$H := \bigcup_{j=1}^{2} \left\{ x \in \mathbb{R} : \|\{\mathrm{M}_{p_j} f_{jk}(x)\}\|_{\ell^{r_j}} > C|F_3|^{-\frac{1}{q_j}} \right\}$$

By Chebychev and Fefferman-Stein inequalities (A.3), $|H| \leq 2^{-12}|F_3|$ provided C is chosen large enough. Then

$$\widetilde{H} := \bigcup_{Q \in \mathcal{Q}} 9Q, \qquad \mathcal{Q} = \left\{\text{max. dyad. int. } Q : |Q \cap H| \geq 2^{-5}|Q|\right\}$$

satisfies $|\widetilde{H}| \leq 9 \cdot 2^5|H| \leq 2^{-3}|F_3|$. Therefore the set $F_3' := F_3 \backslash \widetilde{H}$ is a major subset of F_3. Fixing now any $\mathbf{f}_3 = \{f_{3k}\}$ restricted to F_3' as in (A.10), we apply the domination Theorem 1.3 to each m_k in (A.10), yielding the existence of sparse collections $\mathcal{S}_k$ for which we have the estimate

$$|\Lambda_{m_k}(f_{1k}, f_{2k}, f_{3k})| \lesssim \mathsf{PSF}_{\mathcal{S}_k}^{\vec{p}}(f_{1k}, f_{2k}, f_{3k}).$$

holds true. We claim that for all k

$$|I \cap H| \leq 2^{-5}|I| \qquad \forall I \in \mathcal{S}_k. \tag{A.11}$$

This is because if (A.11) fails for I, I must be contained in $3Q$ for some $Q \in \mathcal{Q}$. But the support of f_{3k} is contained in $\widetilde{H}^c$ which does not intersect $3Q$, whence $\langle f_{3k} \rangle_{I,p_3} = 0$. Relation (A.11) has the consequence that if $\{E_I : I \in \mathcal{S}_k\}$ denote the distinguished pairwise disjoint subsets of $I \in \mathcal{S}_k$ with $|E_I| \geq 2^{-2}|I|$, the sets $\widetilde{E_I} := E_I \cap H^c$ are also pairwise disjoint and $|\widetilde{E_I}| \geq 2^{-3}|I|$. By a similar argument to the one used to get to (A.8) but with $\widetilde{E_I}$ replacing E_I, followed by Hölder's inequality in k with tuple $\vec{r}$, and later by Hölder's inequality for the integral with the tuple $\vec{t}$ from (A.9), the left hand side of (A.10) is bounded by

$$\begin{aligned} &\sum_k \mathsf{PSF}_{\mathcal{S}_k}^{\vec{p}}(f_{1k}, f_{2k}, f_{3k}) \lesssim \int_{H^c} \left(\prod_{j=1}^{3} \left\|\{\mathrm{M}_{p_j} f_{jk}(x)\}\right\|_{\ell^r_j} \right) \mathrm{d}x \\ &\lesssim \prod_{j=1}^{3} \left\|\{\mathrm{M}_{p_j} f_{jk}\}\right\|_{L^{t_j}(H^c; \ell^{r_j})} \end{aligned} \tag{A.12}$$

Now by Fefferman-Stein's inequality since $t_3 > p_3$

$$\left\|\{\mathrm{M}_{p_3} f_{3k}\}\right\|_{L^{t_3}(H^c; \ell^{r_j})} \lesssim \|\mathbf{f}_3\|_{L^{t_3}(\mathbb{R}; \ell^{r_3})} \leq \|\mathbf{1}_{F_3}\|_{t_3} = |F_3|^{\frac{1}{t_3}} \tag{A.13}$$

Further, for $j=1,2$, by log-convexity of L^{t_j}-norms

$$\left\|\{\mathrm{M}_{p_j} f_{jk}\}\right\|_{L^{t_j}(H^c;\ell^{r_j})} \leq \left\|\{\mathrm{M}_{p_j} f_{jk}\}\right\|_{L^{q_j}(\mathbb{R};\ell^{r_j})}^{\frac{q_j}{t_j}} \left\|\{\mathrm{M}_{p_j} f_{jk}\}\right\|_{L^{\infty}(H^c;\ell^{r_j})}^{1-\frac{q_j}{t_j}} \lesssim |F_3|^{\frac{1}{t_j}-\frac{1}{q_j}} \tag{A.14}$$

where, to obtain the final step, we used the Fefferman-Stein inequality to estimate the $L^{q_j}(\mathbb{R};\ell^{r_j})$-norm by $O(1)$ and the definition of H to estimate the $L^{\infty}(H^c;\ell^{r_j})$-norm by $|F_3|^{-\frac{1}{q_j}}$. Using (A.13) and (A.14) for $j=1,2$ to bound the right hand side of (A.12) finally yields (A.10) and completes the proof of the Theorem.

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