T. HARUI, J. KOMEDA AND A. OHBUCHI
KODAI MATH. J.
31 (2008), 257–262

DOUBLE COVERINGS BETWEEN SMOOTH PLANE CURVES

Takeshi Harui, Jiryo Komeda and Akira Ohbuchi*

Abstract

We completely classify the pairs of two smooth plane curves with double coverings between them. More precisely, we show that there exist no double coverings between two smooth plane curves except for several special cases.

1. Introduction and preliminaries

We consider the following problem on smooth plane curves:

Problem. Classify the pairs of smooth plane curves (C, C') such that there exists a covering $\pi : C \to C'$ of degree $n \geq 2$.

In this article we completely classify such pairs in the case $n = 2$. There exist several trivial pairs: $(\mathbf{P}^1, \mathbf{P}^1)$, $(E, \mathbf{P}^1)$ and (E, E'), where E and E' are elliptic curves with an isogeny $\pi : E \to E'$ of degree 2. We give another example.

Example 1.1. Let C be a Fermat quartic defined by $x^4 + y^4 = 1$. It has an involution $\sigma : C \to C((x, y) \mapsto (-x, y))$. Then $\tau : C \to B, ((x, y) \mapsto (x^2, y))$ is a double covering, where B is an irreducible curve defined by $x^2 + y^4 = 1$. Note that the normalization E of B is an elliptic curve and there exists a lifted double covering $\pi : C \to E$. Thus we obtain a pair (C, E) of smooth plane curves with a double covering from C to E.

In fact, we show that pairs of the above type (a smooth plane quartic with an involution and an elliptic curve) are the only non-trivial ones (see Theorem 2.1 and Proposition 2.6).

Notation and Conventions

A curve is a one-dimensional algebraic variety. It is possibly reducible or non-reduced, though it will be smooth and irreducible mainly in this article. A

2000 *Mathematics Subject Classification*. 14H51.
* Partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, #17540030.
Received June 28, 2007.

g_d^r is a linear system of degree d and dimension r on a smooth curve. It is said to be *simple* if the associated rational map to $\mathbf{P}^r$ is birational onto its image. Otherwise it is said to be *compounded*.

For two divisors D and D' on a variety, $D \sim D'$ denotes that they are linearly equivalent.

2. Results

Our main result is the following theorem.

THEOREM 2.1. *Let C, C' be two smooth plane curves of degree d, d', respectively. Then there exist no double coverings from C to C' except for the following cases*:

(1) *C' is rational $(d' \leq 2)$ and C is rational or elliptic $(d \leq 3)$.*
(2) *C and C' are elliptic $(d = d' = 3)$.*
(3) *C' is elliptic and C is a bi-elliptic plane quartic $(d' = 3, d = 4)$.*

In particular, a double covering of a smooth plane curve of degree greater than three cannot be a smooth plane curve.

Our proof of this theorem consists of two parts. First we consider the case of small degree.

LEMMA 2.2. *Let C, C' be two smooth plane curves of d, d', respectively. Assume that $d' \leq 3$ and there exists a double covering $\pi : C \to C'$. Then one of the following holds*:

(1) *C' is rational $(d' \leq 2)$ and C is rational or elliptic $(d \leq 3)$.*
(2) *C and C' are elliptic $(d = d' = 3)$.*
(3) *C' is elliptic and C is a bi-elliptic plane quartic $(d' = 3, d = 4)$.*

Proof. First assume that C' is rational. Then C is hyperelliptic. On the other hand, it is well-known that $\mathrm{gon}(C) = d - 1$. Hence we have $d \leq 3$, which implies that C is rational or elliptic. Secondly, assume that C' is elliptic. Then C is bi-elliptic. It is known that a bi-elliptic curve of genus g does not have a plane model of degree less than $g + 1$ (cf. [CM, 2.2]). Since C is a smooth plane curve of degree d, we have

$$d \geq g(C) + 1 = \frac{1}{2}(d-1)(d-2) + 1,$$

which implies that $(d-1)(d-4) \leq 0$. Hence $d \leq 4$ holds. □

Next we show the non-existence in the case of large degree.

PROPOSITION 2.3. *Let C, C' be two smooth plane curves of d, d', respectively. If $d' \geq 4$, then there exist no double coverings from C to C'.*

Proof. Assume that $d' \geq 4$ and there exists a double covering $\pi : C \to C'$. First note that C' has two pencils of degree $d'-1$ and d'. The pull-back of them by π are pencils of degree $2(d'-1)$ and $2d'$ on C. Since $\mathrm{gon}(C) = d-1$, we have $d-1 \leq 2(d'-1)$, i.e., $2d' \geq d+1$. On the other hand, C has no pencils of degree between $d+1$ and $2d-5$ (cf. [C]). Hence we obtain that $2d' \geq 2d-4$, i.e., $d' \geq d-2$. Since $d > d'$ by our assumption $d' \geq 4$, we have $d' = d-1$ or $d' = d-2$.

Case (i) $d' = d-1$. We then have

$$g(C) = \frac{1}{2}(d-1)(d-2) \quad \text{and}$$
$$g(C') = \frac{1}{2}(d'-1)(d'-2) = \frac{1}{2}(d-2)(d-3).$$

Since $g(C) \geq 2g(C') - 1$ holds by Riemann-Hurwitz formula, we obtain that

$$\frac{1}{2}(d-1)(d-2) \geq (d-2)(d-3) - 1,$$

which implies that $(d-2)(d-5) \leq 2$. Hence we have $d \leq 5$. Thus it suffices to exclude the case $(d, d') = (5, 4)$. In this case there exists a divisor N on C' of degree 1 such that $C = \mathrm{Spec}(\mathcal{O}_{C'} \oplus \mathcal{O}_{C'}(-N))$. Since $g(C') = 3$, the morphism $C'^3 \to \mathrm{Pic}^1(C')((P, Q, R) \mapsto P + Q - R)$ is surjective. Hence $N \sim P + Q - R$ for some $P, Q, R \in C'$. We take an effective divisor D on C' linearly equivalent to $K_{C'} - R$. Then we have

$$\begin{aligned} h^0(C, \mathcal{O}_C(\pi^* D)) &= h^0(C', \mathcal{O}_{C'}(D)) + h^0(C', \mathcal{O}_{C'}(D - N)) \\ &= h^0(C', \mathcal{O}_{C'}(D)) + h^0(C', \mathcal{O}_{C'}(K_{C'} - P - Q)) \\ &= 2 + 1 = 3. \end{aligned}$$

Hence we obtain a net $|\pi^* D| = g_6^2$ on C. If this is compounded, then C is a bi-elliptic plane quintic, which leads to a contradiction by Lemma 2.2. If the g_6^2 is simple, then C has a plane model C_0 of degree 6 with 4 double points as its only singularities because C is a 4-gonal curve of genus 6. We count the number of pencils of degree 4. Take a g_4^1 without base points on C. Applying the base-point-free pencil trick, we have an exact sequence

$$\begin{aligned} 0 \to H^0(C, \mathcal{O}_C(g_6^2 - g_4^1)) \to H^0(C, \mathcal{O}_C(g_6^2)) \otimes H^0(C, \mathcal{O}_C(g_4^1)) \\ \to H^0(C, \mathcal{O}_C(g_6^2 + g_4^1)). \end{aligned}$$

If $H^0(C, \mathcal{O}_C(g_6^2 - g_4^1)) \neq 0$, then the g_4^1 corresponds to a double point of C_0. The number of such pencils are at most four. If $H^0(C, \mathcal{O}_C(g_6^2 - g_4^1)) = 0$, then $\dim|g_6^2 + g_4^1| \geq 5$, which implies that $|g_6^2 + g_4^1|$ is the canonical linear system of C. Hence $g_4^1 = |K_C - g_6^2|$. Thus we obtain that C has at most five g_4^1's. This is again a contradiction, since a smooth plane quintic C has infinitely many g_4^1's. Thus we complete the proof in this case.

Case (ii) $d' = d - 2$. In this case C has a pencil of degree $2(d' - 1) = 2d - 6$. Recall that C has no pencils of degree between $d + 1$ and $2d - 5$. Hence we have $2d - 6 \le d$, which implies that $d \le 6$. Thus we only have to consider the case $(d, d') = (6, 4)$. In this case we take a ramification point P of $\pi : C \to C'$ and set $P' := \pi(P)$. Then one of the following holds for the Weierstrass semigroup $H(P')$ of C' at P':

$$H(P') = \begin{cases} \langle 3, 4 \rangle, \\ \langle 3, 5, 7 \rangle \quad \text{or} \\ \langle 4, 5, 6, 7 \rangle. \end{cases}$$

In the first two cases $H(P')$ contains 3 and 7, which implies that $H(P)$, the Weierstrass semigroup of C at P, contains 6 and 14. In the third case $H(P)$ contains 8, 10, 12 and 14. Then we obtain a contradiction in each case by using the lemma below. Thus we complete the proof. □

Lemma 2.4 ([K], [KK]). *Let C be a smooth plane sextic.*
(1) *If $H(P)$ contains* 6 *for some point $P \in C$, then $H(P) = \langle 5, 6 \rangle$.*
(2) *There exists no point $P \in C$ such that $H(P)$ contains* 8, 10, 12 *and* 14.

Proof. For two plane curves Γ_1 and Γ_2 such that no common component of them contains P, we denote by $I(\Gamma_1 \cap \Gamma_2, P)$ the intersection multiplicity of Γ_1 and Γ_2 at P. Since plane cubics cut out the canonical linear system on C, we have

$$G(P) := \mathbf{N}_0 \backslash H(P) = \{ I(C \cap \Gamma, P) + 1 \mid \Gamma \text{ is a plane cubic} \}.$$

Note that $\# G(P) = 10$. Let T be the tangent line to C at P, L_1 a line in $\mathbf{P}^2$ that is distinct with T and contains P and L_0 a line in $\mathbf{P}^2$ not containing P. For any non-negative integers α and β with $\alpha + \beta \le 3$, we denote by $\Gamma(\alpha, \beta)$ the plane cubic $\alpha T + \beta L_1 + (3 - \alpha - \beta) L_0$. We can write $T|_C = \nu P + D$, where ν is an integer such that $2 \le \nu \le 6$ and D is an effective divisor on C whose support does not contain P.

(1) First note the following.

$$\text{If } \nu = 5, \text{ then we have } I(C \cap \Gamma(1,0), P) = 5.$$
$$\text{If } \nu = 4, \text{ then we have } I(C \cap \Gamma(1,1), P) = 5.$$
$$\text{If } \nu = 3, \text{ then we have } I(C \cap \Gamma(1,2), P) = 5.$$
$$\text{If } \nu = 2, \text{ then we have } I(C \cap \Gamma(2,1), P) = 5.$$

Thus we obtain that $G(P)$ contains 6 if $\nu \le 5$. In other words, if $H(P)$ contains 6, then $\nu = 6$ holds. Then we have $I(C \cap \Gamma(\alpha, \beta), P) = 6\alpha + \beta$ for $0 \le \alpha + \beta \le 3$. It follows that

$$G(P) \supset \{1, 2, 3, 4, 7, 8, 9, 13, 14, 19\}.$$

Hence we obtain that $G(P) = \{1, 2, 3, 4, 7, 8, 9, 13, 14, 19\}$ by comparing the cardinality of these sets. Equivalently, $H(P) = \langle 5, 6 \rangle$ holds.

(2) First we exclude the case where $v \geq 3$.

$$\text{If } v = 6, \text{ then we have } I(C \cap \Gamma(1,1), P) = 7.$$
$$\text{If } v = 5, \text{ then we have } I(C \cap \Gamma(1,2), P) = 7.$$
$$\text{If } v = 4, \text{ then we have } I(C \cap \Gamma(2,1), P) = 9.$$
$$\text{If } v = 3, \text{ then we have } I(C \cap \Gamma(2,1), P) = 7.$$

Thus we obtain that $H(P) \not\ni 8$ or $H(P) \not\ni 10$ in these cases. Suppose that $v = 2$. Then there exists a conic B such that $i_2 := I(C \cap B, P) \geq 5$. Note that $i_2 \leq 12$. If $i_2 = 5$, then $I(C \cap (T + B), P) = 7$, which implies that $H(P) \not\ni 8$. If i_2 is even with $i_2 \leq 12$, then $I(C \cap (L_1 + B), P) = i_2 + 1$, which implies that $H(P) \not\ni i_2 + 2$. If i_2 is odd with $7 \leq i_2 \leq 11$, then $I(C \cap (L_0 + B), P) = i_2$, which implies that $H(P) \not\ni i_2 + 1$. Thus we obtain a contradiction in any case.

Remark 2.5. Weierstrass semigroups of smooth plane curves of low degree are classified (cf. [K], [KK]). Our argument in the proof of the above lemma is based on theirs. □

In the end, we remark a fact on smooth plane quartics.

PROPOSITION 2.6. *Let C be a smooth plane quartic. Then it is bi-elliptic if and only if it has an involution.*

Proof. It is clear that a bi-elliptic curve has an involution. Conversely, assume that C has an involution σ. We then have a double covering $\pi : C \to C'$, where C' is the normalization of the (possibly singular) curve $C^{\langle\sigma\rangle}$. Then $g(C') = 0, 1$ or 2. It is known that a curve of genus 3 with a double covering onto a curve of genus 2 is hyperelliptic (cf. [A]). Since C is non-hyperelliptic, we have $g(C') = 1$. Thus C is bi-elliptic. □

REFERENCES

[A] R. D. M. Accola, On lifting the hyperelliptic involution, Proc. Amer. Math. Soc. **122** (1994), 341–347.

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of algebraic curves, I, Grundlehren **267**, Springer-Verlag, 1985.

[C] M. Coppens, The existence of base point free linear systems on smooth plane curves, J. Alg. Geom. **4** (1995), 1–15.

[CM] M. Coppens and G. Martens, Linear series on 4-gonal curves, Math. Nachr. **213** (2000), 35–55.

[HO] T. Harui and A. Ohbuchi, On the number of pencils of minimal degree on curves with small gonality, preprint.

[K] S. J. Kim, Weierstrass semigroups on a smooth plane curve of degree **4**, **5** and **6**, preprint.

[KK] S. J. Kim and J. Komeda, Weierstrass points on a non-singular plane curve of degree **7**, Research Reports of Kanagawa Institute of Technology **B-31** (2007), 29–34.

Takeshi Harui
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka, 560-0043
Japan
E-mail: takeshi@cwo.zaq.ne.jp, t-harui@cr.math.sci.osaka-u.ac.jp

Jiryo Komeda
Department of Mathematics
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology
Atsugi, Kanagawa, 243-0292
Japan
E-mail: komeda@gen.kanagawa-it.ac.jp

Akira Ohbuchi
Faculty of Integrated Arts and Sciences
Tokushima University
Tokushima, 770-0814
Japan
E-mail: ohbuchi@ias.tokushima-u.ac.jp