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DOUBLE FORMS, CURVATURE STRUCTURES AND THE (p,q)-CURVATURES

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ABSTRACT. We introduce a natural extension of the metric tensor and the Hodge star operator to the algebra of double forms to study some aspects of the structure of this algebra. These properties are then used to study new Riemannian curvature invariants, called the (p,q)-curvatures. They are a generalization of the p-curvature obtained by substituting the Gauss-Kronecker tensor to the Riemann curvature tensor. In particular, for $p=0$, the $(0,q)$-curvatures coincide with the H. Weyl curvature invariants, for $p=1$ the $(1,q)$-curvatures are the curvatures of generalized Einstein tensors, and for $q=1$ the $(p,1)$-curvatures coincide with the p-curvatures.

Also, we prove that the second H. Weyl curvature invariant is nonnegative for an Einstein manifold of dimension $n \geq 4$, and it is nonpositive for a conformally flat manifold with zero scalar curvature. A similar result is proved for the higher H. Weyl curvature invariants.

1. Introduction

Let (M,g) be a smooth Riemannian manifold of dimension n. We denote by $\Lambda^* M = \bigoplus_{p\geq 0} \Lambda^{*p} M$ the ring of differential forms on M. Considering the tensor product over the ring of smooth functions, we define $\mathcal{D} = \Lambda^* M \otimes \Lambda^* M = \bigoplus_{p,q\geq 0} \mathcal{D}^{p,q}$, where $\mathcal{D}^{p,q} = \Lambda^{*p} M \otimes \Lambda^{*q} M$. It is a graded associative ring, and it is called the ring of double forms on M.

The ring of curvature structures on M is the ring $\mathcal{C} = \sum_{p\geq 0} \mathcal{C}^p$, where $\mathcal{C}^p$ denotes the symmetric elements in $\mathcal{D}^{p,p}$.

These notions have been developed by Kulkarni [3], Thorpe [7] and other mathematicians.

The object of this paper is to investigate some properties of these structures in order to study generalized p-curvature functions.

The paper is divided into 5 sections. In section 2, we study the multiplication map by g^l in $\mathcal{D}^{p,q}$. In particular we prove that it is one-to-one if $p+q+l \leq n$. This result will play an important role in simplifying complicated calculations, as shown in section 5. We also deduce some properties of the multiplication map by g.

In section 3, we introduce a natural inner product $\langle\, ,\rangle$ in $\mathcal{D}$ and we extend the Hodge operator $*$ in a natural way to $\mathcal{D}$. Then, we prove two simple relations between the contraction map and the multiplication map by g, namely for all $\omega \in \mathcal{D}$,

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we have

$$g\omega = *c*\omega.$$

Also, we prove that the contraction map is the adjoint of the multiplication by g. Precisely for all $\omega_1, \omega_2 \in \mathcal{D}$, we have

$$\langle g\omega_1, \omega_2\rangle = \langle \omega_1, c\omega_2\rangle,$$

and we deduce some properties of the contraction map c.

At the end of this section, we deduce a canonical orthogonal decomposition of $\mathcal{D}^{p,q}$, and we give explicit formulas for the orthogonal projections onto the different factors.

In section 4, we concentrate on the ring of symmetric double forms satisfying the first Bianchi identity which shall be denoted by $\mathcal{C}_1$. We prove in this context a useful explicit formula for the Hodge star operator. Also, we emphasize its action on the different factors of the previous orthogonal decomposition of double forms.

In section 5, we define new Riemannian curvature invariants, namely the (p,q)-curvature tensors $R_{(p,q)}$ and their sectional curvatures $s_{(p,q)}$. Note that these curvatures include many of the well-known curvatures.

For $q = 1$, the $(p,1)$-curvature coincides with the p-curvature [4]. In particular, $s_{(0,1)}$ is half of the scalar curvature and $s_{(n-2,1)}$ is the sectional curvature of (M,g). For $p = 0$ and $2q = n$, $s_{(0,\frac{n}{2})}$ is, up to a constant, the Killing-Lipshitz curvature. More generally, $s_{(n-2q,q)}(P)$ is, up to a constant, the Killing-Lipshitz curvature of $P^{\perp}$.

For $p = 0$, $s_{(0,q)}$ are scalar functions which generalize the usual scalar curvature. They are, up to a constant, the H. Weyl curvature invariants, that is, the integrands in the Weyl tube formula [8].

Finally, for $p = 1$, $R_{(1,q)}$ are generalized Einstein tensors. In particular, for $q = 1$ we recover the usual Einstein tensor.

This section also contains several examples and properties of these curvature invariants. In particular, by using the $(p,1)$-curvatures we prove a characterization of Einstein metrics and conformally flat metrics with constant scalar curvature. Also, a generalization of the previous result to the higher (p,q)-curvatures is proved.

In the last section, section 6, we prove under certain geometric hypothesis on the metric, a restriction on the sign of the H. Weyl curvature invariants that are the integrands in his well-known tube formula [8]. In particular we prove the following results:

If (M,g) is an Einstein manifold with dimension $n \geq 4$, then $h_4 \geq 0$ and $h_4 \equiv 0$ if and only if (M,g) is flat.

If (M,g) is a conformally flat manifold with zero scalar curvature and dimension $n \geq 4$, then $h_4 \leq 0$ and $h_4 \equiv 0$ if and only if (M,g) is flat.

Here h_4 is the second H. Weyl curvature invariant, which can be defined by

$$h_4 = |R|^2 - |c(R)|^2 + \frac{1}{4}|c^2(R)|^2,$$

where R denotes the Riemann curvature tensor of (M,g).

2. The algebra of double forms

Let (V,g) be a Euclidean real vector space of dimension n. In the following, we shall identify whenever convenient (via their Euclidean structures) the vector spaces with their duals. Let $\Lambda^*V = \bigoplus_{p\geq 0}\Lambda^{*p}V$ (resp. $\Lambda V = \bigoplus_{p\geq 0}\Lambda^{p}V$) denote the exterior algebra of p-forms (resp. p-vectors) on V. Considering the tensor product, we define the space of double forms $\mathcal{D} = \Lambda^*V \otimes \Lambda^*V = \bigoplus_{p,q\geq 0} \mathcal{D}^{p,q}$, where $\mathcal{D}^{p,q} = \Lambda^{*p}V \otimes \Lambda^{*q}V$. It is a bi-graded associative algebra, where the multiplication is denoted by a dot. We shall omit it whenever it is possible.

For $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{r,s}$, we have

$$\omega_1.\omega_2 = (\theta_1 \otimes \theta_2).(\theta_3 \otimes \theta_4) = (\theta_1 \wedge \theta_3) \otimes (\theta_2 \wedge \theta_4) \in \mathcal{D}^{p+r,q+s}. \tag{1}$$

Recall that each element of the tensor product $\mathcal{D}^{p,q} = \Lambda^{*p}V \otimes \Lambda^{*q}V$ can be identified canonically with a bilinear form $\Lambda^p V \times \Lambda^q V \to \mathbf{R}$. That is, a multilinear form which is skew symmetric in the first p-arguments and also in the last q-arguments. Under this identification, we have

$$\begin{aligned} &\omega_1.\omega_2(x_1 \wedge ... \wedge x_{p+r}, y_1 \wedge ... \wedge y_{q+s}) \\ &= (\theta_1 \wedge \theta_3)(x_1 \wedge ... \wedge x_{p+r})(\theta_2 \wedge \theta_4)(y_1 \wedge ... \wedge y_{q+s}) \\ &= \frac{1}{p!r!s!q!} \sum_{\sigma \in S_{p+r}, \rho \in S_{q+s}} \epsilon(\sigma)\epsilon(\rho)\omega_1(x_{\sigma(1)} \wedge ... \wedge x_{\sigma(p)}; y_{\rho(1)} \wedge ... \wedge y_{\rho(q)}) \\ &\qquad \times\ \omega_2(x_{\sigma(p+1)} \wedge ... \wedge x_{\sigma(p+r)}; y_{\rho(q+1)} \wedge ... \wedge y_{\rho(q+s)}). \end{aligned} \tag{2}$$

A similar calculation shows that

$$\begin{aligned} &\omega_1^k(x_1 \wedge ... \wedge x_{kp}, y_1 \wedge ... \wedge y_{kq}) \\ &= (\theta_1 \wedge ... \wedge \theta_1)(x_1 \wedge ... \wedge x_{kp})(\theta_2 \wedge ... \wedge \theta_2)(y_1 \wedge ... \wedge y_{kp}) \\ &= \frac{1}{(p!)^k (q!)^k} \sum_{\sigma \in S_{kp}, \rho \in S_{kq}} \epsilon(\sigma)\epsilon(\rho)\omega_1(x_{\sigma(1)} \wedge ... \wedge x_{\sigma(p)}; y_{\rho(1)} \wedge ... \wedge y_{\rho(q)}) \\ &\qquad ... \,\omega_1(x_{\sigma(p(k-1)+1)} \wedge ... \wedge x_{\sigma(kp)}; y_{\rho(q(k-1)+1)} \wedge ... \wedge y_{\rho(kq)}). \end{aligned} \tag{3}$$

In particular, if $\omega_1 \in \mathcal{D}^{1,1}$ we have

$$\omega_1^k(x_1 \wedge ... \wedge x_k, y_1 \wedge ... \wedge y_k) = k! \det[\omega_1(x_i, y_j)]. \tag{4}$$

We now introduce a basic map on $\mathcal{D}$:

Definition 2.1. The contraction c maps $\mathcal{D}^{p,q}$ into $\mathcal{D}^{p-1,q-1}$. Let $\omega \in \mathcal{D}^{p,q}$, and set $c\omega = 0$ if $p = 0$ or $q = 0$. Otherwise set

$$c\omega(x_1 \wedge ... \wedge x_{p-1}, y_1 \wedge ... \wedge y_{q-1}) = \sum_{j=1}^{n} \omega(e_j \wedge x_1 \wedge ... x_{p-1}, e_j \wedge y_1 \wedge ... \wedge y_{q-1}),$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of V.

The contraction map c together with the multiplication map by g (which shall be denoted also by g) play a very important role in our study.

Letting $\omega \in \mathcal{D}^{p,q}$, the following formula was proved in [3]:

$$c(g\omega) = gc\omega + (n - p - q)\omega. \tag{5}$$

After consecutive applications of the previous formula, we get

$$\begin{aligned} c^k(g\omega) &= gc^k\omega + k(m + k - 1)c^{k-1}\omega, \text{ where } m = n - p - q, \\ c^k(g^2\omega) &= g^2c^k\omega + 2k(m + k - 2)gc^{k-1}\omega + k(k-1)(m + k - 3)(m + k - 2)c^{k-2}\omega, \\ c^k(g^3\omega) &= g^3c^k\omega + 3k(m + k - 3)g^2c^{k-1}\omega + 3k(k-1)(m+k-3)(m+k-4)gc^{k-2}\omega \\ &\quad + k(k-1)(k-2)(m + k - 3)(m + k - 4)(m + k - 5)c^{k-3}\omega. \end{aligned}$$

More generally, we have

Lemma 2.1. *For all $k, l \geq 1$ and $\omega \in D^{p,q}$, we have*

$$c^k(\frac{g^l}{l!}\omega) = \frac{g^l}{l!}c^k\omega + \sum_{r=1}^{\min\{k,l\}} C_r^k \prod_{i=0}^{r-1}(n-p-q+k-l-i)\frac{g^{l-r}}{(l-r)!}.c^{k-r}\omega. \tag{6}$$

Corollary 2.2. *If $n = p+q$ and $\omega \in D^{p,q}$, then for all k we have*

$$c^k(g^k\omega) = g^k(c^k\omega).$$

Proof. After taking $k = l$ and $n = p+q$ in formula (6), we get

$$c^k(\frac{g^k}{k!}\omega) = \frac{g^k}{k!}.c^k\omega + \sum_{r=1}^{k} C_r^k \prod_{i=0}^{r-1}(-i)\frac{g^{k-r}}{(k-r)!}c^{k-r}\omega = \frac{g^k}{k!}c^k\omega.$$

□

As a second consequence of the previous lemma, we get the following result which generalizes another lemma of Kulkarni [3].

Proposition 2.3. *The multiplication by g^l is injective on $D^{p,q}$ whenever $p+q+l < n+1$.*

Proof. This property is true for $l = 0$. Suppose that $g^{l-1}\omega = 0 \Rightarrow \omega = 0$ for $p+q+l-1 < n+1$, and let $g^l\omega = 0$ and $p+q+l < n+1$. Then the contractions $c^k(g^l.\omega) = o$ for all k.

Taking $k = 1, 2, ..., k, ..., \min\{p,q\}, \min\{p,q\}+1$ and after a simplification (if needed) by $g^{l-1}, g^{l-2}, ..., g^{l-k}, ..., g^{l-\min\{p,q\}+1}$, respectively, we get (using the previous lemma)

$$\begin{aligned}
-gc\omega &= l(n-p-q+1-l)\omega,\\
-gc^2\omega &= (l+1)(n-p-q+2-l)c\omega,\\
&\vdots\\
-g.c^k\omega &= (l+k-1)(n-p-q+k-l)c^{k-1}\omega,\\
&\vdots\\
-g.c^{\min\{p,q\}}\omega &= (l+\min\{p,q\}-1)(n-\max\{p,q\}-l)c^{\min\{p,q\}-1}\omega,\\
0 &= (l+\min\{p,q\})(n-\max\{p,q\}+1-l)c^{\min\{p,q\}}\omega.
\end{aligned}$$

Consequently, we have $c^{\min\{p,q\}}\omega = ...c^k\omega... = \omega = 0$. □

Remark 2.1. 1) The previous proposition cannot be obtained directly from Kulkarni's Lemma [3], since consecutive applications of that lemma show that the multiplication by g^l is 1-1 only if $p+q+2l-2 < n$.

2) We deduce from the previous proof that more generally we have

$$g^l\omega = 0 \Rightarrow c^k\omega = 0 \qquad \text{for} \quad l+p+q < n+1+k.$$

Corollary 2.4. 1) *Let $p+q = n-1$. Then for each $i \geq 0$, the multiplication map by g^{2i+1},*

$$g^{2i+1} : D^{p-i,q-i} \to D^{p+i+1,q+i+1},$$

is an isomorphism. In particular, we have

$$D^{p+i+1,q+i+1} = g^{2i+1}D^{p-i,q-i}.$$

2) *Let $p+q=n$. Then for each $i \geq 0$, the multiplication map by g^{2i},*

$$g^{2i} : D^{p-i,q-i} \to D^{p+i,q+i},$$

is an isomorphism. In particular, we have

$$D^{p+i,q+i} = g^{2i}D^{p-i,q-i}.$$

Proof. From the previous proposition it is 1-1, and for dimension reasons (since $C^n_{p-i}C^n_{q-i} = C^n_{p+i+1}C^n_{q+i+1}$ if $p+q=n-1$, and $C^n_{p-i}C^n_{q-i} = C^n_{p+i}C^n_{q+i}$ if $p+q=n$) it is an isomorphism. □

The following proposition gives more detail about the multiplication by g.

Proposition 2.5. *The multiplication map by g on $D^{p,q}$ is*

1) *one-to-one if and only if $p+q \leq n-1$,*
2) *bijective if and only if $p+q = n-1$,*
3) *onto if and only if $p+q \geq n-1$.*

Proof. The "only if" part of the proposition is due simply to dimension reasons, so that parts 1) and 2) are direct consequences of Kulkarni's Lemma.

Now let $i \geq 0$, $p_0+q_0 = n-1$ for some $p_0, q_0 \geq 0$, and

$$g : D^{p_0+i,q_0+i} \to D^{p_0+i+1,q_0+i+1}.$$

Remark that the restriction of the map g to the subspace $g^{2i}D^{p_0-i,q_0-i}$ of D^{p_0+i,q_0+i} is onto. This is because its image is exactly $g^{2i+1}D^{p_0-i,q_0-i} = D^{p_0+i+1,q_0+i+1}$ by the previous proposition. The proof is similar in case where there exists $p_0, q_0 \geq 0$ such that $p_0+q_0 = n$. This completes the proof of the proposition. □

3. The natural inner product and the Hodge star operator on $D^{p,q}$

3.1. **The natural inner product on $D^{p,q}$.** The natural metric on $\Lambda^{*p}V$ induces in a standard way an inner product on $D^{p,q} = \Lambda^{*p}V \otimes \Lambda^{*q}V$. We shall denote it by $\langle\, , \rangle$.

We extend $\langle\, , \rangle$ to $\mathcal{D}$ by declaring that $D^{p,q} \perp D^{r,s}$ if $p \neq r$ or if $q \neq s$.

Theorem 3.1. *If $\omega_1, \omega_2 \in \mathcal{D}$, then*

$$\langle g\omega_1, \omega_2\rangle = \langle \omega_1, c\omega_2\rangle. \tag{7}$$

That is, the contraction map c is the adjoint of the multiplication map by g.

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of V^*. Since the contraction map c and the multiplication by g are linear, it suffices to prove the theorem for

$$\omega_2 = e_{i_1} \wedge ... \wedge e_{i_{p+1}} \otimes e_{j_1} \wedge ... \wedge e_{j_{q+1}} \text{ and } \omega_1 = e_{k_1} \wedge ... \wedge e_{k_p} \otimes e_{l_1} \wedge ... \wedge e_{l_q},$$

where $i_1 < ... < i_{p+1}; j_1 < ... < j_{q+1}; k_1 < ... < k_p$ and $l_1 < ... < l_q$. Since

$$g\omega_1 = \sum_{i=1}^n e_i \wedge e_{k_1} \wedge ... \wedge e_{k_p} \otimes e_i \wedge e_{l_1} \wedge ... \wedge e_{l_q},$$

it follows that

$$\langle g\omega_1, \omega_2\rangle = \sum_{i=1}^n \langle e_i \wedge e_{k_1} \wedge ... \wedge e_{k_p}, e_{i_1} \wedge ... \wedge e_{i_{p+1}}\rangle \langle e_i \wedge e_{l_1} \wedge ... \wedge e_{l_q}, e_{j_1} \wedge ... \wedge e_{j_{q+1}}\rangle.$$

Therefore, it is zero unless

$$e_{k_1}\wedge ...\wedge e_{k_p}=e_{i_1}\wedge ...\hat{e}_{i_r}...\wedge e_{i_{p+1}},$$
$$e_{l_1}\wedge ...\wedge e_{l_q}=e_{j_1}\wedge ...\hat{e}_{j_s}...\wedge e_{j_{q+1}},$$

and $i_r=j_s$ for some r,s, so that in this case we have

$$\begin{aligned}\langle g\omega_1,\omega_2\rangle &=\sum_{i=1}^{n}\langle e_i\wedge e_{i_1}\wedge ...\hat{e}_{i_r}...\wedge e_{i_{p+1}},e_{i_1}\wedge ...\wedge e_{i_{p+1}}\rangle\\ &\qquad\qquad \times\langle e_i\wedge e_{j_1}\wedge ...\hat{e}_{j_s}...\wedge e_{j_{q+1}},e_{j_1}\wedge ...\wedge e_{j_{q+1}}\rangle\\ &=(-1)^{r+s}.\end{aligned}$$

On the other hand, we have

$$c\omega_2=0\ \text{ if } \{i_1,...,i_{p+1}\}\cap\{j_1,...,i_{q+1}\}=\emptyset,$$

otherwise,

$$c\omega_2=\sum_{\substack{i_r=j_s\\ 1\leq r\leq p+1\\ 1\leq s\leq q+1}}(-1)^{r+s}e_{1_1}\wedge ...\hat{e}_{i_r}...\wedge e_{i_{p+1}}\otimes e_{j_1}\wedge ...\hat{e}_{j_s}...\wedge e_{j_{q+1}}.$$

Therefore

$$\begin{aligned}\langle\omega_1,c\omega_2\rangle=&\sum_{\substack{i_r=j_s\\ 1\leq r\leq p+1\\ 1\leq s\leq q+1}}(-1)^{r+s}\langle e_{k_1}\wedge ...\wedge e_{k_p},e_{i_1}\wedge ...\hat{e}_{i_r}...\wedge e_{i_{p+1}}\rangle\\ &\qquad\qquad\times\langle e_{l_1}\wedge ...\wedge e_{l_q},e_{j_1}\wedge ...\hat{e}_{j_s}...\wedge e_{j_{q+1}}\rangle,\end{aligned}$$

which is zero unless

$$e_{k_1}\wedge ...\wedge e_{k_p}=e_{i_1}\wedge ...\hat{e}_{i_r}...\wedge e_{i_{p+1}},$$
$$e_{l_1}\wedge ...\wedge e_{l_q}=e_{j_1}\wedge ...\hat{e}_{j_s}...\wedge e_{j_{q+1}},$$

and $i_r=j_s$ for some r,s. In such case it is $(-1)^{r+s}$. This completes the proof. □

3.2. **Hodge star operator.** The Hodge star operator $*:\Lambda^pV^*\to\Lambda^{n-p}V*$ extends in a natural way to a linear operator $*:\mathcal{D}^{p,q}\to\mathcal{D}^{n-p,n-q}$. If $\omega=\theta_1\otimes\theta_2$, then we define

$$*\omega=*\theta_1\otimes *\theta_2.$$

Note that $*\omega(.,.)=\omega(*.,*.)$ as a bilinear form. Many properties of the ordinary Hodge star operator can be extended to this new operator. We prove some of them below:

Proposition 3.2. *For all $\omega,\theta\in D^{p,q}$, we have*

$$\langle\omega,\theta\rangle=*(\omega.*\theta)=*(*\omega.\theta). \tag{8}$$

Proof. Let $\omega=\omega_1\otimes\omega_2$ and $\theta=\theta_1\otimes\theta_2$. Then

$$\begin{aligned}\omega.*\theta&=(\omega_1\wedge *\theta_1)\otimes(\omega_2\wedge *\theta_2)\\ &=\langle\omega_1,\theta_1\rangle *1\otimes\langle\omega_2,\theta_2\rangle *1\\ &=\langle\omega,\theta\rangle *1\otimes *1.\end{aligned}$$

This completes the proof. □

The proof of the following properties is similar and straightforward.

Proposition 3.3. 1) *For all p,q, on $D^{p,q}$ we have*
$$**=(-1)^{(p+q)(n-p-q)}Id,$$
where Id is the identity map on $D^{p,q}$.

2) *For all $\omega_1\in D^{p,q},\omega_2\in D^{n-p,n-q}$ we have*
$$\langle\omega_1,*\omega_2\rangle=(-1)^{(p+q)(n-p-q)}\langle*\omega_1,\omega_2\rangle.$$

3) *If $\bar\omega:\Lambda^p\to\Lambda^p$ denotes the linear operator corresponding to $\omega\in D^{p,p}$, then*
$$*\bar\omega*:\Lambda^{n-p}\to\Lambda^{n-p}$$
is the linear operator corresponding to $*\omega\in D^{n-p,n-p}$.

Using the Hodge star operator, we can provide a simple formula relating the multiplication by g and the contraction map c, as follows.

Theorem 3.4. *For every $\omega\in D^{p,q}$, we have*
$$g\omega=*c*\omega. \tag{9}$$
That is, the following diagram is commutative for all p,q:
$$\begin{tikzcd} D^{p,q} \arrow[r, "g"] \arrow[d, "*"] & D^{p+1,q+1} \\ D^{n-p,n-q} \arrow[r, "c"] & D^{n-p-1,n-q-1} \arrow[u, "*"] \end{tikzcd}$$

Proof. The proof is similar to the one of Theorem 3.1. Let $\{e_1,...,e_n\}$ be an orthonormal basis of V^*, and let
$$\omega=e_{i_1}\wedge...\wedge e_{i_p}\otimes e_{j_1}\wedge...\wedge e_{j_q}.$$
Then
$$g\omega=\sum_{i=1}^n e_i\wedge e_{i_1}\wedge...\wedge e_{i_p}\otimes e_i\wedge e_{j_1}\wedge...\wedge e_{j_q}.$$
On the other hand, we have
$$*\omega=\epsilon(\rho)\epsilon(\sigma)e_{i_{p+1}}\wedge...\wedge e_{i_n}\otimes e_{j_{q+1}}\wedge...\wedge e_{j_n},$$
so that
$$c*\omega=\sum_{\substack{i_r=j_s\\ p+1\le r\le n\\ q+1\le s\le n}}(-1)^{r+s}\epsilon(\rho)\epsilon(\sigma)e_{i_{p+1}}\wedge...\hat{e}_{i_r}...\wedge e_{i_n}\otimes e_{j_{q+1}}\wedge...\hat{e}_{j_s}...\wedge e_{j_n}.$$
Therefore
$$\begin{aligned}*c*\omega&=\sum_{\substack{i_r=j_s\\ p+1\le r\le n\\ q+1\le s\le n}}(-1)^{r+s}\epsilon(\rho)\epsilon(\sigma)*e_{i_{p+1}}\wedge...\hat{e}_{i_r}...\wedge e_{i_n}\otimes *e_{j_{q+1}}\wedge...\hat{e}_{j_s}...\wedge e_{j_n}\\ &=\sum_{\substack{i_r=j_s\\ p+1\le r\le n\\ q+1\le s\le n}}e_{i_r}\wedge e_{i_1}\wedge...\wedge e_{i_p}\otimes e_{j_s}\wedge e_{j_1}\wedge...\wedge e_{j_q}\\ &=g\omega.\end{aligned}$$
This completes the proof. ∎

As a direct consequence of the previous theorem and Proposition 2.5, we have the following corollaries:

Corollary 3.5. *The contraction map c on $D^{p,q}$ is*

1) *onto if and only if $p+q \leq n-1$,*
2) *bijective if and only if $p+q = n-1$,*
3) *one-to-one if and only if $p+q \geq n-1$.*

Corollary 3.6. *For all $p, q \geq 0$ such that $p+q \leq n-1$, we have the orthogonal decomposition*

$$D^{p+1,q+1} = \operatorname{Ker} c \oplus gD^{p,q},$$

where $c : D^{p+1,q+1} \to D^{p,q}$ is the contraction map.

Proof. First note that if $\omega_1 \in \ker c$ and $g\omega_2 \in gD^{p,q}$, then by formula (7), we have

$$\langle \omega_1, g\omega_2 \rangle = \langle c\omega_1, \omega_2 \rangle = 0.$$

Next, since g is one-to-one and c is onto, we have

$$\dim(gD^{p,q}) = \dim D^{p,q} = \dim(\operatorname{image} c).$$

This completes the proof. □

Remark 3.1. 1) If $p+q > n-1$, then we have $\ker c = 0$ and $D^{p+1,q+1}$ is isomorphic to some $g^r D^{s,t}$ with $s+t \leq n-1$ by Corollary 2.4.

2) Note that in general $\operatorname{Ker} c$ is not irreducible; see [3] for the reduction matter.

3.3. **Orthogonal decomposition of $D^{p,q}$.** Following Kulkarni we call the elements in $\ker c \subset D^{p,q}$ effective elements of $D^{p,q}$, and are denoted by $E^{p,q}$.

So if we apply Corollary 3.6 several times, we obtain the orthogonal decomposition of $D^{p,q}$:

$$D^{p,q} = E^{p,q} \oplus gE^{p-1,q-1} \oplus g^2 E^{p-2,q-2} \oplus \ldots \oplus g^r D^{p-r,q-r}, \tag{10}$$

where $r = \min\{p, q\}$.

In this section, we show how double forms decompose explicitly under this orthogonal decomposition. To simplify the exposition, we shall consider only the case where $p = q$.

First, note that formula (6) for $\omega \in E^{p,p}$ becomes

$$\begin{aligned} c^k\left(\frac{g^l}{l!}\omega\right) &= \prod_{i=1}^{i=k}(n-2p-l+i)\frac{g^{l-k}}{(l-k)!}.\omega \qquad \text{if} \quad l \geq k, \\ c^k(g^l\omega) &= 0 \qquad \text{if} \quad l < k. \end{aligned} \tag{11}$$

With respect to the previous orthogonal decomposition, let $\omega = \sum_{i=0}^{p} g^i \omega_{p-i} \in D^{p,p}$ where $\omega_{p-i} \in E^{p-i,p-i}$. Then using the previous formula (11), we have

$$\begin{aligned} c^k(\omega) &= \sum_{i=0}^{p} c^k(g^i\omega_{p-i}) = \sum_{i=k}^{p} c^k(g^i\omega_{p-i}) \\ &= \sum_{i=k}^{p} i! \prod_{j=1}^{j=k}(n-2(p-i)-i+j)\frac{g^{i-k}}{(i-k)!}\omega_{p-i}. \end{aligned}$$

Therefore, we get

$$c^k(\omega) = \sum_{i=k}^{p} i! \prod_{j=1}^{j=k} (n-2p+i+j) \frac{g^{i-k}}{(i-k)!}\omega_{p-i}. \tag{12}$$

Taking in the previous formula $k = p, p-1, p-2, ..., k, ..., 0$ respectively, and solving for ω_k we get

$$\begin{aligned}
\frac{p!n!}{(n-p)!}\omega_0 &= c^p(\omega),\\
\frac{(p-1)!(n-2)!}{(n-p-1)!}\omega_1 &= c^{p-1}(\omega) - \frac{1}{n}g.c^p(\omega),\\
\frac{(p-2)!(n-4)!}{(n-p-2)!}\omega_2 &= c^{p-2}(\omega) - \frac{1}{n-2}gc^{p-1}(\omega) + \frac{1}{2!(n-2)(n-1)}g^2c^p(\omega),\\
&\vdots\\
\frac{(p-k)!(n-2k)!}{(n-p-k)!}\omega_k &= c^{p-k}(\omega) + \sum_{r=1}^{k}\frac{(-1)^r}{r!\prod_{i=0}^{r-1}(n-2k+2+i)}g^rc^{p-k+r}(\omega),\\
&\vdots\\
\omega_p &= \omega + \sum_{r=1}^{p}\frac{(-1)^r}{r!\prod_{i=0}^{r-1}(n-2p+2+i)}g^rc^r(\omega).
\end{aligned}$$

Note that $\omega_p = \operatorname{con}\omega$ is the conformal component defined by Kulkarni.

We have therefore proved the following theorem (it generalizes a similar classical result in the case where ω is the Riemann curvature tensor).

Theorem 3.7. *With respect to the orthogonal decomposition* (10)*, each* $\omega \in D^{p,p}$ *is decomposed as follows:*

$$\omega = \omega_p + g.\omega_{p-1} + ... + g^p.\omega_0,$$

where $\omega_0 = \frac{(n-p)!}{p!n!}c^p(\omega)$*, and for* $1 \leq k \leq p$ *we have*

$$\omega_k = \frac{(n-p-k)!}{(p-k)!(n-2k)!}\left[c^{p-k}(\omega) + \sum_{r=1}^{k}\frac{(-1)^r}{\prod_{i=0}^{r-1}(n-2k+2+i)}\frac{g^r}{r!}c^{p-k+r}(\omega)\right].$$

In particular, for $\omega = R$, we recover the well-known decomposition of the Riemann curvature tensor

$$R = W + \frac{1}{n-2}(c(R) - \frac{1}{n}g.c^2(R))g + \frac{1}{2n(n-1)}c^2(R).g^2.$$

4. The algebra of curvature structures

Note that from the definition of the product (see formula (1)), we have

$$\omega_1.\omega_2 = (-1)^{pr+qs}\omega_2.\omega_1.$$

Then, following Kulkarni, we define the algebra of curvature structures to be the commutative sub-algebra $\mathcal{C} = \bigoplus_{p\geq 0}\mathcal{C}^p$, where $\mathcal{C}^p$ denotes the symmetric elements of $D^{p,p}$. That is, the sub-algebra of symmetric double forms.

Another basic map in $D^{p,q}$ is the first Bianchi sum, denoted $\mathcal{B}$. It maps $\mathcal{D}^{p,q}$ into $\mathcal{D}^{p+1,q-1}$ and is defined as follows. Let $\omega \in \mathcal{D}^{p,q}$ and set $\mathcal{B}\omega = 0$ if $q = 0$. Otherwise set

$$\mathcal{B}\omega(x_1\wedge\ldots\wedge x_{p+1}, y_1\wedge\ldots\wedge y_{q-1}) = \sum_{j=1}^{p+1}(-1)^j\omega(x_1\wedge\ldots\wedge\hat{x}_j\wedge\ldots x_{p+1}, x_j\wedge y_1\wedge\ldots\wedge y_{q-1}),$$

where $\hat{}$ denotes omission.

It is easy to show that for $\omega \in \mathcal{D}^{p,q}, \theta \in \mathcal{D}^{r,s}$, we have [3]

$$\mathcal{B}(\omega.\theta) = \mathcal{B}\omega.\theta + (-1)^{p+q}\omega.\mathcal{B}\theta.$$

Consequently, ker$\mathcal{B}$ is closed under multiplication in $\mathcal{D}$.

The algebra of curvature structures satisfying the first Bianchi identity is defined to be $\mathcal{C}_1 = \mathcal{C} \cap \ker\mathcal{B}$.

4.1. **Sectional curvature.** Let G_p denote the Grassmann algebra of p-planes in V, and let $\omega \in \mathcal{C}^p$. We define the sectional curvature of ω to be

$$K_\omega(P) = \omega(e_1 \wedge \ldots \wedge e_p, e_1 \wedge \ldots \wedge e_p),$$

where $\{e_1, \ldots, e_p\}$ is any orthonormal basis of V.

Using formula (2), we can evaluate the sectional curvature of the tensors $g^p\omega$. For $\omega \in \mathcal{C}^r$ and $\{e_1, \ldots, e_{p+r}\}$ orthonormal, we get

$$\begin{aligned} &g^p\omega(e_1 \wedge \ldots \wedge e_{p+r}, e_1 \wedge \ldots \wedge e_{p+r}) \\ &= p!\sum_{1\le i_1<i_2<\ldots<i_r\le p+r}\omega(e_{i_1} \wedge \ldots \wedge e_{i_r}, e_{i_1} \wedge \ldots \wedge e_{i_r}) = p!\,\mathrm{trace}\,\omega_{|\Lambda^r P}, \end{aligned} \tag{13}$$

where P denotes the plane spanned by $\{e_1, \ldots, e_{p+r}\}$.

The sectional curvature K_ω generically determines ω. Precisely, for $\omega, \theta \in \mathcal{C}_1^p$, the equality $K_\omega = K_\theta$ implies $\omega = \theta$ (cf. prop. 2.1 in [3]). In particular, we have the following characterization of curvature structures $\omega \in \mathcal{C}_1^p$ with constant sectional curvature:

$$K_\omega \equiv c \qquad \text{if and only if} \qquad \omega = c\frac{g^p}{p!}. \tag{14}$$

Next, we shall prove a useful explicit formula for the Hodge star operator.

Theorem 4.1. *For $\omega \in \mathcal{C}_1^p$ and $1 \le p \le k \le n$ we have*

$$\frac{1}{(k-p)!} * (g^{k-p}\omega) = \sum_{r=\max\{0,p-n+k\}}^{p} \frac{(-1)^{r+p}}{r!}\frac{g^{n-k-p+r}}{(n-k-p+r)!}c^r\omega. \tag{15}$$

In particular, for $k = n$ and $k = n-1$ respectively, we get

$$*(\frac{g^{n-p}\omega}{(n-p)!}) = \frac{1}{p!}c^p\omega \quad \textit{and} \quad *(\frac{g^{n-p-1}\omega}{(n-p-1)!}) = \frac{c^p\omega}{p!}g - \frac{c^{p-1}}{(p-1)!}\omega. \tag{16}$$

Proof. It is not difficult to check that

$$\sum_{1\le i_1,i_2,\dots,i_p\le n}\omega(e_{i_1}\wedge\dots\wedge e_{i_p},e_{i_1}\wedge\dots\wedge e_{i_p})$$
$$=\sum_{1\le i_1,i_2,\dots,i_p\le k}\omega(e_{i_1}\wedge\dots\wedge e_{i_p},e_{i_1}\wedge\dots\wedge e_{i_p})$$
$$+\sum_{r=0}^{p-1}(-1)^{r+p+1}C_r^p\sum_{k+1\le i_{r+1},\dots,i_p\le n}c^r\omega(e_{i_{r+1}}\wedge\dots\wedge e_{i_p},e_{i_{r+1}}\wedge\dots\wedge e_{i_p}).$$

Then using formula (13), the previous formula becomes

$$c^p\omega=p!\frac{g^{k-p}}{(k-p)!}\omega(e_{i_1}\wedge\dots\wedge e_{i_k},e_{i_1}\wedge\dots\wedge e_{i_k})+\sum_{r=\max\{0,p-n+k\}}^{p-1}(-1)^{r+p+1}C_r^p$$
$$\frac{(p-r)!}{(n-k-p+r)!}g^{n-k-p+r}c^r\omega(e_{i_{k+1}}\wedge\dots\wedge e_{i_n},e_{i_{k+1}}\wedge\dots\wedge e_{i_n}).$$

Finally, note that the general term of the previous sum is $c^p\omega$ if $r=p$. This completes the proof, since both sides of the equation satisfy the first Bianchi identity. □

The following corollary is a direct consequence of the previous theorem.

Corollary 4.2. 1) *For $\omega\in\mathcal{C}_1^p$ and $1\le p\le n$ we have*

$$*\omega=\sum_{r=\max\{0,2p-n\}}^{p}\frac{(-1)^{r+p}}{r!}\frac{g^{n-2p+r}}{(n-2p+r)!}c^r\omega. \tag{17}$$

2) *For all $0\le k\le n$ we have*

$$*\frac{g^k}{k!}=\frac{g^{n-k}}{(n-k)!}.$$

Theorem 4.3. *With respect to the decomposition* (10), *we have*

$$*\omega=\sum_{i=0}^{\min\{p,n-p\}}(p-i)!(-1)^i\frac{1}{(n-p-i)!}g^{n-p-i}\omega_i \tag{18}$$

for $\omega=\sum_{i=0}^{p}g^{p-i}\omega_i$. In particular if $n=2p$, we have

$$*\omega=\sum_{i=0}^{p}(-1)^i g^{p-i}\omega_i.$$

Proof. First, let $\omega\in E_1^i$ be effective. Then formula (15) shows that

$$\frac{1}{(k-i)!}*(g^{k-i}\omega)=\begin{cases}0 & \text{if } i-n+k>0,\\ \frac{(-1)^i}{(n-k-i)!}g^{n-k-i}\omega & \text{if } i-n+k\le 0.\end{cases} \tag{19}$$

Next, let $\omega=\sum_{i=0}^{p} g^{p-i}\omega_i$, where $\omega_i\in E_1^i$. Then

$$\begin{aligned} *\omega &= \sum_{i=0}^{p} *(g^{p-i}\omega_i)\\ &= \sum_{i=0}^{\min\{p,n-p\}} (p-i)!\frac{(-1)^i}{(n-p-i)!}g^{n-p-i}\omega_i. \end{aligned}$$

□

Corollary 4.4. *With respect to the decomposition* (10), *we have*

$$*(g^l\omega)=\sum_{i=0}^{\min\{p,n-p-l\}} (p-i+l)!\frac{(-1)^i}{(n-p-l-i)!}g^{n-p-l-i}\omega_i \tag{20}$$

for $\omega=\sum_{i=0}^{p} g^{p-i}\omega_i$.

Proof. First, formula (18) implies that

$$*(g^l\omega)=\sum_{i=0}^{\min\{p+l,n-p-l\}} (p-i+l)!\frac{(-1)^i}{(n-p-l-i)!}g^{n-p-l-i}(g^l\omega)_i.$$

Next, note that

$$g^l\omega=\sum_{i=0}^{p} g^{p+l-i}\omega_i.$$

Consequently,

$$(g^l\omega)_i=\begin{cases}0 & \text{if } i>p,\\ \omega_i & \text{if } i\le p.\end{cases}$$

This completes the proof of the corollary. □

5. The (p,q)-curvatures

Let (M,g) be an n-dimensional Riemannian manifold and let T_mM be its tangent space at a point $m\in M$. Let $D^{p,q},\mathcal{C}^p,\mathcal{C}_1^p$... also denote the vector bundles over M having as fibers at m the spaces $D^{p,q}(T_mM)$, $\mathcal{C}^p(T_mM)$, $\mathcal{C}_1^p(T_mM)$.... Note that all the above algebraic results can be applied to the ring of all global sections of these bundles.

Remark that, since the metric g and the Riemann curvature tensor R both satisfy the first Bianchi identity, then so are all the tensors g^pR^q and $*(g^pR^q)$.

The aim of this section is to study some geometric properties of these tensors. First we start with the case $q=1$.

5.1. **The p-curvature.** Recall that the p-curvature [4], defined for $0\le p\le n-2$ and denoted by s_p, is the sectional curvature of the tensor

$$\frac{1}{(n-2-p)!}*(g^{n-2-p}R).$$

For a given tangent p-plane at $m\in M$, $s_p(P)$ coincides with half of the scalar curvature at m of the totally geodesic submanifold $\exp_m P^{\perp}$. For $p=0$, it is half of the usual scalar curvature, and for $p=n-2$ it coincides with the usual sectional curvature.

In this subsection, using the p-curvature and the previous results, we shall give a short proof for the following properties. Similar results were proved by a long calculation in [6] and [4].

Theorem 5.1. 1) *For each $2 \leq p \leq n-2$, the p-curvature is constant if and only if (M,g) has constant sectional curvature.*

2) *For each $1 \leq p \leq n-1$, the Riemannian manifold (M,g) is Einstein if and only if the function $P \to s_p(P) - s_{n-p}(P^\perp) = \lambda$ is constant. Furthermore, in such a case we have $\lambda = \frac{n-2p}{2n}c^2R$.*

3) *For each $2 \leq p \leq n-2$ and $p \neq \frac{n}{2}$, the function $P \to s_p(P)+s_{n-p}(P^\perp) = \lambda$ is constant if and only if the manifold (M,g) has constant sectional curvature. Furthermore, in such a case we have $\lambda = \frac{2p(p-1)+(n-2p)(n-1)}{2n(n-1)}c^2R$.*

4) *Let $n = 2p$. Then the Riemannian manifold (M,g) is conformally flat with constant scalar curvature if and only if the function $P \to s_p(P)+s_p(P^\perp) = \lambda$ is constant. Furthermore, in such a case we have $\lambda = \frac{n-2}{4(n-1)}c^2R$.*

Proof. First we prove 1). Let $s_p \equiv c$; then the sectional curvature of the tensor $g^{n-2-p}R \in \mathcal{C}_1^{n-p}$ is constant. Therefore we have $\frac{1}{(n-2-p)!}g^{n-2-p}R = c\frac{g^{n-p}}{(n-p)!}$, and so by Proposition 2.3 we have $(n-p)(n-p-1)R = cg^2$. That is, R has constant sectional curvature.

Next we prove 2). Suppose $s_p(P) - s_{n-p}(P^\perp) = c$ for all P. Then

$$\frac{1}{(n-2-p)!}g^{n-2-p}R(*P,*P) - \frac{1}{(p-2)!}g^{p-2}R(P,P) = c \text{ for all } P.$$

Hence using formula (15), we get

$$\sum_{r=0}^{2}\frac{(-1)^r}{r!}\frac{g^{p-2+r}}{(p-2+r)!}c^rR(P,P) - \frac{1}{(p-2)!}g^{p-2}R(P,P) = c \text{ for all } P.$$

The left-hand side is the sectional curvature of a curvature tensor which satisfies the first Bianchi identity. Then

$$-\frac{g^{p-1}}{(p-1)!}cR + \frac{g^p}{2(p!)}c^2R = c\frac{1}{p!}g^p.$$

Using Proposition 2.3, we get

$$-cR + \frac{g}{2p}c^2R = c\frac{1}{p}g,$$

and therefore

$$cR = \frac{c^2R - 2c}{2p}g.$$

Then (M,g) is an Einstein manifold. Furthermore, after taking the trace we get $c = \frac{n-2p}{2n}c^2R$.

Finally we prove 3) and 4). Suppose $s_p(P) + s_{n-p}(P^\perp) = c$ for all P. Then as in part 2) we have

$$2\frac{g^{p-2}}{(p-2)!}R - \frac{g^{p-1}}{(p-1)!}cR + \frac{g^p}{2(p!)}c^2R = c\frac{1}{p!}g^p.$$

Then using Proposition 2.3, we get

$$2R - \frac{g}{(p-1)}cR + \frac{g^2}{2p(p-1)}c^2R = c\frac{1}{p(p-1)}g^2.$$

This implies that

$$2\omega_2 - \frac{n-2p}{p-1} g\omega_1 + (2 + \frac{(n-2p)(n-1)}{p(p-1)}) g^2\omega_0 = \frac{c}{p(p-1)} g^2,$$

where $R = \omega_2 + g\omega_1 + g^2\omega_0$. Then if $n \neq 2p$ we have $\omega_2 = \omega_1 = 0$ and therefore the sectional curvature of (M, g) is constant. In the case $n = 2p$, we have $\omega_2 = 0$ and $c = 2\omega_0 p(p-1) = \frac{n-2}{4(n-1)} c^2 R$ so that (M, g) is conformally flat with constant scalar curvature. $\square$

5.2. **The** (p, q)**-curvatures.** The (p, q)-curvatures are the p-curvatures of the Gauss-Kronecker tensor R^q (that is, the product of the Riemann tensor R with itself q-times in the ring of curvature structures). Precisely, they are defined as follows.

Definition 5.1. The (p, q)-curvature, denoted $s_{(p,q)}$, for $1 \leq q \leq \frac{n}{2}$ and $0 \leq p \leq n - 2q$, is the sectional curvature of the following (p, q)-curvature tensor:

$$R_{(p,q)} = \frac{1}{(n-2q-p)!} * \left(g^{n-2q-p} R^q\right). \tag{21}$$

In other words, $s_{(p,q)}(P)$ is the sectional curvature of the tensor $\frac{1}{(n-2q-p)!} g^{n-2q-p} R^q$ at the orthogonal complement of P.

These curvatures include many of the well-known curvatures.

Note that for $q = 1$, we have $s_{(p,1)} = s_p$, which coincides with the p-curvature. In particular, $s_{(0,1)}$ is half of the scalar curvature and $s_{(n-2,1)}$ is the sectional curvature of (M, g).

For $p = 0$ and $2q = n$, $s_{(0,\frac{n}{2})} = *R^{n/2}$ is, up to a constant, the Killing-Lipshitz curvature. More generally, $s_{(n-2q,q)}(P)$ is, up to a constant, the Killing-Lipshitz curvature of $P^{\perp}$. That is, the $(2p)$-sectional curvatures defined by A. Thorpe in [7].

For $p = 0$, $s_{(0,q)} = *\frac{1}{(n-2q)!} g^{n-2q} R^q = \frac{1}{(2q)!} c^{2q} R^q$ are scalar functions which generalize the usual scalar curvature. They are, up to constants, the integrands in the Weyl tube formula [8].

For $p = 1$, $s_{(1,q)}$ are the curvatures of generalized Einstein tensors. Precisely, let us define the following:

Definition 5.2. 1) The $2q$-scalar curvature function, or the $2q$-H. Weyl curvature invariant, denoted h_{2q}, is the $(0, q)$-curvature. That is,

$$h_{2q} = s_{(0,q)} = \frac{1}{(2q)!} c^{2q} R^q.$$

2) The $2q$-Einstein tensor, denoted T_{2q}, is defined to be the $(1, q)$-curvature tensor. That is,

$$T_{2q} = * \frac{1}{(n-2q-1)!} g^{n-2q-1} R^q.$$

By formula (16), we have

$$T_{2q} = \frac{1}{(2q)!} c^{2q} R^q - \frac{1}{(2q-1)!} c^{2q-1} R^q = h_{2q} - \frac{1}{(2q-1)!} c^{2q-1} R^q.$$

For $q = 1$, we recover the usual Einstein tensor $T_2 = \frac{1}{2} c^2 R - cR$. Note that $c^{2q-1} R^q$ can be considered as a generalization of the Ricci curvature. In a forthcoming paper [5], we prove that the $2q$-Einstein tensor is the gradiant of the total $2q$-scalar curvature function seen as a functional on the space of all Riemannian metrics with volume 1, which generalizes the well-known classical result about the scalar curvature.

Finally, note that in general $s_{(p,q)}(P)$ also coincides with the $2q$-scalar curvature of $P^{\perp}$.

5.3. **Examples.** 1) Let (M,g) be with constant sectional curvature λ. Then

$$R = \frac{\lambda}{2} g^2 \qquad \text{and} \qquad R^q = \frac{\lambda^q}{2^q} g^{2q}.$$

Therefore

$$*\frac{1}{(n-2q-p)!} g^{n-2q-p} R^q = *\frac{\lambda^q}{2^q(n-2q-p)!} g^{n-p} = \frac{\lambda^q (n-p)!}{2^q(n-2q-p)!}\frac{g^p}{p!},$$

so that the (p,q)-curvature is also constant and equal to $\frac{\lambda^q(n-p)!}{2^q(n-2q-p)!}$.

The converse will be discussed in the next section.

2) Let (M,g) be a Riemannian product of two Riemannian manifolds (M_1,g_1) and (M_2,g_2). If we index by i the invariants of the metric g_i for $i=1,2$, then

$$R = R_1 + R_2 \quad \text{and} \quad R^q = (R_1+R_2)^q = \sum_{i=0}^{q} C_i^q R_1^i R_2^{q-i}.$$

Consequently, a straightforward calculation shows that

$$\begin{aligned} h_{2q} = \frac{c^{2q}R^q}{(2q)!} &= \sum_{i=0}^{q} C_i^q \frac{c^{2q}}{(2q)!}(R_1^i R_2^{q-i}) \\ &= \sum_{i=0}^{q} C_i^q \frac{c^{2i}R_1^i}{(2i)!}\frac{c^{2q-2i}R_2^{q-i}}{(2q-2i)!} \\ &= \sum_{i=0}^{q} C_i^q (h_{2i})_1 (h_{2q-2i})_2, \end{aligned}$$

where we used the convention $h_0 = 1$.

3) Let (M,g) be a hypersurface of the Euclidean space. If B denotes the second fundamental form at a given point, then the Gauss equation shows that

$$R = \frac{1}{2}B^2 \qquad \text{and} \quad R^q = \frac{1}{2^q}B^{2q}.$$

Consequently, if $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ denote the eigenvalues of B, then the eigenvalues of R^q are $\frac{(2q)!}{2^q}\lambda_{i_1}\lambda_{i_2}...\lambda_{i_{2q}}$, where $i_1 < ... < i_{2q}$. Therefore all the tensors g^pR^q are diagonalizable, and their eigenvalues have the following form:

$$g^pR^q(e_1...e_{p+2q}, e_1...e_{p+2q}) = \frac{p!(2q)!}{2^q} \sum_{1\leq i_1<...<i_{2q}\leq p+2q} \lambda_{i_1}...\lambda_{i_{2q}},$$

where $\{e_1, ...e_n\}$ is an orthonormal basis of eigenvectors of B. In particular, we have

$$h_{2q} = s_{(0,q)} = \frac{(2q)!}{2^q} \sum_{1\leq i_1<...<i_{2q}\leq n} \lambda_{i_1}...\lambda_{i_{2q}}.$$

So the invariants h_{2q} are, up to a constant, the symmetric functions in the eigenvalues of B. More generally, we have

$$s_{(p,q)}(e_{n-p+1},...,e_n)=\frac{(2q)!}{2^q}\sum_{1\leq i_1<...<i_{2q}\leq n-p}\lambda_{i_1}...\lambda_{i_{2q}}.$$

4) Let (M,g) be a conformally flat manifold. Then it is well known that at each point of M, the Riemann curvature tensor is determined by a symmetric bilinear form h, precisely we have $R=g.h$. Consequently, $R^q=g^qh^q$.

Let $\{e_1,...,e_n\}$ be an orthonormal basis of eigenvectors of h and let $\lambda_1\leq\lambda_2\leq ...\leq\lambda_n$ denote the eigenvalues of h.

Then it is not difficult to see that all the tensors g^pR^q are also diagonalizable. The eigenvalues are given by

$$g^pR^q(e_1...e_{p+2q},e_1...e_{p+2q})=(p+q)!q!\sum_{1\leq i_1<...<i_q\leq p+2q}\lambda_{i_1}...\lambda_{i_q}.$$

In particular, the (p,q)-curvatures are determined by

$$s_{(p,q)}(e_{n-p+1},...,e_n)=\frac{(n-q-p)!q!}{(n-2q-p)!}\sum_{1\leq i_1<...<i_q\leq n-p}\lambda_{i_1}...\lambda_{i_q}.$$

5.4. **Properties.** The following theorem generalizes a similar induction formula [4] for the p-curvature:

Theorem 5.2. *For $1\leq q\leq\frac{n}{2}$ and $1\leq p\leq n-2q$ we have*

$$\sum_{k=p}^{n}s_{(p,q)}(P,e_k)=(n-2q-p+1)s_{(p-1,q)}(P),$$

where P is an arbitrary tangent $(p-1)$-plane and $\{e_p,...,e_n\}$ is any orthonormal basis of $P^{\perp}$. In particular, we have

$$\sum_{i=1}^{n}T_{2q}(e_i,e_i)=(n-2q)h_{2q}.$$

Proof. Using (9), we have

$$\begin{aligned}\frac{1}{(n-2q-p)!}c*(g^{n-2q-p}R^q)&=\frac{1}{(n-2q-p)!}*g(g^{n-2q-p}R^q)\\&=(n-2q-p+1)*(\frac{g^{n-2q-p+1}}{(n-2q-p+1)!}R^q).\end{aligned}$$

To finish the proof just take the sectional curvatures of both sides. □

The following proposition is the only exception in this paper where one needs the use of the second Bianchi identity; see [5] for the proof.

Proposition 5.3 (Schur's theorem)**.** *Let $p\geq 1$ and $q\geq 1$. If at every point $m\in M$ the (p,q)-curvature is constant (that is on the fiber at m), then it is constant.*

The following can be seen as the converse of a Thorpe's result [7].

Proposition 5.4. *If R^s and R^{s+r} are both with constant sectional curvature λ and μ respectively, such that $\lambda\neq 0$ and $s+2r\leq n$, then R^r is also with constant sectional curvature and is equal to $\frac{\mu s!r!}{\lambda(s+r)!}$.*

Proof. Suppose that

$$R^s = \lambda \frac{g^s}{s!} \qquad \text{and} \quad R^{s+r} = \mu \frac{g^{(s+r)}}{(s+r)!}.$$

Then

$$\lambda \frac{g^s}{s!} R^r = \mu \frac{g^{(s+r)}}{(s+r)!}.$$

Since $s + 2r \leq n$, Proposition 2.3 shows that

$$\frac{\lambda}{s!} R^r = \mu \frac{g^r}{(s+r)!}.$$

This completes the proof. □

As in the case of R^s, it is not true in general that if h_{2s} is constant, then the higher scalar curvatures are constants. Nevertheless, we have the following result.

Proposition 5.5. *If for some s, the tensor R^s has constant sectional curvature λ, then for all $r \geq 0$, we have*

$$h_{2s+2r} = \frac{(n-2r)!}{(2s)!(n-2s-2r)!} \lambda h_{2r}.$$

In particular, if n is even, the Gauss-Bonnet integrand is determined by

$$h_n = \lambda h_{n-2s}.$$

Proof. Suppose $R^s = \lambda \frac{g^{2s}}{(2s)!}$. Then

$$\begin{aligned} h_{2s+2r} =& \frac{1}{(n-2s-2r)!} * (g^{n-2s-2r} R^{s+r}) \\ =& \frac{1}{(n-2s-2r)!} * (g^{n-2s-2r} \lambda \frac{g^{2s}}{(2s)!} R^r) \\ =& \frac{\lambda}{(2s)!(n-2s-2r)!} * (g^{n-2r} R^r) = \frac{(n-2r)!\lambda}{(2s)!(n-2s-2r)!} h_{2r}. \end{aligned}$$

□

Theorem 5.6. 1) *For every (p,q) such that $2q \leq p \leq n-2q$, the (p,q)-curvature $s_{(p,q)} \equiv \lambda$ is constant if and only if the sectional curvature of R^q is constant and equal to $\frac{\lambda(2q)!(n-p-2q)!}{(n-p)!}$.*

2) *For every (p,q) such that $p < 2q$, the (p,q)-curvature $s_{(p,q)} \equiv c$ is constant if and only if $c^{2q-p}(R^q)$ is proportional to the metric. That is, $c^{2q-p}(R^q) = const.g^p$.*

Proof. Recall that $s_{(p,q)} \equiv \lambda$ if and only if

$$\frac{g^{n-2q-p}}{(n-2q-p)!} R^q = \lambda \frac{g^{n-p}}{(n-p)!},$$

that is,

$$g^{n-2q-p} \Big(\frac{R^q}{(n-2q-p)!} - \lambda \frac{g^{2q}}{(n-p)!} \Big) = 0.$$

Now, let $2q \leq p \leq n-2q$. Then by Proposition 2.3, the last equation is equivalent to

$$R^q = \lambda \frac{(n-2q-p)!}{(n-p)!} g^{2q}.$$

Next, if $p < 2q$, then by Remark 2.1 in section 2, our condition is equivalent to

$$c^{2q-p}\Big(\frac{R^q}{(n-2q-p)!} - \lambda \frac{g^{2q}}{(n-p)!}\Big) = 0,$$

that is,

$$c^{2q-p}(R^q) = const.g^p,$$

which completes the proof of the theorem. □

The following lemma provides a characterization of the previous condition on R^q and generalizes a similar result in the case of Ricci curvature ($p = q = 1$).

Lemma 5.7. *For $p < 2q$, the tensor $c^{2q-p}(R^q)$ is proportional to the metric g^p if and only if*

$$\omega_i = 0 \text{ for } 1 \leq i \leq \min\{p, n-p\},$$

where $R^q = \sum_{i=0}^{2q} g^{2q-i}\omega_i$.

Proof. Formula (12) shows that

$$c^{2q-p}(R^q) = \sum_{i=2q-p}^{2q} i!\Big(\prod_{j=1}^{2q-p}(n-4q+i+j)\Big)\frac{g^{i-2q+p}}{(i-2q+p)!}\omega_{2q-i},$$

and therefore $c^{2q-p}(R^q) = \lambda g^p$ if and only if

$$\sum_{s=0}^{p}(2q-s)!\Big(\prod_{j=1}^{2q-p}(n-2q-s+j)\Big)\frac{g^{p-s}}{(p-s)!}\omega_s = \lambda g^p,$$

where we changed the index to $s = 2q-i$. Consequently,

$$g^{p-s}\omega_s = 0 \,\text{for}\, 1 \leq s \leq p, \text{and}\, \lambda = \frac{(2q)!}{p!}\Big(\prod_{j=1}^{2q-p}(n-2q+j)\Big)\omega_0.$$

By Proposition 2.3, this is equivalent to $\omega_s = 0$ for $1 \leq s \leq n-p$ and $s \leq p$. Note that $g^{p-s}\omega_s = 0$ if $s > n-p$. This completes the proof of lemma. □

Theorem 5.8. 1) *Let $2q \leq r \leq n-2q$, $n \neq 2r$, and $R^q = \sum_{i=0}^{2q} g^{2q-i}\omega_i$. Then:*

(a) *The function $P \to s_{(r,q)}(P) - s_{(n-r,q)}(P^\perp) \equiv \lambda$ is constant if and only if $\omega_i = 0$ for $1 \leq i \leq 2q-1$ and $\big(\frac{(n-r)!}{(n-2q-r)!} - \frac{r!}{(r-2q)!}\big)\omega_0 = \lambda$.*

(b) *The function $P \to s_{(r,q)}(P) + s_{(n-r,q)}(P^\perp) \equiv \lambda$ is constant if and only if $\omega_i = 0$ for $1 \leq i \leq 2q$ and $\big(\frac{(n-r)!}{(n-2q-r)!} + \frac{r!}{(r-2q)!}\big)\omega_0 = \lambda$. That is, R^q has constant sectional curvature.*

2) *Let $2q \leq r \leq n-2q$ and $n = 2r$. Then:*

(a) *The function $P \to s_{(r,q)}(P) - s_{(r,q)}(P^\perp) \equiv \lambda$ is constant if and only if $\omega_i = 0$ for i odd such that $1 \leq i \leq 2q-1$ and $\lambda = 0$*

(b) *The function $P \to s_{(r,q)}(P) + s_{(r,q)}(P^\perp) \equiv \lambda$ is constant if and only if $\omega_i = 0$ for i even and $2 \leq i \leq 2q$ and $2\frac{r!}{(r-2q)!}\omega_0 = \lambda$.*

Proof. Let $k,l\geq 0$ be such that $k+p=n-l-p$ and $\omega=\sum_{i=0}^{p} g^{p-i}\omega_i\in\mathcal{C}^p$. Then

$$\frac{g^k}{k!}\omega-*(\frac{g^l}{l!}\omega)=\frac{g^k}{k!}\omega-\sum_{i=0}^{\min\{p,n-p-l\}}\frac{(p-i+l)!(-1)^i}{l!(n-p-l-i)!}g^{n-p-l-i}\omega_i$$

$$=\frac{1}{(n-2p-l)!}\sum_{i=0}^{p}\big[1-(-1)^i\frac{(p-i+l)!(n-2p-l)!}{l!(n-p-l-i)!}\big]g^{n-p-l-i}\omega_i.$$

Therefore,

$$\frac{g^k}{k!}\omega-*(\frac{g^l}{l!}\omega)=\lambda\frac{g^{n-l-p}}{(n-l-p)!} \tag{22}$$

if and only if

$$\sum_{i=1}^{p}\big[1-(-1)^i\frac{(p-i+l)!(n-2p-l)!}{l!(n-p-l-i)!}\big]g^{n-p-l-i}\omega_i$$
$$+\big[\omega_0-\frac{(p+l)!(n-2p-l)!}{l!(n-p-l)!}\omega_0-\lambda\frac{(n-2p-l)!}{(n-l-p)!}\big]g^{n-l-p}=0.$$

For $1\leq i\leq p$, let

$$\alpha_i=1-(-1)^i\frac{(p-i+l)!(n-2p-l)!}{l!(n-p-l-i)!}=1-(-1)^i\frac{(s+l)!}{l!}\frac{k!}{(s+k)!},$$

where $s=p-i\leq p-1$. It is clear that $\alpha_{2j+1}>0$, and it is not difficult to check that $\alpha_i\neq 0$ for i even, $1\leq i\leq p-1$ and $k\neq l$. Also, note that $\alpha_p=1-(-1)^p=0$ since p is even.

Therefore, in the case where $k\neq l$, condition (22) is equivalent to

$$\omega_i=0\quad\text{for } 1\leq i\leq p-1\quad\text{and}\quad\lambda=\{\frac{(n-p-l)!}{(n-2p-l)!}-\frac{(p+l)!}{l!}\}\omega_0.$$

In the case $k=l$, we have $\alpha_i=1-(-1)^i$. Condition (22) is therefore equivalent to

$$\omega_i=0\quad\text{for } i \text{ odd},\quad 1\leq i\leq p\quad\text{and}\quad\lambda=0.$$

In a similar way, we have

$$\frac{g^k}{k!}\omega+*(\frac{g^l}{l!}\omega)=\lambda\frac{g^{n-l-p}}{(n-l-p)!} \tag{23}$$

if and only if

$$\sum_{i=1}^{p}\big[1+(-1)^i\frac{(p-i+l)!(n-2p-l)!}{l!(n-p-l-i)!}\big]g^{n-p-l-i}\omega_i$$
$$+\big[\omega_0+\frac{(p+l)!(n-2p-l)!}{l!(n-p-l)!}\omega_0-\lambda\frac{(n-2p-l)!}{(n-l-p)!}\big]g^{n-l-p}=0.$$

In the case $k\neq l$, this is equivalent to

$$\omega_i=0 \text{ for } 1\leq i\leq p \text{ and } \lambda=\{\frac{(n-p-l)!}{(n-2p-l)!}+\frac{(p+l)!}{l!}\}\omega_0,$$

and in the case $k=l$, we have $\alpha_i=1+(-1)^i$. Hence condition (23) is then equivalent to

$$\omega_i=0\quad\text{for } i \text{ even},\quad 1\leq i\leq p\quad\text{and}\quad\lambda=2\frac{(p+l)!}{l!}\omega_0.$$

To complete the proof of the theorem, just note that, for $2q \le r \le n-2q$, the condition $s_{(r,q)} \pm s_{(n-r,q)} \equiv \lambda$ is equivalent to

$$*(\frac{g^{n-2q-r}}{(n-2q-r)!}R^q) \pm \frac{g^{r-2q}}{(r-2q)!}R^q = \lambda\frac{g^r}{r!}.$$

Next, apply the previous result after taking $l = n-2q-r$ and $k = r-2q$ and $p = 2q$. □

Remark. If $r < 2q$, then $s_{(n-r,q)} \equiv 0$. So our condition implies that $s_{(r,q)} \equiv \lambda$ is constant, and such a case was discussed above.

6. Generalized Avez-type formula

The following theorem generalizes a result due to Avez [1] in the case when $n = 4$ and $\omega = \theta = R$.

Theorem 6.1. *Let $n = 2p$ and $\omega, \theta \in \mathcal{C}_1^p$. Then*

$$*(\omega\theta) = \sum_{r=0}^{p} \frac{(-1)^{r+p}}{(r!)^2}\langle c^r\omega, c^r\theta\rangle.$$

In particular if $n = 4q$, then the Gauss-Bonnet integrand is determined by

$$h_{4q} = \sum_{r=0}^{2q} \frac{(-1)^r}{(r!)^2}|c^r R^q|^2.$$

Proof. Let $\theta \in \mathcal{C}_1^p$ and $\omega \in \mathcal{C}_1^{n-p}$. Then using formula (8) and Corollary 4.2 we get

$$*(\omega\theta) = \langle \omega, *\theta\rangle = \sum_{r=\max\{0,2p-n\}}^{p} \frac{(-1)^{r+p}}{(r!)(n-2p+r)!}\langle \omega, g^{n-2p+r}c^r\theta\rangle.$$

To complete the proof, we just take $n = 2p$ and use Theorem 3.1. □

The following corollary is an alternative way to write the previous formula.

Corollary 6.2. *Let $n = 2p$ and $\omega, \theta \in \mathcal{C}_1^p$. Then*

$$*(\omega\theta) = \sum_{r=0}^{p} \frac{(-1)^{r+p}}{(r!)^2}\langle g^r\omega, g^r\theta\rangle.$$

Proof. The proof is a direct consequence of the previous formula and Corollary 2.2. □

The following result is of the same type as the previous one

Theorem 6.3. *With respect to the decomposition* 10*, let $\omega = \sum_{i=0}^{n-p} g^{n-p-i}\omega_i \in \mathcal{C}_1^{n-p}$ and $\theta = \sum_{i=0}^{p} g^{p-i}\theta_i \in \mathcal{C}_1^p$. Then*

$$*(\omega\theta) = \sum_{r=0}^{\min\{p,n-p\}} (-1)^r(n-2r)!\langle \omega_i, \theta_i\rangle.$$

Proof. By formula (18) we have

$$*(\omega\theta) = \langle \omega, *\theta\rangle = \sum_{i=0}^{\min\{p,n-p\}} \frac{(-1)^i (p-i)!}{(n-p-i)!}\langle \omega, g^{n-p-i}\theta_i\rangle,$$

and therefore using Lemma 6.4 below, we get

$$*(\omega\theta) = \sum_{i=0}^{\min\{p,n-p\}} \frac{(-1)^i (p-i)!}{(n-p-i)!}\langle g^{n-p-i}\omega_i, g^{n-p-i}\theta_i\rangle.$$

After separately considering the cases $p < n-p, p = n-p$ and $p > n-p$ and the lemma below, one can complete the proof easily. □

Lemma 6.4. *Let $\omega_1 \in E_1^r, \omega_2 \in E_1^s$ be effective. Then*

$$\langle g^p\omega_1, g^q\omega_2\rangle = 0 \quad \textit{if} \quad (p \neq q) \quad \textit{or} \quad (p = q \quad \textit{and} \quad r \neq s).$$

Furthermore, in the case $p = q \geq 1$ and $r = s$, we have

$$\langle g^p\omega_1, g^p\omega_2\rangle = p!(\prod_{i=0}^{p-1}(n-2r-i))\langle \omega_1, \omega_2\rangle.$$

Proof. Recall that (see formula (11)) $c^p(g^q\omega_2) = 0$ if $p > q$, and $c^q(g^p\omega_1) = 0$ if $p < q$. This proves the first part of the lemma. Also by the same formula and formula (7) we have

$$\langle g^p\omega_1, g^p\omega_2\rangle = \langle \omega_1, c^p(g^p\omega_2)\rangle = \langle \omega_1, p!(\prod_{i=0}^{p-1}(n-2r-i))\omega_2\rangle.$$

□

Corollary 6.5. *Let $q = s + t$. Then*

$$h_{2q} = \frac{1}{(n-2q)!} \sum_{i=0}^{\min\{2s,n-2s\}} (-1)^i (n-2i)! \langle (R^s)_i, (R^t)_i\rangle.$$

In particular, we have

$$h_{4q} = \frac{1}{(n-4q)!} \sum_{i=0}^{\min\{2q,n-2q\}} (-1)^i (n-2i)! \langle (R^q)_i, (R^q)_i\rangle.$$

Proof. Note that

$$h_{2q} = \frac{1}{(n-2q)!} * (g^{n-2q}R^q) = \frac{1}{(n-2q)!} * (g^k R^s g^l R^t),$$

where $k + l = n - 2q$ and $s + t = q$. Then we apply the previous theorem to get

$$h_{2q} = \frac{1}{(n-2q)!} \sum_{i=0}^{\min\{k+2s,l+2t\}} (-1)^i (n-2i)! \langle (g^k R^s)_i, (g^l R^t)_i\rangle.$$

Recall that $(g^k R^s)_i = R_i^s$ if $i \leq 2s$; otherwise it is zero. The same is true for $(g^l R^t)_i$. This completes the proof. □

The case $q = 1$ is of special interest. It provides an obstruction to the existence of an Einstein metric or a conformally flat metric with zero scalar curvature in arbitrary higher dimensions, as follows.

Theorem 6.6. 1) *If (M,g) is an Einstein manifold with dimension $n \geq 4$, then $h_4 \geq 0$ and $h_4 \equiv 0$ if and only if (M,g) is flat.*

2) *If (M,g) is a conformally flat manifold with zero scalar curvature and dimension $n \geq 4$, then $h_4 \leq 0$ and $h_4 \equiv 0$ if and only if (M,g) is flat.*

Proof. Straightforward using the previous corollary and Theorem 5.1. □

The previous theorem can be generalized as below. Its proof is also a direct consequence of the previous corollary and Theorem 5.8.

Theorem 6.7. *Let (M,g) be a Riemannian manifold with dimension $n = 2r \geq 4q$, for some $q \geq 1$.*

1) *If $s_{(r,q)}(P) = s_{(r,q)}(P^{\perp})$ for all r-planes P, then $h_{4q} \geq 0$ and $h_{4q} \equiv 0$ if and only if (M,g) is flat.*
2) *If $s_{(r,q)}(P) = -s_{(r,q)}(P^{\perp})$ for all r-planes P, then $h_{4q} \leq 0$ and $h_{4q} \equiv 0$ if and only if (M,g) is flat.*

The previous two theorems generalize similar results of Thorpe [7] in the case $n = 4q$.

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