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DOUBLE LOGARITHMIC INEQUALITY WITH A SHARP CONSTANT

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ABSTRACT. We prove a Log Log inequality with a sharp constant. We also show that the constant in the Log estimate is "almost" sharp. These estimates are applied to prove a Moser-Trudinger type inequality for solutions of a 2D wave equation.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

By the Sobolev embedding theorem, it is well known that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for $2 \leq p < \infty$ but not in $L^\infty(\mathbb{R}^2)$. Moreover, H^1 functions are in a so-called Orlicz space, i.e. their exponential powers are integrable functions. Precisely, we have the following Moser-Trudinger inequality (see [1, 11, 14, 16]).

Proposition 1.1. *Let $\alpha \in (0, 4\pi)$. A constant c_α exists such that*

$$\int_{\mathbb{R}^2} \left(\exp\left(\alpha u(x)^2\right) - 1\right) dx \leq c_\alpha \|u\|_{L^2}^2 \tag{1}$$

for all u in $H^1(\mathbb{R}^2)$ such that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Moreover, if $\alpha \geq 4\pi$, then (1) is false.

Remark 1.2. We point out that $\alpha = 4\pi$ becomes admissible in (1) if we require $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Precisely, we have

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \left(\exp\left(4\pi u(x)^2\right) - 1\right) dx < \infty, \tag{2}$$

and this is false for $\alpha > 4\pi$.

In this paper, we show that we can control the L^∞ norm by the H^1 norm and a stronger norm with a logarithmic growth or double logarithmic growth. The inequality is sharp for the double logarithmic growth.

Recall that H^1 is the usual Sobolev space endowed with the norm $\|u\|_{H^1}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2$. For any real number $\alpha \in]0, 1[$, we denote by $\dot{\mathcal{C}}^\alpha$ the sub-space of α- Hölder continuous functions endowed with the semi-norm

$$\|u\|_{\dot{\mathcal{C}}^\alpha} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

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Also, we denote $\|u\|_{\mathcal{C}^\alpha} := \|u\|_{\dot{\mathcal{C}}^\alpha} + \|u\|_{L^\infty}$ and define $N_\alpha(u)$ to be the ratio $N_\alpha(u) := \frac{\|u\|_{\dot{\mathcal{C}}^\alpha}}{\|\nabla u\|_{L^2}}$. For any bounded domain Ω in $\mathbb{R}^2$, define $H_0^1(\Omega)$ to be the completion in the Sobolev space $H^1(\Omega)$ of smooth and compactly supported functions.

The main result of this paper is the following theorem.

Theorem 1.3 (Double logarithmic inequality)**.** *Let $\alpha \in]0,1[$ and let B_1 be the unit ball in $\mathbb{R}^2$. Any function in $H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$ is bounded. Moreover, a positive constant C_0 exists such that for any function $u \in H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$, we have*

$$\|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha}\|\nabla u\|_{L^2}^2 \log\Big[e^3 + C_0 N_\alpha(u)\sqrt{\log(2e + N_\alpha(u))}\Big], \tag{3}$$

and the constant $\frac{1}{2\pi\alpha}$ in (3) *is sharp.*

Note that $\log(e) = 1$. Our second result concerns the following logarithmic inequality.

Theorem 1.4 (Logarithmic inequality)**.** *Let α be in $]0,1[$. For any real number $\lambda > \frac{1}{2\pi\alpha}$, a constant C_λ exists such that, for any function $u \in H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$, we have*

$$\|u\|_{L^\infty}^2 \leq \lambda \|\nabla u\|_{L^2}^2 \log\Big(C_\lambda + N_\alpha(u)\Big). \tag{4}$$

Moreover, the above inequality does not hold for $\lambda = \frac{1}{2\pi\alpha}$.

2. A Littlewood-Paley proof

To prove the fundamental theorems, we start by showing that inequality (4) can easily be obtained with an unknown absolute constant instead of $\frac{1}{2\pi\alpha}$. To do so, we give a brief review of the Littlewood-Paley theory, and we refer the reader to [5] for a thorough treatment. Denote by $\mathcal{C}_0$ the annular ring defined by

$$\mathcal{C}_0 = \{\xi \in \mathbb{R}^2 \text{ such that } \frac{3}{4} < |\, \xi \,| < \frac{8}{3}\},$$

and choose two nonnegative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0,4/3))$ and $\mathcal{D}(\mathcal{C}_0)$ such that

$$\forall\, \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1,$$

$$\forall\, \xi \in \mathbb{R}^2 \backslash \{0\}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1.$$

Denote $h = \mathcal{F}^{-1}\varphi$ and define the frequency projectors Δ_j and $\dot{\Delta}_j$ by

$$\begin{aligned}
\text{for } j \in \mathbb{Z}, \quad \dot{\Delta}_j u &= \varphi(2^{-j}D)u = 2^{2j}\int_{\mathbb{R}^2} h(2^j y)u(x-y)dy,\\
\text{if } j \geq 0, \quad \Delta_j u &= \dot{\Delta}_j u,\\
\Delta_{-1} u &= \chi(D)u = \mathcal{F}^{-1}\left(\chi(\xi)\hat{u}(\xi)\right),\\
\text{if } j \leq -2, \quad \Delta_j u &= 0\,.
\end{aligned}$$

Recall that

$$\|\nabla u\|_{L^2} \sim \Big(\sum_{j\in\mathbb{Z}} 2^{2j}\|\dot{\Delta}_j u\|_{L^2}^2\Big)^{1/2}$$

and

$$\|u\|_{\dot{\mathcal{C}}^\alpha} \sim \sup_{j\in\mathbb{Z}} \left(2^{j\alpha}\|\dot{\Delta}_j u\|_{L^\infty}\right).$$

We mention that C will be used to denote a constant which may vary from line to line.

We have the following result in the whole space.

Proposition 2.1. *Let α be in $]0,1[$. For any function $u \in \mathcal{C}^\alpha(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)$, one has*

$$\|u\|^2_{L^\infty(\mathbb{R}^2)} \le C\|u\|^2_{L^2(\mathbb{R}^2)} + C\|\nabla u\|^2_{L^2(\mathbb{R}^2)} \log\Big(e + N_\alpha(u)\Big). \tag{5}$$

Proof. Write

$$u = \Delta_{-1}u + \sum_{j=0}^{\infty} \Delta_j u = \Delta_{-1}u + \sum_{j=0}^{l-1} \Delta_j u + \sum_{j=l}^{\infty} \Delta_j u,$$

where l is a nonnegative integer which will be chosen later.

Using Bernstein's inequality, we get

$$\begin{aligned}\|u\|_{L^\infty} &\le C\|\Delta_{-1}u\|_{L^2} + C\sum_{j=0}^{l-1} 2^j\|\Delta_j u\|_{L^2} + \sum_{j=l}^{\infty} 2^{-j\alpha}\left(2^{j\alpha}\|\Delta_j u\|_{L^\infty}\right)\\ &\le C\|u\|_{L^2} + C\sqrt{l}\,\Big(\sum_{j=0}^{l-1} 2^{2j}\|\Delta_j u\|^2_{L^2}\Big)^{1/2} + C\Big(\sum_{j=l}^{\infty} 2^{-j\alpha}\Big)\|u\|_{\dot{\mathcal{C}}^\alpha}\\ &\le C\left(\|u\|_{L^2} + \sqrt{l}\,\|\nabla u\|_{L^2} + \frac{2^{-\alpha l}}{1-2^{-\alpha}}\,\|u\|_{\dot{\mathcal{C}}^\alpha}\right),\end{aligned}$$

so

$$\|u\|^2_{L^\infty} \le C\left(\|u\|^2_{L^2} + l\,\|\nabla u\|^2_{L^2} + \frac{2^{-2\alpha l}}{(1-2^{-\alpha})^2}\,\|u\|^2_{\dot{\mathcal{C}}^\alpha}\right).$$

Denoting by $]x[$ the integer part of the real number x and choosing

$$l := \mathrm{Max}\Big(1, 1 + \Big]2\log_2\left(N_\alpha(u)^2\right)\Big[\Big),$$

the proof of Proposition 2.1 is achieved. □

Clearly, if u is supported in the unit ball B_1, then using the Poincaré inequality and Proposition 2.1, we get

$$\|u\|^2_{L^\infty} \le C\|\nabla u\|^2_{L^2}\log\Big(C_0 + N_\alpha(u)\Big), \tag{6}$$

for some constant C_0 big enough.

3. Proof of Theorem 1.3

To prove (3) and the fact that the constant is sharp, it is sufficient to show that

$$2\pi\alpha = \inf_{u\in H^1_0(B_1)\cap\dot{\mathcal{C}}^\alpha(B_1)} \frac{\|\nabla u\|^2_{L^2}\log\left[e^3 + C_0 N_\alpha(u)\sqrt{\log(2e+N_\alpha(u))}\right]}{\|u\|^2_{L^\infty}}, \tag{7}$$

for any C_0 big enough. Let us start by proving the sharpness of the constant. Define $u_k(x) = f_k(-2\log|x|)$, where for all nonnegative integer k

$$f_k(t) \quad = \quad \begin{cases} 0 & \text{if} \quad t \le 0, \\ \sqrt{\frac{k}{4\pi}}\ \frac{t}{k} & \text{if} \quad 0 \le t \le k, \\ \sqrt{\frac{k}{4\pi}} & \text{if} \quad t \ge k. \end{cases}$$

These functions were introduced in [11] to show the optimality of the exponent 4π in Trudinger-Moser inequality (see also [1] and [10]). An easy computation shows that $\|\nabla u_k\|_{L^2}^2 = 1$. By interpolation and since f_k is nonnegative, we have

$$\|u_k\|_{\dot{\mathcal{C}}^\alpha} \le \ \|u_k\|_{L^\infty}^{1-\alpha}\|u_k\|_{Lip}^{\alpha},$$

where $\|u_k\|_{Lip} = \sup_{x\neq y}\dfrac{|u_k(x)-u_k(y)|}{|x-y|}$. Hence

$$\|u_k\|_{\dot{\mathcal{C}}^\alpha} \le C\ k^{\frac{1}{2}-\alpha}\exp\left(\frac{\alpha k}{2}\right).$$

Denoting by

$$R(u) := \frac{\|\nabla u\|_{L^2}^2 \log\left[e^3 + C_0 N_\alpha(u)\sqrt{\log(2e+N_\alpha(u))}\ \right]}{\|u\|_{L^\infty}^2},$$

it is clear that

$$R(u_k) \ge \inf_{u\in H_0^1(B_1)\cap\dot{\mathcal{C}}^\alpha(B_1)} \frac{\|\nabla u\|_{L^2}^2 \log\left[e^3 + C_0 N_\alpha(u)\sqrt{\log(2e+N_\alpha(u))}\ \right]}{\|u\|_{L^\infty}^2}.$$

Taking the limit as $k\to\infty$, we deduce that

$$2\pi\alpha \ge \inf_{u\in H_0^1(B_1)\cap\dot{\mathcal{C}}^\alpha(B_1)} \frac{\|\nabla u\|_{L^2}^2 \log\left[e^3 + C_0 N_\alpha(u)\sqrt{\log(2e+N_\alpha(u))}\ \right]}{\|u\|_{L^\infty}^2}.$$

To prove (3), we start by noting that for any function u, the norms $\|\nabla u\|_{L^2}$ and $\|u\|_{\dot{\mathcal{C}}^\alpha}$ are nonincreasing under symmetric nonincreasing rearrangements, while $\|u\|_{L^\infty}$ remains unchanged.

Using the fact that for all $C > 0$

$$t \to f(t) := t^2 \log\left[e^3 + \frac{C}{t}\sqrt{\left[\log(2e+\frac{1}{t})\right]}\ \right]$$

is increasing, it is sufficient to check the minimizer figured in (7) in the class of nonnegative, nonincreasing and radially symmetric functions.

Without loss of generality, we can normalize $\|u\|_{L^\infty}$ to be equal to 1. Since u vanishes on the boundary, we deduce that $\|u\|_{\dot{\mathcal{C}}^\alpha}$ is larger than or equal to 1. Moreover, if $\|u\|_{\dot{\mathcal{C}}^\alpha} = 1$, then necessarily, $u(x) = 1 - |x|^\alpha$ and the inequality is trivial. In the sequel, we will assume that $\|u\|_{\dot{\mathcal{C}}^\alpha} > 1$.

Let $H^1_{0,rad}(B_1)$ be the space of all nonincreasing and radially symmetric functions in $H^1_0(B_1)$. For any parameter $D > 1$, we denote by K_D the closed convex subset of $H^1_{0,rad}(B_1)$ defined by

$$K_D = \{u \in H^1_{0,rad}(B_1) : u(r) \ge 1 - Dr^\alpha, \quad r\in\,]0,1]\}. \tag{8}$$

Note that the set of radially symmetric functions which satisfy $\|u\|_{\dot{\mathcal{C}}^\alpha} \le D$ is included in K_D. Hence, to get the result, it is sufficient to prove that

$$2\pi\alpha \le \inf_{D\ge 1} \inf_{\{u\in K_D,\ \|u\|_{L^\infty}=1,\ \|u\|_{\dot{\mathcal{C}}^\alpha}=D\}} \|\nabla u\|_{L^2}^2 \log\left[e^3 + \frac{C_0 D}{\|\nabla u\|_{L^2}}\sqrt{\log(2e+\frac{D}{\|\nabla u\|_{L^2}})}\ \right]$$

or just that

$$2\pi\alpha \quad \le \quad \inf_{D\ge 1} \inf_{\{u\in K_D\}} \|\nabla u\|_{L^2}^2 \log\left[e^3 + \frac{C_0 D}{\|\nabla u\|_{L^2}}\sqrt{\log(2e+\frac{D}{\|\nabla u\|_{L^2}})}\ \right].$$

Consider the problem of minimizing

$$I[u] := \|\nabla u\|_{L^2(B_1)}^2 \tag{9}$$

among all the functions belonging to the set K_D. This is a variational problem with obstacle. It is well known (see, for example, Kinderlehrer-Stampacchia [9] and L. C. Evans [6]) that it has a unique minimizer u^* which is variationally characterized by

$$\int_{B_1} \nabla u^* \cdot \nabla v \; dx \ge \|\nabla u^*\|_{L^2(B_1)}^2, \tag{10}$$

for any $v \in K_D$. Moreover u^* is in the Sobolev space $W^{2,\infty}(B_1)$. Hence the radially symmetric set

$$\mathcal{O} := \{x \in B_1 : u^*(x) > 1 - D|x|^\alpha\}$$

is open and u^* is harmonic in $\mathcal{O}$. On the other hand, note that any radially symmetric harmonic functions in $\mathbb{R}^2$ can only coincide in a unique tangent point with the function $r \to 1 - Dr^\alpha$. Note also that because of the boundary condition at $r = 1$, u^* cannot start to be harmonic near $r = 0$. Therefore there exists a unique $a \in\,]0, 1[$ such that

$$\begin{aligned} u^*(r) &= 1 - Dr^\alpha \text{ if } r \in [0, a], \\ u^*(r) &= (1 - Da^\alpha)\frac{\log r}{\log a} \text{ if } r \in [a, 1], \end{aligned} \tag{11}$$

also satisfy the tangent condition

$$a^\alpha = \frac{1 - Da^\alpha}{D|\log(a^\alpha)|}. \tag{12}$$

Note that if $D \to 1$, then $a \to 1$, and therefore (12) still makes sense in the limit case.

In particular, note that $\|u^*\|_{L^\infty} = 1$, $\|u^*\|_{\dot{\mathcal{C}}^\alpha} = D$, and

$$\|\nabla u^*\|_{L^2}^2 = \pi\alpha D^2 a^{2\alpha} - 2\pi(\frac{1 - Da^\alpha}{\log(a)})^2 \log(a). \tag{13}$$

Substituting D from (12) into (13), we get

$$\|\nabla u^*\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(a^\alpha)}{(1 - \log(a^\alpha))^2}.$$

Denoting by $x := a^\alpha \in]0, 1[$, we have

$$\|\nabla u^*\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(x)}{(1 - \log(x))^2} \tag{14}$$

and

$$\|u^*\|_{\dot{\mathcal{C}}^\alpha} = \frac{1}{x(1-\log(x))}. \tag{15}$$

Setting

$$g(x) := \frac{1}{x\sqrt{2\pi\alpha(1/2-\log(x))}}$$

and

$$F_C(x) := \frac{\frac{1}{2}-\log(x)}{(1-\log(x))^2}\log\left[e^3 + Cg(x)\sqrt{\log(2e+g(x))}\right],$$

it is sufficient to show that a constant C_0 exists such that for all $0 < x \leq 1$, the function F_{C_0} satisfies

$$F_{C_0}(x) \geq 1. \tag{16}$$

First, observe that for every $0 < x \leq 1$

$$\frac{\frac{1}{2}-\log(x)}{(1-\log(x))^2} \geq \frac{1}{(2-\log(x))}.$$

Hence for any $C > 0$, (16) holds if $2 - \log x \leq 3$, namely if $x \geq 1/e$.

In the sequel, we suppose that $x \leq 1/e$, hence

$$\begin{aligned} F(x) &\geq \frac{1}{(2-\log(x))}\Big[-\log(x) + \log(\frac{C_0}{\sqrt{2\pi\alpha}}) - \frac{1}{2}\log(1/2-\log(x)) \\ &\qquad\qquad + \frac{1}{2}\log(\log(2e+g(x)))\Big] \\ &\geq 1 + \frac{1}{(2-\log(x))}\Big[\log(\frac{C_0}{e^2\sqrt{2\pi\alpha}}) + \frac{1}{2}\log\Big(\frac{\log(2e+g(x))}{(1/2-\log(x))}\Big)\Big]. \end{aligned} \tag{17}$$

The function $h(x) = \frac{\log(2e+g(x))}{(1/2-\log(x))}$ is bounded away from zero on $(0, 1/e)$. Hence, we can find C_0 big enough such that the second term on the right-hand side of (17) is non-negative. This achieves the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

The proof of Theorem 1.4 is similar to that of Theorem 1.3. Indeed, consider u^* the minimizer of the Dirichlet norm (9) among all functions in K_D defined in (8). Note that according to (14) and (15), we have

$$\|\nabla u^*\|_{L^2}^2 \log\Big(C_\lambda + N_\alpha(u^*)\Big) := H(x),$$

where

$$H(x) = 2\pi\alpha\frac{1/2-\log(x)}{(1-\log(x))^2}\log\left(C_\lambda + \frac{1}{x\sqrt{2\pi\alpha(1/2-\log(x))}}\right).$$

Taking $C_\lambda = e$ in $H(x)$, we see that $H(x)$ goes to $2\pi\alpha$ as x goes to 0. Hence, for any $\lambda > \frac{1}{2\pi\alpha}$, there exists $x_\lambda > 0$ such that $\lambda H(x) \geq 1$, for any $0 < x < x_\lambda$ and $C_\lambda \geq e$. Now, if $x \in [x_\lambda, 1]$, choosing the constant $C_\lambda > e$ big enough such that

$$\frac{1/2}{(1-\log(x_\lambda))^2}\log(C_\lambda) \geq 1,$$

we see that $\lambda H(x) \geq 1$. Hence, by this choice of C_λ, we see that $\lambda H(x) \geq 1$ for all $0 < x \leq 1$. This achieves the proof of (4).

Now, let us prove that (4) does not hold for $\lambda = \frac{1}{2\pi\alpha}$. More precisely, we will prove that a sequence of functions $(u_n)_n$ exists such that $u_n \in H_0^1(B_1) \cap \dot{\mathcal{C}}^\alpha(B_1)$ and for n big enough the following holds:

$$\|u_n\|_{L^\infty}^2 > \frac{1}{2\pi\alpha}\|\nabla u_n\|_{L^2}^2 \log\Big(n^{1/4} + n^{1/4}N_\alpha(u_n)\Big). \tag{18}$$

Let u_n be the radially symmetric function defined by

$$u_n(r) = 1 - e^n r^\alpha \text{ if } r \in [0, a_n], \text{ and } u_n(r) = (1 - e^n a_n^\alpha)\frac{\log r}{\log a_n} \text{ if } r \in [a_n, 1],$$

where a_n is chosen such that $a_n^\alpha := x_n$ is the unique solution in $(0,1)$ of the equation $x = \frac{1-e^n x}{e^n|\log(x)|}$. Note indeed that the function $h(x) = e^n(x + x|\log(x)|)$ is increasing on $(0,1)$. Hence, we see easily that

$$\frac{e^{-n}}{n\log(n)} \leq x_n \leq \frac{e^{-n}}{n}. \tag{19}$$

Obviously, this construction is inspired from the minimizer of the variational problem with obstacle described in Section 3 where we have chosen $D_n = e^n$. Hence, according to (14) and (15), we have

$$\|\nabla u_n\|_{L^2}^2 = 2\pi\alpha\frac{1/2 - \log(x_n)}{(1-\log(x_n))^2}$$

and

$$\|u_n\|_{\dot{\mathcal{C}}^\alpha} = \frac{1}{x_n(1-\log(x_n))}.$$

Now to prove (18), it is sufficient to prove that for n big enough we have

$$h_n := \frac{\frac{1}{2} - \log(x_n)}{(1-\log(x_n))^2}\log\left[n^{1/4} + \frac{n^{1/4}}{x_n\sqrt{2\pi\alpha(1/2 - \log(x_n))}}\right] < 1.$$

Note that using (19), we have

$$h_n < \frac{\frac{1}{2} + n + \log(n) + \log\log n}{(1 + \log(n) + n)^2}\log\left[n^{1/4} + \frac{n^{1/4}e^n n\log n}{\sqrt{2\pi\alpha n}}\right].$$

Hence $h_n < 1 - \frac{1}{4}\frac{\log n}{n} + o(\frac{\log n}{n})$, which is strictly less than 1 if n is sufficiently large. The proof of (18) is achieved. □

5. Case of the whole space

Theorems 1.3 and 1.4 were stated in the ball of radius one. If the function u is supported in a bigger ball $B_R = B(0,R)$, then a simple scaling argument shows that

$$\|u\|_{L^\infty(B_R)}^2 \leq \frac{1}{2\pi\alpha}\|\nabla u\|_{L^2(B_R)}^2 \log\left[e^3 + C_0 R^\alpha N_\alpha(u)\sqrt{\log\left(2e + R^\alpha N_\alpha(u)\right)}\right].$$

Remark 5.1. Using symmetric nonincreasing rearrangement of functions, the results of Theorem 1.3 and Theorem 1.4 remain true for any bounded and regular domain Ω of $\mathbb{R}^2$. Precisely, if $f \in H_0^1(\Omega) \cap \dot{\mathcal{C}}^\alpha(\Omega)$, then its corresponding symmetric nonincreasing function, usually denoted by $f^\star$, is in $H_0^1(B_R) \cap \dot{\mathcal{C}}^\alpha(B_R)$, where $R = \sqrt{\frac{|\Omega|}{2\pi}}$. We refer to [15], [2] for the definition, the properties and applications of rearrangements of functions. Applying the results of Theorem 1.3 and Theorem 1.4 to $f^\star$ and using the fact that

$$\|f^\star\|_{L^\infty} = \|f\|_{L^\infty},$$
$$\|\nabla f^\star\|_{L^2} \le \|\nabla f\|_{L^2}, \quad \|f^\star\|_{\dot{\mathcal{C}}^\alpha} \le \|f\|_{\dot{\mathcal{C}}^\alpha},$$

we get the result for a general domain Ω.

Note that this estimate cannot be extended to the whole space since R^α diverges. Instead, we have the following result concerning the whole space.

Corollary 5.2. *Let $\alpha \in \,]0,1[$. For any $\lambda > \frac{1}{2\pi\alpha}$ and any $0 < \mu \le 1$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$*

$$\|u\|_{L^\infty}^2 \le \lambda \|u\|_\mu^2 \log\left(C_\lambda + \frac{8^\alpha \mu^{-\alpha}\|u\|_{\mathcal{C}^\alpha}}{\|u\|_\mu}\right), \tag{20}$$

where $\|u\|_\mu^2 = \|\nabla u\|_{L^2}^2 + \mu^2\|u\|_{L^2}^2$.

Proof. Let u be a function in $H^1(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$, $\lambda > \frac{1}{2\pi\alpha}$ and $0 < \mu \le 1$. Fix a radially symmetric function φ in $\mathcal{C}_0^\infty(B_4)$ satisfying $0 \le \varphi \le 1$, $\varphi \equiv 1$ for r near 0, $|\partial_r \varphi| \le 1$ and $|\Delta\varphi| \le 1$. Define φ_μ by $\varphi_\mu(x) = \varphi(\frac{\mu}{2}|x|)$.

Without loss of generality, we can assume that $\|u\|_{L^\infty} = |u(0)|$. Note that in particular one has

$$\|\varphi_\mu u\|_{\dot{\mathcal{C}}^\alpha} \le \|u\|_{\mathcal{C}^\alpha},$$
$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 + \frac{\mu^2}{4}\|u\|_{L^2}^2 + 2\int_{\mathbb{R}^2} \varphi_\mu u \nabla\varphi_\mu \nabla u dx.$$

Integrating by parts,

$$2\int_{\mathbb{R}^2} \varphi_\mu u \nabla\varphi_\mu \nabla u dx = -\frac{1}{2}\int_{R^2} \Delta\varphi_\mu^2 u^2 dx = -\frac{\mu^2}{8}\int_{R^2} \Delta\varphi^2(\frac{\mu}{2}x)\ u^2 dx.$$

Hence,

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 + \mu^2\|u\|_{L^2}^2.$$

Applying Theorem 1.4 in the ball $B_{8/\mu}$ and using the fact that for any constant $C > 0$ the function $x \to x^2 \log(C_\lambda + \frac{C}{x})$ is increasing, the proof of Corollary 5.2 is achieved. □

We also have the following result.

Corollary 5.3. *Let $\alpha \in \,]0,1[$. For any $\lambda > \frac{1}{2\pi\alpha}$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$,*

$$\|u\|_{L^\infty} \le \|u\|_{L^2} + \|\nabla u\|_{L^2}\sqrt{\lambda \log\left(e + C_\lambda \frac{\|u\|_{\mathcal{C}^\alpha}}{\|\nabla u\|_{L^2}}\right)}. \tag{21}$$

For the proof of Corollary 5.3, we take the Littlewood-Paley decomposition of u, $u = \Delta_{-1}u + v$, where $v = \sum_{j=0}^\infty \Delta_j u$. Hence $\|v\|_{L^2} \le C\|\nabla v\|_{L^2}$ and $\|v\|_{\mathcal{C}^\alpha} \le \|u\|_{\mathcal{C}^\alpha}$. So

$$\|u\|_{L^\infty} \le \|\Delta_{-1}u\|_{L^\infty} + \|v\|_{L^\infty}.$$

Then, we apply Corollary 5.2 to v with λ' and μ' such that $\lambda'(1 + C^2\mu'^2) < \lambda$. □

Of course, we have similar inequalities for the Log Log inequality (3) in $\mathbb{R}^2$ with the sharp constant $\frac{1}{2\pi\alpha}$.

6. Application to the wave equation

Corollary 5.2 is useful in the study of the Cauchy problem associated with the following type of 2D-nonlinear wave equation

$$\partial_t^2 u - \Delta u + u + u\left(\exp(4\pi u^2) - 1\right) = 0, \tag{22}$$

with initial data $u(0,\cdot) = f,\ \partial_t u(0,\cdot) = g$, where $(f,g) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ (see [8] for more details). For such a problem, only global (in time) wellposedness for small data or local wellposedness for radially symmetric data $(0,g)$ satisfying $\|g\|_{L^2} \leq 1$ are known so far. See [12], [13] and [3]. To establish an energy estimate for solutions of (22), we need to estimate the source term $u(\exp(4\pi u^2) - 1)$ in $L^1_t(L^2_x)$ (or any other dual Strichartz norm). The problem with taking the L^2_x norm is that the factor 4π appearing in the exponential will be doubled, and hence, we cannot apply the Moser-Trudinger inequality if $\|u\|_{H^1} > \frac{1}{\sqrt{2}}$.

In the following, we show how Corollary 5.2 enables us to overcome this difficulty and allows us to deal with solutions such that $\|u\|_{H^1} \leq 1$. This seems to be optimal [8]. For simplicity, we assume that u solves the "linearized problem"; this corresponds to the first iteration in a proof based on the Picard scheme.

In the sequel, we assume that $(f,g) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that

$$\|f\|^2_{H^1} + \|g\|^2_{L^2} \leq 1. \tag{23}$$

Denote by v the solution of the 2D linear Klein-Gordon equation

$$\begin{aligned} \partial_t^2 v - \Delta v + v = &\quad 0, \\ v(0,\cdot) = f, \quad \partial_t v(0,\cdot) = &\quad g. \end{aligned} \tag{24}$$

Since the energy $\|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} + \|v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} + \|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)}$ is conserved, $v(t,\cdot)$ remains in the unit ball of H^1 uniformly in time. So according to (2) we have

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^2} \left(e^{4\pi v(t,x)^2} - 1\right) dx \leq C,$$

which means that $\exp(4\pi v^2) - 1 \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^2))$. For any $\mu > 0$, denote

$$E_\mu(t) := \|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} + \mu^2 \|v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)}.$$

The following result will enable us to estimate $\exp(4\pi v^2) - 1$ in $L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$.

Proposition 6.1. *Let v be the solution of* (24) *with initial data satisfying* (23)*. For any $T > 0$ and $0 < \mu < 1$, a nonnegative constant C exists such that*

$$\int_0^T \|\exp(4\pi v^2(t,\cdot)) - 1\|_{L^2(\mathbb{R}^2)}\, dt \leq C.$$

Proof. Recall that since $v \in \mathcal{C}(\mathbb{R},\ H^1) \cap \mathcal{C}^1(\mathbb{R},\ L^2)$, the function $t \longrightarrow E_\mu(t)$ is continuous. The energy conservation satisfied by v shows that

$$\|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} + E_1(t) = E_1(0) + \|g\|^2_{L^2} \leq 1.$$

Now, fix $\mu < 1$ and $T > 0$. There exists a time $\tau = \tau(\mu, T) \leq T$ such that

$$\sup_{t\in[0,T]} E_\mu(t) = E_\mu(\tau) < 1.$$

For almost every t we have

$$\int_{\mathbb{R}^2} \left(\exp(4\pi v^2(t,x)) - 1\right)^2 dx \leq \|\exp(4\pi v^2(t,\cdot)) - 1\|_{L^1} \exp(4\pi \|v(t,\cdot)\|^2_{L^\infty}). \tag{25}$$

Note that, thanks to the conservation of the energy and the Moser-Trudinger inequality, the first factor in the above inequality is uniformly bounded. On the other hand, choosing $\alpha = \frac{1}{4}$ in (20) we obtain, for any $\lambda > \frac{2}{\pi}$,

$$\exp(2\pi\|v(t,\cdot)\|_{L^\infty}^2) \leq \big(C + \frac{\|v(t,\cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\big)^{2\pi\lambda E_\mu(\tau)}. \tag{26}$$

Using the fact that the bound given on the right-hand side of (26) is increasing in $E_\mu(\tau)$, we can assume that $E_\mu(\tau) > 1/2$. Since $E_\mu(\tau) < 1$, one can choose $\lambda > \frac{2}{\pi}$ such that $\beta := 2\pi\lambda E_\mu(\tau) < 4$. Hence, we have

$$\begin{aligned}\int_0^T \exp(2\pi\|v(t,\cdot)\|_{L^\infty}^2)dt &\leq C\int_0^T \big(C + \|v(t,\cdot)\|_{\mathcal{C}^{1/4}}\big)^\beta dt \\ &\leq CT^{1-\frac{\beta}{4}}\Big(\int_0^T \big(C + \|v(t,\cdot)\|_{\mathcal{C}^{1/4}}\big)^4 dt\Big)^{\frac{\beta}{4}}.\end{aligned}$$

Now, thanks to the so-called Strichartz estimates (see [4, 7]), we have $v \in L^4(\mathbb{R}, \mathcal{C}^{1/4}(\mathbb{R}^2))$, and therefore Proposition 6.1 is proved. □

Remark 6.2. To study the Cauchy problem for (22), we need a bound in $L^1_T(L^2)$ for $u\big(\exp(4\pi u^2) - 1\big)$. Using Hölder inequality we have

$$\|v(\exp(4\pi v^2(t,\cdot)) - 1)\|_{L^2(\mathbb{R}^2)} \leq \|\exp(4\pi v^2(t,\cdot)) - 1\|_{L^{2(1+\varepsilon)}(\mathbb{R}^2)}\|v(t,\cdot)\|_{L^{2(1+\frac{1}{\varepsilon})}(\mathbb{R}^2)}.$$

Following the same proof as that of the above proposition and suitably choosing $\varepsilon > 0$, we can prove a bound for $\|\exp(4\pi v^2) - 1\|_{L^1_T(L^{2+2\varepsilon})}$.

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