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DOUBLE QUINTIC SYMMETROIDS, REYE CONGRUENCES, AND THEIR DERIVED EQUIVALENCE

Shinobu Hosono & Hiromichi Takagi

Dedicated to Professor Yujiro Kawamata on the occasion of his 60th birthday

Abstract

Let $\mathscr{Y}$ be the double cover of the quintic symmetric determinantal hypersurface in $\mathbb{P}^{14}$. We consider Calabi-Yau threefolds Y defined as smooth linear sections of $\mathscr{Y}$. In our previous works [**HoTa1**, 2, 3], we have shown that these Calabi-Yau threefolds Y are naturally paired with Reye congruence Calabi-Yau threefolds X by the projective duality of $\mathscr{Y}$, and observed that these Calabi-Yau threefolds have several interesting properties from the viewpoint of mirror symmetry and also projective geometry. In this paper, we prove the derived equivalence between the linear sections Y of $\mathscr{Y}$ and the corresponding Reye congruences X.

1. Introduction

Non-birational smooth Calabi-Yau threefolds which have an equivalent derived category are of considerable interest from the viewpoint of the homological mirror symmetry due to Kontsevich [**Ko**]. As such an example, it has been proved in [**BC**] that smooth Calabi-Yau threefolds which are given by respective smooth linear sections of $\mathrm{G}(2,7)$ and Pfaff(7) (see [**Ro**]) are derived equivalent. To our best knowledge, this pair is the first example of derived equivalent, but non-birational smooth Calabi-Yau threefolds with Picard number one. In [**Ku2**], the derived equivalence has been understood as a corollary of a more general statement that a non-commutative resolution of Pfaff(7) is homologically projective dual to $\mathrm{G}(2,7)$. The homological projective duality is a framework proposed in [**Ku1**], which sheds light on the classical projective duality in the theory of derived category. In the case of $\mathrm{G}(2,7)$ and Pfaff(7), the classical projective geometries involved are that of the projective space of skew symmetric matrices $\mathbb{P}(\wedge^2\mathbb{C}^7)$ and its dual projective space $\mathbb{P}(\wedge^2(\mathbb{C}^*)^7)$.

In the previous work [**HoTa1**], by studying mirror symmetry of Calabi-Yau threefold X of a Reye congruence, we naturally came to the

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projective duality of $\mathbb{P}(\mathsf{S}^2\mathbb{C}^5)$, the projective space of symmetric matrices, and its dual projective space $\mathbb{P}(\mathsf{S}^2(\mathbb{C}^*)^5)$. We have observed that X should be paired with another smooth Calabi-Yau threefold Y, which is given as a linear section of the double quintic symmetroid. We are also led to the prediction [ibid. Conj.2] of their derived equivalence. The main result of this paper is an affirmative proof for this prediction.

Let $V = \mathbb{C}^5$ and X be a smooth linear section of the second symmetric product $\mathscr{X} = \mathsf{S}^2\mathbb{P}(V)$ in $\mathbb{P}(\mathsf{S}^2V)$. Then Y is given by the orthogonal linear section of the double symmetroid $\mathscr{Y}$, which is the double cover of the determinantal symmetroid $\mathscr{H}$ in $\mathbb{P}(\mathsf{S}^2V^*)$. $\mathscr{X}$ has a natural resolution $\check{\mathscr{X}}$ given by the Hilbert scheme of two points. As for $\mathscr{Y}$, a nice desingularization $\widetilde{\mathscr{Y}}$ has been obtained in our recent paper [**HoTa3**]. Furthermore, it has been found that a finite collection of sheaves $(\mathcal{F}_i)_{i\in I}$ on $\check{\mathscr{X}}$ defines a dual Lefschetz collection in the derived categories $\mathcal{D}^b(\check{\mathscr{X}})$ [ibid. Thm.3.4.4], and correspondingly a collection of sheaves $(\mathcal{E}_i)_{i\in I}$ on $\widetilde{\mathscr{Y}}$ defines a Lefschetz collection in $\mathcal{D}^b(\widetilde{\mathscr{Y}})$ [ibid. Thm.8.1.1]. In this paper, we study a certain closed subscheme Δ in $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ and construct a locally free resolution of its ideal sheaf $\mathcal{I}$ in terms of the sheaves $(\mathcal{F}_i)_{i\in I}$ and $(\mathcal{E}_i)_{i\in I}$. Considering a Fourier-Mukai functor with its kernel I being the restriction of the sheaf $\mathcal{I}$ to $Y \times X$ in $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$, we prove the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ (**Theorem 8.0.1**).

We introduce the subscheme Δ from the flag variety $\Delta_0 := \mathrm{F}(2,3,V)$ in $\mathrm{G}(3,V) \times \mathrm{G}(2,V)$. As we summarize in Subsection 2.2, $\check{\mathscr{X}}$ is a $\mathbb{P}^2$-bundle over $\mathrm{G}(2,V)$ and hence there is a morphism to $\mathrm{G}(2,V)$. In contrast to this, the geometry of the double quintic symmetroid $\mathscr{Y}$ is more involved. It turns out that $\mathscr{Y}$ describes the connected families of planes contained in singular quadrics, which, in the case of quadrics of rank 4, are represented by conics on $\mathrm{G}(3,V)$ (Subsection 2.3 and Section 4). In particular, we see that there is a generically conic bundle $\mathscr{Z}$ over $\mathscr{Y}$ which parameterizes pairs of singular quadrics and planes therein, and hence there exists a natural morphism $\mathscr{Z} \to \mathrm{G}(3,V)$. Roughly speaking, the subscheme Δ is constructed by pulling back Δ_0 by the product of the above morphisms $\mathscr{Z} \times \check{\mathscr{X}} \to \mathrm{G}(3,V) \times \mathrm{G}(2,V)$, pushing forward by the morphism $\mathscr{Z} \times \check{\mathscr{X}} \to \mathscr{Y} \times \check{\mathscr{X}}$ and taking the transform by the birational morphism $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}} \to \mathscr{Y} \times \check{\mathscr{X}}$. The fact that the connected families of planes in rank 4 quadrics can be described by conics on $\mathrm{G}(3,V)$ implies that $\mathscr{Y}$ is birational to the Hilbert scheme of conics on $\mathrm{G}(3,V)$ studied by [**IM**].

The choice of the subscheme Δ comes from our observation on the so-called BPS numbers of Y listed in [**HoTa1**, Table 3]. Let I_x be the restriction of the ideal sheaf I to $Y \times \{x\}$. We will show that I_x defines a curve C_x on Y of arithmetic genus 3 and degree 5 parameterized by X, and C_x is smooth if X and x are general (Propositions 3.1.3 and

7.2.2). Our observation/discovery about this family of curves is that this family can be identified in the table of BPS numbers $n_g^Y(d)$ at genus 3 and degree 5 as

$$n_3^Y(5) = 100 = (-1)^{\dim X} e(X) \times 2,$$

where $e(X) = -50$ is the Euler number of X. We note that the signed Euler number $(-1)^{\dim X} e(X)$ is in accord with the counting rule of the BPS numbers for a family of curves [**GV**]. The factor 2 indicates that there should be another family of curves parameterized by X. In fact it turns out that there is a "shadow" curve C'_x of the same genus and degree as C_x (Fig.1 in Subsection 3.1). The BPS numbers are integers which we calculate by using mirror symmetry, and its mathematical ground is still opened in general. We believe, however, that our observation above may be justified by the Donaldson-Thomas (DT) invariants or the Pandharipande-Thomas (PT) invariants associated with a suitable moduli problem of ideal sheaves or stable pairs (see [**PT**] and reference therein also [**HST**] for the BPS invariants). Namely we expect that the Calabi-Yau threefold X appears as a suitable moduli space of the ideal sheaf of curves on Y and the derived equivalence between the two is a consequence of this.

Here we should remark some similarities of our construction to that of the Grassmann-Pfaffian case. The proof of derived equivalence due to [**BC**] (and also [**Ku2**]) is based on a certain incidence relation on $\mathrm{G}(2,7) \times \mathrm{Pfaff}(7)$ which gives rise to the kernel of a Fourier-Mukai functor. Our proof using Δ_0 is basically parallel to this, although the formulation of our incidence relation Δ in $\widetilde{\mathscr{Y}} \times X$ and the corresponding ideal sheaf are much more involved and requires the resolution $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ [**HoTa3**]. In the Grassmann-Pfaffian case, the restriction I_y to $\{y\} \times X$ of the corresponding ideal sheaf I defines a generically smooth family of curves of genus 6 and degree 14 on X. In this case, however, we read the corresponding BPS number as $n_6^X(14) = 123676$ (see [**HoTa1**, (4.2)]) and there seems to be some complications in the possible moduli interpretation in terms of DT/PT invariants.

Recently, the derived equivalence of the Grassmann-Pfaffian Calabi-Yau threefolds has been formulated in the framework of categorical geometric invariant theory [**BDFIK**, **Ha**, **DS**]. Our case has been argued in [**Hor**, **HorKn**] in the language of physics called gauged linear sigma model, which is closely related to geometric invariant theory. It seems interesting to see how the derived equivalence in our case will be formulated in the categorical geometric invariant theory.

Here we outline the present paper. In Section 2, we summarize some basic results on which our arguments are based. In Section 3, we construct a family of curves on Y parameterized by X (Proposition 3.1.3) and show that it comes with another family of "shadow" curves. Also we calculate the Brauer group of Y as a corollary. In Section 4, based on the results [**HoTa3**], we summarize the birational geometry of $\mathscr{Y}$ and introduce the desingularization $\widetilde{\mathscr{Y}}$. We will also introduce generically conic bundles over some varieties which are birational to $\mathscr{Y}$. In Section 5, we introduce the subscheme Δ representing the incidence relation on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ and its ideal sheaf $\mathcal{I}$. Dividing the process into four major steps, we obtain a locally free resolution of $\mathcal{I}$ in terms of the collections of locally free sheaves $(\mathcal{E}_i)_{i\in I}$ and $(\mathcal{F})_{i\in I}$ (Theorem 5.1.3). In section 6, we show that the subscheme Δ is contained in (the pullback of) the universal family of hyperplane sections $\mathcal{V}$. In Section 7, restricting the ideal sheaf to $Y \times X$, we show that this defines a family of curve on Y which is flat over X, and coincides with the family obtained in Section 3 (Proposition 7.2.2). In Section 8, using the properties of the (dual) Lefschetz collections $(\mathcal{E}_i)_{i\in I}$ and $(\mathcal{F}_i)_{i\in I}$ [**HoTa3**, Thm.8.1.1], we show the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$. Some further discussions are presented in Section 9.

Notation: Throughout the present paper, we work over $\mathbb{C}$, the complex number field. We will use the following notation to simplify lengthy formulas:

V: a (fixed) 5 dimensional complex vector space. $\mathbb{P}^4 := \mathbb{P}(V)$.
V_i: an i-dimensional vector subspace of V.
$\Omega(1) := \Omega_{\mathbb{P}(V)}(1)$.
$\Omega(1)^{\wedge i} := \bigwedge^i(\Omega_{\mathbb{P}(V)}(1))$ for $i \geq 2$.
$T(-1) := T_{\mathbb{P}(V)}(-1)$.
$T(-1)^{\wedge i} := \bigwedge^i(T(-1))$ for $i \geq 2$.
$\mathcal{O}(i) := \mathcal{O}_{\mathbb{P}(V)}(i)$ for $i \in \mathbb{Z}$.

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2. Preliminaries

Here we summarize the basic results on which our proof of the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ relies. We also summarize the construction of the Calabi-Yau threefolds X and Y which have been described in detail in [**HoTa1**, **HoTa3**].

2.1. Basic general results. For the computations of cohomology groups which appear in this paper, we use extensively the following Bott theorem about the cohomology groups of Grassmann bundles.

For a locally free sheaf $\mathcal{E}$ of rank r on a variety and a nonincreasing sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ of integers, we denote by $\Sigma^\beta \mathcal{E}$ the associated locally free sheaf with the Schur functor Σ^β.

Theorem 2.1.1. (Bott Theorem) *Let $\pi\colon \mathrm{G}(r, \mathcal{A}) \to X$ be a Grassmann bundle for a locally free sheaf $\mathcal{A}$ on a variety X of rank n and $0 \to \mathcal{S} \to \mathcal{A} \to \mathcal{Q} \to 0$ the universal exact sequence. For $\beta := (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ $(\alpha_1 \geq \cdots \geq \alpha_r)$ and $\gamma := (\alpha_{r+1}, \ldots, \alpha_n) \in \mathbb{Z}^{n-r}$ $(\alpha_{r+1} \geq \cdots \geq \alpha_n)$, we set $\alpha := (\beta, \gamma)$ and $\mathcal{V}(\alpha) := \Sigma^\beta \mathcal{S}^* \otimes \Sigma^\gamma \mathcal{Q}^*$. Define $\rho := (n, \ldots, 1)$ and for an element σ of the n-th symmetric group $\mathfrak{S}_n$, we set $\sigma \cdot (\alpha) := \sigma(\alpha + \rho) - \rho$. Then the followings hold:*

(1) *If $\alpha + \rho$ contains two equal integers, then $R^i \pi_* \mathcal{V}(\alpha) = 0$ for any $i \geq 0$.*
(2) *If there exists an element $\sigma \in \mathfrak{S}_n$ such that $\sigma(\alpha + \rho)$ is strictly decreasing, then $R^i \pi_* \mathcal{V}(\alpha) = 0$ for any $i \geq 0$ except $R^{l(\sigma)} \pi_* \mathcal{V}(\alpha) = \Sigma^{\sigma \cdot (\alpha)} \mathcal{A}^*$, where $l(\sigma)$ represents the length of $\sigma \in \mathfrak{S}_n$.*

Proof. See [**Bo**], [**D**], or [**W**, (4.19) Corollary]. q.e.d.

Theorem 2.1.2. (Grothendieck-Verdier duality) *Let $f\colon X \to Y$ be a proper morphism of smooth varieties X and Y. Set $n := \dim X - \dim Y$. We have the following functorial isomorphism: For $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ and $\mathcal{E}^\bullet \in \mathcal{D}^b(Y)$,*

$$Rf_* R\mathcal{H}om(\mathcal{F}^\bullet, Lf^* \mathcal{E}^\bullet \otimes \omega_{X/Y}[n]) \simeq R\mathcal{H}om(Rf_* \mathcal{F}^\bullet, \mathcal{E}^\bullet).$$

In particular, if $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are locally free (we write them simply $\mathcal{E}$ and $\mathcal{F}$) and if $R^\bullet f_* \mathcal{F} = 0$ $(\bullet > 0)$, then

$$R^{\bullet + n} f_*(\mathcal{F}^* \otimes f^* \mathcal{E} \otimes \omega_{X/Y}) \simeq \mathcal{E}xt^\bullet(f_* \mathcal{F}, \mathcal{E}).$$

Proof. See [**Huy**, Theorem 3.34]. q.e.d.

For our proof of the derived equivalence, we use the following theorem:

Theorem 2.1.3. *Let X and Y be smooth projective varieties and $\mathscr{P}$ a coherent sheaf on $X \times Y$ flat over X. Then the Fourier-Mukai transform $\Phi_{\mathcal{P}}\colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ ($\mathcal{P}$ is called the kernel of $\Phi_{\mathcal{P}}$) is fully faithful if and only if the following two conditions are satisfied:*

(i) *For any point $x \in X$, it holds $\mathrm{Hom}(\mathcal{P}_x, \mathcal{P}_x) \simeq \mathbb{C}$, and*
(ii) *if $x_1 \neq x_2$, then $\mathrm{Ext}^i(\mathcal{P}_{x_1}, \mathcal{P}_{x_2}) = 0$ for any i.*

Moreover, under these conditions, $\Phi_{\mathcal{P}}$ is an equivalence of triangulated categories if and only if $\dim X = \dim Y$ and $\mathcal{P} \otimes \mathrm{pr}_1^* \omega_X \simeq \mathcal{P} \otimes \mathrm{pr}_2^* \omega_Y$.

In particular, if $\dim X = \dim Y$, $\omega_X \simeq \mathcal{O}_X$ and $\omega_Y \simeq \mathcal{O}_Y$, then $\Phi_{\mathcal{P}}$ is fully faithful if and only if it is an equivalence.

Proof. See [**BO**, Theorem 1.1], [**B**, Theorem 1.1], [**Huy**, Corollary 7.5 and Proposition 7.6]. q.e.d.

In this paper, we adopt the following definition of Calabi-Yau variety and also Calabi-Yau manifold.

Definition 2.1.4. We say a normal projective variety X *a Calabi-Yau variety* if X has only Gorenstein canonical singularities, the canonical bundle of X is trivial, and $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$. If X is smooth, then X is called *a Calabi-Yau manifold.* A smooth Calabi-Yau threefold is abbreviated as a Calabi-Yau threefold.

2.2. The Hilbert scheme $\check{\mathscr{X}}$ of two points on $\mathbb{P}(V)$. Let $\mathscr{X}$ be the Chow variety of two points on $\mathbb{P}(V)$ embedded by the Chow form into $\mathbb{P}(\mathsf{S}^2V)$. Denote by $\mathscr{X}_0$ the second Veronese variety $v_2(\mathbb{P}(V))$. It is a well-known fact that $\mathscr{X}_0 = \operatorname{Sing} \mathscr{X}$ and $\mathscr{X}$ is the secant variety of $\mathscr{X}_0$. If we take a coordinate of S^2V so that it represents a generic 5×5 symmetric matrix, then $\mathscr{X}_0$ (resp. $\mathscr{X}$) is characterized as the locus of rank 1 (resp. rank ≤ 2) symmetric matrices.

Let $\check{\mathscr{X}}$ be the Hilbert scheme of 0-dimensional subschemes of length two in $\mathbb{P}(V)$, which will be called the Hilbert scheme of two points in $\mathbb{P}(V)$ hereafter. A 0-dimensional subscheme of length two may be determined from the corresponding 0-cycle η of length two on $\mathbb{P}(V)$ and a line $l \subset \mathbb{P}(V)$ containing η, and vice versa. Hence, we have an isomorphism $\check{\mathscr{X}} \simeq \mathbb{P}(\mathsf{S}^2\mathcal{F})$, where $\mathcal{F}$ is the universal subbundle of rank two on $\mathrm{G}(2, V)$. Let

$$0 \to \mathcal{F} \to V \otimes \mathcal{O}_{\mathrm{G}(2,V)} \to \mathcal{G} \to 0 \tag{2.1}$$

be the universal exact sequence on $\mathrm{G}(2, V)$. Using the induced injection $\mathsf{S}^2\mathcal{F} \hookrightarrow \mathsf{S}^2V \otimes \mathcal{O}_{\mathrm{G}(2,V)}$, we obtain a (composite) morphism $\check{\mathscr{X}} \to \mathbb{P}(\mathsf{S}^2V) \times \mathrm{G}(2, V) \to \mathbb{P}(\mathsf{S}^2V)$. Then the tautological divisor of $\mathbb{P}(\mathsf{S}^2\mathcal{F})$ is the pull-back of $\mathcal{O}_{\mathbb{P}(\mathsf{S}^2V)}(1)$. The image of this morphism is nothing but $\mathscr{X}$ and the induced morphism $f \colon \check{\mathscr{X}} \to \mathscr{X}$ coincides with the Hilbert-Chow morphism. Moreover, f is the blow-up along $\mathscr{X}_0$.

$$\begin{tikzcd} & \check{\mathscr{X}} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathscr{X} & & \mathrm{G}(2, V). \end{tikzcd}$$

We define the following divisors on $\check{\mathscr{X}}$:

$$H_{\check{\mathscr{X}}} = f^*\mathcal{O}_{\mathscr{X}}(1) \text{ and } L_{\check{\mathscr{X}}} = g^*\mathcal{O}_{\mathrm{G}(2,V)}(1).$$

We also denote by E_f the f-exceptional divisor.

2.3. The double quintic symmetroid $\mathscr{Y}$. Hereafter we assume $V \simeq \mathbb{C}^5$ and write by V^* the dual vector space of V. Let $\mathscr{H} \subset \mathbb{P}(\mathsf{S}^2 V^*)$ be the locus of singular quadrics in $\mathbb{P}(V)$, which will be called the (generic) *quintic symmetroid* since it is the hypersurface defined by the determinant of the (generic) 5×5 symmetric matrix. It is the locus of 5×5 symmetric matrices of rank ≤ 4.

Let $\mathscr{U} := \{(t, [Q]) \mid t \in \operatorname{Sing} Q\} \subset \mathbb{P}(V) \times \mathbb{P}(\mathsf{S}^2 V^*)$. Then the natural morphism $p\colon \mathscr{U} \to \mathscr{H}$ is a desingularization of $\mathscr{H}$ (see [**HoTa3**, Subsect.4.1]).

To construct the double cover $\mathscr{Y}$ of $\mathscr{H}$ branched along the locus of symmetric matrices of rank ≤ 3, we introduce the variety $\mathscr{Z}$ which parameterizes the pair of quadrics Q in $\mathbb{P}(V)$ and planes $\mathbb{P}(\Pi)$ such that $\mathbb{P}(\Pi) \subset Q$. To describe $\mathscr{Z}$ more explicitly, consider the universal exact sequence on $\mathrm{G}(3, V)$;

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathcal{W} \to 0. \tag{2.2}$$

The surjection $\mathsf{S}^2 V^* \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathsf{S}^2 \mathcal{U}^*$ follows from the dual sequence. Then we define a locally free sheaf $\mathcal{E}^*$ on $\mathrm{G}(3, V)$ by

$$0 \to \mathcal{E}^* \to \mathsf{S}^2 V^* \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathsf{S}^2 \mathcal{U}^* \to 0. \tag{2.3}$$

Then, since the fiber of $\mathbb{P}(\mathsf{S}^2 \mathcal{U}^*)$ over a point $[\Pi] \in \mathrm{G}(3, V)$ may be identified with the linear system of quadrics on $\mathbb{P}(\Pi)$, the fiber of $\mathbb{P}(\mathcal{E}^*)$ represents the quadrics which contain the plane $\mathbb{P}(\Pi)$. Namely we have

$$\mathscr{Z} = \mathbb{P}(\mathcal{E}^*) \subset \mathrm{G}(3, V) \times \mathbb{P}(\mathsf{S}^2 V^*).$$

Note that the image of the naturally induced morphism $\mathscr{Z} \to \mathbb{P}(\mathsf{S}^2 V^*)$ coincides with the singular quadrics $\mathscr{H}$, since a smooth quadric does not contain a plane.

Let $\mathscr{Z} \overset{\pi_{\mathscr{Z}}}{\to} \mathscr{Y} \overset{\rho_{\mathscr{Y}}}{\to} \mathscr{H}$ be the Stein factorizabtion of the natural morphism $\mathscr{Z} \to \mathscr{H}$. Then $\rho_{\mathscr{Y}}\colon \mathscr{Y} \to \mathscr{H}$ turns out to be the finite double covering branched along the locus of quadrics of rank less than or equal to three [**HoTa3**, Prop. 4.2.2]. $\mathscr{Y}$ is called the (generic) *double quintic symmetroid*. We say that $y \in \mathscr{Y}$ is a *rank i point* if $\rho_{\mathscr{Y}}(y) \in \mathscr{H}$ corresponds to a quadric of rank i. $G_{\mathscr{Y}} := \operatorname{Sing} \mathscr{Y}$ is the subset consisting of rank 1 and 2 points ([ibid, Prop. 5.7.2]). We introduce divisors on $\mathscr{Y}$ and $\mathscr{Z}$, respectively, by

$$M_{\mathscr{Y}} = \rho_{\mathscr{Y}}^* \mathcal{O}_{\mathscr{H}}(1) \text{ and } M_{\mathscr{Z}} = \pi_{\mathscr{Z}}^* \circ \rho_{\mathscr{Y}}^* \mathcal{O}_{\mathscr{H}}(1),$$

where $\mathcal{O}_{\mathscr{H}}(1) := \mathcal{O}_{\mathbb{P}(\mathsf{S}^2 V^*)}(1)|_{\mathscr{H}}$.

Consider the fiber $\mathscr{Z}_{[Q]}$ of the morphism $\mathscr{Z} \to \mathscr{H}$ over a quadric $[Q] \in \mathscr{H}$. If $\operatorname{rank} Q = 4$, then Q is a cone over the smooth quadric ($\simeq \mathbb{P}^1 \times \mathbb{P}^1$) in $\mathbb{P}(V/V_1)$ with the vertex $[V_1]$, and the planes in Q consist of two different $\mathbb{P}^1$-families which correspond to the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. If $\operatorname{rank} Q = 3$, then Q is a cone over the smooth quadric in $\mathbb{P}(V/V_2)$ with the vertex $\mathbb{P}(V_2) \simeq \mathbb{P}^1$, and in this case there is only one $\mathbb{P}^1$-family of planes in Q. Thus the morphism $\pi_{\mathscr{Z}}\colon \mathscr{Z} \to \mathscr{Y}$ of the Stein factorization is a generically conic bundle; generic points of $\mathscr{Y}$ are represented by pairs (Q,q) of quadrics Q of rank 4 (or 3) and connected families q of planes contained in Q, where q represent conics in $\mathrm{G}(3,V)$ ([**HoTa3**, Prop. 4.2.5]). It turns out that several birational models of $\mathscr{Z} \to \mathscr{Y}$ play crucial roles to construct the kernel of a Fourier-Mukai functor giving an equivalence of $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$.

The following computations of the Chern classes of the locally free sheaf $\mathcal{E}$ on $\mathrm{G}(3,V)$ will be used in the next section.

Lemma 2.3.1. *We have* $c_1(\mathcal{E}) = c_1(\mathcal{O}_{\mathrm{G}(3,V)}(4))$ *and also* $c_2(\mathcal{E}) = 5c_2(\mathcal{W}) + 6c_1(\mathcal{O}_{\mathrm{G}(3,V)}(1))^2$.

Proof. This follows from standard computations of Chern classes by using the exact sequences (2.2) and (2.3). Note that $c_1(\mathcal{O}_{\mathrm{G}(3,V)}(1))$ is given by the Schubert cycle σ_1, which is $c_1(\mathcal{U}^*)$ in our notation. Since $\mathrm{rk}\,\mathcal{U} = 3$, the first relation follows from $c_1(\mathcal{E}) = c_1(\mathsf{S}^2\mathcal{U}^*) = 4c_1(\mathcal{U}^*)$. For the second relation, we derive $c_2(\mathsf{S}^2\mathcal{U}^*) = 5c_1(\mathcal{U}^*)^2 + 5c_2(\mathcal{U}^*)$ and use $c_2(\mathcal{W}) = c_1(\mathcal{U}^*)^2 - c_2(\mathcal{U}^*)$, $c_2(\mathcal{E}) = c_1(\mathsf{S}^2\mathcal{U}^*)^2 - c_2(\mathsf{S}^2\mathcal{U}^*)$. q.e.d.

2.4. Calabi-Yau threefolds X and Y. Let us identify $\mathbb{P}(\mathsf{S}^2V^*)$ with the space of quadrics in $\mathbb{P}(V)$. Then a 4-plane $P \subset \mathbb{P}(\mathsf{S}^2V^*)$ may be regarded as a 4-dimensional linear system of quadrics $P = |Q_1, Q_2, \cdots, Q_5|$ with the quadrics Q_i defined by the corresponding 5×5 symmetric matrices A_i. Let $P^\perp \subset \mathbb{P}(\mathsf{S}^2V)$ be the orthogonal space of P with respect to the dual pairing between S^2V and S^2V^*, and define $X = \mathscr{X} \cap P^\perp$. Dually, we may construct also Y in $\mathscr{Y}$ as the pull-back of the quintic symmetroid $\mathscr{H} \cap P$. The linear system P is called *regular* if i) it is base point free and ii) any line $l \subset \mathrm{Sing}\,Q$ for some $Q \in P$ is not contained in a linear subsystem of dimension ≥ 2. X is smooth if and only if P is regular [**HoTa1**, Prop.2.1]. It has been shown that Y is also smooth when P is regular [ibid. Prop.3.11]. We say that X and Y defined for such a choice of P are *orthogonal* to each other.

Proposition 2.4.1. *X and Y constructed as above are Calabi-Yau threefolds with the following invariants*:

1) $\deg(X) = 35,\ \ c_2.D = 50,\ \ h^{2,1}(X) = 26,\ h^{1,1}(X) = 1,$

where D is the restriction to X of a hyperplane section of $\mathscr{X}$, $\deg(X) := D^3$ *and c_2 is the second Chern class of X.*

2) $\deg(Y) = 10,\ \ c_2.M = 40,\ \ h^{2,1}(Y) = 26,\ h^{1,1}(Y) = 1,$

where M is the restriction of $M_{\mathscr{Y}}$ to Y, $\deg(Y) := M^3$, *and c_2 is the second Chern class of Y.*

Proof. The invariants of X are easy to be determined, see [**HoTa1**, Prop.2.1]. The invariants of Y are determined in [**HoTa3**, Prop.4.3.4] (see also [**HoTa1**, Prop.3.11 and 3.12]). q.e.d.

Hereafter we consider the Calabi-Yau threefolds X and Y which are orthogonal to each other.

Since X is smooth, X is disjoint from $\operatorname{Sing}\mathscr{X}$, and hence we can consider X to be contained in $\check{\mathscr{X}}$. Moreover, X is mapped by $g\colon \check{\mathscr{X}} \to \mathrm{G}(2,V)$ onto its image isomorphically, and hence we can also consider X to be contained in $\mathrm{G}(2,V)$. By the existence of this embedding into $\mathrm{G}(2,V)$, X is called *a (generalized) Reye congruence*. As a subvariety of $\mathrm{G}(2,V)$, X is characterized as the subset of lines l in $\mathbb{P}(V)$ such that quadrics which contain l form a 2-dimensional linear system (net) in P ([**HoTa3**, Prop.3.5.2]).

Y will be called the *(3-dimensional) double quintic symmetroid orthogonal to X*. Since Y is smooth, Y is disjoint from $\operatorname{Sing}\mathscr{Y} = G_{\mathscr{Y}}$.

3. A family of curves on Y parameterized by X

In this section, using the generically conic bundle $\pi_{\mathscr{Z}}\colon \mathscr{Z} \to \mathscr{Y}$, we construct a family of curves on Y of degree 5 parameterized by X, and show that its general member is a smooth curve of genus 3 for a general P. Later in Section 7 (see also Section 9.1), we will show that this family is flat and explains the BPS number of curves of genus 3 and degree 5 on Y. The ideal sheaf of this family of curves in $Y \times X$ will be related with the birational model $\mathrm{G}(3,T(-1)^{\wedge 2})$ of $\mathscr{Y}$ and will give the kernel of the Fourier-Mukai functor which shows that $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

3.1. Constructing the family of curves. Recall our definition of basic morphisms;

$$\mathscr{H} \xleftarrow{\rho_{\mathscr{Y}}} \mathscr{Y} \xleftarrow{\pi_{\mathscr{Z}}} \mathscr{Z} \xrightarrow{\rho_{\mathscr{Z}}} \mathrm{G}(3,V),$$

where $\rho_{\mathscr{Z}} : \mathscr{Z} \to \mathrm{G}(3,V)$ is a $\mathbb{P}^8$-bundle since the fiber over a point $[\Pi]$ consists of (singular) quadrics which contain the plane $\mathbb{P}(\Pi)$.

We consider X in $\mathrm{G}(2,V)$, and denote by l_x the line in $\mathbb{P}(V)$ which corresponds to a point $x \in X$. We set $P_x = \{[Q] \in P \mid l_x \subset Q\}$, the linear subsystem of P consisting of quadrics which contain the line l_x. Then $\dim P_x = 2$ holds [**HoTa3**, Prop.3.5.2]).

Lemma 3.1.1. *For any $x \in X$, the plane P_x is not contained in the quintic symmetroid $H := \mathscr{H} \cap P$. Moreover, for a general regular P and a general $x \in X$, the intersection $H \cap P_x$ is a plane quintic curve with only three nodes.*

Proof. If $P_x \subset H$, then it is a divisor on H and $\rho_{\mathscr{Y}}^{-1}(P_x) = aM$ with some integer a, where M is the generator of $\mathrm{Pic}(Y)$ and satisfies $M^3 = 10$. We set $M_H := \mathcal{O}_P(1)|_H$. By pulling back the intersection relation $1 = M_H \cdot M_H \cdot P_x$ to Y, we have $2 = M \cdot M \cdot (aM) = 10a$, which is absurd.

Note that H is a quintic hypersurface while $P_x \simeq \mathbb{P}^2$ is a linear subspace of P. Therefore $H \cap P_x$ is a plane quintic curve in P_x. The final part follows from an explicit calculation of the plane curve $H \cap P_x$ by *Macaulay2*. We verify in the example below that, for a general P and a general x, the curve $H \cap P_x$ has three nodes as singularities. q.e.d.

Example 3.1.2. (Nodal quintic curve $H \cap P_x$) We fix a general 4 dimensional (regular) linear system of quadrics $P = |Q_1, Q_2, ..., Q_5|$ giving quadratic forms $q_i(\boldsymbol{z}) = {}^t\boldsymbol{z}A_i\boldsymbol{z}$ on $\mathbb{P}(V)$ by 5×5 symmetric matrices. Explicitly we give them by

$$A_\lambda := \sum_{i=1}^{5} \lambda_i A_i = \begin{pmatrix} \lambda_1 & \lambda_4 & \lambda_3 & \lambda_5 & \lambda_2 \\ \lambda_4 & -\lambda_3 & \lambda_2-\lambda_5 & \lambda_2 & \lambda_4 \\ \lambda_3 & \lambda_2-\lambda_5 & \lambda_2 & \lambda_4 & \lambda_1+2\lambda_2 \\ \lambda_5 & \lambda_2 & \lambda_4 & \lambda_1 & \lambda_4 \\ \lambda_2 & \lambda_4 & \lambda_1+2\lambda_2 & \lambda_4 & \lambda_1+\lambda_2 \end{pmatrix}.$$

We identify $[A_\lambda]$ with the corresponding point $[\lambda] = [\sum_i \lambda_i Q_i]$ in P. Then it is easy to verify that $[\boldsymbol{z}] = [-1, 0, 0, 1, 2]$ and $[w] = [-1, 2, 0, -1, 0]$ satisfies ${}^t\boldsymbol{z}A_\lambda\boldsymbol{w} = 0$ for any $[\lambda] \in P$, hence defines a point x in X and also the corresponding line $l_x = \langle \boldsymbol{z}, \boldsymbol{w} \rangle$. The plane P_x is determined by the linear equations ${}^t\boldsymbol{z}A_\lambda\boldsymbol{z} = {}^t\boldsymbol{w}A_\lambda\boldsymbol{w} = 0$ as $P_x = \{3\lambda_1 + 2\lambda_4 - \lambda_5 = 2\lambda_1 - \lambda_2 - \lambda_3 = 0\} \subset P$. Then the curve $H \cap P_x$ is given by the quintic equation representing $P_x \cap \{\det A_\lambda = 0\}$. By calculating the Jacobian, it is straightforward to see that $H \cap P_x$ has three singular points at $[\lambda] = [1, \frac{2}{9}(2\alpha^2 + 3\alpha + 1), -\frac{2}{9}(2\alpha^2 + 3\alpha - 8), \alpha, 3 + 2\alpha]$ for the three distinct roots α of the cubic $4x^3 - x^2 - 13x - 26 = 0$. By writing the local equation of the curve, we verify that all these singularities are nodal.

The symmetroid $H = \mathscr{H} \cap P$ is written by $\{\det A_\lambda = 0\} \subset P$. By using *Macaulay2*, we verify that Sing H is a smooth curve of genus 26 and degree 20 as noted in [**HoTa1**]. We also verify that Sing $H \cap (H \cap P_x) = \emptyset$.

Finally, consider the set $\{[\lambda] \in H \cap P_x \mid A_\lambda(a\boldsymbol{z} + b\boldsymbol{w}) = \boldsymbol{0}, \exists [a\boldsymbol{z} + b\boldsymbol{w}] \in l_x\}$ of quadrics which contain l_x with a point on l_x passing through their vertices. Note that by the regularity condition ii), there is no quadric that contains l_x in its vertex. We can verify that the three nodes on $H \cap P_x$ exactly correspond to this set (see Fig.1).

From Lemma 3.1.1, we see that the normalization of $H \cap P_x$ is a smooth curve of genus three for a general P and a general $x \in X$. We show that the normalization exists as curves on $Z := \pi_{\mathscr{Z}}^{-1}(Y)$ and $Y = \rho_{\mathscr{Y}}^{-1}(H)$. To describe the result precisely, let us define

$$G_x := \{[\Pi] \in \mathrm{G}(3, V) \mid l_x \subset \mathbb{P}(\Pi)\}$$

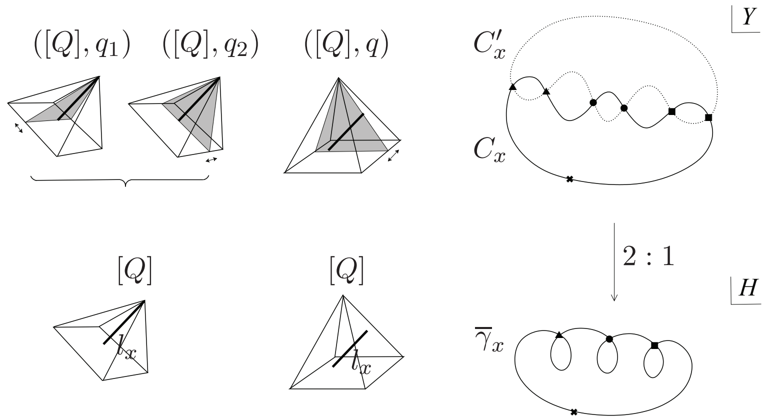

Fig.1. Curve C_x and its "shadow" C'_x. The line l_x and quadrics which contain l_x. If l_x passes through the vertex of Q, then two points $([Q], q_1), ([Q], q_2)$ map to $[Q]$. Otherwise, $([Q], q) \in Y$ is uniquely determined by $[Q]$.

and also

$$\mathscr{Z}_x := \{([\Pi], [Q]) \mid l_x \subset \mathbb{P}(\Pi) \subset Q\} = \rho_{\mathscr{Z}}^{-1}(G_x) \subset \mathscr{Z}.$$

G_x is a plane in $\mathrm{G}(3, V)$ and $\mathscr{Z}_x$ is a $\mathbb{P}^8$-bundle over G_x under the natural projection $\mathscr{Z}_x \to G_x$. Set

$$\gamma_x := \mathscr{Z}_x \cap \pi_{\mathscr{Z}}^{-1}(Y) = \{([\Pi], [Q]) \mid l_x \subset \mathbb{P}(\Pi) \subset Q, [Q] \in P\}$$

and denote by C_x its image on Y. Now we show

Proposition 3.1.3. *For smooth Calabi-Yau threefolds X and Y which are orthogonal to each other, $\{\gamma_x\}_{x\in X}$ is a family of curves of arithmetic genus 3 and of degree 5 with respect to $M_{\mathscr{Z}}$, and its images $\{C_x\}_{x\in X}$ on Y is a family of curves of degree 5 with respect to M. Moreover, if X and Y are general, then a general member C_x is a smooth curve of genus 3.*

Proof. Consider the projections $\overline{\gamma}_x := \rho_{\mathscr{Y}} \circ \pi_{\mathscr{Z}}(\gamma_x)$ and $\mathscr{H}_x := \rho_{\mathscr{Y}} \circ \pi_{\mathscr{Z}}(\mathscr{Z}_x)$. We define $\mathbb{P}_x := \{[Q] \in \mathbb{P}(\mathsf{S}^2 V^*) \mid l_x \subset Q\}$. If we write $x = w_{\boldsymbol{xy}} \in X$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{P}(V)$ as a point of $\mathsf{S}^2\mathbb{P}(V)$, then $\mathbb{P}_x = \{[Q] \in \mathbb{P}(\mathsf{S}^2 V^*) \mid {}^t\boldsymbol{x}A_Q\boldsymbol{x} = {}^t\boldsymbol{y}A_Q\boldsymbol{y} = {}^t\boldsymbol{x}A_Q\boldsymbol{y} = 0\}$, where A_Q is a 5×5 symmetric matrix defining the quadric Q. In particular, $\mathbb{P}_x$ is isomorphic to $\mathbb{P}^{11}$. Then we have $\mathscr{H}_x = \{[Q] \mid l_x \subset {}^\exists\mathbb{P}(\Pi) \subset Q\}$ and $\overline{\gamma}_x = \mathscr{H}_x \cap P = \mathscr{H}_x \cap P_x$. If a singular quadric $[Q] \in \mathscr{H}$ contains a line l, then there always exists at least one plane $\mathbb{P}(\Pi)$ such that $l \subset \mathbb{P}(\Pi) \subset Q$. Hence we have $\mathscr{H}_x = \mathscr{H} \cap \mathbb{P}_x$ and $\overline{\gamma}_x = \mathscr{H}_x \cap P_x = \mathscr{H} \cap P_x \subset \mathscr{H} \cap P$.

Since $H = \mathscr{H} \cap P$, we have $\overline{\gamma}_x = H \cap P_x$, which is a plane quintic curve by Lemma 3.1.1.

Now we note that if a line l is contained in a singular quadric Q but $l \not\subset \operatorname{Sing} Q$, then there are at most two planes satisfying $l \subset \mathbb{P}(\Pi) \subset Q$. For the lines l_x of $x \in X$, we have $\dim P_x = 2$ as we saw above. By the condition ii) of the regularity of P (see the beginning of Section 2.4), there is no quadric $[Q] \in P$ which contains the line l_x in $\operatorname{Sing} Q$. Therefore $\gamma_x \to \overline{\gamma}_x$ is finite of degree at most two. In particular, γ_x is a curve.

By [**HoTa3**, Prop.3.5.2], P_x is a plane in $\mathbb{P}_x$ and hence P_x is of codimension 9 in $\mathbb{P}_x$. Therefore, since $\overline{\gamma}_x = \mathscr{H}_x \cap P_x \subset \mathscr{H}_x \cap \mathbb{P}_x = \mathscr{H}_x$, we see that $\overline{\gamma}_x$ is also a complete intersection in $\mathscr{H}_x = \rho_{\mathscr{Y}} \circ \pi_{\mathscr{Z}}(\mathscr{Z})$ by 9 hyperplane sections. Corresponding to this, we also see that γ_x is a complete intersection of 9 elements of $|M_{\mathscr{Z}_x}| := |M_{\mathscr{Z}}|_{\mathscr{Z}_x}|$ in $\mathscr{Z}_x$ since γ_x is the pull-back of $\overline{\gamma}_x$. Using this, we compute the degree and the arithmetic genus of γ_x as follows: The degree of γ_x with respect to $M_{\mathscr{Z}}$ is evaluated by using $\mathscr{Z}_x = \rho_{\mathscr{Z}}^{-1}(G_x)$ and the Segre class of the projective bundle $\mathscr{Z} = \mathbb{P}(\mathcal{E}^*) \to \mathrm{G}(3, V)$ as

$$M_{\mathscr{Z}} \cdot (\mathscr{Z}_x \cdot M_{\mathscr{Z}}^9) = M_{\mathscr{Z}}^{10} \cdot \mathscr{Z}_x = s_2(\mathcal{E}|_{G_x}) = (c_1(\mathcal{E})^2 - c_2(\mathcal{E}))G_x,$$

which is equal to $(c_1(\mathcal{E})^2 - c_2(\mathcal{E}))G_x = (10c_1(\mathcal{O}_{\mathrm{G}(3,V)}(1))^2 - 5c_2(\mathcal{W}))G_x$ by Lemma 2.3.1. Since G_x is a plane, we have $c_1(\mathcal{O}_{\mathrm{G}(3,V)}(1))^2 G_x = 1$. We note that, by definition, $c_2(\mathcal{W}) = \sigma_2$ which represents the 4-cycle $\{[\Pi] \mid t \in \mathbb{P}(\Pi)\}$ in $\mathrm{G}(3, V)$ parameterizing 2-planes containing a fixed point t of $\mathbb{P}^4$. Therefore choosing such a point $t \in \mathbb{P}^4$ so that $t \notin l_x$, we see $c_2(\mathcal{W})G_x = 1$. Hence we have $M_{\mathscr{Z}}^{10} \cdot \mathscr{Z}_x = M_{\mathscr{Z}} \cdot \gamma_x = 5$. Since $\deg \gamma_x = \deg \overline{\gamma}_x = 5$, we see that $\gamma_x \to \overline{\gamma}_x$ is birational. The canonical divisor of γ_x is the restriction of $K_{\mathscr{Z}_x} + 9M_{\mathscr{Z}_x}$. From the relative Euler sequence of the projective bundle $\mathscr{Z}_x = \mathbb{P}(\mathcal{E}^*|_{G_x})$ over $G_x \simeq \mathbb{P}^2$ and $c_1(\mathcal{E}) = c_1(\mathcal{O}_{\mathrm{G}(3,V)}(4))$, we have $K_{\mathscr{Z}_x} = (-9M_{\mathscr{Z}} + N_{\mathscr{Z}})|_{\mathscr{Z}_x}$, where $N_{\mathscr{Z}} := \rho_{\mathscr{Z}}^* \mathcal{O}_{\mathrm{G}(3,V)}(1)$. Thus $K_{\gamma_x} = N_{\mathscr{Z}}|_{\gamma_x}$. Using the Segre class again, we evaluate

$$\rho_{\mathscr{Z}*}(M_{\mathscr{Z}}^9 \cdot \mathscr{Z}_x) = s_1(\mathcal{E}|_{G_x}) = c_1(\mathcal{E}|_{G_x}) = 4N_{\mathscr{Z}}|_{G_x},$$

and obtain $\deg K_{\gamma_x} = N_{\mathscr{Z}} M_{\mathscr{Z}}^9 \cdot \mathscr{Z}_x = 4(N_{\mathscr{Z}})^2|_{G_x} = 4$. Therefore the arithmetic genus of γ_x is 3.

Now we consider the image C_x on Y of γ_x. Note that a point $([\Pi], [Q])$ of γ_x satisfying $l_x \subset \mathbb{P}(\Pi) \subset Q$ $([Q] \in P)$ is mapped to a point $([Q], q)$ in Y, where q represents a connected family of planes contained in Q. Then $\gamma_x \to C_x$ is injective since once we fix a quadric Q of rank 3 or 4 and a connected family q in Q, there exists at most one point $([\Pi], [Q])$ which satisfies $[\Pi] \in q$ and $l_x \subset \mathbb{P}(\Pi)$. In particular, the degree of C_x is 5 with respect to M.

Now we assume that X and Y are general. By Lemma 3.1.1, the geometric genus of $\overline{\gamma}_x$ is three for a general x. Since the arithmetic genus of γ_x is three, γ_x is the normalization of $\overline{\gamma}_x$. Since $\overline{\gamma}_x$ has only nodes as its singularities and $\gamma_x \to C_x$ is injective, we conclude $C_x \simeq \gamma_x$ and C_x is a smooth curve of genus 3 for general $x \in X$. q.e.d.

Assume that X and Y are general. For a general point x in X, as we have verified in Example 3.1.2, the plane quintic curve $\overline{\gamma}_x$ has only three nodes and does not intersect with the singular locus Sing H. Since $Y \to H$ is a double cover branched along $\operatorname{Sing} H$ and also the smooth curve C_x of degree 5 covers $\overline{\gamma}_x$, the inverse image $q^{-1}(\overline{\gamma}_x)$ is the union of C_x and another curve C'_x, which we called "shadow" curve of C_x in Introduction (see Fig.1). As shown in Fig.1, we note that the shadow curve is also a smooth curve of genus 3 and degree 5 for a general x and intersects at 6 points with C_x. These 6 points are the inverse images of the three nodal points on $\overline{\gamma}_x$.

3.2. The Brauer group of Y. As an interesting corollary to the existence of the curves γ_x on Z, we show that Y has a non-trivial Brauer group. Let $N_Z := N_{\mathscr{Z}}|_Z$ for $N_{\mathscr{Z}} = \rho_{\mathscr{Z}}^* \mathcal{O}_{\mathrm{G}(3,V)}(1)$.

Proposition 3.2.1. *The $\mathbb{P}^1$-fibration $Z \to Y$ is not associated to a locally free sheaf of rank two on Y. In particular, the Brauer group of Y contains a non-trivial 2-torsion element.*

Proof. Assume by contradiction that $Z = \mathbb{P}(\mathcal{A})$ for some locally free sheaf $\mathcal{A}$ of rank two on Y. Since fibers of $Z \to Y$ are of degree two with respect to N_Z, we can write $N_Z \equiv 2H_{\mathbb{P}(\mathcal{A})} + aM_Y$, where a is an integer (here we use $\rho(Y) = 1$). Since $N_Z \cdot \gamma_x = 4$ and $M_Y \cdot \gamma_x = 5$ by the proof of Proposition 3.1.3, we have $4 = 2H_{\mathbb{P}(\mathcal{A})} \cdot \gamma'_x + 5a$. Thus a is even and $\frac{1}{2}N_Z$ is numerically equivalent to the Cartier divisor $H_{\mathbb{P}(\mathcal{A})} + \frac{1}{2}aM_Y$. Note that

$$
\begin{aligned}
(N_Z)^4 = N_{\mathscr{Z}}^4 M_{\mathscr{Z}}^{10} &= s_2(\mathcal{E}) c_1(\mathcal{O}_{\mathrm{G}(3,V)}(1))^4 \\
&= (c_1(\mathcal{E})^2 - c_2(\mathcal{E})) c_1(\mathcal{O}_{\mathrm{G}(3,V)}(1))^4.
\end{aligned}
$$

By Lemma 2.3.1, we have $(N_Z)^4 = 40$. Then $(\frac{1}{2}N_Z)^4$ is not an integer, a contradiction. q.e.d.

We present a further discussion on the Brauer groups of X and Y in Subsection 9.2.

4. Birational geometry of $\mathscr{Y}$ and generically conic bundles

Let us consider $\mathscr{Y}_3 := \mathrm{G}(3, T(-1)^{\wedge 2})$, which is a $\mathrm{G}(3,6)$-bundle over $\mathbb{P}(V)$. The fiber of $\mathscr{Y}_3 \to \mathbb{P}(V)$ over a point $[V_1] \in \mathbb{P}(V)$ parameterizes planes in $\mathbb{P}(\wedge^2(V/V_1))$. $\mathscr{Y}_3$ appeared naturally in the construction of a resolution $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ and played important roles to describe a Lefschetz collection in $\mathcal{D}^b(\widetilde{\mathscr{Y}})$ [**HoTa3**].

4.1. Birational geometry of $\mathscr{Y}$. Here we briefly describe the construction of the resolution $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ which can be summarized in the diagram:

$$
\begin{tikzcd}
& \mathscr{Y}_2 \arrow[dl] \arrow[dr] & & \mathscr{Z} \arrow[r] \arrow[d] & \mathrm{G}(3,V) \\
\mathscr{Y}_3 \arrow[rr, dashed, "\text{(anti-)flip}"] \arrow[dr] & & \widetilde{\mathscr{Y}} \arrow[r, "\rho_{\widetilde{\mathscr{Y}}}"] \arrow[dl] & \mathscr{Y} \arrow[d, "\rho_{\mathscr{Y}}"] & \\
& \overline{\mathscr{Y}} & & \mathscr{H} &
\end{tikzcd}
\tag{4.1}
$$

Note that $\mathscr{Y}_3$ defined above is equivalently described by

$$\mathscr{Y}_3 = \left\{([U],[V_1]) \in \mathrm{G}(3,\wedge^3 V) \times \mathbb{P}(V) \mid U \wedge V_1 = 0\right\},$$

since $U \wedge V_1 = 0$ implies $[U] = [\bar{U} \wedge V_1]$ for some $[\bar{U}] \in \mathrm{G}(3,\wedge^2(V/V_1))$ and there is a bijective correspondence between $([U],[V_1])$ and $([\bar{U}],[V_1]) \in \mathrm{G}(3, T(-1)^{\wedge 2})$. With this definition of $\mathscr{Y}_3$, $\overline{\mathscr{Y}}$ is defined by the projection to the first factor with the reduced structure [**HoTa3**, Def. 5.3.1]. $\mathscr{Y}_2$ and $\widetilde{\mathscr{Y}}$ are described as blow-ups of $\mathscr{Y}_3$ and $\overline{\mathscr{Y}}$, respectively, and the fibers of the resolution $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ are described in detail [ibid, Sect.5.6].

We can see the birational correspondence between $\mathscr{Y}_3$ and $\mathscr{Y}$ as follows: As described in Subsection 2.3, the fiber of $\mathscr{Z} \to \mathscr{Y}$ over $y \in \mathscr{Y} \setminus G_{\mathscr{Y}}$ is a smooth conic in $\mathrm{G}(3,V)$ which parametrizes a family of planes contained in the corresponding quadric $\rho_{\mathscr{Y}}(y) = [Q_y] \in \mathscr{H}$. Observing this, we write each point $y \in \mathscr{Y} \setminus G_{\mathscr{Y}}$ by the pair $([Q_y], q_y)$ as in the proof of Proposition 3.1.3. Consider the Plücker embedding $\mathrm{G}(3,V) \hookrightarrow \mathbb{P}(\wedge^3 V)$. In $\mathbb{P}(\wedge^3 V)$, a conic q on $\mathrm{G}(3,V)$ is realized as a conic in the corresponding plane $\mathbb{P}^2_q$ in $\mathbb{P}(\wedge^3 V)$. If $\operatorname{rank} y = 4$ (i.e., $\operatorname{rank} Q_y = 4$), then $\mathbb{P}^2_{q_y}$ takes the form

$$[U_y] = [\bar{U}_y \wedge V_1]$$

with some $[\bar{U}_y] \in \mathrm{G}(3,\wedge^2(V/V_1))$, since the planes in Q_y parametrized by q_y contain the vertex $[V_1]$ of Q_y in common. In this case, the intersection $\mathbb{P}(U_y) \cap \mathrm{G}(3,V)$ in $\mathbb{P}(\wedge^3 V)$ recovers the conic q_y, since

$$\mathbb{P}(U_y) \cap \mathrm{G}(3,V) \text{ in } \mathbb{P}(\wedge^3 V) \simeq \mathbb{P}(\bar{U}_y) \cap \mathrm{G}(2,V/V_1) \text{ in } \mathbb{P}(\wedge^2(V/V_1)) \tag{4.2}$$

and $\mathbb{P}(\bar{U}_y) \not\subset \mathrm{G}(2,V/V_1)$ holds. Namely the intersection with the quadric $\mathrm{G}(2,V/V_1)$ determines a conic $\bar{q}_y$ in $\mathrm{G}(2,V/V_1)$ and also q_y in $\mathrm{G}(3,V)$, and in turn the quadric Q_y, hence $y \in \mathscr{Y}$ [**HoTa3**, Sect. 5.1, 5.2]. Therefore we see the bijective correspondence $([Q_y], q_y) \longleftrightarrow ([\bar{U}_y], V_1)$ when $\operatorname{rank} y = 4$. Since general points $([U],[V_1]) \in \mathscr{Y}_3$ determine conics by the intersection $\mathbb{P}(\bar{U}) \cap \mathrm{G}(2,V/V_1)$, the correspondence defines a birational map $\mathscr{Y}_3 \dashrightarrow \mathscr{Y}$.

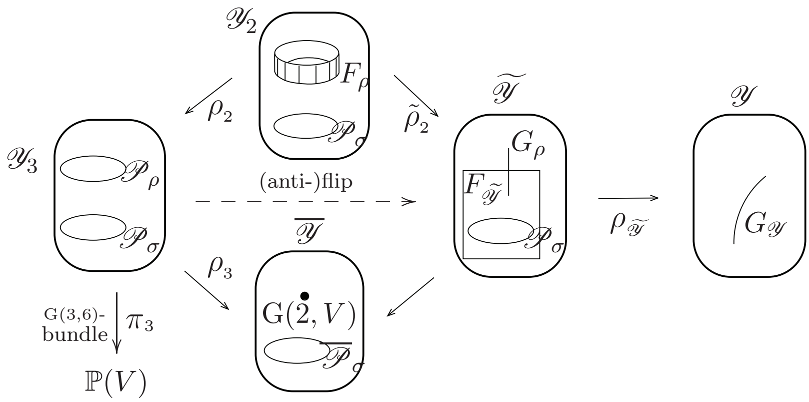

Fig.2. Birational geometries of $\mathscr{Y}$. The singular locus $\overline{\mathscr{P}}_\rho \simeq \mathrm{G}(2,V)$ of $\overline{\mathscr{Y}}$ and also ρ_3-exceptional set $\mathscr{P}_\rho = \mathrm{F}(1,2,V)$ in $\mathscr{Y}_3$ and the exceptional set G_ρ of $\widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}$ are depicted. F_ρ and $F_{\widetilde{\mathscr{Y}}}$ are exceptional divisors which are contracted to G_ρ and $G_{\mathscr{Y}}$, respectively. $G_{\mathscr{Y}}$ is the singular locus of $\mathscr{Y}$.

The above correspondence of the planes $\mathbb{P}(U) \subset \mathbb{P}(\wedge^3 V)$ to conics does not work obviously when $\mathbb{P}(U) \subset \mathrm{G}(3,V)$. There are two types of planes contained in $\mathrm{G}(3,V)$. The first one is

$$\mathrm{P}_{V_2} = \{[\Pi] \in \mathrm{G}(3,V) \mid V_2 \subset \Pi\} \simeq \mathbb{P}^2$$

with some V_2, and the second one is

$$\mathrm{P}_{V_1 V_4} = \{[\Pi] \in \mathrm{G}(3,V) \mid V_1 \subset \Pi \subset V_4\} \simeq \mathbb{P}^2$$

with some $V_1 \subset V_4$. They are called *ρ-plane* and *σ-plane*, respectively, and conics on $\mathrm{G}(3,V)$ contained in these planes are called *ρ-conic* and *σ-conic*. Other types of conics on $\mathrm{G}(3,V)$ are called *τ-conic*, and they are determined by the intersection $\mathbb{P}(U) \cap \mathrm{G}(3,V)$.

The above two types of planes in $\mathrm{G}(3,V)$ determine the corresponding planes in $\mathbb{P}(\wedge^3 V)$ under the Plücker embedding $\mathrm{G}(3,V) \hookrightarrow \mathbb{P}(\wedge^3 V)$ and determine the corresponding loci $\overline{\mathscr{P}}_\rho$ and $\overline{\mathscr{P}}_\sigma$ in $\overline{\mathscr{Y}}$, where

$$\overline{\mathscr{P}}_\rho \simeq \mathrm{G}(2,V), \ \ \overline{\mathscr{P}}_\sigma \simeq F(1,4,V).$$

It has been shown that the singular locus in $\overline{\mathscr{Y}}$ is exactly along $\overline{\mathscr{P}}_\rho$ with the singularity being given by the affine cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^5$ for each point. The resolution $\mathscr{Y}_3 \to \overline{\mathscr{Y}}$ is given by the blow-up along $\overline{\mathscr{P}}_\rho$ in one direction giving the exceptional set $\mathscr{P}_\rho = F(1,2,V)$. The blow-up along $\overline{\mathscr{P}}_\rho$ in the other direction gives the resolution $\widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}$, where the exceptional set G_ρ is described by $\mathbb{P}(\mathsf{S}^2\mathcal{G}^*)$ with the universal quotient bundle $\mathcal{G}$ of $\mathrm{G}(2,V)$ (cf. [**HoTa3**, (5.11)]). $\mathscr{Y}_2$ is obtained by further blow-up of $\mathscr{Y}_3$ (or $\widetilde{\mathscr{Y}}$) with the exceptional divisor F_ρ (see Fig. 2).

Planes contained in a quadric rank $Q = 3$ determine a (smooth)ρ-conic, since they contain the vertex V_2 of Q in common. Hence points $y = ([Q], q) \in \mathscr{Y}$ with rank Q=3 correspond to general points in the exceptional set $G_\rho = \mathbb{P}(\mathsf{S}^2\mathcal{G}^*)$. The resolution $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ contracts a prime divisor $F_{\widetilde{\mathscr{Y}}}$ to $G_{\mathscr{Y}}$ which corresponds to quadrics $[Q] \in \mathscr{H}$ with rank $Q \leq 2$. The divisor consists of τ- and ρ-conics in $\mathrm{G}(3, V)$ with rank ≤ 2 and also the σ-planes (see [**HoTa3**, Sect. 5.6] for the complete description). In particular, the fiber of $\rho_{\widetilde{\mathscr{Y}}}$ over a point $[Q] \in G_{\mathscr{Y}}$ with rank $Q = 2$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$, which generically parametrizes τ-conics of rank two, and ρ-conics of rank two appear on the diagonal.

To describe the birational geometry (4.1) of $\mathscr{Y}$, we will use the following conventions (note that $H_{\tilde{\mathscr{X}}} = f^*\mathcal{O}_{\mathscr{X}}(1), L_{\tilde{\mathscr{X}}} = g^*\mathcal{O}_{\mathrm{G}(2,V)}(1)$ introduced in Subsection 2.2 do not subject to these conventions since these are related to the birational geometry of $\mathscr{X}$) without mentioning at each time:

K_Σ: canonical divisor on a normal variety Σ .
L_Σ: pull back on a variety Σ of $\mathcal{O}(1)$ if there is a morphism $\Sigma \to \mathbb{P}(V)$.
M_Σ: pull back on a variety Σ of $\mathcal{O}_{\mathscr{H}}(1)$ if there is a morphism $\Sigma \to \mathscr{H}$.
N_Σ: pull back on a variety Σ of $\mathcal{O}_{\mathrm{G}(3,V)}(1)$ if there is a morphism $\Sigma \to \mathrm{G}(3, V)$.
$H_{\mathbb{P}(\mathcal{E})}$: tautological divisor of the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$, namely $H_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ with the property $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}^*$.

We denote the universal exact sequence on $\mathscr{Y}_3 = \mathrm{G}(3, T(-1)^{\wedge 2})$ by

$$0 \to \mathcal{S} \to \pi_3^*(T(-1)^{\wedge 2}) \to \mathcal{Q} \to 0, \tag{4.3}$$

where $\mathcal{S}$ is the relative universal subbundle of rank three and $\mathcal{Q}$ is the relative universal quotient bundle of rank three. We note that $\mathscr{P}_\rho = F(1, 2, V) \simeq \mathbb{P}(T(-1))$.

Proposition 4.1.1. *The following relations among divisors hold*:

(1) $\det \mathcal{Q} = \det \mathcal{S}^* + 3L_{\mathscr{Y}_3} = \det\{\mathcal{S}^*(L_{\mathscr{Y}_3})\}$,
(2) $K_{\mathscr{Y}_3} = -6 \det \mathcal{Q} + 4L_{\mathscr{Y}_3}$,
(3) $\mathcal{Q}|_{\mathscr{P}_\rho} \simeq \mathcal{S}^*(L_{\mathscr{Y}_3})|_{\mathscr{P}_\rho}$,
(4) $\det \mathcal{Q}|_{\mathscr{P}_\rho} = 2(H_{\mathscr{P}_\rho} + L_{\mathscr{P}_\rho})$,
(5) $M_{\mathscr{Y}_2} = \rho_2^*(\det \mathcal{Q}) - L_{\mathscr{Y}_2} - F_\rho$.

Proof. (1) is immediate from the exact sequence (4.3). (2) follows from $T_{\mathscr{Y}_3/\mathbb{P}(V)} = \mathcal{S}^* \otimes \mathcal{Q}$ and $K_{\mathscr{Y}_3} = -\det T_{\mathscr{Y}_3/\mathbb{P}(V)} + 5L_{\mathscr{Y}_3}$. (3) and (4) are obtained in [**HoTa3**, Props. 6.3.1, 6.3.2]. See [ibid, Prop. 6.4.1] for (5). q.e.d.

4.2. Generically conic bundle $\mathscr{X}_3 \to \mathscr{Y}_3$. Consider the universal subbundle $\mathcal{S}$ and define the projective bundle $\mathscr{X}_3^u := \mathbb{P}(\mathcal{S}) \subset \mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3$ over $\mathscr{Y}_3 = \mathrm{G}(3, T(-1)^{\wedge 2})$, where the superscript u stands for universal. Since the fiber of $\mathbb{P}(\mathcal{S})$ over $([\bar{U}], [V_1])$ is the plane $\mathbb{P}(\bar{U}) \subset \mathbb{P}(\wedge^2(V/V_1))$, the intersection with $\mathrm{G}(2, V/V_1)$ in (4.2) can be described by

$$\mathscr{X}_3^u \cap \left(\mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3\right),$$

using the inclusion $\mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3 \hookrightarrow \mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3$. We denote this restriction by $\mathscr{X}_3$, and note that it can be written explicitly by

$$\mathscr{X}_3 = \left\{\left([\bar{a} \wedge \bar{b}], ([\bar{U}], [V_1])\right) \mid [\bar{a} \wedge \bar{b}] \in \mathbb{P}(\bar{U}) \cap \mathrm{G}(2, V/V_1), ([\bar{U}], [V_1]) \in \mathscr{Y}_3\right\}.$$

By definition, $\mathscr{X}_3$ fits into the following diagram with the natural morphisms:

$$\begin{tikzcd}[column sep=small] & \mathscr{X}_3 \arrow[dl, "\pi_{G'}"'] \arrow[dr, "\pi_{3'}"] & \\ \mathrm{G}(2, T(-1)) \arrow[dr, "\pi_G"'] & & \mathscr{Y}_3 = \mathrm{G}(3, T(-1)^{\wedge 2}) \arrow[dl, "\pi_3"] \\ & \mathbb{P}(V). & \end{tikzcd} \tag{4.4}$$

Note that $\pi_{G'}$ maps the point $\left([\bar{a} \wedge \bar{b}], ([\bar{U}], [V_1])\right)$ to $([\bar{a} \wedge \bar{b}], [V_1]) \in \mathrm{G}(2, T(-1))$, which may be considered as a point $([a \wedge b \wedge v_1], [V_1])$ with $V_1 = \mathbb{C}v_1$ in $\mathrm{G}(3, V) \times \mathbb{P}(V)$. Hence we naturally have

$$\mathscr{X}_3 \subset \mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3 \subset \mathrm{G}(3, V) \times \mathscr{Y}_3,$$

where we use $\mathrm{G}(2, T(-1)) \subset \mathrm{G}(3, V) \times \mathbb{P}(V)$ (actually, $\mathrm{G}(2, T(-1)) \simeq F(1, 3, V)$ is the universal family of planes in $\mathbb{P}(V)$ parameterized by $\mathrm{G}(3, V)$). As explained in (4.2), the intersection $\mathbb{P}(\bar{U}) \cap \mathrm{G}(2, V/V_1)$ generically defines a conic on $\mathrm{G}(2, V/V_1)$ or $\mathrm{G}(3, V)$. Therefore we have;

Proposition 4.2.1. $\pi_{3'} \colon \mathscr{X}_3 \to \mathscr{Y}_3$ *is a generically conic bundle. More precisely, fibers over* $\mathscr{Y}_3 \setminus (\mathscr{P}_\rho \sqcup \mathscr{P}_\sigma)$ *are conics in* $\mathrm{G}(3, V)$.

Below are some properties of $\mathscr{X}_3$ which will be used in later sections.

Proposition 4.2.2. $\pi_{G'} \colon \mathscr{X}_3 \to \mathrm{G}(2, T(-1))$ *is a* $\mathrm{G}(2, 5)$*-bundle. In particular,* $\mathscr{X}_3$ *is smooth.*

Proof. By definition, the fiber over $([\bar{a} \wedge \bar{b}], [V_1]) \in \mathrm{G}(2, T(-1))$ consists of $\bar{U} \simeq \mathbb{C}^3$ (or points $([\bar{a} \wedge \bar{b}], ([\bar{U}], [V_1])) \in \mathscr{X}_3$) satisfying

$$\mathbb{C}\bar{a} \wedge \bar{b} \subset \bar{U} \subset \wedge^2(V/V_1).$$

The claim is immediate if we write the above condition as $\mathbb{C} \subset \bar{U} \subset \mathbb{C}^6$. q.e.d.

We define

$$\mathscr{Z}_\rho := \pi_{3'}^{-1}(\mathscr{P}_\rho) = \mathbb{P}(\mathcal{S}|_{\mathscr{P}_\rho}) \text{ and } \mathscr{Z}_\sigma := \pi_{3'}^{-1}(\mathscr{P}_\sigma) = \mathbb{P}(\mathcal{S}|_{\mathscr{P}_\sigma}).$$

Then the fibers of $\mathscr{Z}_\rho \to \mathscr{P}_\rho$ and $\mathscr{Z}_\sigma \to \mathscr{P}_\sigma$ are the family of planes parameterized by $\mathscr{P}_\rho$ and $\mathscr{P}_\sigma$ respectively. The diagram (4.4) naturally restricts to $\mathscr{P}_\rho$ and $\mathscr{P}_\sigma$, respectively.

Proposition 4.2.3. $\pi_{G'}|_{\mathscr{Z}_\rho} \colon \mathscr{Z}_\rho \to \mathrm{G}(2, T(-1))$ *is a* $\mathbb{P}^1$*-bundle.*

Proof. Consider a point $([\bar{a} \wedge \bar{b}], [V_1]) \in \mathrm{G}(2, T(-1))$. We set $V_3 = \langle a, b, v_1 \rangle$ with $V_1 = \langle v_1 \rangle$ and $\bar{a} = a \bmod V_1$, $\bar{b} = b \bmod V_1$. By definition, the fiber of $\pi_{G'}|_{\mathscr{Z}_\rho}$ over the point consists of $[\bar{U}] \subset \mathrm{G}(3, \wedge^2(V/V_1))$ satisfying

$$\mathbb{C}\bar{a} \wedge \bar{b} \subset \bar{U} \subset \wedge^2(V/V_1) \text{ and } [U] = [\bar{U} \wedge V_1] \text{ is a } \rho\text{-plane.}$$

We note that the former condition is rephrased by $\wedge^3 V_3 \subset U$, and the latter by $[U] = \mathrm{P}_{V_2} = [(V/V_2) \wedge (\wedge^2 V_2)]$ for some V_2 with $V_1 \subset V_2$. From these, we see that the fiber is described by $\{[\bar{U}] = [(V/V_2) \wedge (V_2/V_1)] \mid V_1 \subset V_2 \subset V_3\} \simeq \mathbb{P}(V_3/V_1)$. q.e.d.

Finally, we observe a fact about the relative Euler sequence for the projective bundle $\pi_\rho \colon \mathscr{Z}_\rho = \mathbb{P}(\mathcal{S}|_{\mathscr{P}_\rho}) \to \mathscr{P}_\rho$;

$$0 \to \mathcal{O}_{\mathscr{Z}_\rho}(-1) \to \pi_\rho^*(\mathcal{S}|_{\mathscr{P}_\rho}) \to \mathcal{R}_\rho \to 0, \tag{4.5}$$

where we set $\mathcal{R}_\rho := T_{\mathscr{Z}_\rho/\mathscr{P}_\rho} \otimes \mathcal{O}_{\mathscr{Z}_\rho}(-1)$.

Lemma 4.2.4. *Let $\mathcal{W}_\rho$ be the pull-back of the universal quotient bundle $\mathcal{W}$ on $\mathrm{G}(3, V)$ by the composition $\rho_G \circ \pi_{G'}|_{\mathscr{Z}_\rho} : \mathscr{Z}_\rho \to \mathrm{G}(2, T(-1)) \to \mathrm{G}(3, V)$, where ρ_G is the natural map $\rho_G : \mathrm{G}(2, T(-1)) = F(1, 3, V) \to \mathrm{G}(3, V)$. Then*

$$\mathcal{W}_\rho \simeq \mathcal{R}_\rho \otimes \pi_\rho{}^* \mathcal{O}_{\mathbb{P}(T(-1))}(1),$$

where $\mathcal{O}_{\mathbb{P}(T(-1))}(1)$ is the tautological sheaf on $\mathbb{P}(T(-1)) = F(1, 2, V) = \mathscr{P}_\rho$.

Proof. Let $([\bar{U}], [V_1])$ with $[\bar{U}] = [(V/V_2) \wedge (V_2/V_1)]$ be a point of $\mathscr{P}_\rho$. Then a point $z_\rho = ([\bar{U}_1], ([\bar{U}], [V_1]))$ in the fiber $\pi_\rho^{-1}(\mathscr{P}_\rho) \subset \mathscr{Z}_\rho$ may be written by $\bar{U}_1 = (V_3/V_2) \wedge (V_2/V_1)$ with $V_2 \subset V_3$. Since the composition $\rho_G \circ \pi_{G'}|_{\mathscr{Z}_\rho}$ sends z_ρ to $\wedge^3 V_3$, we have $\mathcal{W}_\rho|_{z_\rho} = V/V_3$. On the other hand, the Euler sequence (4.5) restricts at z_ρ to $0 \to \bar{U}_1 \to \bar{U} \to \bar{U}/\bar{U}_1 \to 0$. Hence we have

$$\mathcal{R}_\rho \otimes \pi_\rho^* \mathcal{O}_{\mathbb{P}(T(-1))}(1)|_{z_\rho} = \bar{U}/\bar{U}_1 \otimes (V_2/V_1)^* \simeq V/V_3,$$

which shows the claim. q.e.d.

4.3. Generically conic bundle $\pi_{\mathscr{Z}_2} : \mathscr{Z}_2 \to \mathscr{Y}_2$. Let us recall that $\mathscr{Z}_3^u = \mathbb{P}(\mathcal{S})$ is the projective bundle over $\mathscr{Y}_3$ and denote the natural projection by $\pi_{3^u} : \mathscr{Z}_3^u \to \mathscr{Y}_3$. The generically conic bundle $\pi_{3'} : \mathscr{Z}_3 \to \mathscr{Y}_3$ is the fiberwise intersection of $\mathscr{Z}_3^u$ with the Grassmann bundle $\mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3$. $\mathscr{Z}_\rho$ is the inverse image $\pi_{3'}^{-1}(\mathscr{P}_\rho)$ of $\mathscr{P}_\rho \simeq F(1,2,V)$. $\mathscr{Y}_2$ is the blow-up of $\mathscr{Y}_3$ along $\mathscr{P}_\rho$ with the exceptional divisor F_ρ. We will consider the blow-up $\rho_{2'} : \mathscr{Z}_2 \to \mathscr{Z}_3$ of $\mathscr{Z}_3$ along $\mathscr{Z}_\rho$ and denote by E_ρ its exceptional divisor. Then there is a projection $\pi_{2'} : \mathscr{Z}_2 \to \mathscr{Y}_2$ with the following commutative diagram:

$$\begin{tikzcd}
(\mathscr{Z}_\rho \subset \mathscr{Z}_3) \arrow[d, "\pi_{3'}"] & (E_\rho \subset \mathscr{Z}_2) \arrow[l, "\rho_{2'}"'] \arrow[d, "\pi_{2'}"] \\
(\mathscr{P}_\rho \subset \mathscr{Y}_3) & (F_\rho \subset \mathscr{Y}_2) \arrow[l, "\rho_2"']
\end{tikzcd}$$

By construction, the fibers of $\pi_{2'}$ over $\mathscr{Y}_2 \setminus (F_\rho \cup \rho_2^{-1}(\mathscr{P}_\sigma))$ are τ-conics in the fibers of the Grassmann bundle $\mathrm{G}(2, T(-1))$. We will show that the fibers of $\pi_{2'}$ over F_ρ are ρ-conics.

We first describe the exceptional set F_ρ (see also [**HoTa3**, Sect. 5.4]).

Lemma 4.3.1. $\mathrm{G}(2, T(-1)) \in |2H_{\mathbb{P}(T(-1)^{\wedge 2})} + L_{\mathbb{P}(T(-1)^{\wedge 2})}|$.

Proof. Note that $\mathrm{G}(2, T(-1))$ is a divisor in $\mathbb{P}(T(-1)^{\wedge 2})$. Fix a point $[V_1] \in \mathbb{P}(V)$. The defining equation of $\mathrm{G}(2, V/V_1)$ in $\mathbb{P}(\wedge^2(V/V_1))$ is the Plücker quadric, which defines a symmetric bilinear form $\wedge^2(V/V_1) \times \wedge^2(V/V_1) \to \wedge^4(V/V_1)$. Since this globalizes to $\wedge^2 T(-1) \times \wedge^2 T(-1) \to \wedge^4 T(-1) \simeq \mathcal{O}(1)$, the defining equation of $\mathrm{G}(2, T(-1))$ in $\mathbb{P}(T(-1)^{\wedge 2})$ is an element of $H^0(\mathbb{P}(V), \mathsf{S}^2(\Omega(1)^{\wedge 2}) \otimes \mathcal{O}(1))$. This proves the claim. q.e.d.

Lemma 4.3.2. *The normal bundles of $\mathscr{P}_\rho$ in $\mathscr{Y}_3$ and $\mathscr{P}_\sigma$ in $\mathscr{Y}_3$, respectively, are given by*

$$\mathcal{N}_{\mathscr{P}_\rho/\mathscr{Y}_3} = \mathsf{S}^2\mathcal{S}^* \otimes L_{\mathscr{Y}_3}|_{\mathscr{P}_\rho} \text{ and } \mathcal{N}_{\mathscr{P}_\sigma/\mathscr{Y}_3} = \mathsf{S}^2\mathcal{S}^* \otimes L_{\mathscr{Y}_3}|_{\mathscr{P}_\sigma}.$$

Proof. Recall that both $\mathscr{P}_\rho$ and $\mathscr{P}_\sigma$ consist of points $([\bar{U}], [V_1]) \in \mathscr{Y}_3$ satisfying $\mathbb{P}(\bar{U}) \cap \mathrm{G}(2, V/V_1) = \mathbb{P}(\bar{U})$ in $\mathbb{P}(\wedge^2(V/V_1))$. As described in the above Lemma 4.3.1, the defining equation of $\mathrm{G}(2, T(-1)) \subset \mathbb{P}(T(-1)^{\wedge 2})$ is given by a section of $\mathsf{S}^2(T(-1)^{\wedge 2})^* \otimes \mathcal{O}(1)$ over $\mathbb{P}(V)$. Pulling this back by $\pi_3 : \mathscr{Y}_3 \to \mathbb{P}(V)$ and composing with the surjection $\pi_3^*(T(-1)^{\wedge 2})^* \to \mathcal{S}^*$, we obtain a section of $\mathsf{S}^2\mathcal{S}^* \otimes L_{\mathscr{Y}_3}$ over $\mathscr{Y}_3$. $\mathscr{P}_\rho \sqcup \mathscr{P}_\sigma$ is exactly the scheme of the zeros of this section (which is isomorphic to the orthogonal Grassmann bundle $\mathrm{OG}(3, T(-1)^{\wedge 2})$). The claimed forms of the normal bundles follow from this. q.e.d.

Proposition 4.3.3. $F_\rho = \mathbb{P}(\mathsf{S}^2\mathcal{S}^* \otimes L_{\mathscr{Y}_3}|_{\mathscr{P}_\rho})$. *The fibers of $F_\rho \to \mathscr{P}_\rho$ are conics in the ρ-planes parametrized by $\mathscr{P}_\rho$.*

Proof. The first claim follows from $F_\rho = \mathbb{P}(\mathcal{N}_{\mathscr{P}_\rho/\mathscr{Y}_3})$ with the above Lemma 4.3.2. For the second claim, let us recall that $\mathscr{P}_\rho \subset \mathscr{Y}_3$ consists of points $([\bar{U}],[V_1])$ with $[\bar{U}] = [(V/V_2)\wedge(V_2/V_1)]$ parametrized by $[V_1 \subset V_2] \in F(1,2,V)$. Then the fiber of $F_\rho \to \mathscr{P}_\rho$ over a point $([\bar{U}],[V_1])$ is given by

$$\mathbb{P}(\mathsf{S}^2\mathcal{S}^* \otimes L_{\mathscr{Y}_3}|_{([\bar{U}],[V_1])}) = \mathbb{P}(\mathsf{S}^2\bar{U}^* \otimes V_1^*) \simeq \mathbb{P}(\mathsf{S}^2\bar{U}^*),$$

which we can identify with conics on the ρ-plane $\mathbb{P}(\bar{U})$ as claimed. q.e.d.

Under the composition map $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}} \to \mathscr{Y}$, general points on F_ρ are mapped to points $([Q],q) \in \mathscr{Y}$ with $\operatorname{rank} Q = 3$ and the corresponding (smooth) ρ-conic q in $\mathrm{G}(3,V)$, i.e., the $\mathbb{P}^1$-family of planes contained in the quadric Q. More precisely, the image of F_ρ in $\widetilde{\mathscr{Y}}$ has a bijective correspondence to the set of pairs $([Q],q)$ of a quadric Q ($\operatorname{rank} Q \le 3$) and a ρ-conic ($\operatorname{rank} q \le 3$) which parametrizes planes contained in the quadric Q. See [**HoTa3**, Sect. 5.5, 5.6] for more complete descriptions.

Proposition 4.3.4. *$E_\rho \to F_\rho$ is the universal family of conics parametrized by F_ρ. Hence $\mathscr{X}_2 \to \mathscr{Y}_2$ is a generically conic bundle with fibers over $\mathscr{Y}_2 \setminus \mathscr{P}_\sigma$ being conics on* $\mathrm{G}(3,V)$ *and the fibers over $\mathscr{P}_\sigma$ being σ-planes on* $\mathrm{G}(3,V)$. (*For notational simplicity, we write the transform of $\mathscr{P}_\sigma \subset \mathscr{Y}_3$ on $\mathscr{Y}_2$ by the same $\mathscr{P}_\sigma$.*)

Proof. We describe the blow-up $\rho_{2'} : \mathscr{X}_2 \to \mathscr{X}_3$ along $\mathscr{X}_\rho \subset \mathscr{X}_3$. For this, we consider the subvariety $\mathscr{X}_\rho = \pi_{3'}^{-1}(\mathscr{P}_\rho) = \mathbb{P}(\mathcal{S}|_{\mathscr{P}_\rho})$ in $\mathscr{X}_3^u = \mathbb{P}(\mathcal{S})$ with the following normal bundle sequence:

$$0 \to \mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3} \to \mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3^u} \to \mathcal{N}_{\mathscr{X}_3/\mathscr{X}_3^u}|_{\mathscr{X}_\rho} \to 0.$$

We note that since $\pi_{3^u} : \mathscr{X}_3^u \to \mathscr{Y}_3$ is a projective bundle and hence is flat, we have

$$\mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3^u} = (\pi_{3^u}|_{\mathscr{X}_\rho})^*\mathcal{N}_{\mathscr{P}_\rho/\mathscr{Y}_3} = (\pi_{3^u}|_{\mathscr{X}_\rho})^*\mathsf{S}^2\mathcal{S}^*(L_{\mathscr{Y}_3})|_{\mathscr{P}_\rho},$$

where we use Lemma 4.3.2 for the normal bundle $\mathcal{N}_{\mathscr{P}_\rho/\mathscr{Y}_3}$. From Lemma 4.3.1 and the definition $\mathscr{X}_3 = \mathrm{G}(2,T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3 \cap \mathscr{X}_3^u$, we have

(4.6)
$$0 \to \mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3} \to (\pi_{3^u}|_{\mathscr{X}_\rho})^*\mathsf{S}^2\mathcal{S}^*(L_{\mathscr{Y}_3})|_{\mathscr{P}_\rho} \to \mathcal{O}_{\mathscr{X}_3^u}(2H_{\mathscr{X}_3^u} + L_{\mathscr{X}_3^u})|_{\mathscr{X}_\rho} \to 0,$$

where $H_{\mathscr{X}_3^u} = \pi_{G'}^*\mathcal{O}_{\mathrm{G}(2,T(-1))}(1)$ and $L_{\mathscr{X}_3^u} = (\pi_3 \circ \pi_{3^u})^*\mathcal{O}(1)$. The exceptional set (divisor) E_ρ of the blow-up is given by $\mathbb{P}(\mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3})$. Now, take a point $([\bar{U}],[V_1]) \in \mathscr{P}_\rho$ with $[\bar{U}] = [(V/V_2)\wedge(V_2/V_1)]$ and $[V_1 \subset V_2] \in F(1,2,V)$. Then the fiber of $\mathscr{X}_\rho \to \mathscr{P}_\rho$ over the point $([\bar{U}],[V_1])$ is the plane $\mathbb{P}(\bar{U})$ in $\mathrm{G}(2,V/V_1) \subset \mathbb{P}(\wedge^2(V/V_1))$. Restricting the sequence (4.6) to $\mathbb{P}(\bar{U})$, we obtain

$$0 \to \mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3}|_{\mathbb{P}(\bar{U})} \to \mathsf{S}^2\bar{U}^* \otimes V_1^* \otimes \mathcal{O}_{\mathbb{P}(\bar{U})} \to \mathcal{O}_{\mathbb{P}(\bar{U})}(2) \otimes V_1^* \to 0.$$

If we further restrict this to a point $x \in \mathbb{P}(\bar{U})$, we see that the stalk $(\mathcal{N}_{\mathscr{X}_\rho/\mathscr{X}_3})_x$ at x is given by the kernel of the map $\mathsf{S}^2\bar{U}^* \to \mathcal{O}_{\mathbb{P}(\bar{U})}(2)_x$, which we identify with the conics in $\mathbb{P}(\bar{U})$ passing through x. Now we recall that the fiber of $F_\rho \to \mathscr{P}_\rho$ over $([\bar{U}], [V_1])$ is given by $\mathbb{P}(\mathsf{S}^2\bar{U})$ which parametrizes the conics in $\mathbb{P}(\bar{U})$. Therefore, we see that $E_\rho = \mathbb{P}(\mathcal{N}_{\mathscr{Z}_\rho/\mathscr{Z}_3}|_{\mathbb{P}(\bar{U})})$ describes the conics in $\mathbb{P}(\bar{U})$ which correspond to each point of F_ρ, i.e., the universal family of conics in the ρ-planes as claimed.

The rest of the claims is clear since $\mathscr{Z}_3 \to \mathscr{Y}_3$ is a generically conic bundle over $\mathscr{Y}_3 \setminus \mathscr{P}_\rho \sqcup \mathscr{P}_\sigma$. q.e.d.

Definition 4.3.5. For later use in Subsection 5.6.1, we introduce

(i) $\mathrm{G}(2,4)$-bundle $B(2,4,\mathscr{Y}_2) := \mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_2$ over $\mathscr{Y}_2$,

(ii) $\mathbb{P}^2$-bundle $\mathscr{Z}_2^u := \mathscr{Z}_3^u \times_{\mathscr{Y}_3} \mathscr{Y}_2 = \mathbb{P}(\rho_2^*\mathcal{S})$ over $\mathscr{Y}_2$,

(iii) $\mathscr{Z}_2^t := B(2,4,\mathscr{Y}_2) \cap \mathscr{Z}_2^u$, the intersection of (i) and (ii) over $\mathscr{Y}_2$ and

(iv) natural morphisms $\pi_{B'} : B(2,4,\mathscr{Y}_2) \to \mathscr{Y}_2, \pi_{2^t} : \mathscr{Z}_2^t \to \mathscr{Y}_2$ and also

$$\mu_B : B(2,4,\mathscr{Y}_2) \to \mathrm{G}(3,V), \ \ \mu_{2^t} : \mathscr{Z}_2^t \to \mathrm{G}(3,V).$$

Since $\pi_{3^u} : \mathscr{Z}_3^u = \mathbb{P}(\mathcal{S}) \to \mathscr{Y}_3$ is a flat fibration, the morphism $\mathscr{Z}_3^u \times_{\mathscr{Y}_3} \mathscr{Y}_2 \to \mathscr{Z}_3^u$ is the blow-up along $\pi_{3^u}^{-1}(\mathscr{P}_\rho)$ with its exceptional divisor $\mathscr{Z}_3^u \times_{\mathscr{Y}_3} F_\rho = \mathbb{P}(\rho_2^*\mathcal{S}|_{F_\rho})$. The definition $\mathscr{Z}_2^t$ corresponds to the intersection $\mathscr{Z}_3 = (\mathrm{G}(2, T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3) \cap \mathscr{Z}_3^u$. We note that $\mathscr{Z}_2^t$ is the total transform of $\mathscr{Z}_3 \subset \mathscr{Z}_3^u$ under the blow-up $\mathscr{Z}_2^u \to \mathscr{Z}_3^u$ since it contains the exceptional divisor. The superscript t is used to remind this. $\mathscr{Z}_2^t$ is reduced since $\mathscr{Z}_3$ is smooth by Proposition 4.2.2. $\mathscr{Z}_2^t$ will play important roles in our proof of Proposition 5.6.4 (see Lemma 5.6.2).

We now summarize the generically conic bundles in the following diagam (and introduce morphisms between them):

$$\begin{tikzcd}
 & & \mathscr{Z}_3 \arrow[dl, "\pi_{G'}"'] \arrow[d, "\pi_{3'}"] & \mathscr{Z}_2 \arrow[l, "\rho_{2'}"'] \arrow[d, "\pi_{2'}"'] & & \mathscr{Z}_2^o \arrow[ll, hook', "\iota_{2'}"'] \arrow[d, "\pi_{2'}^o"] & \\
 & \mathrm{G}(2,T(-1)) \arrow[dl, "\rho_G"'] \arrow[dr, "\pi_G"] & \mathscr{Y}_3 \arrow[d, "\pi_3"] & \mathscr{Y}_2 \arrow[l, "\rho_2"'] \arrow[d, "\pi_2"'] \arrow[dr, "\tilde{\rho}_2"] & & \mathscr{Y}_2^o \arrow[ll, hook', "\iota_2"'] \arrow[dr, "\tilde{\rho}_2^o"] & \\
G(3,V) & & \mathbb{P}(V) \arrow[r, equal] & \mathbb{P}(V) & \widetilde{\mathscr{Y}} & & \widetilde{\mathscr{Y}}^o \arrow[ll, hook', "\tilde{\iota}"']
\end{tikzcd} \tag{4.7}$$

Here we set $\mathscr{Y}_2^o = \mathscr{Y}_2 \setminus \mathscr{P}_\sigma$, $\widetilde{\mathscr{Y}}^o = \widetilde{\mathscr{Y}} \setminus \mathscr{P}_\sigma$ and $\mathscr{Z}_2^o = \mathscr{Z}_2 \setminus \pi_{2'}^{-1}(\mathscr{P}_\sigma)$ and define $\iota_2, \tilde{\iota}$ and $\iota_{2'}$ to be the respective inclusions as in the above diagram. This diagram will be used extensively in our construction of the ideal sheaf $\mathcal{I}$ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ in the next section. In the rest of this section, we prepare some prelininary results which will be used in later sections.

4.4. The Grassmann bundle $\mathrm{G}(2,T(-1))\to\mathbb{P}(V)$**.** Fix a point $[V_1]\in\mathbb{P}(V)$. Then the Euler sequence $0\to\mathcal{O}(-1)\to V\otimes\mathcal{O}\to T(-1)\to 0$ over $[V_1]$ is represented by $0\to V_1\to V\to V/V_1\to 0$. We denote the projection by $\pi_{V_1}:V\to V/V_1$. We consider the dual vector space V^* to V and identify the dual space $(V/V_1)^*$ as $\{\varphi\in V^*\mid\varphi|_{V_1}=0\}$. Then it is easy to deduce the following isomorphisms:

$$\begin{tikzcd} {\scriptstyle([\overline{V}_2],[V_1])\in}\,\mathrm{G}(2,T(-1)) \arrow[r, "\simeq"] \arrow[d, "\rho_G"] & \mathrm{G}(2,\Omega(1))\,{\scriptstyle\ni([(\overline{V}_2)^{\perp}],[V_1])} \arrow[d, "\rho"] \\ {\scriptstyle[\pi_{V_1}^{-1}(\overline{V}_2)]\in}\,\mathrm{G}(3,V) \arrow[r, "\simeq"] & \mathrm{G}(2,V^*)\,{\scriptstyle\ni[(\pi_{V_1}^{-1}(\overline{V}_2))^{\perp}],} \end{tikzcd} \tag{4.8}$$

where $\overline{V}_2$ is a two dimensional subspace in V/V_1.

Note that Grassmann bundles $\mathrm{G}(2,T(-1))$ and $\mathrm{G}(2,\Omega(1))$, respectively, are embedded into the projective bundles $\mathbb{P}(T(-1)^{\wedge 2})$ and $\mathbb{P}(\Omega(1)^{\wedge 2})$, which have the following relations:

Lemma 4.4.1. *There is a natural isomorphism* $\mathbb{P}(T(-1)^{\wedge 2})\simeq\mathbb{P}(\Omega(1)^{\wedge 2})$, *and under this isomorphism we have the following identifications*:

$$H_{\mathbb{P}(\Omega(1)^{\wedge 2})}=H_{\mathbb{P}(T(-1)^{\wedge 2})}+L_{\mathbb{P}(T(-1)^{\wedge 2})}\ \textit{and}\ L_{\mathbb{P}(\Omega(1)^{\wedge 2})}=L_{\mathbb{P}(T(-1)^{\wedge 2})}.$$

Proof. From the natural isomorphism $T(-1)^{\wedge 2}\simeq\Omega(1)^{\wedge 2}\otimes\wedge^4 T(-1)\simeq\Omega(1)^{\wedge 2}\otimes\mathcal{O}(1)$ and our definitions of $H_{\mathbb{P}(\mathcal{E})}$ and L_{Σ} introduced above (4.3), the claimed identifications follow. q.e.d.

We note that $H_{\mathbb{P}(\Omega(1)^{\wedge 2})}|_{\mathrm{G}(2,\Omega(1))}=\rho^*\mathcal{O}_{\mathrm{G}(2,V^*)}(1)$. Since we identify $\mathcal{O}_{\mathrm{G}(2,V^*)}(1)=\mathcal{O}_{\mathrm{G}(3,V)}(1)$ in (4.8), we have $H_{\mathbb{P}(\Omega(1)^{\wedge 2})}|_{\mathrm{G}(2,T(-1))}=\rho_G^*\mathcal{O}_{\mathrm{G}(3,V)}(1)$. Hence, using the above lemma and setting $N_{\mathrm{G}(2,T(-1))}:=\rho_G^*\mathcal{O}_{\mathrm{G}(3,V)}(1)$, we have

$$N_{\mathrm{G}(2,T(-1))}=\left(H_{\mathbb{P}(T(-1)^{\wedge 2})}+L_{\mathbb{P}(T(-1)^{\wedge 2})}\right)\big|_{\mathrm{G}(2,T(-1))}. \tag{4.9}$$

4.5. Canonical divisors $K_{\mathscr{Z}_i/\mathscr{Y}_i}$**.** In this subsection, we calculate $N_{\mathscr{Z}_3}=(\rho_G\circ\pi_{G'})^*\mathcal{O}_{G(3,V)}(1)$ and the relative canonical divisors $K_{\mathscr{Z}_i/\mathscr{Y}_i}:=K_{\mathscr{Z}_i}-\pi_{i'}^*K_{\mathscr{Y}_i}$ $(i=2,3)$ as follows:

Proposition 4.5.1. (1) $N_{\mathscr{Z}_3}=H_{\mathbb{P}(\mathcal{S})}|_{\mathscr{Z}_3}+L_{\mathscr{Z}_3}$,
(2) $K_{\mathscr{Z}_3/\mathscr{Y}_3}=\pi_{3'}^*(\det\mathcal{Q}-L_{\mathscr{Y}_3})-N_{\mathscr{Z}_3}$,
(3) $K_{\mathscr{Z}_2/\mathscr{Y}_2}=M_{\mathscr{Z}_2}-N_{\mathscr{Z}_2}$.

Proof. (1) follows from (4.9) and Lemma 4.4.1 since $H_{\mathbb{P}(\mathcal{S})}|_{\mathscr{Z}_3}$ is given by the restriction $H_{\mathbb{P}(T(-1)^{\wedge 2})}|_{\mathscr{Z}_3}$ (see (4.3)). For (2), recall $\mathscr{Z}_3^u=\mathbb{P}(\mathcal{S})$ and the projection $\pi_{3^u}:\mathscr{Z}_3^u\to\mathscr{Y}_3$. Then we have $K_{\mathscr{Z}_3^u/\mathscr{Y}_3}=-3H_{\mathbb{P}(\mathcal{S})}+\pi_{3^u}^*\det\mathcal{S}^*$ from the Euler sequence for $\mathscr{Z}_3^u=\mathbb{P}(\mathcal{S})$

$$0\to\mathcal{O}_{\mathscr{Z}_3^u}(-1)\to\pi_{3^u}^*\mathcal{S}\to T_{\mathscr{Z}_3^u/\mathscr{Y}_3}(-1)\to 0.$$

Note the inclusion $i : \mathrm{G}(2, T(-1)) \hookrightarrow \mathbb{P}(T(-1)^{\wedge 2})$ and the definition $\mathscr{Z}_3 = \mathscr{Z}_3^u \cap i(\mathrm{G}(2, T(-1)))$. Then, by Proposition 4.3.1, we have $\mathscr{Z}_3 \in |2H_{\mathbb{P}(\mathcal{S})} + L_{\mathscr{Z}_3^u}|$. Hence, by the adjunction formula and Proposition 4.1.1 (2), we have

$$K_{\mathscr{Z}_3/\mathscr{Y}_3} = \{K_{\mathscr{Z}_3^u/\mathscr{Y}_3} + (2H_{\mathbb{P}(\mathcal{S})} + L_{\mathscr{Z}_3^u})\}\big|_{\mathscr{Z}_3} = -H_{\mathbb{P}(\mathcal{S})}|_{\mathscr{Z}_3} + L_{\mathscr{Z}_3} + \pi_{3'}^* \det \mathcal{S}^*.$$

Now by (1) and Proposition 4.1.1 (1), we obtain (2). For (3), recall that $E_\rho = \pi_{2'}^{-1}(F_\rho)$ is the exceptional divisor of $\rho_{2'} \colon \mathscr{Z}_2 \to \mathscr{Z}_3$. Since the codimension of $\mathscr{P}_\rho$ in $\mathscr{Y}_3$ is three and that of $\pi_{3'}^{-1}(\mathscr{P}_\rho)$ in $\mathscr{Z}_3$ is two, we have

$$K_{\mathscr{Z}_2/\mathscr{Y}_2} = \rho_{2'}^* K_{\mathscr{Z}_3/\mathscr{Y}_3} - E_\rho.$$

Using (2), we obtain $K_{\mathscr{Z}_2/\mathscr{Y}_2} = \pi_{2'}^*(\rho_2^* \det \mathcal{Q} - L_{\mathscr{Y}_2} - F_\rho) - N_{\mathscr{Z}_2}$. Now, by Proposition 4.1.1 (5), the claimed relation follows. q.e.d.

4.6. The subvariety $\mathscr{Z}_3^u$ in $\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3$. The univeral family of planes $\mathscr{Z}_3^u$ is contained in $\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3$, which is a $\mathbb{P}^5$-bundle over $\mathscr{Y}_3$. In later sections, we will be required to describe the ideal sheaf of $\mathscr{Z}_3^u$ in $\mathbb{P}(T(-1)^2) \times_{\mathbb{P}(V)} \mathscr{Y}_3$ as a projective subbundle. Here we recall the following standard lemma:

Lemma 4.6.1. *Let X be a variety, and $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ a short exact sequence of locally free $\mathcal{O}_X$-modules. Associated to the surjection $\mathcal{B} \to \mathcal{C}$, we may regard $\mathbb{P}(\mathcal{C}^*)$ as a subbundle of $\mathbb{P}(\mathcal{B}^*)$. As such, the subvariety $\mathbb{P}(\mathcal{C}^*)$ is the complete intersection with respect to a section of $\pi^*\mathcal{A}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)$, where π is the natural projection $\mathbb{P}(\mathcal{B}^*) \to X$.*

Proof. Let $\mathcal{I}$ be the ideal sheaf of $\mathbb{P}(\mathcal{C}^*)$ in $\mathbb{P}(\mathcal{B}^*)$. We have a natural exact sequence $0 \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1) \to \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1) \to \mathcal{O}_{\mathbb{P}(\mathcal{C}^*)}(1) \to 0$. Pushing forward this on X, we have $0 \to \pi_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)) \to \mathcal{B} \to \mathcal{C} \to 0$. Therefore $\mathcal{A} \simeq \pi_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1))$. Moreover, by a natural map $\pi^*\pi_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)) \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)$, we obtain $\pi^*\mathcal{A} \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{B}^*)}(1)$. Investigating this along fibers, we see that this is surjective. q.e.d.

Proposition 4.6.2. *The subvariety $\mathscr{Z}_3^u$ in $\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3$ is the complete intersection with respect to a section of $\mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2})}(1) \boxtimes \mathcal{Q}$, hence $\mathcal{O}_{\mathscr{Z}_3^u}$ has the following Koszul resolution as a $\mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3}$-module*:

$$0 \to \bigwedge^3 \{\mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2})}(-1) \boxtimes \mathcal{Q}^*\} \to \bigwedge^2 \{\mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2})}(-1) \boxtimes \mathcal{Q}^*\} \to \\ \mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2})}(-1) \boxtimes \mathcal{Q}^* \to \mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_3} \to \mathcal{O}_{\mathscr{Z}_3^u} \to 0.$$

Proof. Since $\mathscr{Z}_3^u = \mathbb{P}(\mathcal{S})$, the assertion follows by applying Lemma 4.6.1 to the dual of (4.3). q.e.d.

5. The ideal sheaf $\mathcal{I}$ of a closed subscheme Δ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$

We formulate a natural incidence correspondence between $\widetilde{\mathscr{Y}}$ and $\check{\mathscr{X}}$ and consider its ideal sheaf $\mathcal{I}$ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$. We obtain a locally free resolution of the ideal sheaf, which will play a central role for our proof of the derived equivalence between X and Y in Section 8.

5.1. The ideal sheaf $\mathcal{I}$ and its locally free resolution. Let us consider an incident relation in $\mathrm{G}(3,V) \times \mathrm{G}(2,V)$ by

$$\Delta_0 = \{([V_3],[V_2]) \mid V_3 \supset V_2\},$$

which defines the flag variety $F(2,3,V)$. We note that $\mathrm{G}(3,V)$ is connected to $\mathscr{Z}_2, \mathscr{Z}_3$ and also $\mathrm{G}(2,T(-1))$ by the morphisms in the diagram (4.7). Considering the products of these morphisms with $g : \check{\mathscr{X}} \to \mathrm{G}(2,V)$, we denote the pull-backs of $\Delta_0 \subset \mathrm{G}(3,V)\times\mathrm{G}(2,V)$ in $\mathscr{Z}_2 \times \check{\mathscr{X}}$, $\mathscr{Z}_3 \times \check{\mathscr{X}}$ and $\mathrm{G}(2,T(-1)) \times \check{\mathscr{X}}$ by

$$\Delta_2 := (\mu_2 \times g)^*(\Delta_0), \quad \Delta_3 := (\mu_3 \times g)^*(\Delta_0), \; \Delta_G := (\rho_G \times g)^*(\Delta_0),$$

with $\mu_2 := \rho_G \circ \pi_{G'} \circ \rho_{2'}$, $\mu_3 := \rho_G \circ \pi_{G'}$, and denote their ideal sheaves by $\mathcal{I}_{\Delta_2}, \mathcal{I}_{\Delta_3}$ and $\mathcal{I}_{\Delta_G}$, respectively. Recall the following definitions of the universal sheaves:

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathcal{W} \to 0, \;\; 0 \to \mathcal{F} \to V \otimes \mathcal{O}_{\mathrm{G}(2,V)} \to \mathcal{G} \to 0.$$

Proposition 5.1.1. *The ideal sheaf $\mathcal{I}_{\Delta_0}$ of Δ_0 has the following Koszul resolution*:

(5.1)
$$0 \to \bigwedge^4(\mathcal{W}^* \boxtimes \mathcal{F}) \to \bigwedge^3(\mathcal{W}^* \boxtimes \mathcal{F}) \to \bigwedge^2(\mathcal{W}^* \boxtimes \mathcal{F}) \to \mathcal{W}^* \boxtimes \mathcal{F} \to \mathcal{I}_{\Delta_0} \to 0.$$

Proof. Tensoring the two surjections $V \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathcal{W}$ and $V^* \otimes \mathcal{O}_{\mathrm{G}(2,V)} \to \mathcal{F}^*$, we obtain a map $(V \otimes V^*) \otimes \mathcal{O}_{\mathrm{G}(3,V)\times\mathrm{G}(2,V)} \to \mathcal{W} \boxtimes \mathcal{F}^*$. Associated to the identity element in $\mathrm{Hom}(V,V) \simeq V \otimes V^*$, we obtain a map $\mathcal{O}_{\mathrm{G}(3,V)\times\mathrm{G}(2,V)} \to \mathcal{W}\boxtimes\mathcal{F}^*$. We show that Δ_0 is the scheme of zeros of the section associated to this map. Indeed, at a point $([V_3],[V_2])$ of $\mathrm{G}(3,V)\times\mathrm{G}(2,V)$, the fiber of $\mathcal{W}$ is V/V_3 and the fiber of $\mathcal{F}^*$ is V_2^*. Then it is easy to see that the identity in $V \otimes V^*$ is contained in the kernel of the natural map $V \otimes V^* \to (V/V_3) \otimes V_2^*$ if and only if $V_2 \subset V_3$, namely, $([V_3],[V_2]) \in \Delta_0$. q.e.d.

Pulling back (5.1), we obtain the following locally free resolution of $\mathcal{I}_{\Delta_2}$:

$$\begin{aligned} 0 \to &\bigwedge^4(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \bigwedge^3(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \\ &\bigwedge^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F} \to \mathcal{I}_{\Delta_2} \to 0. \end{aligned} \tag{5.2}$$

We extend all morphisms in the diagram (4.7) by taking the product $\times \mathrm{id}_{\check{\mathscr{X}}}$ to the corresponding morphisms between the products with $\check{\mathscr{X}}$

and indicate these by $\check{}$, e.g., $\check{\pi}_{3'} = \pi_{3'} \times \mathrm{id}_{\check{\mathscr{X}}}$. We consider the restriction $\mathcal{I}^o_{\Delta_2} = \check{\iota}_2^* \mathcal{I}_{\Delta_2}$ of the ideal sheaf $\mathcal{I}_{\Delta_2}$ to $\mathscr{Z}_2^o \times \check{\mathscr{X}}$. In what follows, for simplicity, we abuse $\check{\pi}^o_{2'*}$ for $\check{\pi}^o_{2'*} \circ \check{\iota}_2^*$ when it is clear from the context.

Proposition 5.1.2. *Define $\mathcal{I}_2^o := \check{\pi}^o_{2'*}\mathcal{I}_{\Delta_2}$, which is an ideal sheaf on $\mathscr{Y}_2^o \times \check{\mathscr{X}}$. There is an exact sequence on $\mathscr{Y}_2^o \times \check{\mathscr{X}}$:*

$$
\begin{aligned}
0 \to & R^1\check{\pi}^o_{2'*}{\textstyle\bigwedge}^4(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to R^1\check{\pi}^o_{2'*}{\textstyle\bigwedge}^3(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \\
& R^1\check{\pi}^o_{2'*}{\textstyle\bigwedge}^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \mathcal{I}_2^o \to 0.
\end{aligned}
\tag{5.3}
$$

We prove this proposition in Subsection 5.4. Our strategy to obtain the ideal sheaf $\mathcal{I}$ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ consists of the following four steps:

- S1. We evaluate each term of (5.3) by Grothendieck-Verdier duality (Subsection 5.5).
- S2. We further simplify the sheaves which result from (S1) to obtain a locally free resolution of $\mathcal{I}_2^o$ (Subsection 5.6).
- S3. We define the push forward $\mathcal{I}^o := (\tilde{\rho}_2 \times \mathrm{id})_*\mathcal{I}_2^o$ and obtain its locally free resolution on $\widetilde{\mathscr{Y}^o} \times \check{\mathscr{X}}$ (Subsection 5.7).
- S4. Finally, we define the ideal sheaf $\mathcal{I} = (\tilde{\iota} \times \mathrm{id})_*\mathcal{I}^o$ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ with its locally free resolution (Subsection 5.8).

Our final results can then be summerized in the following form:

Theorem 5.1.3. *The ideal sheaf $\mathcal{I}$ has the following* $\mathrm{SL}(V)$*-equivariant locally free resolution*:

$$
\begin{aligned}
0 \to \widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}} \; & \to \widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^* \\
& \to \left(\mathcal{O}_{\widetilde{\mathscr{Y}}} \boxtimes g^*\mathsf{S}^2\mathcal{F}^*\right) \oplus \left(\widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}})\right) \\
& \to \mathcal{I} \otimes \left(\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\right) \;\; \to \;\; 0
\end{aligned}
\tag{5.4}
$$

with the locally free sheaves $\widetilde{\mathcal{S}}_L = (\widetilde{\mathcal{S}}_L^*)^*$, $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{T}}$ on $\widetilde{\mathscr{Y}}$ introduced in [**HoTa3**, Subsect. 6.1, 6.2]. *Let Δ be the closed subscheme defined by $\mathcal{I}$. Then Δ is an* $\mathrm{SL}\,(V)$*-invariant, normal, and Cohen-Macaulay variety.*

We recall that the locally free sheaves $\widetilde{\mathcal{S}}_L = (\widetilde{\mathcal{S}}_L^*)^*$, $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{T}}$ on $\widetilde{\mathscr{Y}}$ are defined by the following relations on $\mathscr{Y}_2$;

$$
\rho_2^*\mathcal{S}^*(L_{\mathscr{Y}_2}) = \tilde{\rho}_2^*\widetilde{\mathcal{S}}_L^*, \quad \rho_2^*\mathcal{Q} = \tilde{\rho}_2^*\widetilde{\mathcal{Q}}, \quad \mathcal{T}_2 = \tilde{\rho}_2^*\widetilde{\mathcal{T}},
$$

where $\mathcal{S}$ and $\mathcal{Q}$ are the universal sheaves on $\mathscr{Y}_3$ and $\mathcal{T}_2$ is defined by the exact sequence

$$
0 \to \mathcal{T}_2^* \to \pi_2^*T(-1)^* \to (\rho_2|_{F_\rho})^*\mathcal{O}_{\mathbb{P}(T(-1))}(1) \to 0.
\tag{5.5}
$$

5.2. Relations to the (dual) Lefschetz collections. Let us introduce the following ordered collection of sheaves on $\widetilde{\mathscr{Y}}$ and $\check{\mathscr{X}}$, respectively:

$$(5.6)\qquad \begin{aligned} (\mathcal{E}_3,\mathcal{E}_2,\mathcal{E}_{1a},\mathcal{E}_{1b}) &= \Big(\widetilde{\mathcal{S}}_L,\ \widetilde{\mathcal{T}}^*,\ \mathcal{O}_{\widetilde{\mathscr{Y}}},\ \widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}})\Big),\\ (\mathcal{F}_3,\mathcal{F}_2,\mathcal{F}'_{1a},\mathcal{F}_{1b}) &= \Big(\mathcal{O}_{\check{\mathscr{X}}},\ g^*\mathcal{F}^*,\ g^*\mathsf{S}^2\mathcal{F}^*,\ \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}})\Big), \end{aligned}$$

and write the locally free resolution (5.4) as

$$\begin{aligned} 0\to\mathcal{E}_3\boxtimes\mathcal{F}_3\to\mathcal{E}_2\boxtimes\mathcal{F}_2\to\mathcal{E}_{1a}\boxtimes\mathcal{F}'_{1a}\oplus\mathcal{E}_{1b}\boxtimes\mathcal{F}_{1b}&\\ \to\mathcal{I}\otimes\Big(\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}})\boxtimes\mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\Big)&\to 0. \end{aligned}$$

The ordered collection $(\mathcal{F}_i)_{i\in I}=(\mathcal{F}_3,\mathcal{F}_2,\mathcal{F}_{1a},\mathcal{F}_{1b})$ for the dual Lefschetz collection of $\check{\mathscr{X}}$ ([**HoTa3**, Thm. 3.4.5]) and $(\mathcal{E}_i)_{i\in I}=(\mathcal{E}_3,\mathcal{E}_2,\mathcal{E}_{1a},\mathcal{E}_{1b})$ for the Lefschetz collection of $\widetilde{\mathscr{Y}}$ ([*ibid*, Thm. 8.1.1]) are defined from (5.6) with a slight modification $\mathcal{F}_{1a}:=\mathcal{F}'_{1a}/\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}+2L_{\check{\mathscr{X}}})$. The modification of $\mathcal{F}'_{1a}$ will be explained in Proposition 6.0.1, where we will consider the ideal sheaf $\mathcal{I}$ in the (pull-back of the) universal family of hyperplane sections in $\mathbb{P}(\mathsf{S}^2V^*)\times\mathbb{P}(\mathsf{S}^2V)$. Here we explain the obvious duality in the following the quiver diagrams (determined in [**HoTa3**]) which represents the non-vanishing Hom's among the ordered collections $(\mathcal{E}_i)_{i\in I}$ and $(\mathcal{F}_i)_{i\in I}$:

$$(5.7)\qquad \begin{tikzcd} & & \mathcal{E}_{1a} \\ \mathcal{E}_3 \arrow[r, "V"] \arrow[urr, bend left, "\wedge^2 V"] \arrow[drr, bend right, "\mathsf{S}^2 V"'] & \mathcal{E}_2 \arrow[ur, "V"] \arrow[dr, "V"] & \\ & & \mathcal{E}_{1b} \end{tikzcd} \qquad \begin{tikzcd} & & \mathcal{F}_{1a} \\ \mathcal{F}_3 \arrow[r, "V^*"] \arrow[urr, bend left, "\mathsf{S}^2 V^*"] \arrow[drr, bend right, "\wedge^2 V^*"'] & \mathcal{F}_2 \arrow[ur, "V^*"] \arrow[dr, "V^*"] & \\ & & \mathcal{F}_{1b} \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{O}_{\widetilde{\mathscr{Y}}}}(\mathcal{E}_i,\mathcal{E}_j) \qquad\qquad \mathrm{Hom}_{\mathcal{O}_{\check{\mathscr{X}}}}(\mathcal{F}_i,\mathcal{F}_j)$$

The duality in the above diagrams is related to the $\mathrm{SL}(V)$-equivariance of the resolution (5.4) (or corresponding resolution (6.1)) as follows: Let us consider two pairs $(\mathcal{A}_1,\mathcal{A}_2)$ and $(\mathcal{B}_1,\mathcal{B}_2)$ of $\mathrm{SL}(V)$-equivariant, locally free sheaves on an $\mathrm{SL}(V)$-variety in general. By tensoring the evaluation maps $\mathrm{Hom}(\mathcal{A}_1,\mathcal{A}_2)\otimes\mathcal{A}_1\to\mathcal{A}_2$ and $\mathrm{Hom}(\mathcal{B}_1,\mathcal{B}_2)\otimes\mathcal{B}_1\to\mathcal{B}_2$, we have

$$(\mathrm{Hom}(\mathcal{A}_1,\mathcal{A}_2)\otimes\mathrm{Hom}(\mathcal{B}_1,\mathcal{B}_2))\otimes(\mathcal{A}_1\otimes\mathcal{B}_1)\to\mathcal{A}_2\otimes\mathcal{B}_2.$$

Suppose that we have the following duality as $\mathrm{SL}(V)$-modules:

$$(5.8)\qquad \mathrm{Hom}(\mathcal{A}_1,\mathcal{A}_2)\simeq\mathrm{Hom}(\mathcal{B}_1,\mathcal{B}_2)^*.$$

Then corresponding to the identity element in

$$\mathrm{Hom}(\mathcal{A}_1,\mathcal{A}_2)\otimes\mathrm{Hom}(\mathcal{B}_1,\mathcal{B}_2)\simeq\mathrm{Hom}(\mathrm{Hom}(\mathcal{B}_1,\mathcal{B}_2),\mathrm{Hom}(\mathcal{B}_1,\mathcal{B}_2)),$$

there is a unique $\mathrm{SL}(V)$-equivariant morphism

$$\mathcal{A}_1\otimes\mathcal{B}_1\to\mathcal{A}_2\otimes\mathcal{B}_2. \tag{5.9}$$

Now we read each morphism in the resolution (5.4) (a) $\mathcal{E}_3\boxtimes\mathcal{F}_3\to\mathcal{E}_2\boxtimes\mathcal{F}_2$, (b) $\mathcal{E}_2\boxtimes\mathcal{F}_2\to\mathcal{E}_{1a}\boxtimes\mathcal{F}'_{1a}$, (c) $\mathcal{E}_2\boxtimes\mathcal{F}_2\to\mathcal{E}_{1b}\boxtimes\mathcal{F}_{1b}$, (d) $\mathcal{E}_{1a}\boxtimes\mathcal{F}'_{1a}\to\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}})\boxtimes\mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})$ and (e) $\mathcal{E}_{1b}\boxtimes\mathcal{F}_{1b}\to\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}})\boxtimes\mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})$ in order. For (a) and (c), we can verify the relation (5.8) directly from the diagrams. For (b), we read the isomorphism $\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_{1a})\simeq V$ from (5.7). To compute $\mathrm{Hom}(\mathcal{F}_2,\mathcal{F}'_{1a})$, we use the exact sequence $0\to\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}+2L_{\check{\mathscr{X}}})\to\mathcal{F}'_{1a}\to\mathcal{F}_{1a}\to 0$. The vanishing $H^\bullet(\check{\mathscr{X}},g^*\mathcal{F}\otimes\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}+2L_{\check{\mathscr{X}}}))=0$ follows from the fact that $g:\check{\mathscr{X}}\to\mathrm{G}(2,V)$ is a $\mathbb{P}^2$-bundle. Therefore we read $\mathrm{Hom}(\mathcal{F}_2,\mathcal{F}'_{1a})\simeq\mathrm{Hom}(\mathcal{F}_2,\mathcal{F}_{1a})\simeq V^*$ from (5.7). For (d) and (e), respectively, we use the following isomorphisms:

$$\mathrm{Hom}(\mathcal{E}_{1a},\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}))=H^0(\widetilde{\mathscr{Y}},\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}))\simeq\mathsf{S}^2V,$$
$$\mathrm{Hom}(\mathcal{F}'_{1a},\mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}}))=H^0(\mathrm{G}(2,V),\mathsf{S}^2\mathcal{F}\otimes\mathcal{O}(2L_{\check{\mathscr{X}}}))\simeq\mathsf{S}^2V^*,$$

and

$$\mathrm{Hom}(\mathcal{E}_{1b},\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}))=H^0(\mathscr{Y}_3,\mathcal{Q})=H^0(\mathbb{P}(V),T(-1)^2)\simeq\textstyle\bigwedge^2V^*,$$
$$\mathrm{Hom}(\mathcal{F}_{1b},\mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}}))=H^0(\mathrm{G}(2,V),\mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}}))\simeq\textstyle\bigwedge^2V.$$

5.3. Closed subschemes $\Delta_3\subset\mathscr{Z}_3\times\check{\mathscr{X}}$ and $\Delta_2\subset\mathscr{Z}_2\times\check{\mathscr{X}}$. For a point $x\in\check{\mathscr{X}}$, we denote by $\Delta_{3,x}$(resp. $\Delta_{G,x}$) the fiber of Δ_3(resp. Δ_G) $\to\check{\mathscr{X}}$ over x. Let l_x be the line in $\mathbb{P}(V)$ corresponding to $g(x)\in\mathrm{G}(2,V)$.

Proposition 5.3.1. *For any point $x\in\check{\mathscr{X}}$, the variety $\Delta_{3,x}$ is a $\mathrm{G}(2,5)$-bundle over $\Delta_{G,x}$, which can be identified with the blow-up of $\mathbb{P}(V)$ along l_x.*

Proof. Recall that Δ_3 is the pull-back of Δ_G by $\check{\pi}_{G'}:\mathscr{Z}_3\times\check{\mathscr{X}}\to\mathrm{G}(2,T(-1))\times\check{\mathscr{X}}$. Then by Proposition 4.2.2, $\Delta_{3,x}$ is a $\mathrm{G}(2,5)$-bundle over $\Delta_{G,x}$. We show that $\Delta_{G,x}$ is isomorphic to the blow-up of $\mathbb{P}(V)$ along l_x. We describe $\mathrm{G}(2,T(-1))$ as the universal family of planes on $\mathbb{P}(V)$, namely,

$$\mathrm{G}(2,T(-1))=\{([V_3],[V_1])\mid V_1\subset V_3\}\subset\mathrm{G}(3,V)\times\mathbb{P}(V).$$

Then it holds that

$$\Delta_{G,x}=\{([V_3],[V_1])\mid V_1\subset V_3,l_x\subset\mathbb{P}(V_3)\}\subset\mathrm{G}(2,T(-1)).$$

We see that the natural projection morphism from $\Delta_{G,x}$ to $\mathbb{P}(V)$ sending $([V_3],[V_1])$ to $[V_1]$ is the blow-up of $\mathbb{P}(V)$ along the line l_x. q.e.d.

Define $\Delta_{\mathscr{Y}_3} := \breve{\pi}_{3'}(\Delta_3)$ in $\mathscr{Y}_3 \times \check{\mathscr{X}}$ with its reduced structure, and consider $\Delta_{\mathscr{Y}_3,x}$ as above. We consider the natural projection $\mathrm{pr}_1 \circ \breve{\pi}_3|_{\Delta_{\mathscr{Y}_3 \times \{x\}}} : \Delta_{\mathscr{Y}_3,x} \to \mathbb{P}(V)$.

Proposition 5.3.2. *Let $[V_1]$ be a point of $\mathbb{P}(V)$. If $[V_1] \notin l_x$, then the fiber of $\Delta_{\mathscr{Y}_3,x} \to \mathbb{P}(V)$ is isomorphic to $\mathrm{G}(2,5)$. If $[V_1] \in l_x$, then the fiber of $\Delta_{\mathscr{Y}_3,x} \to \mathbb{P}(V)$ is isomorphic to the 8-dimensional Schubert cycle $\{[\bar{U}] \mid \mathbb{P}(\bar{U}) \cap \bar{P}_x \neq \emptyset\} \subset \mathrm{G}(3, \wedge^2 V/V_1)$, where $\bar{P}_x$ is a fixed plane of $\mathbb{P}(\wedge^2 V/V_1)$. In particular, $\Delta_{3,x} \to \Delta_{\mathscr{Y}_3,x}$ is and hence $\Delta_3 \to \Delta_{\mathscr{Y}_3}$ is birational.*

Proof. We can write

$$\Delta_{3,x} = \left\{ ([\bar{a} \wedge \bar{b}], ([\bar{U}], [V_1])) \in \mathscr{Z}_3 \;\middle|\; l_x \subset \mathbb{P}(V_3), \begin{array}{l} V_1 = \langle v \rangle, \\ V_3 = \langle a, b, v \rangle \end{array} \right\}, \tag{5.10}$$

where $a \in V$ is determined by $\bar{a} = a \bmod V_1$ and similarly for b. Also, we have

$$\Delta_{\mathscr{Y}_3,x} = \left\{ ([\bar{U}], [V_1]) \in \mathscr{Y}_3 \;\middle|\; {}^{\exists}V_3 \text{ such that } \begin{array}{c} \wedge^2(V_3/V_1) \subset \bar{U} \\ \text{and } l_x \subset \mathbb{P}(V_3) \end{array} \right\}. \tag{5.11}$$

If $[V_1] \notin l_x$, then V_3 in (5.11) is unique by the condition $l_x \subset \mathbb{P}(V_3)$, i.e., $V_3 = V_1 \oplus \mathbb{C}l_x$. Then the three dimensional subspaces $\bar{U}$ satisfying $\wedge^2(V_3/V_1) \subset \bar{U} \subset \wedge^2(V/V_1)$ determines the fiber of $\Delta_{\mathscr{Y}_3,x} \to \mathbb{P}(V)$ over $[V_1]$, which is isomorphic to $\mathrm{G}(2,5)$. In particular, identifying $[\bar{a} \wedge \bar{b}]$ with $[\wedge^2(V_3/V_1)]$, we see that the morphism $\Delta_{3,x} \to \Delta_{\mathscr{Y}_3,x}$ is birational, and so is $\Delta_3 \to \Delta_{\mathscr{Y}_3}$. Now, assume that $[V_1] \in l_x$. Then V_3 satisfying $l_x \subset \mathbb{P}(V_3)$ form a ρ-plane $\mathrm{P}_{V_{2,x}} = \{[\Pi] \mid V_{2,x} \subset \Pi\} \subset \mathrm{G}(3,V)$, where $V_{2,x}$ is determined by $l_x = \mathbb{P}(V_{2,x})$. Writing the corresponding plane $\bar{P}_x = \{[\wedge^2(\Pi/V_1)] \mid [\Pi] \in \mathrm{P}_{V_{2,x}}\} \simeq \mathbb{P}^2$ in $\mathbb{P}(\wedge^2(V/V_1))$, we can rephrase the condition ${}^{\exists}V_3$ s.t. $\wedge^2(V_3/V_1) \subset \bar{U}$ by $\bar{P}_x \cap \mathbb{P}(\bar{U}) \neq \emptyset$. Then the fiber of $\Delta_{\mathscr{Y}_3,x} \to \mathbb{P}(V)$ over $[V_1]$ can be identified with the Schubert cycle, $\{[\bar{U}] \mid \mathbb{P}(\bar{U}) \cap \bar{P}_x \neq \emptyset\} \subset \mathrm{G}(3, \wedge^2 V/V_1)$. q.e.d.

Remark. The Schubert cycle $\{[\bar{U}] \mid \mathbb{P}(\bar{U}) \cap \bar{P}_x \neq \emptyset\}$ has a natural resolution of singularities; $\{([\wedge^2(V_3/V_1)], [\bar{U}]) \mid [\wedge^2(V_3/V_1)] \in \mathbb{P}(\bar{U}), l_x \subset \mathbb{P}(V_3)\}$, which is contained in $\Delta_{3,x}$ and has a $\mathrm{G}(2,5)$-bundle structure over $\{[V_3] \mid l_x \subset \mathbb{P}(V_3)\} \simeq \mathbb{P}^2$.

Proposition 5.3.3. *$\Delta_{2,x}$ is the blow-up of $\Delta_{3,x}$ along $\mathscr{Z}_\rho \cap \Delta_{3,x}$. In particular, $\Delta_{2,x}$ is and hence Δ_2 is smooth.*

Proof. Recall that $\mathscr{Z}_2 \to \mathscr{Z}_3$ is the blow-up along $\mathscr{Z}_\rho = \pi_{3'}^{-1}(\mathscr{P}_\rho) \subset \mathscr{Z}_3$. By Proposition 4.2.3, we see that $\mathscr{Z}_\rho \cap \Delta_{3,x}$ is a $\mathbb{P}^1$-bundle over the smooth variety $\Delta_{G,x} \subset \mathrm{G}(2, T(-1))$ (see Proposition 5.3.1), and hence $\mathscr{Z}_\rho \cap \Delta_{3,x}$ is smooth. Therefore, $\Delta_{2,x}$ is smooth and so is Δ_2. q.e.d.

5.4. Locally free resolution (5.3) of $\mathcal{I}_2^o$ on $\mathscr{Y}_2^o \times \check{\mathscr{X}}$. We start preparing the following two lemmas:

Lemma 5.4.1. *Let* $\mathrm{P} = \mathrm{P}_{\mathrm{V}_2}$ *be the* ρ*-plane in* $\mathrm{G}(3, V)$ *associated to some two dimensional vector space* V_2 *in* V *(cf. Subsection* 4.1*). Then* $\mathcal{W}|_{\mathrm{P}} \simeq T_{\mathrm{P}}(-1)$.

Proof. From the natural surjection $V \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathcal{W}$, we obtain the surjection $V/V_2 \otimes \mathcal{O}_{\mathrm{P}} \to \mathcal{W}|_{\mathrm{P}}$. Since $\mathrm{P} \simeq \mathbb{P}(\mathrm{V}/\mathrm{V}_2)$, this surjection can be identified in the Euler sequence of P. Therefore $\mathcal{W}|_{\mathrm{P}} \simeq T_{\mathrm{P}}(-1)$.

q.e.d.

Lemma 5.4.2. *Let* q *be a* τ*- or* ρ*-conic on* $\mathrm{G}(3, V)$*. Then* $H^\bullet(\mathcal{W}^*|_q) = 0$*. Moreover, if* q *is smooth, then* $\mathcal{W}|_q \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$.

Proof. We denote by $\mathbb{P}_q^2$ the plane spanned by q. Let us first assume that q is a τ-conic. As we can see in (4.2), there exists an $S \simeq \mathrm{G}(2, 4)$ in $\mathrm{G}(3, V)$ such that $q \subset S$. The conic q is a complete intersection in S since $\mathbb{P}_q^2 \cap S = q$, and then $\mathcal{O}_q$ has the following Koszul resolution as a $\mathcal{O}_S$-module:

$$0 \to \mathcal{O}_S(-3) \to \mathcal{O}_S(-2)^{\oplus 3} \to \mathcal{O}_S(-1)^{\oplus 3} \to \mathcal{O}_S \to \mathcal{O}_q \to 0.$$

Tensoring this exact sequence with $\mathcal{W}^*|_S$ and using Theorem 2.1.1, it is easy to derive $H^\bullet(\mathcal{W}^*|_q) = 0$.

Now consider the case where q is a ρ-conic. By Lemma 5.4.1, it holds that $\mathcal{W}|_{\mathbb{P}_q^2} \simeq T_{\mathbb{P}_q^2}(-1)$. Tensoring $\mathcal{W}^*|_{\mathbb{P}_q^2}$ with the exact sequence $0 \to \mathcal{O}_{\mathbb{P}_q^2}(-2) \to \mathcal{O}_{\mathbb{P}_q^2} \to \mathcal{O}_q \to 0$, we have

$$0 \to \Omega^1_{\mathbb{P}_q^2}(-1) \to \Omega^1_{\mathbb{P}_q^2}(1) \to \mathcal{W}^*|_q \to 0.$$

Since all the cohomology groups of $\Omega^1_{\mathbb{P}_q^2}(-1)$ and $\Omega^1_{\mathbb{P}_q^2}(1)$ vanish, we have $H^\bullet(\mathcal{W}^*|_q) = 0$. The rest of the claims is easy. q.e.d.

The following argument to obtain (5.3) is inspired by [**Ku2**, Lemma 8.2].

Proof of Proposition 5.1.2. Let us split (5.2) into the short exact sequences:

$$0 \to \bigwedge^4(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \bigwedge^3(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \mathcal{K}_1 \to 0. \tag{5.12}$$

$$0 \to \mathcal{K}_1 \to \bigwedge^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to \mathcal{K}_2 \to 0. \tag{5.13}$$

$$0 \to \mathcal{K}_2 \to \mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F} \to \mathcal{I}_{\Delta_2} \to 0. \tag{5.14}$$

By Lemma 5.4.2, we have

$$\check{\pi}^o_{2'*}(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) = R^1\check{\pi}^o_{2'*}(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) = 0. \tag{5.15}$$

Hence from (5.14) we have $\breve{\pi}^o_{2'*}\mathcal{I}_{\Delta_2} = R^1\breve{\pi}^o_{2'*}\mathcal{K}_2$ and $\breve{\pi}^o_{2'*}\mathcal{K}_2 = 0$. Then by (5.13), we obtain

$$0 \to R^1\breve{\pi}^o_{2'*}\mathcal{K}_1 \to R^1\breve{\pi}^o_{2'*}\bigwedge^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \to R^1\breve{\pi}^o_{2'*}\mathcal{K}_2 = \breve{\pi}^o_{2'*}\mathcal{I}_{\Delta_2} \to 0.$$

Since $\wedge^2(\mu_2^*\mathcal{W}^*\boxtimes g^*\mathcal{F})|_q \simeq \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6}$ on a smooth fiber q by Lemma 5.4.2 and $\breve{\pi}^o_{2'*}\wedge^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F})$ is torsion free, we have $\breve{\pi}^o_{2'*}\wedge^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) = 0$. This implies that $\breve{\pi}^o_{2'*}\mathcal{K}_1 = 0$ by (5.13). Then by (5.12), we have

$$0 \to R^1\breve{\pi}^o_{2'*}\bigwedge^4(\mu_2^*\mathcal{W}^*\boxtimes g^*\mathcal{F}) \to R^1\breve{\pi}^o_{2'*}\bigwedge^3(\mu_2^*\mathcal{W}^*\boxtimes g^*\mathcal{F}) \to R^1\breve{\pi}^o_{2'*}\mathcal{K}_1 \to 0.$$

Eliminating $R^1\hat{\pi}^o_{2'*}\mathcal{K}_1$ from the above exact sequences, we obtain (5.3).

Let Δ_2^o and $\Delta^o_{\mathscr{Y}_2}$ be the closed subschemes of $\mathscr{Z}_2^o \times \breve{\mathscr{X}}$ and $\mathscr{Y}_2^o \times \breve{\mathscr{X}}$ defined by $\mathcal{I}^o_{\Delta_2}$ and $\mathcal{I}^o_2$ respectively. We set $\breve{\pi}_{\Delta_2^o} := \breve{\pi}_{2'}|_{\Delta_2^o}$. The following lemma will be used in Subsection 5.9.

Lemma 5.4.3. *$\breve{\pi}_{\Delta_2^o*}\mathcal{O}_{\Delta_2^o} = \mathcal{O}_{\Delta^o_{\mathscr{Y}_2}}$ and $R^1\breve{\pi}_{\Delta_2^o*}\mathcal{O}_{\Delta_2^o} = 0$.*

Proof. By (5.14) and (5.15), we have $R^1\breve{\pi}^o_{2'*}\mathcal{I}^o_{\Delta_2} = 0$. Taking the higher direct image of the exact sequence $0 \to \mathcal{I}^o_{\Delta_2} \to \mathcal{O}_{\mathscr{Z}_2^o\times\breve{\mathscr{X}}} \to \mathcal{O}_{\Delta_2^o} \to 0$, we obtain the exact sequence $0 \to \mathcal{I}^o_2 \to \mathcal{O}_{\mathscr{Y}_2^o\times\breve{\mathscr{X}}} \to \breve{\pi}_{\Delta_2^o*}\mathcal{O}_{\Delta_2^o} \to 0$ and $R^1\breve{\pi}_{\Delta_2^o*}\mathcal{O}_{\Delta_2^o} = 0$ since $R^1\breve{\pi}^o_{2'*}\mathcal{I}_{\Delta_2^o} = 0$ and $R^1\breve{\pi}^o_{2'*}\mathcal{O}_{\mathscr{Z}_2^o\times\breve{\mathscr{X}}} = 0$. Hence $\breve{\pi}_{\Delta_2^o*}\mathcal{O}_{\Delta_2^o} = \mathcal{O}_{\Delta^o_{\mathscr{Y}_2}}$. q.e.d.

5.5. Step 1: Evaluating (5.3).

In this step, we rewrite each term of the resolution (5.3) by the Grothendieck-Verdier duality.

Lemma 5.5.1. *Let q be a τ- or ρ-conic on $\mathrm{G}(3,V)$. Then $H^1(q, \wedge^i\mathcal{W}^{\oplus 2} \otimes \mathcal{O}_q(-1)) = 0$ $(1 \leq i \leq 4)$.*

Proof. Assume that q is a τ-conic. As in the proof of Lemma 5.4.2, we take $S = \{[U] \mid V_1 \subset U \subset V\} \simeq \mathrm{G}(2,4)$ in $\mathrm{G}(3,V)$ such that $q \subset S$ and consider the Koszul resolution of $\mathcal{O}_q$ on S. Tensoring this exact sequence with $\wedge^i\mathcal{W}|_S^{\oplus 2} \otimes \mathcal{O}_S(-1)$, we obtain

$$\begin{aligned} 0 \to (\textstyle\bigwedge^i\mathcal{W}|_S^{\oplus 2})(-4) \to (\bigwedge^i\mathcal{W}|_S^{\oplus 2})(-3)^{\oplus 3} \to (\bigwedge^i\mathcal{W}|_S^{\oplus 2})(-2)^{\oplus 3} \to \\ \textstyle\bigwedge^i\mathcal{W}|_S^{\oplus 2}(-1) \to \bigwedge^i\mathcal{W}^{\oplus 2} \otimes \mathcal{O}_q(-1) \to 0. \end{aligned}$$

Using Theorem 2.1.1, it is easy to derive the assertion.

When q is a ρ-conic, as in the proof of Lemma 5.4.2, tensoring $(\wedge^i\mathcal{W}^{\oplus 2}|_{\mathbb{P}^2_q})(-1) \simeq (\wedge^i T_{\mathbb{P}^2_q}(-1)^{\oplus 2})(-1)$ with the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2_q}(-2) \to \mathcal{O}_{\mathbb{P}^2_q} \to \mathcal{O}_q \to 0$, we have

$$\begin{aligned} 0 \to (\textstyle\bigwedge^i T_{\mathbb{P}^2_q}(-1)^{\oplus 2})(-3) \to (\bigwedge^i T_{\mathbb{P}^2_q}(-1)^{\oplus 2})(-1) \\ \to \textstyle\bigwedge^i\mathcal{W}^{\oplus 2} \otimes \mathcal{O}_q(-1) \to 0. \end{aligned}$$

Now computing the cohomology groups of $(\wedge^i T_{\mathbb{P}^2_q}(-1)^{\oplus 2})(-3)$ and $(\wedge^i T_{\mathbb{P}^2_q}(-1)^{\oplus 2})(-1)$ by Theorem 2.1.1, we have the assertion. q.e.d.

By Lemma 5.5.1, we may apply the Grothendieck-Verdier duality 2.1.2 to the morphism $\breve{\pi}^o_{2'} : \mathscr{Z}^o_2 \times \breve{\mathscr{X}} \to \mathscr{Y}^o_2 \times \breve{\mathscr{X}}$, and then we have

$$R^1\breve{\pi}^o_{2'*}\bigwedge^i(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \simeq \left(\breve{\pi}^o_{2'*}\left\{\bigwedge^i(\mu_2^*\mathcal{W} \boxtimes g^*\mathcal{F}^*) \otimes \omega_{\mathscr{Z}_2 \times \breve{\mathscr{X}}/\mathscr{Y}_2 \times \breve{\mathscr{X}}}\right\}\right)^*.$$

Note that $\omega_{\mathscr{Z}_2 \times \breve{\mathscr{X}}/\mathscr{Y}_2 \times \breve{\mathscr{X}}} = \mathrm{pr}_1^*\omega_{\mathscr{Z}_2/\mathscr{Y}_2} = \omega_{\mathscr{Z}_2/\mathscr{Y}_2} \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}$. By Proposition 4.5.1, we have $K_{\mathscr{Z}_2/\mathscr{Y}_2} = M_{\mathscr{Z}_2} - N_{\mathscr{Z}_2}$ for $\omega_{\mathscr{Z}_2/\mathscr{Y}_2}$. Thus we have

$$\begin{aligned} &R^1\breve{\pi}^o_{2'*}\bigwedge^i(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \simeq \\ &\Big(\breve{\pi}^o_{2'*}(\bigwedge^i(\mu_2^*\mathcal{W} \boxtimes g^*\mathcal{F}^*) \otimes (\mathcal{O}_{\mathscr{Z}^o_2}(-N_{\mathscr{Z}^o_2}) \boxtimes \mathcal{O}_{\breve{\mathscr{X}}})) \\ &\qquad \otimes (\mathcal{O}_{\mathscr{Y}^o_2}(M_{\mathscr{Y}^o_2}) \boxtimes \mathcal{O}_{\breve{\mathscr{X}}})\Big)^*. \end{aligned} \tag{5.16}$$

To simplify this, we use the following formula (see [**FH**, Exercise 6.11]):

$$\bigwedge^i(\mu_2^*\mathcal{W} \boxtimes g^*\mathcal{F}^*) \simeq \bigoplus_\lambda \Sigma^\lambda\mu_2^*\mathcal{W} \boxtimes \Sigma^{\lambda'}g^*\mathcal{F}^*, \tag{5.17}$$

where λ are partitions of i with at most 2 rows and column, and λ' is the dual partition to λ.

Proposition 5.5.2. *The exact sequence* (5.3) *twisted by* $\mathcal{O}_{\mathscr{Y}^o_2}(M_{\mathscr{Y}^o_2}) \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(2L_{\breve{\mathscr{X}}})$ *is evaluated as*

$$\begin{aligned} 0 \to &\{\pi^o_{2'*}\mathcal{O}_{\mathscr{Z}^o_2}(N_{\mathscr{Z}^o_2})\}^* \boxtimes \mathcal{O}_{\breve{\mathscr{X}}} \to \{\breve{\pi}^o_{2'*}(\mu_2^*\mathcal{W})\}^* \boxtimes g^*\mathcal{F}^* \to \\ &\mathcal{O}_{\mathscr{Y}^o_2} \boxtimes g^*\mathsf{S}^2\mathcal{F}^* \oplus \{\pi^o_{2'*}(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}^o_2}))\}^* \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(L_{\breve{\mathscr{X}}}) \to \\ &\qquad \mathcal{I}^o_2 \otimes \mathcal{O}_{\mathscr{Y}^o_2}(M_{\mathscr{Y}^o_2}) \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(2L_{\breve{\mathscr{X}}}) \to 0 \end{aligned} \tag{5.18}$$

Proof. By using (5.17), we calculate $\wedge^4(\mu_2^*\mathcal{W} \boxtimes g^*\mathcal{F}^*) \simeq \Sigma^{(2,2)}\mu_2^*\mathcal{W} \boxtimes \Sigma^{(2,2)}g^*\mathcal{F}^* = \mathcal{O}_{\mathscr{Z}_2}(2N_{\mathscr{Z}_2}) \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(2L_{\breve{\mathscr{X}}})$. Then we have

$$\begin{aligned} &R^1\breve{\pi}^o_{2'*}\bigwedge^4(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \\ &\simeq (\{\pi^o_{2'*}\mathcal{O}_{\mathscr{Z}^o_2}(N_{\mathscr{Z}^o_2})\}^* \otimes \mathcal{O}_{\mathscr{Y}^o_2}(-M_{\mathscr{Y}^o_2})) \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(-2L_{\breve{\mathscr{X}}}). \end{aligned}$$

Similarly, we evaluate $\wedge^3(\mu_2^*\mathcal{W} \boxtimes g^*\mathcal{F}^*) \simeq \mu_2^*\mathcal{W}(N_{\mathscr{Z}_2}) \boxtimes g^*\mathcal{F}^*(L_{\breve{\mathscr{X}}})$ with $\lambda = \lambda' = (2,1)$ and have

$$R^1\breve{\pi}^o_{2'*}\bigwedge^3(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F}) \simeq \{\pi^o_{2'*}(\mu_2^*\mathcal{W})^* \otimes \mathcal{O}_{\mathscr{Y}_2}(-M_{\mathscr{Y}_2})\} \boxtimes (g^*\mathcal{F}^*(-2L_{\breve{\mathscr{X}}})),$$

where we use $\mathcal{F} = \mathcal{F}^*(-L_{\breve{\mathscr{X}}})$. Finally, we have

$$\begin{aligned} \bigwedge^2(\mu_2^*\mathcal{W} \boxtimes g^*\mathcal{F}^*) &\simeq \bigwedge^2\mu_2^*\mathcal{W} \boxtimes \mathsf{S}^2(g^*\mathcal{F}^*) \oplus \mathsf{S}^2(\mu_2^*\mathcal{W}) \boxtimes \bigwedge^2 g^*\mathcal{F}^* \\ &\simeq \mathcal{O}_{\mathscr{Z}_2}(N_{\mathscr{Z}_2}) \boxtimes g^*\mathsf{S}^2\mathcal{F}^* \oplus \mu_2^*\mathsf{S}^2\mathcal{W} \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(L_{\breve{\mathscr{X}}}). \end{aligned}$$

Using this we evaluate $R^1\breve{\pi}^o_{2'*}\wedge^2(\mu_2^*\mathcal{W}^* \boxtimes g^*\mathcal{F})$ as

$$\begin{aligned} &\mathcal{O}_{\mathscr{Y}^o_2}(-M_{\mathscr{Y}^o_2}) \boxtimes g^*\mathsf{S}^2\mathcal{F} \oplus \\ &\Big\{\pi^o_{2'*}(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}^o_2}))^* \otimes \mathcal{O}_{\mathscr{Y}^o_2}(-M_{\mathscr{Y}^o_2})\Big\} \boxtimes \mathcal{O}_{\breve{\mathscr{X}}}(-L_{\breve{\mathscr{X}}}), \end{aligned}$$

and use $\mathcal{F} = \mathcal{F}^*(-L_{\breve{\mathscr{X}}})$ again to obtain (5.18). q.e.d.

5.6. Step 2: A locally free resolution of $\mathcal{I}_2^o$ on $\mathscr{Y}_2^o \times \check{\mathscr{X}}$. We will characterize the following sheaves:

$$\pi^o_{2'*}\mathcal{O}_{\mathscr{Z}_2^o}(N_{\mathscr{Z}_2^o}),\ \ \pi^o_{2'*}(\mu_2^*\mathcal{W}),\ \ \pi^o_{2'*}(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2^o})), \tag{5.19}$$

which have appeared in the resolution (5.18). Below is a preliminary result.

Lemma 5.6.1. *All the sheaves in* (5.19) *are locally free on* $\mathscr{Y}_2^o$.

Proof. By the Grauert theorem, it suffices to check that the dimensions of H^0-terms of the restrictions to fibers are constant on $\mathscr{Y}_2^o$. Let q be the fiber of $\mathscr{Z}_2^o \to \mathscr{Y}_2^o$ over a point of $\mathscr{Y}_2^o$. Let $[V_1] \in \mathbb{P}(V)$ be the image of q by $\mathscr{Z}_2^o \to \mathbb{P}(V)$. We may consider q as a conic on $\mathrm{G}(2, V/V_1)$. Let $\mathbb{P}^2_q$ be the plane spanned by q.

For $\pi^o_{2'*}\mathcal{O}_{\mathscr{Z}_2^o}(N_{\mathscr{Z}_2^o})$, we have

$$H^0(q, N_{\mathscr{Z}_2^o}|_q) \simeq H^0(q, \mathcal{O}_q(1)) \simeq H^0(\mathbb{P}^2_q, \mathcal{O}_{\mathbb{P}^2_q}(1)) \simeq \mathbb{C}^3.$$

For the sheaf $\pi^o_{2'*}(\mu_2^*\mathcal{W})$, note that the sheaf $\mu_2^*\mathcal{W}|_q$ is generated by global sections and its degree is two. Therefore, by the Riemann-Roch theorem, $H^0(q, \mu_2^*\mathcal{W}|_q) \simeq \mathbb{C}^4$. Finally, for the sheaf $\pi^o_{2'*}(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2^o}))$, we can show

$$H^0(q, (\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2^o}))|_q) \simeq \mathbb{C}^3$$

by similar computations to those in the proof of Lemma 5.5.1. Here we present the calculations only for the case where q is a ρ-conic: Tensoring $(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2^o}))|_{\mathbb{P}^2_q} \simeq (\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-1)$ with the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2_q}(-2) \to \mathcal{O}_{\mathbb{P}^2_q} \to \mathcal{O}_q \to 0$, we have

$$0 \to (\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-3) \to (\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-1) \to (\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-1)|_q \to 0.$$

We compute the cohomology groups by Theorem 2.1.1. It turns out that all the cohomology groups of $(\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-1)$ vanish, and the only nonvanishing cohomology group of $(\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-3)$ is H^1 with

$$H^1(\mathbb{P}^2_q, (\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-3)) \simeq \bar{U}^* \otimes \bigwedge^3 \bar{U}^*, \tag{5.20}$$

where $\bar{U}$ is the three dimensional subspace of $\wedge^2(V/V_1)$ such that $\mathbb{P}^2_q = \mathbb{P}(\bar{U})$. Consequently, we have

$$H^0(q, (\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2^o}))|_q) \simeq H^1(\mathbb{P}^2_q, (\mathsf{S}^2 T_{\mathbb{P}^2_q}(-1))(-3)) \simeq \mathbb{C}^3.$$

q.e.d.

5.6.1. Part 1: Here we consider sheaves on $\mathscr{Z}_2^t$ (see Definition 4.3.5) which have similar forms to each of the sheaves in (5.19). Recall that we have defined $\mathscr{Z}_2^t = B(2,4,\mathscr{Y}_2) \cap \mathbb{P}(\rho_2^*\mathcal{S})$ in Definition 4.3.5 with the Grassmann bundle $B(2,4,\mathscr{Y}_2)$. An advantage of working on $\mathscr{Z}_2^t$ is that its structure sheaf $\mathcal{O}_{\mathscr{Z}_2^t}$ has a nice Koszul resolution as $\mathcal{O}_{B(2,4)}$-module, where and hereafter we abbreviate $B(2,4,\mathscr{Y}_2)$ in subscripts to $B(2,4)$.

It should be useful to summarize the generically conic bundles $\mathscr{Z}_2, \mathscr{Z}_3$ and the corresponding universal planes $\mathscr{Z}_2^u$ and $\mathscr{Z}_3^u$ in the following diagram:

$$\begin{array}{rcl} \mathscr{Z}_3^u & \longleftarrow & \mathscr{Z}_2^u = \mathbb{P}(\rho_2^*\mathcal{S}) \\ & & \cup \\ \cup & & \mathscr{Z}_2^t = B(2,4,\mathscr{Y}_2) \cap \mathscr{Z}_2^u \\ & & \cup \\ (\mathrm{G}(2,T(-1)) \times_{\mathbb{P}(V)} \mathscr{Y}_3) \cap \mathscr{Z}_3^u = \mathscr{Z}_3 & \longleftarrow & \mathscr{Z}_2. \end{array}$$

By Proposition 4.6.2 and the definition $\mathscr{Z}_2^u := \mathbb{P}(\rho_2^*\mathcal{S})$, the variety $\mathscr{Z}_2^u$ is a complete intersection in $\mathbb{P}(T(-1)^{\wedge 2}) \times_{\mathbb{P}(V)} \mathscr{Y}_2$ with respect to a section of $\mathcal{O}_{\mathbb{P}(T(-1)^2)}(1) \boxtimes \rho_2^*\mathcal{Q}$. Therefore $\mathscr{Z}_2^t = B(2,4,\mathscr{Y}_2) \cap \mathscr{Z}_2^u$ is the complete intersection in $B(2,4,\mathscr{Y}_2)$ by a section of $(\mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2})}(1) \boxtimes \rho_2^*\mathcal{Q})|_{B(2,4)}$. By (4.9) and Proposition 4.4.1, we have an isomorphism

$$\begin{aligned} &(\mathcal{O}_{\mathbb{P}(T(-1)^{\wedge 2})}(1) \boxtimes \rho_2^*\mathcal{Q})|_{B(2,4)} \\ &\simeq \mathcal{O}_{\mathrm{G}(2,T(-1))}(N_{\mathrm{G}(2,T(-1))} - L_{\mathrm{G}(2,T(-1))}) \boxtimes \rho_2^*\mathcal{Q}. \end{aligned}$$

From Proposition 4.6.2, we see that the sheaf $\mathcal{O}_{\mathscr{Z}_2^t}$ has the following Koszul resolution as a $\mathcal{O}_{B(2,4)}$-module:

$$0 \to \mathcal{A}_3 \to \mathcal{A}_2 \to \mathcal{A}_1 \to \mathcal{O}_{B(2,4)} \to \mathcal{O}_{\mathscr{Z}_2^t} \to 0, \tag{5.21}$$

where we set

$$\mathcal{A}_i := \mathcal{O}_{\mathrm{G}(2,T(-1))}(-iN_{\mathrm{G}(2,T(-1))} + iL_{\mathrm{G}(2,T(-1))}) \boxtimes \textstyle\bigwedge^i \rho_2^*\mathcal{Q}^*$$

for $i = 0, 1, 2, 3$. Recall $\pi_{2^t} \colon \mathscr{Z}_2^t \to \mathscr{Y}_2$ and $\mu_{2^t} \colon \mathscr{Z}_2^t \to \mathrm{G}(3,V)$, and our convention $N_{\mathscr{Z}_2^t}$ for the pull-back of $N_{\mathrm{G}(2,T(-1))}$. Using the Koszul resolution (5.21), we have

Lemma 5.6.2. (i) $\pi_{2^t *}\mathcal{O}_{\mathscr{Z}_2^t}(N_{\mathscr{Z}_2^t}) \simeq \rho_2^*\mathcal{S}^*(L_{\mathscr{Y}_2})$,
(ii) $\pi_{2^t *}(\mu_{2^t}^*\mathcal{W}) \simeq \pi_2^* T(-1)$, *and*
(iii) $\pi_{2^t *}(\{\mu_{2^t}^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2^t})) \simeq \rho_2^*\mathcal{Q}(-M_{\mathscr{Y}_2} - F_\rho)$.

Proof. We show (i)–(iii) using Theorem 2.1.1 and noting that $\mathrm{pr}_2 \colon B(2,4,\mathscr{Y}_2) \to \mathscr{Y}_2$ is a $\mathrm{G}(2,4)$-bundle. Let Γ be a fiber of pr_2.
(i) We tensor (5.21) with $\mathcal{O}_{B(2,4)}(N_{B(2,4)})$, which is the pull-back to $B(2,4,\mathscr{Y}_2)$ of $\mathcal{O}_{\mathbb{P}(\Omega(1)^{\wedge 2})}(1)|_{\mathrm{G}(2,T(-1))}$ by (4.9) and Lemma 4.4.1. It is easy to see that

$$\begin{aligned} &R^\bullet \mathrm{pr}_{2*}(\mathcal{O}_{B(2,4)}(N_{B(2,4)}) \otimes \mathcal{A}_1) = 0 \text{ for } \bullet > 0, \\ &R^\bullet \mathrm{pr}_{2*}(\mathcal{O}_{B(2,4)}(N_{B(2,4)}) \otimes \mathcal{A}_i) = 0 \text{ for } \bullet \geq 0 \text{ for } i = 2, 3, \end{aligned}$$

since $(\mathcal{O}_{B(2,4)}(N_{B(2,4)}) \otimes \mathcal{A}_i)|_\Gamma \simeq \mathcal{O}_\Gamma(-i+1) \otimes \wedge^i \mathbb{C}^3$. Moreover, we have

$$\begin{aligned} &\mathrm{pr}_{2*}(\mathcal{O}_{B(2,4)}(N_{B(2,4)}) \otimes \mathcal{A}_1) = \rho_2^*\mathcal{Q}^*(L_{\mathscr{Y}_2}), \\ &\mathrm{pr}_{2*}\mathcal{O}_{B(2,4)}(N_{B(2,4)}) = \pi_2^* T(-1)^{\wedge 2}. \end{aligned}$$

Therefore we obtain the short exact sequence

$$0 \to \rho_2^*\mathcal{Q}^*(L_{\mathscr{Y}_2}) \to \pi_2^*T(-1)^{\wedge 2} \to \pi_{2^t*}\mathcal{O}_{\mathscr{X}_2^t}(N_{\mathscr{X}_2^t}) \to 0,$$

which we identify with the pull-back of the dual of the universal exact sequence (4.3) twisted by $L_{\mathscr{Y}_2}$ by the proof of Lemma 4.4.1. Hence $\pi_{2^t*}\mathcal{O}_{\mathscr{X}_2^t}(N_{\mathscr{X}_2^t}) \simeq \rho_2^*\mathcal{S}^*(L_{\mathscr{Y}_2})$ as claimed.
(ii) We tensor (5.21) with the pull back $\mu_B^*\mathcal{W}$ on $B(2,4,\mathscr{Y}_2)$ of $\mathcal{W}$ by $\mu_B^* : B(2,4,\mathscr{Y}_2) \to \mathrm{G}(3,V)$. We see that $R^\bullet \mathrm{pr}_{2*}(\mu_B^*\mathcal{W} \otimes \mathcal{A}_i) = 0$ for $\bullet \geq 0$ and $i = 1,2,3$ by Theorem 2.1.1 since $(\mu_B^*\mathcal{W} \otimes \mathcal{A}_i)|_\Gamma \simeq \mathcal{W}_\Gamma(-i)$, where $\mathcal{W}_\Gamma$ is the universal quotient bundle of rank 2 on $\Gamma \simeq \mathrm{G}(2,4)$. Moreover, $\mathrm{pr}_{2*}\mu_B^*\mathcal{W} \simeq \pi_2^*T(-1)$ by Theorem 2.1.1 and the universal sequence

$$0 \to \mathcal{U}_{\mathrm{G}(2,T(-1))} \to \pi_G^*T(-1) \to \rho_G^*\mathcal{W} \to 0$$

of $\pi_G : \mathrm{G}(2,T(-1)) \to \mathbb{P}(V)$. Therefore we have $\pi_{2^t*}(\mu_{2^t}^*\mathcal{W}) \simeq \pi_2^*T(-1)$ as claimed.
(iii) We tensor (5.21) with $\mathsf{S}^2\mu_B^*\mathcal{W}(-N_{B(2,4)})$. We see that

$$R^\bullet \mathrm{pr}_{2*}(\mathsf{S}^2\mu_B^*\mathcal{W}(-N_{B(2,4)}) \otimes \mathcal{A}_i) = 0 \text{ for } \bullet \geq 0 \text{ and } i = 0,1,3,$$
$$R^\bullet \mathrm{pr}_{2*}(\mathsf{S}^2\mu_B^*\mathcal{W}(-N_{B(2,4)}) \otimes \mathcal{A}_2) = 0 \text{ for } \bullet \neq 2$$

by Theorem 2.1.1 since $(\mathsf{S}^2\mu_B^*\mathcal{W}(-N_{B(2,4)}) \otimes \mathcal{A}_i)|_\Gamma \simeq \mathsf{S}^2\mathcal{W}_\Gamma(-i-1)$. Moreover,

$$R^2\mathrm{pr}_{2*}(\mathsf{S}^2\mu_B^*\mathcal{W}(-N_{B(2,4)}) \otimes \mathcal{A}_2)$$
$$\simeq \{\textstyle\bigwedge^2 \rho_2^*\mathcal{Q}^*\}(L_{\mathscr{Y}_2}) \simeq \rho_2^*(\mathcal{Q} \otimes \det \mathcal{Q}^*)(L_{\mathscr{Y}_2}).$$

By Proposition 4.1.1 (5), we also have

$$\rho_2^*(\mathcal{Q} \otimes \det \mathcal{Q}^*)(L_{\mathscr{Y}_2}) \simeq \rho_2^*\mathcal{Q}(-M_{\mathscr{Y}_2} - F_\rho).$$

Therefore we have

$$\pi_{2^t*}(\{\mu_{2^t}^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{X}_2^t})) \simeq R^2\mathrm{pr}_{2*}(\mathsf{S}^2\mu_B^*\mathcal{W}(-N_{B(2,4)}) \otimes \mathcal{A}_2),$$

which is isomorphic to $\rho_2^*\mathcal{Q}(-M_{\mathscr{Y}_2} - F_\rho)$ as claimed. q.e.d.

Proposition 5.6.3. *There exist the following injective morphisms:*
(i) $\pi_{2^t*}\mathcal{O}_{\mathscr{X}_2^t}(N_{\mathscr{X}_2^t}) \hookrightarrow \pi_{2'*}\mathcal{O}_{\mathscr{X}_2}(N_{\mathscr{X}_2})$,
(ii) $\pi_{2^t*}(\mu_{2^t}^*\mathcal{W}) \hookrightarrow \pi_{2'*}(\mu_2^*\mathcal{W})$,
(iii) $\pi_{2^t*}(\{\mu_{2^t}^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{X}_2^t}) \hookrightarrow \pi_{2'*}(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{X}_2})$.

Proof. Since $\mathscr{X}_2^t$ is the total transform of the blow-up $\mathscr{X}_3 \times_{\mathscr{Y}_3} \mathscr{Y}_2 \to \mathscr{X}_3$ along $\pi_{3'}^{-1}(\mathscr{P}_\rho)$ and $\mathscr{X}_2$ is the strict transform of $\mathscr{X}_3$, we have a natural morphism $\pi_{2^t*}(\mathcal{B}|_{\mathscr{X}_2^t}) \to \pi_{2'*}(\mathcal{B}|_{\mathscr{X}_2})$ for the sheaves on $\mathscr{Y}_2$ associated to any sheaf $\mathcal{B}$ on $\mathscr{X}_2^u = \mathbb{P}(\rho_2^*\mathcal{S})$. We note that the morphism is injective if the sheaf $\pi_{2^t*}(\mathcal{B}|_{\mathscr{X}_2^t})$ is locally free. Lemma 5.6.2 indicates that this is the case for sheaves $\mathcal{B}$ such that $\mathcal{B}|_{\mathscr{X}_2^t} = \pi_{2^t*}\mathcal{O}_{\mathscr{X}_2^t}(N_{\mathscr{X}_2^t}), \pi_{2^t*}(\mu_{2^t}^*\mathcal{W})$ and $\pi_{2^t*}\big(\mu_{2^t}^*\mathsf{S}^2\mathcal{W}(-N_{\mathscr{X}_2^t})\big)$. q.e.d.

5.6.2. Part 2: Here we determine the sheaves in (5.19) and complete our construction of the locally free resolution of $\mathcal{I}_2^o \otimes \{\mathcal{O}_{\mathscr{Y}_2^o}(M_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\}$ as follows:

$$
\begin{aligned}
0 &\to \iota_2^*\rho_2^*\mathcal{S}(-L_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}} \to \iota_2^*\mathcal{T}_2^* \boxtimes g^*\mathcal{F}^* \\
&\to \mathcal{O}_{\mathscr{Y}_2^o} \boxtimes g^*\mathsf{S}^2\mathcal{F}^* \oplus \iota_2^*\rho_2^*\mathcal{Q}^*(M_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}}) \\
&\to \mathcal{I}_2^o \otimes \{\mathcal{O}_{\mathscr{Y}_2^o}(M_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\} \to 0.
\end{aligned}
\tag{5.22}
$$

Proposition 5.6.4. *It holds that*

(1) $\pi_{2'*}\mathcal{O}_{\mathscr{Z}_2}(N_{\mathscr{Z}_2}) \simeq \rho_2^*\mathcal{S}^*(L_{\mathscr{Y}_2})$.
(2) $\pi_{2'*}(\mu_2^*\mathcal{W}) \simeq \mathcal{T}_2$ *(cf.* (5.5)*), and*
(3) $\pi_{2'*}(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2})) \simeq \rho_2^*\mathcal{Q}(-M_{\mathscr{Y}_2})$.

Proof. From Lemma 5.6.3, we have a natural injection $\pi_{\mathscr{Z}_2^t *}(\mathcal{B}|_{\mathscr{Z}_2^t}) \hookrightarrow \pi_{2'*}(\mathcal{B}|_{\mathscr{Z}_2})$ for the sheaves $\mathcal{B}$ described there. Note that the injection is isomorphic outside F_ρ. Let y be a point of F_ρ and q the fiber of $\mathscr{Z}_2 \to \mathscr{Y}_2$ over y. Let $[V_1]$ be the image of y on $\mathbb{P}(V)$. We write $\mathbb{P}_q^2 = \mathbb{P}(\bar{U})$ with $\bar{U}$ a three-dimensional subspace of $\wedge^2(V/V_1)$. Since q is a ρ-conic, there exists a 2-dimensional subspace V_2 such that $V_1 \subset V_2$ and $\bar{U} = (V/V_2)\wedge(V_2/V_1) \simeq V/V_2$, namely $\rho_2(y) = ([(V/V_2)\wedge(V_2/V_1)], [V_1]) \in \mathscr{Y}_3$.

To see (1), we note the injection in Proposition 5.6.3 (i),

$$\rho_2^*\mathcal{S}^*(L_{\mathscr{Y}_2}) \simeq \pi_{2^t *}\mathcal{O}_{\mathscr{Z}_2^t}(N_{\mathscr{Z}_2^t}) \hookrightarrow \pi_{2'*}\mathcal{O}_{\mathscr{Z}_2}(N_{\mathscr{Z}_2}).$$

with the first isomorphism in Lemma 5.6.2 (i). This injection must be an isomorphism since both the fibers of $\rho_2^*\mathcal{S}^*(L_{\mathscr{Y}_2})$ and $\pi_{2'*}\mathcal{O}_{\mathscr{Z}_2}(N_{\mathscr{Z}_2})$ at y are isomorphic to $H^0(\mathbb{P}_q^2, \mathcal{O}_{\mathbb{P}_q^2}(1))$ (note that $N_{\mathscr{Z}_2} = (H_{\mathbb{P}(T(-1)^{\wedge 2})} + L_{\mathbb{P}(T(-1)^{\wedge 2})})|_{\mathscr{Z}_2}$ from (4.9)).

We show (2). By Lemma 5.6.2 (ii), we have $\pi_2^*T(-1) \simeq \pi_{2^t *}(\mu_{2^t}^*\mathcal{W})$. We now compute the map

$$\pi_2^*T(-1) \otimes k(y) \simeq \pi_{2^t *}(\mu_{2^t}^*\mathcal{W}) \otimes k(y) \to \pi_{2'*}(\mu_2^*\mathcal{W}) \otimes k(y). \tag{5.23}$$

We first write $\pi_2^*T(-1) \otimes k(y) \simeq V/V_1$ and $\pi_{2'*}(\mu_2^*\mathcal{W}) \otimes k(y) \simeq H^0(q, \mu_2^*\mathcal{W}|_q)$ by the Grauert theorem. Then, by Lemma 5.4.1, the map $V/V_1 \to H^0(q, \mu_2^*\mathcal{W}|_q)$ factors through $H^0(\mathbb{P}_q^2, T_{\mathbb{P}_q^2}(-1)) \simeq V/V_2$. Therefore the cokernel of the dual of the map (5.23) is isomorphic to $(V_2/V_1)^*$, which can be identified with the fiber of $\rho_2|_{F_\rho}^*\mathcal{O}_{\mathbb{P}(T(-1))}(1)$ at y (where $\rho_2|_{F_\rho} : F_\rho \to \mathscr{P}_\rho \simeq \mathbb{P}(T(-1))$). Since both the cokernel of

$$(\{\pi_{2'*}(\mu_2^*\mathcal{W})\}^*) \otimes k(y) \to (\{\pi_{2^t *}(\mu_{2^t}^*\mathcal{W})\}^*) \otimes k(y)$$

and that of

$$(\mathcal{T}_2^*) \otimes k(y) \to (\{\pi_{2^t *}(\mu_{2^t}^*\mathcal{W})\}^*) \otimes k(y)$$

are the same one-dimensional subspace, the images of them must coincide, i.e., $\{\pi_{2'*}(\mu_2^*\mathcal{W})\}^* = \mathcal{T}_2^*$ as claimed.

Finally we show (3). Let $E_u := \mathbb{P}(\rho_2^*\mathcal{S}|_{F_\rho})$ which is the restrictions of $\mathscr{Z}_2^u$ over F_ρ, and recall our definition $E_\rho = \pi_{2'}^{-1}(F_\rho)$ in Subsection 4.3. Note that E_u is the exceptional divisor of the blow-up $\mathscr{Z}_2^u \to \mathscr{Z}_3^u$ discussed after Definition 4.3.5. We summarize the geometry of the blow-up and morphisms in the following diagram:

(5.24)
$$\begin{tikzcd}
\mathscr{Z}_3 & \subset & \mathscr{Z}_3^u & & \mathscr{Z}_2^u \arrow[ll] & \supset & \mathscr{Z}_2 \\
\cup & & \cup & & \cup & & \cup \\
\mathscr{Z}_\rho \arrow[drr, "\pi_\rho=\pi_{3'}|_\rho"'] & = & \mathscr{Z}_\rho^u \arrow[d, "\pi_{3^u}|_\rho"] & & E_u = \mathbb{P}(\rho_2^*\mathcal{S}|_{F_\rho}) \arrow[ll] \arrow[d, "\pi_{2^u}|_\rho"'] & & \supset E_\rho \arrow[dll, "\pi_{2'}|_\rho"] \\
& & \mathscr{P}_\rho & & F_\rho \arrow[ll, "\rho_{F_\rho}=\rho_2|_{F_\rho}"] & &
\end{tikzcd}$$

Let $\mathcal{W}_{E_u}$ be the pull-back of $\mathcal{W}_\rho$ (see Proposition 4.2.4) by $E_u \to \mathscr{Z}_\rho^u = \mathscr{Z}_\rho$. Then, by the proof of Lemma 5.6.1, we have

(5.25)
$$\begin{aligned}
&\pi_{2'|_\rho *}\big(\{\mu_2^*\mathsf{S}^2\mathcal{W}\}(-N_{\mathscr{Z}_2})|_{E_\rho}\big) \\
&\simeq R^1\pi_{2^u|_\rho *}\big(\mathsf{S}^2\mathcal{W}_{E_u}(-N_{\mathscr{Z}_2^u}|_{E_u} - \mathscr{Z}_2|_{E_u})\big).
\end{aligned}$$

In Lemma 5.6.5 below, we rewrite the r.h.s. of (5.25) in terms of the pull-back $\mathcal{R}_{E_\rho^u}$ of $\mathcal{R}_\rho$ (introduced in (4.5)) to E_u. The first factor $R^1\pi_{2^u|_\rho *}(\mathsf{S}^2\mathcal{R}_{E_u}(-3\det\mathcal{R}_{E_u})) = R^1\pi_{2^u|_\rho *}(\Sigma\mathcal{R}_{E_u}^{(3,1)*})$ of (5.26) can be evaluated by the Bott theorem 2.1.1 applied to the projective bundle $E_u \to F_\rho$ with its Euler sequence; $0 \to \mathcal{O}_{E_u}(-1) \to \rho_{F_\rho}^*(\mathcal{S}|_{\mathscr{P}_\rho}) \to R_{E_u} \to 0$. It turns out that the first factor is isomorphic to

$$\rho_{F_\rho}^*(\mathcal{S}^* \otimes \det\mathcal{S}^*|_{\mathscr{P}_\rho}) \simeq \rho_{F_\rho}^*(\mathcal{Q} \otimes \det\mathcal{Q}|_{\mathscr{P}_\rho})(-4L_{F_\rho}),$$

where we use Proposition 4.1.1 (1) and (3). Now evaluating (5.26) for the r.h.s. of (5.25), we obtain $\rho_{F_\rho}^*\mathcal{Q}(-M_{\mathscr{Y}_2}|_{F_\rho})$. q.e.d.

Lemma 5.6.5. *The r.h.s. of* (5.25) *is isomorphic to*

(5.26)
$$\begin{aligned}
&R^1\pi_{2^u|_\rho *}(\mathsf{S}^2\mathcal{R}_{E_u}(-3\det\mathcal{R}_{E_u})) \\
&\otimes \mathcal{O}_{F_\rho}(-\rho_{F_\rho}^*\det\mathcal{Q}|_{\mathscr{P}_\rho} + 4L_{F_\rho} - M_{\mathscr{Y}_2}|_{F_\rho}).
\end{aligned}$$

Proof. Let us first note the following relations which follows from Lemma 4.2.4,

$$\mathsf{S}^2\mathcal{W}_\rho \simeq \mathsf{S}^2\mathcal{R}_\rho \otimes \pi_\rho^*\mathcal{O}_{\mathbb{P}(T(-1))}(2) \simeq \mathsf{S}^2\mathcal{R}_\rho(\pi_\rho^*(\det\mathcal{Q}|_{\mathscr{P}_\rho}) - 2L_{\mathscr{Z}_\rho}),$$

where for the second isomorphism we use Proposition 4.1.1 (4) and the equality $H_{\mathscr{P}_\rho} = \mathcal{O}_{\mathbb{P}(T(-1))}(1)$. Then, pulling this back to E_u, we obtain

(5.27)
$$\mathsf{S}^2\mathcal{W}_{E_u} \simeq \mathsf{S}^2\mathcal{R}_{E_u}(\pi_{2^u|_\rho}^*\rho_{F_\rho}^*(\det\mathcal{Q}|_{\mathscr{P}_\rho}) - 2L_{E_u}).$$

Second, we take the determinants of (4.5) to have

(5.28)
$$\begin{aligned}
H_{\mathbb{P}(\mathcal{S}|_{\mathscr{P}_\rho})} &= \det\mathcal{R}_\rho + \pi_\rho^*(\det\mathcal{S}^*|_{\mathscr{P}_\rho}) \\
&= \det\mathcal{R}_\rho + \pi_\rho^*(\det\mathcal{Q}|_{\mathscr{P}_\rho}) - 3L_{\mathscr{Z}_\rho^u},
\end{aligned}$$

where Proposition 4.1.1 (1) is used for the second equality. Therefore, using Proposition 4.5.1 (1), we obtain

$$N_{\mathscr{Z}_3^u}|_{\mathscr{Z}_\rho^u} = \det \mathcal{R}_\rho + \pi^*_{3^u|_\rho}(\det \mathcal{Q}|_{\mathscr{P}_\rho}) - 2L_{\mathscr{Z}_\rho^u},$$

and then pulling this back to E_ρ^u, we have

$$N_{\mathscr{Z}_2^u}|_{E_u} = \det \mathcal{R}_{E_\rho^u} + \pi^*_{2^u|_\rho}(\rho^*_{F_\rho} \det \mathcal{Q}|_{\mathscr{P}_\rho}) - 2L_{E_u}. \tag{5.29}$$

Now, from (5.27) and (5.29), we have

$$\mathsf{S}^2\mathcal{W}_{E_u}(-N_{\mathscr{Z}_2^u}|_{E_u}) \simeq \mathsf{S}^2\mathcal{R}_{E_u}(-\det \mathcal{R}_{E_u}). \tag{5.30}$$

Let us compute the class of the divisor $\mathscr{Z}_2|_{E_u}(= E_\rho)$ in E_u. We see that $\mathscr{Z}_2 \in |2H_{\mathbb{P}(\rho_2^*\mathcal{S})} + L_{\mathscr{Z}_2^u} - \pi^*_{2^u}F_\rho|$ since $\mathscr{Z}_2$ is the strict transform of $\mathscr{Z}_3$ by the blow-up $\mathscr{Z}_2^u \to \mathscr{Z}_3^u \simeq \mathbb{P}(\mathcal{S})$ and $\mathscr{Z}_3 \in |2H_{\mathbb{P}(\mathcal{S})} + L_{\mathscr{Z}_3^u}|$ by the proof of Proposition 4.5.1. Using Proposition 4.1.1 (5), we now have

$$2H_{\mathbb{P}(\rho_2^*\mathcal{S})}+L_{\mathscr{Z}_2^u}-\pi^*_{2^u}F_\rho = 2H_{\mathbb{P}(\rho_2^*\mathcal{S})}+2L_{\mathscr{Z}_2^u}-\pi^*_{2^u}(\rho^*_{F_\rho}\det \mathcal{Q}|_{\mathscr{P}_\rho}-M_{\mathscr{Y}_2}|_{F_\rho}).$$

Therefore, using (5.28) for $H_{\mathbb{P}(\rho_2^*\mathcal{S})}$, we obtain

$$\mathscr{Z}_2|_{E_u} \in |2\det \mathcal{R}_{E_u} + \pi^*_{2^u}(\rho^*_{F_\rho}\det \mathcal{Q}|_{\mathscr{P}_\rho}) - 4L_{E_u} + \pi^*_{2^u}(M_{\mathscr{Y}_2}|_{F_\rho})|. \tag{5.31}$$

From (5.30) and (5.31), we obtain the claimed form for (5.25). q.e.d.

5.7. Step 3: A locally free resolution of $\mathcal{I}^o$ on $\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}$.

We calculate the pushforward $(\tilde{\rho}_2^o \times \mathrm{id})_*$ of the exact sequence (5.22). To do this, we split (5.22) as follows:

$$0 \to \iota_2^*\rho_2^*\mathcal{S}(-L_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}} \to \iota_2^*\mathcal{T}_2^* \boxtimes g^*\mathcal{F}^* \to \mathcal{C} \to 0 \tag{5.32}$$

and

$$\begin{aligned} 0 \to \mathcal{C} \to \mathcal{O}_{\mathscr{Y}_2^o}\boxtimes g^*\mathsf{S}^2\mathcal{F}^* \oplus \iota_2^*\rho_2^*\mathcal{Q}^*(M_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}}) \to \\ \mathcal{I}_2^o \otimes \{\mathcal{O}_{\mathscr{Y}_2^o}(M_{\mathscr{Y}_2^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\} \to 0. \end{aligned} \tag{5.33}$$

Since $\rho_2^*\mathcal{S}(-L_{\mathscr{Y}_2})$, $\mathcal{T}_2^*$ and $\rho_2^*\mathcal{Q}^*(M_{\mathscr{Y}_2})$ are the pull-backs of locally free sheaves $\widetilde{\mathcal{S}}_L$, $\widetilde{\mathcal{T}}^*$, and $\widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}})$ on $\widetilde{\mathscr{Y}}$, the higher direct images of $\iota_2^*\rho_2^*\mathcal{S}(-L_{\mathscr{Y}_2}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}$ and $\iota_2^*\mathcal{T}_2^* \boxtimes g^*\mathcal{F}$ vanish. Therefore the pushforward of (5.32) is still exact and the higher direct images of $\mathcal{C}$ vanish. Then the pushforward of (5.32) is also exact. Therefore we obtain the following exact sequence on $\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}$:

$$\begin{aligned} 0 \to \tilde{\iota}^*\widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}} &\to \tilde{\iota}^*\widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^* \\ &\to \mathcal{O}_{\widetilde{\mathscr{Y}}^o} \boxtimes g^*\mathsf{S}^2\mathcal{F}^* \oplus \tilde{\iota}^*\widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}}) \\ &\to \mathcal{I}^o \otimes \{\mathcal{O}_{\widetilde{\mathscr{Y}}^o}(M_{\widetilde{\mathscr{Y}}^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\} \to 0, \end{aligned} \tag{5.34}$$

where $\mathcal{I}^o := \tilde{\rho}'_{2*}\mathcal{I}_2^o$.

Lemma 5.7.1. *Let Δ^o be the closed subscheme of $\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}$ defined by $\mathcal{I}_2^o$. Define $\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}} := \tilde{\rho}_2^o \times \mathrm{id}|_{\Delta^o_{\mathscr{Y}_2}}$, then $\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}*}\mathcal{O}_{\Delta^o_{\mathscr{Y}_2}} = \mathcal{O}_{\Delta^o}$ and $R^1\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}*}\mathcal{O}_{\Delta_{\mathscr{Y}_2}{}^o} = 0$.*

Proof. By (5.32) and (5.33), we have $R^1(\tilde{\rho}_2^o \times \mathrm{id})_*\mathcal{I}_2{}^o = 0$ since $\rho_2^*\mathcal{S}(-L_{\mathscr{Y}_2})$, $\mathcal{T}^*$ and $\rho_2^*\mathcal{Q}^*(M_{\mathscr{Y}_2})$ are the pull-backs of locally free sheaves on $\widetilde{\mathscr{Y}}$. Taking the higher direct image of the exact sequence $0 \to \mathcal{I}_2{}^o \to \mathcal{O}_{\mathscr{Y}_2^o \times \check{\mathscr{X}}} \to \mathcal{O}_{\Delta^o_{\mathscr{Y}_2}} \to 0$, we obtain the exact sequence $0 \to \mathcal{I}^o \to \mathcal{O}_{\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}} \to \tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}*}\mathcal{O}_{\Delta^o_{\mathscr{Y}_2}} \to 0$ and also $R^1\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}*}\mathcal{O}_{\Delta^o_{\mathscr{Y}_2}} = 0$ since $R^1(\tilde{\rho}_2^o \times \mathrm{id})_*\,\mathcal{I}_2^o = 0$ and $R^1(\tilde{\rho}_2^o \times \mathrm{id})_*\,\mathcal{O}_{\mathscr{Y}_2^o \times \check{\mathscr{X}}} = 0$, respectively. Hence $\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}*}\mathcal{O}_{\Delta^o_{\mathscr{Y}_2}} = \mathcal{O}_{\Delta^o}$. q.e.d.

5.8. Step 4: Ideal sheaf $\mathcal{I}$ and its resolution (5.4) . Let $\mathcal{I} := \tilde{\iota}_*\mathcal{I}^o$ and set $\Gamma_{\widetilde{\mathscr{Y}}} := \widetilde{\mathscr{Y}} \times \check{\mathscr{X}} \setminus \widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}$. We note that $\operatorname{codim}\Gamma_{\widetilde{\mathscr{Y}}} = 6$ since the codimension of $\mathcal{P}_\sigma$ in $\widetilde{\mathscr{Y}}$ is 6. Define a sheaf $\mathcal{A}$ on $\widetilde{\mathscr{Y}}^o$ by

$$0 \to \tilde{\iota}^*\widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}} \to \tilde{\iota}^*\widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^* \to \mathcal{A} \to 0,$$

then, we have the following exact sequence

$$\begin{aligned} 0 \to \widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}} \to \widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^* \to \tilde{\iota}_*\mathcal{A} \to R^1\tilde{\iota}_*(\tilde{\iota}^*\widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}}) \\ \to R^1\tilde{\iota}_*(\tilde{\iota}^*\widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^*) \to R^1\tilde{\iota}_*\mathcal{A} \to R^2\tilde{\iota}_*(\tilde{\iota}^*\widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}}), \end{aligned} \tag{5.35}$$

and also from (5.34),

$$\begin{aligned} 0 \to \tilde{\iota}_*\mathcal{A} \to & \mathcal{O}_{\widetilde{\mathscr{Y}}} \boxtimes g^*\mathsf{S}^2\mathcal{F}^* \oplus \widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}}) \to \\ & \mathcal{I} \otimes \{\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\} \to R^1\tilde{\iota}_*\mathcal{A}, \end{aligned} \tag{5.36}$$

where we note that $\tilde{\iota}_*(\mathcal{E}|_{\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}}) = \mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ since $\operatorname{codim}\Gamma_{\widetilde{\mathscr{Y}}} \geq 2$, and $\tilde{\iota}_*(\mathcal{I}^o \otimes \{\mathcal{O}_{\widetilde{\mathscr{Y}}^o}(M_{\widetilde{\mathscr{Y}}^o}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\}) = \mathcal{I} \otimes \{\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\}$ by definition. For a sheaf $\mathcal{E}$ on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$, it holds that $R^i\tilde{\iota}_*(\mathcal{E}|_{\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}}) = \mathcal{H}^{i+1}_{\Gamma_{\widetilde{\mathscr{Y}}}}(\mathcal{E})$ for $i > 0$ by [**H**, p.9, Corollary 1.9]. Moreover, if $\mathcal{E}$ is locally free, then $\mathcal{H}^{i+1}_{\Gamma_{\widetilde{\mathscr{Y}}}}(\mathcal{E}) = 0$ for $i+1 < 4$ by [**H**, p.44, Theorem 3.8] since $\operatorname{codim}\Gamma_{\widetilde{\mathscr{Y}}} = 6$. Therefore,

$$R^1\tilde{\iota}_*(\tilde{\iota}^*\widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}}) \simeq R^1\tilde{\iota}_*(\tilde{\iota}^*\widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^*) \simeq R^2\tilde{\iota}_*(\tilde{\iota}^*\widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}}) = 0.$$

From (5.35), we see that $R^1\tilde{\iota}_*\mathcal{A} = 0$. Consequently, we obtain the claim that the sequence (5.4) in Theorem 5.1.3 is exact on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ by (5.35) and (5.36).

5.9. Δ is normal and Cohen-Macaulay. Let $\Delta \subset \widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ be the variety determined by the ideal sheaf $\mathcal{I}$. We show that Δ is normal and is Cohen-Macaulay to complete our proof of Theorem 5.1.3.

We see that Δ_2 is smooth by Proposition 5.3.3, and also $\Delta_2 \to \Delta_{\mathscr{Y}_2}$ is birational since $\Delta_3 \to \Delta_{\mathscr{Y}_3}$ is birational by Proposition 5.3.2 and so are $\Delta_2 \to \Delta_3$ and $\Delta_{\mathscr{Y}_2} \to \Delta_{\mathscr{Y}_3}$. Recall that we have set $\breve{\pi}_{\Delta_2^o} := \breve{\pi}_{2'}|_{\Delta_2^o}$ and $\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}} := (\tilde{\rho}_2^o \times \mathrm{id})_*|_{\Delta^o_{\mathscr{Y}_2}}$. We check the claimed properties above separately on $\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}$ and $\mathscr{P}_\sigma \times \check{\mathscr{X}}$.

Let us first consider the properties on $\widetilde{\mathscr{Y}}^o \times \check{\mathscr{X}}$. We show that

$$\begin{aligned} R^i(\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}} \circ \breve{\pi}_{\Delta_2^o})_*\mathcal{O}_{\Delta_2^o} = 0 \text{ for } i > 0 \\ \text{and } (\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}} \circ \breve{\pi}_{\Delta_2^o})_*\mathcal{O}_{\Delta_2^o} \simeq \mathcal{O}_{\Delta^o}. \end{aligned} \tag{5.37}$$

The latter shows that $\Delta^o = \tilde{\iota}^*\Delta$ is normal since Δ_2^o is smooth, and the former show that Δ^o has only rational singularities, and then is Cohen-Macaulay. By using the Leray spectral sequence, the relations in (5.37) follow from Lemma 5.4.3 for $\breve{\pi}_{\Delta_2^o}$ and Lemma 5.7.1 for $\tilde{\rho}_{\Delta^o_{\mathscr{Y}_2}}$.

Now we consider the properties on $\mathscr{P}_\sigma \times \check{\mathscr{X}}$. By the result above, we see that Δ is regular in codimension one since the codimension of $\Delta \cap (\mathscr{P}_\sigma \times \check{\mathscr{X}})$ is greater than two. Therefore it suffices to show that Δ is Cohen-Macaulay at any point of $\Delta \cap (\mathscr{P}_\sigma \times \check{\mathscr{X}})$. This follows from taking the local cohomology sequence of the locally free resolution (5.4) since $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ is smooth and the length of the locally free part of (5.4) is three.

Remark. Since we have shown that Δ is reduced, Δ is the closure of Δ^o.

6. The universal family of hyperplane sections

Let $\mathscr{V} \subset \widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ be the pull-back of the universal family of hyperplane sections in $\mathbb{P}(\mathsf{S}^2V^*) \times \mathbb{P}(\mathsf{S}^2V)$. We can regard $\mathscr{V}$ as the family of the pull-backs of hyperplanes in $\mathscr{X}$ parameterized by $\widetilde{\mathscr{Y}}$, and also as the family of the pull-backs of hyperplanes in $\mathscr{H}$ parameterized by $\check{\mathscr{X}}$. Retaining these two meanings, we say that $\mathscr{V}$ is the universal family of hyperplane sections of $\widetilde{\mathscr{Y}}$ and $\check{\mathscr{X}}$. Note that the fiber of $\mathscr{V}$ over a point $x \in X \subset \check{\mathscr{X}}$ is the pull-back of the hyperplane section $w_{\boldsymbol{xy}}^{\perp} \cap \mathscr{H}$ of $\mathscr{H}$, where $x = w_{\boldsymbol{xy}}$ as a point of $\mathsf{S}^2\mathbb{P}(V)$.

Let $\mathcal{I}_{\mathscr{V}} = \mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}})$ be the ideal sheaf of $\mathscr{V} \subset \widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$. We show, in the followng proposition, that $\Delta \subset \widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ is a closed subscheme of $\mathscr{V}$ and give a locally free resolution of the ideal sheaf of Δ in $\mathscr{V}$ as an $\mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}$-module. It should be noted that the locally free sheaves in this resolution are exactly those used to construct the (dual) Lefschetz collections in $\mathcal{D}^b(\widetilde{\mathscr{Y}})$ and $\mathcal{D}^b(\check{\mathscr{X}})$ in [**HoTa3**].

Proposition 6.0.1. *$\mathcal{I}$ contains $\mathcal{I}_{\mathscr{V}}$,i.e., the subvariety Δ is contained in $\mathscr{V}$. Define $\mathcal{I}_{\Delta/\mathscr{V}} := \mathcal{I}/\mathcal{I}_{\mathscr{V}}$, which is the ideal sheaf of Δ in $\mathscr{V}$, and denote by $\iota_{\mathscr{V}}$ the closed immersion $\mathscr{V} \hookrightarrow \widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$. Then $\iota_{\mathscr{V}*}\mathcal{I}_{\Delta/\mathscr{V}}$ has the following locally free resolution on $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$:*

$$\begin{aligned} &0 \to \widetilde{\mathcal{S}}_L \boxtimes \mathcal{O}_{\check{\mathscr{X}}} \to \widetilde{\mathcal{T}}^* \boxtimes g^*\mathcal{F}^* \to \\ &\mathcal{O}_{\widetilde{\mathscr{Y}}} \boxtimes \{g^*\mathsf{S}^2\mathcal{F}^*/\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}} + 2L_{\check{\mathscr{X}}})\} \oplus \widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(L_{\check{\mathscr{X}}}) \\ &\qquad \to \iota_{\mathscr{V}*}\mathcal{I}_{\Delta/\mathscr{V}} \otimes \{\mathcal{O}_{\widetilde{\mathscr{Y}}}(M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(2L_{\check{\mathscr{X}}})\} \to 0. \end{aligned} \tag{6.1}$$

where the quotient $g^*\mathsf{S}^2\mathcal{F}^*/\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}} + 2L_{\check{\mathscr{X}}})$ is defiend by the inclusion $\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}) \subset g^*\mathsf{S}^2\mathcal{F} = g^*\mathsf{S}^2\mathcal{F}^*(-2L_{\check{\mathscr{X}}})$ in the Euler sequence $0 \to \mathcal{O}_{\mathbb{P}(\mathsf{S}^2\mathcal{F})}(-1) \to g^*\mathsf{S}^2\mathcal{F} \to T_{\mathbb{P}(\mathsf{S}^2\mathcal{F})/\mathrm{G}(2,V)}(-1) \to 0$ for $\check{\mathscr{X}} = \mathbb{P}(\mathsf{S}^2\mathcal{F})$.

Proof. Note that $\mathscr{V}$ admits a natural $\mathrm{SL}(V)$-action, and hence the injection $\mathcal{I}_{\mathscr{V}} = \mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}) \to \mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}$ is $\mathrm{SL}(V)$-equivariant. We apply the construction described in Subsection 5.2 to this $\mathrm{SL}(V)$-equivariant injection noting $\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}), \mathcal{O}_{\widetilde{\mathscr{Y}}})) \simeq \mathsf{S}^2V$ and $\mathrm{Hom}(\mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}), \mathcal{O}_{\check{\mathscr{X}}}) \simeq \mathsf{S}^2V^*$. Then we see that the injection is unique corresponding to the fact that $\mathsf{S}^2V \otimes \mathsf{S}^2V^*$contains a unique one-dimensional representation of $\mathrm{SL}\,(V)$.

We now turn to the $\mathrm{SL}(V)$-equivariant map

$$\mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}) \to \mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathsf{S}^2(g^*\mathcal{F}),$$

which we have from the inclusion $\mathcal{O}_{\check{\mathscr{X}}}(-1) \to g^*\mathsf{S}^2\mathcal{F}$. From the resolution (5.4) and the relation $\mathcal{F}^* = \mathcal{F}(L_{\check{\mathscr{X}}})$, we obtain an $\mathrm{SL}(V)$-equivariant inclusion

$$\begin{aligned} &\mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}) \\ &\to \mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathsf{S}^2(g^*\mathcal{F}) \oplus \widetilde{\mathcal{Q}}^* \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(-L_{\check{\mathscr{X}}}) \to \mathcal{I} \hookrightarrow \mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}. \end{aligned}$$

By the uniqueness of the embedding $\mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}}) \boxtimes \mathcal{O}_{\check{\mathscr{X}}}(-H_{\check{\mathscr{X}}}) \to \mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}$, the image coincides with $\mathcal{I}_{\mathscr{V}}$ and hence $\mathcal{I}_{\mathscr{V}} \subset \mathcal{I}$.

The proof of the remaining assertion follows from the above discussion and Theorem 5.1.3. q.e.d.

In Proposition 6.0.2 below, we show that any hyperplane section of $\widetilde{\mathscr{Y}}$ corresponding to a point of $\check{\mathscr{X}}$ has only canonical singularities. This property is important to apply the Kawamata-Viehweg vanishing theorem in our proof of the derived equivalence (Lemma 8.0.9).

To describe the statement, recall that $f\colon \check{\mathscr{X}} \to \mathscr{X}$ is the Hilbert-Chow morphism and E_f is the f-exceptional divisor as in Subsection 2.2. Let x be a point of $\mathbb{P}(V)$ and e any point of E_f such $f(e) = [2x]$.

Then the fiber of $\mathscr{V} \to \check{\mathscr{X}}$ over e is the pull-back of the hyperplane section of $\mathscr{H}$ parameterizing singular quadrics which contain the point x. In particular, the fiber is independent of a choice of e once we fix a point x, and hence we denote it by V_x.

Proposition 6.0.2. *Any fiber of $\mathscr{V} \to \check{\mathscr{X}}$ is normal and has only canonical singularities.*

Proof. The proof is similar to the argument in Subsection 5.9.

It suffices to show the assertion for V_x $(x \in \mathbb{P}(V))$ since it is a special fiber of $\mathscr{V} \to \check{\mathscr{X}}$. Let V_x^t is the strict transform of V_x on $\mathscr{Y}_2$, which is also the total transform since V_x does not contains the center $G_{\widetilde{\mathscr{Y}}}$ of the birational morphism $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$. Hence $V_x^t \in |M_{\mathscr{Y}_2}|$. We see that $-K_{V_x^t}$ is $(\widetilde{\rho}_2|_{V_x^t})$-ample since $V_x^t \in |M_{\mathscr{Y}_2}|$ and $-K_{\mathscr{Y}_2} = 10M_{\mathscr{Y}_2}$ is $\widetilde{\rho}_2$-ample. Therefore it suffices to show similar assertions for V_x^t. Let W_G, W_3 and W_2 be the pull-backs on $\mathrm{G}(2, T(-1))$, $\mathscr{Z}_3$, and $\mathscr{Z}_2$, respectively, of the subvariety $W_0 := \{\Pi \mid x \in \Pi\} \simeq \mathrm{G}(2,4)$ in $\mathrm{G}(3, V)$. It is easy to see that V_x^t is the image of W_2 set-theoretically and $W_2 \to V_x^t$ is birational (note that, once we fix a general quadric Q and a $\mathbb{P}^1$-family q of planes in Q, there is only one plane in q which contains x). We show that W_2 is smooth, therefore $W_2 \to V_x^t$ is a resolution of singularities. Indeed, W_3 is smooth since $W_0 \simeq \mathrm{G}(2,4)$, $W_G \to W_0$ is a $\mathbb{P}^2$-bundle, and $W_3 \to W_G$ is a $\mathrm{G}(2,5)$-bundle by Proposition 4.2.2. Then W_2 is also smooth since $W_2 \to W_3$ is the blow-up along $\pi_{3'}^{-1}(\mathscr{P}_\rho) \cap W_3$, which is a $\mathbb{P}^1$-bundle over W_G by Proposition 4.2.3, and hence is smooth.

Similarly to the proof of Proposition 5.1.1, we note that the ideal sheaf $\mathcal{I}_{W_0}$ of W_0 has the following Koszul resolution:

$$0 \to \bigwedge^2 \mathcal{W}^* \to \mathcal{W}^* \to \mathcal{I}_{W_0} \to 0. \tag{6.2}$$

Now we check the assertions on $\mathscr{Y}_2^o$. Since V_x^t is Gorenstein, we have only to show that V_x^t is normal and has only rational singularities. In the same way to (5.15), we obtain $\pi^o_{2'*}\iota_2^*\mu_2^*\mathcal{W}^* = R^1\pi^o_{2'*}\iota_2^*\mu_2^*\mathcal{W}^* = 0$. Therefore we have

$$\pi^o_{2'*}\mathcal{I}^o_{W_2} \simeq R^1\pi^o_{2'*}\big(\bigwedge^2 \iota_2^*\mu_2^*\mathcal{W}^*\big), \tag{6.3}$$

where $\mathcal{I}^o_{W_2}$ is the ideal sheaf of W_2 in $\mathscr{Y}_2^o$. In the same way to (5.16), using the Grothendieck-Verdier duality 2.1.2, we obtain

$$\begin{aligned} R^1\pi^o_{2'*}\bigwedge^2 \iota_2^*\mu_2^*\mathcal{W}^* &\simeq \Big(\pi^o_{2'*}\{\bigwedge^2 \iota_2^*\mu_2^*\mathcal{W} \otimes \mathcal{O}_{\mathscr{Z}_2^o}(-N_{\mathscr{Z}_2^o})\} \otimes \mathcal{O}_{\mathscr{Y}_2^o}(M_{\mathscr{Y}_2^o})\Big)^* \\ &\simeq \mathcal{O}_{\mathscr{Y}_2^o}(-M_{\mathscr{Y}_2^o}). \end{aligned}$$

Therefore $\pi^o_{2'*}\mathcal{I}^o_{W_2}$ is nothing but the ideal sheaf $\mathcal{I}^o_{V_x^t}$ of V_x^t since V_x^t is the image of W_2 set-theoretically. In other words, V_x^t is the scheme-theoretic pushforward of W_2. We set $\pi_W^0 := \pi^o_{2'}|_{W_2} \colon W_2 \to V_x^t$. In the same way to show Lemma 5.7.1, we also have

$$R^k\pi^o_{W*}\mathcal{O}_{W_2}=0 \text{ for } k>0 \text{ and } \pi^o_{W*}\mathcal{O}_{W_2}\simeq\mathcal{O}_{V^t_x}. \tag{6.4}$$

Since W_2 is smooth, the latter shows that V^t_x is normal and the former shows that V^t_x has only rational singularities.

Second we show the assertions on the whole $\mathscr{Y}_2$. By the above argument on $\mathscr{Y}_2^o$, we see that V^t_x is regular in codimension one since the codimension of $V^t_x\cap\mathscr{P}_\sigma$ in V^t_x is greater than two. Therefore V^t_x is normal since V^t_x is Gorenstein. To check V^t_x has only canonical singularities, we have only to show that $W_2\to V^t_x$ is crepant since W_2 is smooth. This follows by calculating the canonical divisor of W_2. Note that $\mathcal{N}_{W_2/\mathscr{Z}_2}\simeq\mu_2^*\mathcal{W}|_{W_2}$ by (6.2). Therefore $\det\mathcal{N}_{W_2/\mathscr{Z}_2}\simeq\mathcal{O}_{\mathscr{Z}_2}(-N_{\mathscr{Z}_2})|_{W_2}$ since $\det\mathcal{W}=\mathcal{O}_{\mathrm{G}(3,V)}(1)$. Thus we have

$$\begin{aligned}K_{W_2}&=K_{\mathscr{Z}_2}|_{W_2}+\det\mathcal{N}_{W_2/\mathscr{Z}_2}\\&=\{\pi^*_{2'}(K_{\mathscr{Y}_2}+M_{\mathscr{Y}_2})-N_{\mathscr{Z}_2}\}|_{W_2}+N_{\mathscr{Z}_2}|_{W_2}=\pi^*_{2'}(K_{\mathscr{Y}_2}+M_{\mathscr{Y}_2})|_{W_2},\end{aligned}$$

where we use Proposition 4.5.1 (3) for the second equality. q.e.d.

7. The family of curves on Y parameterized by X revisited

Let X and Y be smooth Calabi-Yau threefolds which are mutually orthogonal linear sections of $\mathscr{X}$ and $\mathscr{Y}$ respectively, as described in Subsection 2.4 and Section 3. In this section, we show that a family of curves on Y parameterized by X comes out from $\Delta\to\check{\mathscr{X}}$ and this family coincides with one constructed in Section 3. We prove also the family is flat.

7.1. Flatness of $\Delta\to\check{\mathscr{X}}$ and the locally free resolution of $\mathcal{I}_x$.

Below, we denote by (-1) the tensor product of $\mathcal{O}_{\widetilde{\mathscr{Y}}}(-M_{\widetilde{\mathscr{Y}}})$.

Proposition 7.1.1. (1) *The scheme Δ is flat over $\check{\mathscr{X}}$.*
(2) *Let Δ_x be the fiber of $\Delta\to\check{\mathscr{X}}$ over a point $x\in\check{\mathscr{X}}$. Then the ideal sheaf $\mathcal{I}_x$ of Δ_x in $\widetilde{\mathscr{Y}}$ is $\mathcal{I}\otimes_{\mathcal{O}_{\widetilde{\mathscr{Y}}\times\check{\mathscr{X}}}}\mathcal{O}_{\widetilde{\mathscr{Y}}_x}$, where $\widetilde{\mathscr{Y}}_x\simeq\widetilde{\mathscr{Y}}$ is the fiber of $\widetilde{\mathscr{Y}}\times\check{\mathscr{X}}\to\check{\mathscr{X}}$ over x. Moreover, the exact sequence* (5.4) *remains exact after restricting on $\widetilde{\mathscr{Y}}_x$ and gives the following locally free resolution of $\mathcal{I}_x$:*

$$0\to\widetilde{\mathcal{S}}_L(-1)\to\widetilde{\mathcal{T}}^*(-1)^{\oplus 2}\to\mathcal{O}_{\widetilde{\mathscr{Y}}}(-1)^{\oplus 3}\oplus\widetilde{\mathcal{Q}}^*\to\mathcal{I}_x\to 0. \tag{7.1}$$

Proof. By Proposition 5.3.2, $\Delta\to\check{\mathscr{X}}$ is equi-dimensional (actually all fibers are isomorphic). Therefore Δ is flat over $\check{\mathscr{X}}$ since Δ is Cohen-Macaulay and $\check{\mathscr{X}}$ is smooth.

Tensoring the exact sequence $0\to\mathcal{I}\to\mathcal{O}_{\widetilde{\mathscr{Y}}\times\check{\mathscr{X}}}\to\mathcal{O}_\Delta\to 0$ with $k(x)$, we obtain the exact sequence $\mathcal{I}\otimes k(x)\to\mathcal{O}_{\widetilde{\mathscr{Y}}_x}\to\mathcal{O}_{\Delta_x}\to 0$. Then, applying [**M**, Theorem 22.5 (1)$\Rightarrow$(2)], we see that $\mathcal{I} \otimes k(x) \to \mathcal{O}_{\widetilde{\mathscr{Y}}_x}$ is injective since Δ is flat over $\check{\mathscr{X}}$. Note that, for any coherent $\mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}$-module $\mathcal{A}$ and a point $x \in \check{\mathscr{X}}$, it holds that $\mathcal{A} \otimes_{\mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}} \mathcal{O}_{\widetilde{\mathscr{Y}}_x} \simeq \mathcal{A} \otimes_{\mathcal{O}_{\check{\mathscr{X}}}} k(x)$ since $\mathcal{O}_{\widetilde{\mathscr{Y}}_x} \simeq \mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}} \otimes_{\mathcal{O}_{\check{\mathscr{X}}}} k(x)$. Therefore $\mathcal{I}_x \simeq \mathcal{I} \otimes k(x) \simeq \mathcal{I} \otimes_{\mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}} \mathcal{O}_{\widetilde{\mathscr{Y}}_x}$.

Proof of the exactness of (7.1) is similar. q.e.d.

7.2. Cutting a family of curves $\mathscr{C} \to X$ from Δ. The generically conic bundle $\mathscr{Z}_2 \to \mathscr{Y}_2$ in Proposition 4.3.4 defines a conic bundle $\mathscr{Z}_2^o \to \mathscr{Y}_2^o$ over $\mathscr{Y}_2^o = \mathscr{Y}_2 \setminus \mathscr{P}_\sigma$, which we can write explicitly as

$$\mathscr{Z}_2^o = \left\{ ([\bar{c}], (\bar{q}, [\bar{U}], [V_1])) \,\middle|\, [\bar{c}] \in \bar{q}, \begin{array}{r} \bar{q} \text{ is a conic in } \mathbb{P}(\bar{U}) \cap \mathrm{G}(2, V/V_1) \\ \text{and } ([\bar{U}], [V_1]) \in \mathscr{Y}_3 \setminus \mathscr{P}_\sigma \end{array} \right\}.$$

Let us recall $\overline{\mathscr{Y}} = \left\{ [U] \in \mathrm{G}(3, \wedge^3 V) \mid [U] = [\bar{U} \wedge V_1] \text{ for some } V_1 \right\}$ and define

$$\widetilde{\mathscr{Z}^o} = \left\{ ([c], (q, [U])) \,\middle|\, [c] \in q, \begin{array}{r} q \text{ is a conic in } \mathbb{P}(U) \cap \mathrm{G}(3, V) \\ \text{and } [U] \in \overline{\mathscr{Y}} \setminus \overline{\mathscr{P}}_\sigma \end{array} \right\}.$$

Then there is a conic bundle $\widetilde{\mathscr{Z}^o} \to \widetilde{\mathscr{Y}^o}$ with the following commutative diagram:

$$\begin{tikzcd} \mathscr{Z}_2^o \arrow[r, "\tilde{\rho}_{2'}^o"'] \arrow[d, "\pi_{2'}^o"'] & \widetilde{\mathscr{Z}^o} \arrow[d, "\pi_{\widetilde{\mathscr{Z}}}^o"] \\ \mathscr{Y}_2^o \arrow[r, "\tilde{\rho}_2^o"] & \widetilde{\mathscr{Y}^o} \end{tikzcd}$$

We note that the conic bundle $\widetilde{\mathscr{Z}^o} \to \widetilde{\mathscr{Y}^o}$ over $\widetilde{\mathscr{Y}^o} \setminus F_{\widetilde{\mathscr{Y}}}$ is isomorphic to the conic bundle $\mathscr{Z} \to \mathscr{Y}$ over $\mathscr{Y} \setminus G_{\mathscr{Y}}$ in Subsection 2.3 (see [**HoTa3**, Prop. 4.2.5, 5.2.1]).

As we noted in Subsection 2.4, we can assume that $X \subset \check{\mathscr{X}}$ and Y is disjoint from $G_{\mathscr{Y}} = \operatorname{Sing} \mathscr{Y}$. Since $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ is isomorphic outside $G_{\mathscr{Y}}$, we can also assume $Y \subset \widetilde{\mathscr{Y}}$. Note that the subvariety $Y \times X$ of $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ is contained in $\mathscr{V}$ since X and Y are mutually orthogonal.

Definition 7.2.1. We define the restriction of $\Delta \subset \widetilde{\mathscr{Y}} \times \check{\mathscr{X}}$ to $Y \times X$ by

$$\mathscr{C} = \Delta|_{Y \times X}.$$

We denote by I the ideal sheaf of $\mathscr{C}$ in $Y \times X$ and by I_x the ideal sheaf of C_x in Y_x, where $Y_x = Y$ is the fiber of $Y \times X \to X$ over x.

Proposition 7.2.2. *The scheme $\mathscr{C}$ is flat over X and its fiber over $x \in X$ coincides with C_x defined in Subsection* 3.1. *Moreover, it holds that $I \simeq \mathcal{I}_{\Delta/\mathscr{V}} \otimes_{\mathcal{O}_{\mathscr{V}}} \mathcal{O}_{Y \times X}$ and $I_x \simeq \mathcal{I}_{\Delta_x/\mathscr{V}_x} \otimes_{\mathcal{O}_{\widetilde{\mathscr{Y}}_x}} \mathcal{O}_{Y_x}$.*

Proof. Recall that the curve $C_x \subset Y$ in Subsection 3.1 is defined through $\gamma_x = G_x \cap \pi_{\mathscr{X}}^{-1}(Y)$, which consists of $([\Pi],[Q]) \in \mathscr{X}$ satisfying $l_x \subset \mathbb{P}(\Pi)$ and $[Q] \in P$. Let us define $\Delta_{\widetilde{\mathscr{X}}^o} \subset \widetilde{\mathscr{X}^o} \times \check{\mathscr{X}}$ to be the image of $\Delta_2^o \subset \widetilde{\mathscr{Z}_2}^o \times \check{\mathscr{X}}$ under $\tilde{\rho}_{2'}^o \times \mathrm{id}_{\check{\mathscr{X}}}$, and denote by $\Delta_{\widetilde{\mathscr{X}}^o,x}$ the fiber over $x \in X$. Then, by definition, the intersection $\Delta_{\widetilde{\mathscr{X}}^o,x} \cap \pi^{o\,-1}_{\widetilde{\mathscr{X}}}(Y)$ consists of points $([c],(q,[U])) \in \widetilde{\mathscr{X}^o}$ such that $\pi^o_{\widetilde{\mathscr{X}}}(([c],(q,[U]))) \in Y$ and $l_x \subset \mathbb{P}(V_c)$ with the three dimensional subspace $V_c \subset V$ representing $[c] = [\wedge^3 V_c]$. Here we note that $\pi^o_{\widetilde{\mathscr{X}}}(([c],(q,[U]))) \in Y$ is equivalent to $[Q_y] \in P$ for the quadric Q_y which corresponds to $y = \pi_{\widetilde{\mathscr{Y}}}((q,[U]))$. Therefore γ_x can be identified with $\Delta_{\widetilde{\mathscr{X}}^o,x} \cap \pi^{o\,-1}_{\widetilde{\mathscr{X}}}(Y)$. Now, due to the isomorphism $\widetilde{\mathscr{X}^o} \to \widetilde{\mathscr{Y}^o}$ over $\widetilde{\mathscr{Y}^o} \setminus F_{\widetilde{\mathscr{Y}}}$ to the conic bundle $\mathscr{X} \to \mathscr{Y}$ over $\mathscr{Y} \setminus G_{\mathscr{Y}}$, the curve $C_x = \pi_{\mathscr{Y}}(\gamma_x)$ can be identified with the curve that is defined by the fiber of the scheme $\mathscr{C}$ over $x \in X$. Hereafter we denote by C_x the fiber of $\mathscr{C} \to X$ over $x \in X$.

Recall that the fiber $\mathscr{V}_x$ of $\mathscr{V} \to \check{\mathscr{X}}$ over x contains Δ_x by Proposition 6.0.1. Since $\mathscr{V}_x$ contains also Y, we see that Y is the complete intersection in $\mathscr{V}_x$ of 9 members $M_1, \ldots, M_9$ of $|M_{\widetilde{\mathscr{Y}}}|_{\mathscr{V}_x}|$. Therefore C_x is cut out from Δ_x in $\mathscr{V}_x$ by $M_1, \ldots, M_9$. By [**M**, Corollary to Theorem 23.3], Δ_x is Cohen-Macaulay for any $x \in \check{\mathscr{X}}$ since Δ is Cohen-Macaulay by Theorem 5.1.3 and is flat over $\check{\mathscr{X}}$ by Proposition 7.1.1. Since C_x is one-dimensional for any $x \in X$, the divisors $M_1, \ldots, M_9$ form a regular sequence [**M**, Theorem 17.4 iii)]. Note that $\Delta|_{\widetilde{\mathscr{Y}} \times X} \to X$ is flat by Proposition 7.1.1 and its ideal sheaf is $\mathcal{I} \otimes \mathcal{O}_{\widetilde{\mathscr{Y}} \times X}$ by [**M**, Theorem 22.5 (2)⇒(1)]. Therefore, by [**M**, Corollary to Theorem 22.5 (2)⇒(1)], $\mathscr{C}$ is flat over X and is cut out in $\mathscr{V}$ from $\Delta|_{\widetilde{\mathscr{Y}} \times X}$ by a regular sequence. The latter implies that $\mathcal{I}_{\Delta/\mathscr{V}} \otimes_{\mathcal{O}_{\widetilde{\mathscr{Y}} \times \check{\mathscr{X}}}} \mathcal{O}_{Y \times X}$ is the ideal sheaf of $\mathscr{C}$ in $X \times Y$. Similarly, we have $I_x = \mathcal{I}_{\Delta_x/\mathscr{V}_x} \otimes_{\mathcal{O}_{\widetilde{\mathscr{Y}}_x}} \mathcal{O}_{Y_x}$ since the scheme C_x is cut out in $\mathscr{V}_x$ from Δ_x by a regular sequence. q.e.d.

8. The Derived equivalence

In this section, we derive the main result of this article:

Theorem 8.0.1. *Let X and Y be smooth Calabi-Yau threefolds which are mutually orthogonal linear sections of $\mathscr{X}$ and $\mathscr{Y}$ respectively. Let I be the ideal sheaf as in Proposition* 7.2.2. *Then the Fourier-Mukai functor Φ_I with I as its kernel is an equivalence between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$.*

Since the cohomology groups related to the locally free resolution of $\mathcal{I}$ have been computed in [**HoTa3**, Thm.8.1.1], the rest of our proof of the derived equivalence between X and Y proceeds in the same way as that of [**BC**]. Namely, we show that the functor $\Phi_I : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is an equivalence by verifying the conditions (i) and (ii) of Theorem 2.1.3. The condition (i) may be verified by the following general lemma [**BC**, Proposition 4.5]. We include the proof for completeness.

Lemma 8.0.2. *Let Y be a smooth projective threefold and I an ideal sheaf of $\mathcal{O}_Y$ such that the closed subscheme C defined by I is of (not necessarily pure) dimension less than or equal to one. Then* $\mathrm{Hom}(I, I) \simeq \mathbb{C}$.

Proof. Taking $\mathrm{Hom}(I, -)$ of the exact sequence

$$0 \to I \to \mathcal{O}_Y \to \mathcal{O}_C \to 0, \tag{8.1}$$

we obtain an injection $\mathrm{Hom}(I, I) \to \mathrm{Hom}(I, \mathcal{O}_Y)$. We have only to show $\mathrm{Hom}(I, \mathcal{O}_Y) \simeq \mathbb{C}$ since $\mathrm{Hom}(I, I)$ contains at least constant maps. To compute $\mathrm{Hom}(I, \mathcal{O}_Y)$, we take $\mathrm{Hom}(-, \mathcal{O}_Y)$ of (8.1). Then we obtain the exact sequence $0 \to \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \to \mathrm{Hom}(I, \mathcal{O}_Y) \to \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_Y)$. By the Serre duality, we have $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_Y) \simeq H^2(Y, \mathcal{O}_C \otimes \omega_Y)$, where the r.h.s. is 0 since $\dim C \leq 1$. Therefore we have $\mathrm{Hom}(I, \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \mathbb{C}$. q.e.d.

In what follows, we show the property (ii) of Theorem 2.1.3, i.e., the vanishing:

$$\mathrm{Ext}^\bullet(I_{x_1}, I_{x_2}) = 0 \text{ for any two distinct points } x_1 \text{ and } x_2 \text{ of } X. \tag{8.2}$$

We denote by $(-t)$ the tensor product of $\mathcal{O}_{\widetilde{\mathscr{Y}}}(-tM_{\widetilde{\mathscr{Y}}})$. Then the result (3) of [**HoTa3**, Thm.8.1.1] may be read as:

Theorem 8.0.3. $H^\bullet(\mathcal{A}^* \otimes \mathcal{B}(-t)) = 0\,(1 \leq t \leq 9)$, *where $\mathcal{A}$ and $\mathcal{B}$, respectively, are one of the sheaves $\widetilde{\mathcal{S}}_L$, $\widetilde{\mathcal{T}}^*$, $\mathcal{O}_{\widetilde{\mathscr{Y}}}$, $\widetilde{\mathcal{Q}}^*(M_{\widetilde{\mathscr{Y}}})$.*

It is standard to derive the following proposition from Proposition 7.1.1 and Theorem 8.0.3.

Proposition 8.0.4. *For any two points x_1 and x_2 of $\check{\mathscr{X}}$, it holds*

$$\mathrm{Ext}^\bullet(\mathcal{I}_{x_1}, \mathcal{I}_{x_2}(-t)) = 0\,(1 \leq t \leq 9).$$

We derive the vanishing (8.2) from Proposition 8.0.4 following [**BC**, Subsection 5.6]. For the derivation, it is important to choose a suitable sequence of the complete intersections by the members of $|M_{\widetilde{\mathscr{Y}}}|$ containing Y.

Lemma 8.0.5. *There exists a tower of complete intersections in $\widetilde{\mathscr{Y}}$*

$$Y_0 \subset Y_1 \subset \cdots \subset Y_9 \subset Y_{10}\ (\dim Y_i = 3 + i),$$

by the members of $|M_{\widetilde{\mathscr{Y}}}|$, which satisfies the following conditions: Set $\Delta_{x_i;j} := \Delta_{x_i}|_{Y_j}$ and denote by $\mathcal{I}_{x_i;j}$ the ideal sheaf of $\Delta_{x_i;j}$ in Y_j and by ι_{Y_j} the embedding $Y_j \hookrightarrow Y_{j+1}$. Then

(1) *$Y_0 = Y$ and $Y_{10} = \widetilde{\mathscr{Y}}$.*

(2) *$Y_9 = \mathscr{V}_{x_1}$, where $\mathscr{V}_{x_1}$ is the fiber of $\mathscr{V} \to \check{\mathscr{X}}$ over x_1 (cf. Section 6). In particular, Y_9 contains Δ_{x_1} (Proposition 6.0.1), and hence $\iota_{Y_9 *}\mathcal{I}_{x_1;9} = \mathcal{I}_{x_1}/\mathcal{O}_{\widetilde{\mathscr{Y}}}(-Y_9) \simeq \mathcal{I}_{x_1}/\mathcal{O}_{\widetilde{\mathscr{Y}}}(-1)$. The ideal sheaf $\mathcal{I}_{x_2;9}$ is equal to $\mathcal{I}_{x_2} \otimes \mathcal{O}_{Y_9} \simeq \iota_{Y_9}^*\mathcal{I}_{x_2}$.*

(3) *$Y_8 = \mathscr{V}_{x_1} \cap \mathscr{V}_{x_2}$, where the intersection is taken in $\widetilde{\mathscr{Y}}$. In particular, Y_8 contains $\Delta_{x_2}|_{Y_9}$, and hence $\iota_{Y_8 *}\mathcal{I}_{x_2;8} = \mathcal{I}_{x_2;9}/\mathcal{O}_{Y_9}(-Y_8) \simeq \mathcal{I}_{x_2;9}/\mathcal{O}_{Y_9}(-1)$. The ideal sheaf $\mathcal{I}_{x_1;8}$ is equal to $\mathcal{I}_{x_1;9} \otimes \mathcal{O}_{Y_8} \simeq \iota_{Y_8}^*\mathcal{I}_{x_1;9}$.*

(4) *For any $j \leq 7$, the ideal sheaf $\mathcal{I}_{x_i;j}$ is equal to $\mathcal{I}_{x_i;j+1} \otimes \mathcal{O}_{Y_j} \simeq \iota_{Y_j}^*\mathcal{I}_{x_i;j+1}$.*

Proof. We take $Y_9 = \mathscr{V}_{x_1}$ and $Y_8 = \mathscr{V}_{x_1} \cap \mathscr{V}_{x_2}$ as in the statement and let $Y_7, \dots, Y_0$ be general complete intersections containing Y (recall that Y is contained in any fiber of $\mathscr{V} \to \check{\mathscr{X}}$ over a point of X). Note that $\mathscr{V}_{x_1}$ is irreducible by Proposition 6.0.2, and also $\mathscr{V}_{x_1} \neq \mathscr{V}_{x_2}$ for $x_1 \neq x_2$. The last property follows from the duality between $\mathbb{P}(\mathsf{S}^2 V)$ and $\mathbb{P}(\mathsf{S}^2 V^*)$ and the fact that X is embedded in $\mathscr{X} \subset \mathbb{P}(\mathsf{S}^2 V)$.

The descriptions of $\mathcal{I}_{x_2;9}$, $\mathcal{I}_{x_1;8}$ and $\mathcal{I}_{x_i;j}$ $(i = 1, 2,\ 0 \leq j \leq 7)$ follow from the last part of the proof of Proposition 7.2.2. q.e.d.

The choices of Y_9 and Y_8 in the above lemma turns out to be crucial in Steps 1 and 2 of the arguments below.

Step 1 (from $\widetilde{\mathscr{Y}}$ to Y_9).

Proposition 8.0.6.

$$\mathrm{Ext}^{\bullet -1}_{Y_9}(\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;9}(-t+1)) = 0 \ (1 \leq t \leq 9).$$

Proof. By $\mathcal{I}_{x_2;9} \simeq \iota_{Y_9}^*\mathcal{I}_{x_2}$, we have

$$\mathrm{Ext}^{\bullet -1}_{Y_9}(\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2;9}(-t+1)) \simeq \mathrm{Ext}^{\bullet -1}_{Y_9}(\mathcal{I}_{x_1;9}, \iota_{Y_9}^*\mathcal{I}_{x_2}(-t+1)) \tag{8.3}$$

By applying the Grothendieck-Verdier duality 2.1.2 to the embedding $\iota_{Y_9} \colon Y_9 \hookrightarrow \widetilde{\mathscr{Y}}$ with $\omega_{Y_9/\widetilde{\mathscr{Y}}} = \mathcal{O}_{\widetilde{\mathscr{Y}}}(-1)$, we have

$$\mathrm{Ext}^{\bullet -1}_{Y_9}(\mathcal{I}_{x_1;9}, \iota_{Y_9}^*\mathcal{I}_{x_2}(-t+1)) \simeq \mathrm{Ext}^{\bullet}_{\widetilde{\mathscr{Y}}}(\iota_{Y_9 *}\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2}(-t)).$$

Taking $\mathrm{Hom}(-, \mathcal{I}_{x_2}(-t))$ of the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathscr{Y}}}(-1) \to \mathcal{I}_{x_1} \to \iota_{Y_9 *}\mathcal{I}_{x_1;9} \to 0$$

(cf. Lemma 8.0.5 (2)), we obtain the exact sequence

$$H^{\bullet -1}(\mathcal{I}_{x_2}(-t+1)) \to \mathrm{Ext}^{\bullet}_{\widetilde{\mathscr{Y}}}(\iota_{Y_9 *}\mathcal{I}_{x_1;9}, \mathcal{I}_{x_2}(-t)) \to \mathrm{Ext}^{\bullet}_{\widetilde{\mathscr{Y}}}(\mathcal{I}_{x_1}, \mathcal{I}_{x_2}(-t)),$$

where the first term vanishes by the following lemma and the last term vanishes by Proposition 8.0.4. q.e.d.

Lemma 8.0.7. *$H^{\bullet -1}(\mathcal{I}_{x_2}(-t+1)) = 0$ $(1 \leq t \leq 9)$.*

Proof. The assertion follows from Proposition 7.1.1 and Theorem 8.0.3. q.e.d.

Step 2 (from Y_9 to Y_8).

Proposition 8.0.8.

$$\mathrm{Ext}^{\bullet-1}_{Y_8}(\mathcal{I}_{x_1;8},\mathcal{I}_{x_2;8}(-t+1))=0\ (1\le t\le 9).$$

Proof. Since $\mathcal{I}_{x_1;8}\simeq\iota^*_{Y_8}\mathcal{I}_{x_1;9}$, we have

$$\begin{aligned}\mathrm{Ext}^{\bullet-1}_{Y_8}(\mathcal{I}_{x_1;8},\mathcal{I}_{x_2;8}(-t+1))&\simeq\mathrm{Ext}^{\bullet-1}_{Y_8}(\iota^*_{Y_8}\mathcal{I}_{x_1;9},\mathcal{I}_{x_2;8}(-t+1))\\&\simeq\mathrm{Ext}^{\bullet-1}_{Y_9}(\mathcal{I}_{x_1;9},\iota_{Y_8*}\mathcal{I}_{x_2;8}(-t+1)).\end{aligned}\tag{8.4}$$

From (8.4) and $\mathrm{Hom}(\mathcal{I}_{x_1;9}(t-1),-)$ of the exact sequence

$$0\to\mathcal{O}_{Y_9}(-1)\to\mathcal{I}_{x_2;9}\to\iota_{Y_8*}\mathcal{I}_{x_2;8}\to 0$$

(cf. Lemma 8.0.5 (3)) we obtain the exact sequence

$$\begin{aligned}\mathrm{Ext}^{\bullet-1}_{Y_9}(\mathcal{I}_{x_1;9},\mathcal{I}_{x_2;9}(-t+1))&\to\mathrm{Ext}^{\bullet-1}_{Y_8}(\mathcal{I}_{x_1;8},\mathcal{I}_{x_2;8}(-t+1))\\&\to\mathrm{Ext}^{\bullet}_{Y_9}(\mathcal{I}_{x_1;9},\mathcal{O}_{Y_9}(-t)),\end{aligned}\tag{8.5}$$

where the first term vanishes from Proposition 8.0.6 and the last term vanishes by the following lemma. q.e.d.

Lemma 8.0.9. $\mathrm{Ext}^{\bullet}_{Y_9}(\mathcal{I}_{x_1;9},\mathcal{O}_{Y_9}(-t))=0\ (1\le t\le 9)$.

Proof. Set $F_{Y_9}:=F_{\widetilde{\mathscr{Y}}}|_{Y_9}$. By the Serre-Grothendieck duality, it holds

$$\mathrm{Ext}^{\bullet}_{Y_9}(\mathcal{I}_{x_1;9},\mathcal{O}_{Y_9}(-t))\simeq H^{12-\bullet}(Y_9,\mathcal{I}_{x_1;9}((t-9)M_{Y_9}+2F_{Y_9}))^*\tag{8.6}$$

since $K_{Y_9}=-9M_{Y_9}+2F_{Y_9}$. Let us first show that

$$H^{12-\bullet}(Y_9,\mathcal{I}_{x_1;9}((t-9)M_{Y_9}+2F_{Y_9}))\simeq H^{12-\bullet}(\widetilde{\mathscr{Y}},\mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}})).\tag{8.7}$$

Since $\Delta_{x_1;9}=\Delta_{x_1}$, we have the following two exact sequences on $\widetilde{\mathscr{Y}}$ and Y_9 respectively:

$$\begin{aligned}0\to\mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}})&\to\mathcal{O}_{\widetilde{\mathscr{Y}}}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}})\\&\to\mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}})\to 0,\end{aligned}\tag{8.8}$$

and

$$\begin{aligned}0\to\mathcal{I}_{x_1;9}((t-9)M_{Y_9}+2F_{Y_9})&\to\mathcal{O}_{Y_9}((t-9)M_{Y_9}+2F_{Y_9})\\&\to\mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}})\to 0.\end{aligned}\tag{8.9}$$

Since $(t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}}=(t+1)M_{\widetilde{\mathscr{Y}}}+K_{\widetilde{\mathscr{Y}}}$ and $(t+1)M_{\widetilde{\mathscr{Y}}}$ is nef and big, the cohomology groups $H^{12-\bullet}(\widetilde{\mathscr{Y}},\mathcal{O}_{\widetilde{\mathscr{Y}}}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}}))$ vanish for $12-\bullet>0$ by the Kawamata-Viewheg vanishing theorem. Moreover, it holds that $H^0(\widetilde{\mathscr{Y}},\mathcal{O}_{\widetilde{\mathscr{Y}}}((t-9)M_{\widetilde{\mathscr{Y}}}+2F_{\widetilde{\mathscr{Y}}}))\simeq H^0(\mathscr{Y},\mathcal{O}_{\mathscr{Y}}((t-9)M_{\mathscr{Y}}))$, and the latter vanishes if $t\le 8$, and is isomorphic to $\mathbb{C}$ if $t=9$. If $t=9$ and $12-\bullet=0$, then $H^0(\widetilde{\mathscr{Y}},\mathcal{O}_{\widetilde{\mathscr{Y}}}(2F_{\widetilde{\mathscr{Y}}}))\simeq H^0(\Delta_{x_1},\mathcal{O}_{\Delta_{x_1}}(2F_{\widetilde{\mathscr{Y}}}))$ and hence

$H^0(\widetilde{\mathscr{Y}}, \mathcal{I}_{x_1}(2F_{\widetilde{\mathscr{Y}}})) = 0$ by (8.8). For the cases $12 - \bullet > 0$ or $1 \le t \le 8$, we have $H^{12-\bullet}(\widetilde{\mathscr{Y}}, \mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}})) \simeq H^{11-\bullet}(\Delta_{x_1}, \mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}}))$ by (8.8). Similarly, by (8.9) and the Kawamata-Viewheg vanishing theorem, we have $H^0(Y_9, \mathcal{I}_{x_1;9}(2F_{Y_9})) = 0$, and for the cases $12 - \bullet > 0$ or $1 \le t \le 8$, $H^{12-\bullet}(Y_9, \mathcal{I}_{x_1;9}((t-9)M_{Y_9} + 2F_{Y_9})) \simeq H^{11-\bullet}(\Delta_{x_1}, \mathcal{O}_{\Delta_{x_1}}((t-9)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}}))$, where we use the fact that $Y_9 = \mathscr{V}_{x_1}$ has only canonical singularities (see Proposition 6.0.2).

Therefore we have the isomorphisms (8.7).

Now we note that the vanishings of $H^{12-\bullet}(\widetilde{\mathscr{Y}}, \mathcal{I}_{x_1}((t-9)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}}))$ follow from the vanishings of

$$\begin{aligned}
&H^{12-\bullet}(\widetilde{\mathscr{Y}}, \widetilde{\mathcal{S}}_L((t-10)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}})),\\
&H^{12-\bullet}(\widetilde{\mathscr{Y}}, \widetilde{\mathcal{T}}^*((t-10)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}})),\\
&H^{12-\bullet}(\widetilde{\mathscr{Y}}, \mathcal{O}_{\widetilde{\mathscr{Y}}}, ((t-10)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}})),\\
&H^{12-\bullet}(\widetilde{\mathscr{Y}}, \widetilde{\mathcal{Q}}^*((t-9)M_{\widetilde{\mathscr{Y}}} + 2F_{\widetilde{\mathscr{Y}}}))
\end{aligned}$$

by the locally free resolution (7.1) for $\mathcal{I}_{x_1}$. Note that these cohomology groups are Serre-dual to

$$\begin{aligned}
H^{\bullet+1}(\widetilde{\mathscr{Y}}, \widetilde{\mathcal{S}}_L^*(-t)), &\quad H^{\bullet+1}(\widetilde{\mathscr{Y}}, \widetilde{\mathcal{T}}(-t)),\\
H^{\bullet+1}(\widetilde{\mathscr{Y}}, \mathcal{O}_{\widetilde{\mathscr{Y}}}(-t)), &\quad H^{\bullet+1}(\widetilde{\mathscr{Y}}, \widetilde{\mathcal{Q}}(-t-1)),
\end{aligned}$$

and vanish by Theorem 8.0.3. Therefore, we have $\mathrm{Ext}^{\bullet}_{Y_9}(\mathcal{I}_{x_1;9}, \mathcal{O}_{Y_9}(-t)) = 0$ by (8.6) and (8.7). q.e.d.

Step 3 (from Y_8 to Y_7).

Proposition 8.0.10.

$$\mathrm{Ext}^{\bullet-1}_{Y_7}(\mathcal{I}_{x_1;7}, \mathcal{I}_{x_2;7}(-t+1)) = 0 \ (1 \le t \le 8). \tag{8.10}$$

Proof. Since $\mathcal{I}_{x_1;7} \simeq \iota^*_{Y_7}\mathcal{I}_{x_1;8}$, we have

$$\begin{aligned}
\mathrm{Ext}^{\bullet-1}_{Y_7}(\mathcal{I}_{x_1;7}, \mathcal{I}_{x_2;7}(-t+1)) &\simeq \mathrm{Ext}^{\bullet-1}_{Y_7}(\iota^*_{Y_7}\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;7}(-t+1))\\
&\simeq \mathrm{Ext}^{\bullet-1}_{Y_8}(\mathcal{I}_{x_1;8}, \iota_{Y_7*}\mathcal{I}_{x_2;7}(-t+1)).
\end{aligned} \tag{8.11}$$

Note that a defining equation of Y_7 restricts to a regular element in each local ring of $\Delta_{x_2;8}$. Hence the sequence

$$0 \to \mathcal{I}_{x_2;8}(-1) \to \mathcal{I}_{x_2;8} \to \iota_{Y_7*}\mathcal{I}_{x_2;7} \to 0 \tag{8.12}$$

is exact. Therefore, by (8.11), and $\mathrm{Hom}(\mathcal{I}_{x_1;8}(t-1), -)$ of (8.12), we have the following exact sequence:

$$\begin{aligned}
\mathrm{Ext}^{\bullet-1}_{Y_8}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t+1)) &\to \mathrm{Ext}^{\bullet-1}_{Y_7}(\mathcal{I}_{x_1;7}, \mathcal{I}_{x_2;7}(-t+1))\\
&\to \mathrm{Ext}^{\bullet}_{Y_8}(\mathcal{I}_{x_1;8}, \mathcal{I}_{x_2;8}(-t)),
\end{aligned} \tag{8.13}$$

where $\mathrm{Ext}^{\bullet-1}_{Y_8}(\mathcal{I}_{x_1;8},\mathcal{I}_{x_2;8}(-t+1))$ and also $\mathrm{Ext}^{\bullet}_{Y_8}(\mathcal{I}_{x_1;8},\mathcal{I}_{x_2;8}(-t))$ vanish by Proposition 8.0.8 if $1\le t\le 8$, and then we obtain the claim. q.e.d.

Step 4 (from Y_7 to Y_6, ..., Y_1 to Y).
Here we complete our proof of (8.2). We can choose the defining equations of Y_i so that each of them restricts to a regular element in the local ring for both $\Delta_{x_1;i+1}$ and $\Delta_{x_2;i+1}$. Then the argument of Step 3 applies inductively, and we have the following vanishings for $i\in 0,1,2,\cdots,6$:

$$\mathrm{Ext}^{\bullet-1}_{Y_i}(\mathcal{I}_{x_1;i},\mathcal{I}_{x_2;i}(-t+1))=0\,(1\le t\le i+1). \tag{8.14}$$

In particular, we have $\mathrm{Ext}^{\bullet-1}_{Y}(I_{x_1},I_{x_2})=0$, which is (8.2).

9. Further discussions

In our proof of the derived equivalence, we have used (the ideal sheaf of) a family of curves $\{C_x\}_{x\in X}$ which arises from the restriction $\mathcal{C}=\Delta|_{Y\times X}$. Obviously, the other choice of a family $\{C_y\}_{y\in Y}$ could be possible for that purpose. In this section, we obtain a flat family of curves on X from Δ for the latter choice, and remark, however, that a technical problem prevent us to complete a proof by using this family. We also make a comment on non-invariances of the fundamental groups and the Brauer groups under the derived equivalence.

9.1. A family of curves on X. For a point $y\in\widetilde{\mathscr{Y}}$, we denote by Q_y the quadric in $\mathbb{P}^4$ corresponding to the image of y on $\mathscr{H}$. We also denote by Δ_y the fiber of $\Delta\to\widetilde{\mathscr{Y}}$ over y, which is a closed subscheme of $\check{\mathscr{X}}$.

Let $V_{\widetilde{\mathscr{Y}}}$ be the open subset of $\widetilde{\mathscr{Y}}$ consisting of points y such that $\operatorname{rank}Q_y=3$ or 4. For a point $y\in V_{\widetilde{\mathscr{Y}}}$, let $q_y\subset \mathrm{G}(3,V)$ be the conic corresponding to y (q_y is one of the connected components of the families of planes in Q_y). We describe Δ_y for $y\in V_{\widetilde{\mathscr{Y}}}$.

Proposition 9.1.1. *Δ_y is the restriction of $\check{\mathscr{X}}=\mathbb{P}(\mathsf{S}^2\mathcal{F})\to \mathrm{G}(2,V)$ over the subset*

$$G_y:=\{l\mid l \text{ is contained in a plane belonging to } q_y\}\subset \mathrm{G}(2,V).$$

If $\operatorname{rank}Q_y=4$, then G_y is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^1}(-2))$ and $\mathcal{O}_{\mathrm{G}(2,V)}(1)|_{G_y}$ is the tautological divisor. If $\operatorname{rank}Q_y=3$, then G_y is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2})$ and $\mathcal{O}_{\mathrm{G}(2,V)}(1)|_{G_y}$ is the tautological divisor. In particular, Δ_y is smooth if $\operatorname{rank}Q_y=4$.

Proof. The first part easily follows from the definition of Δ. We describe G_y. By definition, G_y is nothing but the restriction of $\Delta_0\to \mathrm{G}(3,V)$ over q_y. Since $\Delta_0=\mathrm{F}(2,3,V)$, we can write $\Delta_0=\mathbb{P}(\mathcal{U}^*)$ as a

$\mathbb{P}^2$-bundle over $\mathrm{G}(3,V)$. We denote by p_1 the projection $\Delta_0 \to \mathrm{G}(3,V)$ and by p_2 the projection $\Delta_0 \to \mathrm{G}(2,V)$. We see that

$$\mathcal{O}_{\mathbb{P}(\mathcal{U}^*)}(1) \simeq p_2^*\mathcal{O}_{\mathrm{G}(2,V)}(1) \otimes p_1^*\mathcal{O}_{\mathrm{G}(3,V)}(-1). \tag{9.1}$$

Indeed, by $\mathcal{U}^* \simeq \wedge^2\mathcal{U} \otimes \mathcal{O}_{\mathrm{G}(3,V)}(1)$, we have $\mathbb{P}(\mathcal{U}^*) \simeq \mathbb{P}(\wedge^2\mathcal{U})$ and $\mathcal{O}_{\mathbb{P}(\mathcal{U}^*)}(1) \simeq \mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{U})}(1) \otimes p_1^*\mathcal{O}_{\mathrm{G}(3,V)}(-1)$. Moreover, by the universal exact sequence $0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathrm{G}(3,V)} \to \mathcal{W} \to 0$, we obtain the injection $\wedge^2\mathcal{U} \to \wedge^2 V \otimes \mathcal{O}_{\mathrm{G}(3,V)}$. Therefore, $\mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{U})}(1) = p_2^*\mathcal{O}_{\mathrm{G}(2,V)}(1)$, which implies (9.1). Note that (9.1) is equivalent to $\mathcal{O}_{\mathbb{P}(\mathcal{U}^*(-1))}(1) \simeq p_2^*\mathcal{O}_{\mathrm{G}(2,V)}(1)$.

By the discussion in the proof of [**HoTa3**, Prop. 5.2.1], $\mathcal{U}^*(-1)|_{q_y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ if $\operatorname{rank} Q_y = 4$, and $\mathcal{U}^*(-1)|_{q_y} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}$ if $\operatorname{rank} Q_y = 3$.

Therefore, G_y is as in the statement. q.e.d.

Similarly to the proof of Proposition 7.1.1, we can prove the following:

Proposition 9.1.2. *The scheme Δ is flat over $V_{\widetilde{\mathscr{Y}}}$. The ideal sheaf $\mathcal{I}_y$ of Δ_y in $\check{\mathscr{X}}$ is $\mathcal{I} \otimes \mathcal{O}_{\check{\mathscr{X}}_y}$ for $y \in V_{\widetilde{\mathscr{Y}}}$, where $\check{\mathscr{X}}_y \simeq \check{\mathscr{X}}$ is the fiber of $\widetilde{\mathscr{Y}} \times \check{\mathscr{X}} \to \widetilde{\mathscr{Y}}$ over y. Moreover, the exact sequence* (5.4) *remains to be exact after restricting on $\check{\mathscr{X}}_y$ and gives the following locally free resolution of $\mathcal{I}_y$:*

$$\begin{aligned} 0 \to & g^*\mathcal{O}_{\mathrm{G}(2,V)}(-2)^{\oplus 3} \to g^*(\mathcal{F}^*(-2))^{\oplus 4} \to \\ & g^*\mathsf{S}^2\mathcal{F} \oplus g^*\mathcal{O}_{\mathrm{G}(2,V)}(-1)^{\oplus 3} \to \mathcal{I}_y \to 0. \end{aligned} \tag{9.2}$$

Proof. By Proposition 9.1.1, $\Delta \to \widetilde{\mathscr{Y}}$ is equi-dimensional over $V_{\widetilde{\mathscr{Y}}}$. Therefore all the assertions can be proved in the same way of the proof of Proposition 7.1.1. q.e.d.

The derivations of the following relations are standard:

Lemma 9.1.3. $c_1(\mathsf{S}^2\mathcal{F}^*) = c_1(\mathcal{O}_{\mathrm{G}(2,V)}(3)), c_2(\mathsf{S}^2\mathcal{F}^*) = 2c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))^2 + 4c_2(\mathcal{F}^*)$, *and* $c_3(\mathsf{S}^2\mathcal{F}^*) = 4c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))c_2(\mathcal{F}^*)$.

Now we have

Proposition 9.1.4. *The scheme $\mathscr{C}$ defined as in Subsection* 7.2 *is flat over Y. Let C_y be the fiber of $\mathscr{C}$ over a point $y \in Y$. Then C_y is a curve of arithmetic genus* 14 *and degree* 20. *Moreover, if X and Y are general, then a general C_y is smooth.*

Proof. Note that $Y \subset V_{\widetilde{\mathscr{Y}}}$. Take a point $y \in Y$. We only consider the case of $\operatorname{rank} Q_y = 4$ since the other case can be studied similarly.

First we show that $\deg \Delta_y = 20$ with respect to $H_{\check{\mathscr{X}}} = H_{\mathbb{P}(\mathsf{S}^2\mathcal{F})}$. The degree of Δ_y is evaluated by the Segre class of $\mathsf{S}^2\mathcal{F}^*$ as $s_3(\mathsf{S}^2\mathcal{F}^*)G_y$. By the formula $s_3(\mathsf{S}^2\mathcal{F}^*) = c_3(\mathsf{S}^2\mathcal{F}^*) - 2c_1(\mathsf{S}^2\mathcal{F}^*)c_2(\mathsf{S}^2\mathcal{F}^*) + c_1(\mathsf{S}^2\mathcal{F}^*)^3$ and Lemma 9.1.3, we have

$$\deg \Delta_y = \big(15c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))^3 - 20c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))c_2(\mathcal{F}^*)\big)G_y.$$

By Proposition 9.1.1, we have $\deg G_y = c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))^3 G_y = 4$. Note that $c_2(\mathcal{F}^*) = [\mathrm{G}(2, V_4)]$ as codimension 2 cycle with some 4-dimensional space $V_4 \subset V$. Then we see that $c_2(\mathcal{F}^*)G_y$ is represented by a conic, which parameterizes a $\mathbb{P}^1$-family of rulings in the smooth quadric surface $Q_y \cap \mathbb{P}(V_4)$. Therefore we have $c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))c_2(\mathcal{F}^*)G_y = 2$. Consequently, we have $\deg \Delta_y = 20$.

We show that C_y is a curve. Similarly to the proof of Proposition 7.2.2, C_y is a complete intersection in Δ_y by 4 members of $|H_{\check{\mathscr{X}}}|$. Therefore $\dim C_y \geq 1$. Assume that $\dim C_y = 2$. Then $\deg C_y \leq 20$ since $\deg \Delta_y = 20$. This is, however, impossible since $H_{\check{\mathscr{X}}}|_X$ generates $\operatorname{Pic} X$ modulo torsion and $(H_{\check{\mathscr{X}}}|_X)^3 = 35$. Therefore C_y is a curve, and then $\mathscr{C}$ is flat over Y as in the proof of Proposition 7.2.2.

Now we compute the canonical divisor of C_y. Since $\Delta_y \to G_y$ is a projective bundle, we have $K_{\Delta_y} = -3H_{\check{\mathscr{X}}}|_{\Delta_y} + (g|_{\Delta_y})^*(c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y})$. Since we have seen that C_y is a complete intersection in Δ_y by 4 members of $|H_{\check{\mathscr{X}}}|$, we have $K_{C_y} = H_{\check{\mathscr{X}}}|_{C_y} + (g|_{\Delta_y})^*(c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y})|_{C_y}$. Therefore $\deg K_{C_y} = H^5_{\check{\mathscr{X}}}\Delta_y + H^4_{\check{\mathscr{X}}}(g|_{\Delta_y})^*(c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y})$. We have already computed $H^5_{\check{\mathscr{X}}}\Delta_y = 20$. By using the Segre class of $\mathsf{S}^2\mathcal{F}^*$, we have

$$H^4_{\check{\mathscr{X}}}(g|_{\Delta_y})^*(c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y}) = s_2(\mathsf{S}^2\mathcal{F}^*)|_{G_y}(c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y}).$$

By Lemma 9.1.3, we have $s_2(\mathsf{S}^2\mathcal{F}^*) = c_1(\mathsf{S}^2\mathcal{F}^*)^2 - c_2(\mathsf{S}^2\mathcal{F}^*) = 7c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))^2 - 4c_2(\mathcal{F}^*)$. By Proposition 9.1.1, we have $\mathcal{O}_{G_y}(K_{G_y}) \simeq \mathcal{O}_{\mathrm{G}(2,V)}(-3)|_{G_y} \otimes p^*\mathcal{O}_{\mathbb{P}^1}(2)$, where p is the natural morphism $p\colon G_y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \to q_y \simeq \mathbb{P}^1$. Thus $c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y} = p^*\mathcal{O}_{\mathbb{P}^1}(2)$. Hence

$$\begin{aligned} &H^4_{\check{\mathscr{X}}}(g|_{\Delta_y})^*(c_1(\mathsf{S}^2\mathcal{F}^*)|_{G_y} + K_{G_y}) \\ &= (7c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))^2 - 4c_2(\mathcal{F}^*))|_{G_y} p^*\mathcal{O}_{\mathbb{P}^1}(2). \end{aligned}$$

Recall that $c_2(\mathcal{F}^*)|_{G_y}$ is represented as a conic on $\mathrm{G}(2, V)$, which is a section of $p\colon G_y \to q_y$. Therefore $(7c_1(\mathcal{O}_{\mathrm{G}(2,V)}(1))^2 - 4c_2(\mathcal{F}^*))|_{G_y} p^*\mathcal{O}_{\mathbb{P}^1}(2) = 6$. Consequently, we have $\deg K_{C_y} = 26$, equivalently, $p_a(C_y) = 14$.

Let y be a point of $\mathscr{Y}$ such that $\operatorname{rank} Q_y = 4$. Take a general 4-plane P in $\mathbb{P}(\mathsf{S}^2V^*)$ containing $[Q_y]$ and define X and Y as before. Note that $y \in Y$. Write $P = \langle Q_y, Q_1, \ldots, Q_4\rangle$. Let H_i be the member of $|H_{\check{\mathscr{X}}}|$ corresponding to Q_i. Note that $\Delta_y \subset \mathscr{V}_y$ by Proposition 6.0.1. Since P is general, H_i are general members of $|H_{\check{\mathscr{X}}}|$. Therefore, C_y is smooth since so is Δ_y and C_y is a complete intersection in Δ_y by $H_1, \ldots, H_4$.

q.e.d.

One might consider that we can show the derived equivalence by the Fourier-Mukai functor with the ideal sheaves of the family $\{C_y\}_{y\in Y}$. Actually, computations of Ext groups among the sheaves appearing in the resolution (9.2) are easier than Theorem 8.0.3 (see [**HoTa3**, Thm.3.4.5]). However, one technical problem is involved as follows: When we follow the argument of Section 8, it is crucial to have the property that, for *any* distinct two points y_1 and y_2 of Y, the hyperplane sections $\mathscr{V}_{y_1}$ and $\mathscr{V}_{y_2}$ of X are different (cf. Lemma 8.0.5). This, however, does not hold for y_1 and y_2 such that their images on H coincides. This is the reason for our choice of $\{C_x\}_{x\in X}$ in our proof.

Remark. It should be interesting to find the curves $C_y\,(y \in Y)$ of genus 14 and degree 20 in the table of the BPS numbers of X [**HoTa1**, Table 2]. In fact, we read from the table the counting number of the curves of genus 14 and degree 20 as

$$n_{14}^X(20) = 500.$$

In a similar way to the BPS number $n_3^Y(5) = 100$ discussed in Introduction, we may arrange this number as $n_{14}^X(20) = (-1)^{\dim Y} e(Y) \times 10$. This time, however, it is not clear whether the factor 10 has a nice interpretation from the geometry of X. Nevertheless, we expect that the number $n_{14}^X(20) = 500$ is 'counting' the Euler numbers of the parameter spaces of generically smooth family of curves on X by general properties of the BPS numbers [**GV**], since we were able to verify $n_{14}^X(21) = 0$ after rather heavy calculations using mirror symmetry.

Remark. In the Grassmann-Pfaffian case due to [**BC**, **Ku2**], the constructions of curves $\{C_x\}_{x\in X}$ and $\{C_y\}_{y\in Y}$ are more straightforward. Assume X is a smooth linear section Calabi-Yau threefold of $\mathrm{G}(2,7)$, which is embedded in $\mathbb{P}(\wedge^2\mathbb{C}^7)$, and Y is the corresponding (smooth) orthogonal linear section Calabi-Yau threefolds of $\mathrm{Pf}(7)$ in $\mathbb{P}(\wedge^2(\mathbb{C}^*)^7)$. Let us write by $[\xi_x] = [\xi_x^{(1)}, \xi_x^{(2)}] \simeq \mathbb{P}^1$ the line corresponding to a point $x \in \mathrm{G}(2,7)$, and by $[\eta_y]$ a skew symmetric matrix $\operatorname{rank}\eta_y \le 4$ corresponding to $y \in \mathrm{Pf}(7)$. Then the incidence relation used in [**BC**, **Ku2**] is $\Delta = \{([\xi_x],[\eta_y]) | \dim(\xi_x \cap \operatorname{Ker}(\eta_y)) \ge 1\}$. In this case, the fiber Δ_y of $\Delta \to \mathrm{Pf}(7)$ over y is given by a Schubert cycle σ_3 in $\mathrm{G}(2,7)$ of codimension 3 if $\operatorname{rank}(\eta_y) = 4$, and simplifies the proof of the derived equivalence using the family of curves $\{C_y\}_{y\in Y} = \{\Delta \cap (X \times \{y\})\}_{y\in Y}$. As discussed in Introduction, C_y is generically a smooth curve on X of genus 6 and degree 14. The fiber Δ_x of the other fibration $\Delta \to \mathrm{G}(2,7)$ is easy to be described. It turns out that

$$\Delta_x = \{([\xi_x],[\eta_y]) \mid (\eta_y\, \xi_x^{(1)}) \wedge (\eta_y\, \xi_x^{(2)}) = 0\}.$$

When X and Y are generic and smooth, we can verify by *Macaulay2* that $C_x = \Delta \cap (\{x\} \times Y)$ $(x \in X)$ is generically a smooth curve on Y of genus

11 and degree 14. The corresponding BPS number is, unfortunately, outside of the tables available in literatures (see [**HoTa1**, Section 4]).

9.2. The fundamental groups and the Brauer groups of X and Y.

Finally, it should be worth while discussing about non-invariance of the fundamental groups and the Brauer groups by the derived equivalence between a Reye congruence X and the double symmetroid Y orthogonal to X.

As for the fundamental groups, we have $\pi_1(X) \simeq \mathbb{Z}_2$ by [**HoTa3**, Prop.3.5.3], and $\pi_1(Y) \simeq 0$ by [ibid. Prop.4.3.4]. To our best knowledge, this seems to be the second example of pairs of derived equivalent Calabi-Yau threefolds with different fundamental groups (see [**S**]).

As for the Brauer groups, we follow the argument of [**A**]: By [**BK**], the Atiyah-Hirzebruch spectral sequence gives a short exact sequence for any Calabi-Yau threefold Σ:

$$0 \to H_1(\Sigma, \mathbb{Z}) \to K^1_{\mathrm{top}}(\Sigma)_{\mathrm{tors}} \to \mathrm{Br}\,(\Sigma) \to 0, \tag{9.3}$$

where $K_{\mathrm{top}}(\Sigma) = K^0_{\mathrm{top}}(\Sigma) \oplus K^1_{\mathrm{top}}(\Sigma)$ is the topological K-group and the subscript 'tors' means the torsion part. As we mentioned above, we have $H_1(X, \mathbb{Z}) \simeq \pi_1(X) \simeq \mathbb{Z}_2$ and $H_1(Y, \mathbb{Z}) \simeq 0$. By [**AT**, §2.2], $K^1_{\mathrm{top}}(X) \simeq K^1_{\mathrm{top}}(Y)$ since X and Y are derived equivalent. Therefore, by (9.3), we have the following relation between the Brauer groups of X and Y:

$$0 \to \mathbb{Z}_2 \to \mathrm{Br}\,(Y) \to \mathrm{Br}\,(X) \to 0. \tag{9.4}$$

We have shown $\mathrm{Br}\,(Y)$ contains a nonzero 2-torsion element in Proposition 3.2.1. If $\mathrm{Br}\,(Y) \simeq \mathbb{Z}_2$, we will have $\mathrm{Br}\,(X) \simeq 0$ by (9.4), which indicates that the Brauer groups are not invariant under the derived equivalence.

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Department of Mathematics
Gakushuin Univeristy
Toshima-Ku
Tokyo 171-8588, Japan

E-mail address: hosono@math.gakushuin.ac.jp

Graduate School of Mathematical Sciences
University of Tokyo
Meguro-ku
Tokyo 153-8914, Japan

E-mail address: takagi@ms.u-tokyo.ac.jp