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Double Stage Shrinkage Estimator of Two Parameters Generalized Exponential Distribution

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Abstract

This paper is concerned with double stage shrinkage estimator (DSSE) for lowering the mean squared error of classical estimator (MLE) for the shape parameter (α) of generalized Exponential (GE) distribution in a region (R) around available prior knowledge (α_0) about the actual value (α) as initial estimate in case when a scale parameter (λ) is known as well as to reduce the cost of experimentations.

In situation where the experimentations are time consuming or very costly, a double stage procedure can be used to reduce the expected sample size needed to obtain the estimator.

This estimator is shown to have smaller mean squared error for certain choice of the shrinkage weight factor $\psi(\cdot)$ and for acceptance mentioned region R.

Expressions for Bias, Mean square error (MSE), Expected sample size [E(n/α,R)], Expected sample size proportion [E(n/α,R)/n], probability for avoiding the second sample $[p(\hat{\alpha}_1 \in R)]$ and percentage of overall sample saved $[\frac{n_2}{n} p(\hat{\alpha}_1 \in R) * 100]$ for the proposed estimator are derived.

Numerical results and conclusions are established when the consider estimator (DSSE) are estimator of level of significance Δ.

Comparisons with the classical estimator and with the last studies shown the usefulness of the proposed estimator

Keywords: Generalized Exponential Distribution; double stage shrinkage estimator; Maximum Likelihood Estimator; mean squared error

1. Introduction

Gupta and Kundu (1999) proposed the generalized exponential (GE) distribution as an alternative to the well known Weibull or gamma distributions. It is observed that the proposed two-parameter GE distribution has several desirable properties and in many situations it may fit better than the Weibull or gamma distribution. Extensive work has been done since then to establish several properties of the generalized exponential distribution. The readers are referred to the recent review article by Gupta and Kundu (2007) for a current account of it.[1].Generalized exponential (GE) distribution has been proposed and studied quite extensively recently by Gupta and Kundu [2, 3, 4, 5, 6]. The readers may be referred to some of the related literature on ‐GE‐ $distribution$.[7].The two-parameters ‐GE‐distribution has the following distribution function:

$$F(x;\alpha,\lambda) = [1-e^{-(\lambda x)}]^{\alpha} \quad \text{for } x > 0,\ \alpha > 0,\ \lambda > 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Thus, the probability density function (p.d.f.) of (GE) distribution is

$$f(x;\alpha,\lambda) = \begin{cases} \alpha\lambda x\, e^{-(\lambda x)}(1-e^{-(\lambda x)})^{\alpha-1} & \text{for } x > 0, \alpha, \lambda > 0 \\ 0 & \text{o.w.} \end{cases} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where, α and λ are the shape and scale parameters respectively.

In this paper we introduce the problem of estimating of the shape parameter (α) of GE distribution with known scale parameter (λ) when some prior information (α_0) regarding the actual value (α) available due past experiences such a prior estimate may arise for any one of an umber of reasons [8], e.g., we are estimating α and;

i. We believe α_0 is close to true value of α, or

ii. We fear that α_0 may be near the true value of α, i.e.; something bad happens if $\alpha = \alpha_0$ and we do not know about it.

In such a situation it is natural to start with an estimator $\hat{\alpha}$ (e.g. MLE) of α and modify it by moving it closer to α_0, so that the resulting estimator, though perhaps biased, has smaller mean square error than that of $\hat{\alpha}$ in some interval around α_0. This method of constructing an estimator of α that incorporates the prior value α_0 leads to what is known as a shrinkage estimator.

It is an important aspect of estimation that one should be able to get an estimator quickly using minimum experimentation. This also economizes cost of experimentation. To achieve this, double stage shrinkage estimator were introduced.

A double stage shrinkage estimator procedure is defined as follows:

Let x_{1i}; i = 1, 2, …, n_1 be a random sample of n_1 from GE distribution and $\hat{\alpha}_1$ be a "good" estimator of α based on these n_1 observation. Construct a preliminary test region R in the parameter space based on α_0 and an appropriate criterion.

If $\hat{\alpha}_1 \in$ R shrink $\hat{\alpha}_1$ towards α_0 by shrinkage weight factor $\psi(\hat{\alpha}) = e^{\frac{-10}{n1}}$ and use the shrinkage estimator $\psi(\hat{\alpha}_1)\hat{\alpha}_1 + (1-\psi(\hat{\alpha}_1))\alpha_0$, for estimate α.

If $\hat{\alpha}_1 \notin$R, obtain x_{2i}; i = 1, 2,…, n_2, an additional sample of size n_2 and use a pooled estimator $\hat{\alpha}_p$ of α based on combined sample of size n = n_1 + n_2, i.e.; $\hat{\alpha}_p = \frac{n_1\hat{\alpha}_1 + n_2\hat{\alpha}_2}{n}$.

Thus, the double stage shrinkage estimator (DSSE) of α will be:

$$\tilde{\alpha} = \begin{cases} \psi_1(\hat{\alpha}_1)\hat{\alpha}_1 + (1-\psi_1(\hat{\alpha}_1))\alpha_0 & ,\text{if } \hat{\alpha}_1 \in R \\ \hat{\alpha}_p & ,\text{if } \hat{\alpha}_1 \notin R \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The motivation of this study was provided by the work of [9], [10] and [11].

The aim of this paper is to employ the double stage shrinkage estimator (DSSE) $\tilde{\alpha}$ defined by (3) for estimate the shape parameter (α) of two parameters generalized Exponential (GE) distribution when the scale parameter (λ) is known.

The expression of Bias, Mean squared error (MSE), Relative Efficiency [R.Eff(·)], Expected sample size, Expected sample size proportion, probability for avoiding the second sample and percentage of overall sample saved are derived and obtained for the estimator $\tilde{\alpha}$.

Numerical results and conclusions due mentioned expressions including some constants are performed and displayed in annexed tables.

Comparisons between the proposed estimator with the classical estimator $(\hat{\alpha})$ and with some of the last studies are demonstrated.

2. Unbiased - Maximum Likelihood Estimator of α

In this section, we consider the maximum likelihood estimator (MLE) of GE distribution with shape and scale parameter α and λ respectively i.e. GE (α,λ).

Assume $x_{11}, x_{12}, \ldots, x_{1n_1}$ be a random sample of size n_1 from GE(α,λ) then the log-likelihood function L (α,λ) can be written as:

$$L(\alpha,\lambda) = n_1 \ln\alpha + n_1 \ln\lambda + (\alpha-1)\sum_{i=1}^{n_1} \ln(1-e^{-(\lambda x_{1i})}) - \lambda\sum_{i=1}^{n_1} x_{1i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In this paper we take λ = 1 (λ is known).

So, $$\frac{\partial L}{\partial\alpha} = \frac{n_1}{\alpha} + \sum_{i=1}^{n_1} \ln(1-e^{-x_{1i}}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

Then, the MLE of α, say $\hat{\alpha}_{1\,\text{MLE}}$ is

$$\hat{\alpha}_{1_{MLE}} = -\frac{n_1}{\sum_{i=1}^{n_1} \ln(1-e^{-x_{1i}})} \quad \text{...(6)}$$

Note that, if $x_{1i} \overset{iid}{\in}$ GE (α, 1), then $-\alpha \sum_{i=1}^{n_1} \ln(1-e^{-x_{1i}}) \sim$ G (n_1, 1), see [4].

i.e.; $E(\hat{\alpha}_{1_{MLE}}) = \frac{n_1}{n_1 - 1}\alpha$ and $\text{var}(\hat{\alpha}_{1_{MLE}}) = \frac{n_1^2\alpha^2}{(n_1-1)^2(n_1-2)}$.

Using (6), an unbiased estimator $\hat{\alpha}_1$ of α can be easily obtained as:

$$\hat{\alpha}_1 = \frac{n_1 - 1}{n_1}\hat{\alpha}_{1_{MLE}} = -\frac{n_1 - 1}{\sum_{i=1}^{n_1} \ln(1-e^{-x_{1i}})} \quad \text{... (7)}$$

$$E(\hat{\alpha}_1) = \alpha \text{ and } \text{var}(\hat{\alpha}_1) = \text{MSE}(\hat{\alpha}_1) = \frac{\alpha^2}{n_1 - 2} \quad \text{...(8)}$$

3. Double Stage Shrinkage Estimator (DSSE) $\tilde{\alpha}$

In this section, we consider the (DSSE) $\tilde{\alpha}$ which is defined in (3) using $\hat{\alpha}_1$ defined by (7), when $\psi(\hat{\alpha}_1) = k$ is a constant weight factor ($k = e^{\frac{-10}{n1}}$) for estimate the shape parameter α of GE distribution when $\lambda = 1$.

$$\tilde{\alpha} = \begin{cases} k\hat{\alpha}_1 + (1-k)\alpha_0 & , \text{if } \hat{\alpha}_1 \in R \\ \hat{\alpha}_p = \frac{n_1\hat{\alpha}_1 + n_2\hat{\alpha}_2}{n} & , \text{if } \hat{\alpha}_1 \notin R \end{cases} \quad \text{... (9)}$$

Where R is a pretest region for testing the hypothesis H_0: $\alpha = \alpha_0$ vs H_A: $\alpha \neq \alpha_0$ with level of significance (Δ) using test statistic function $T(\hat{\alpha}_1 / \alpha_0) = \frac{2(n_1-1)\alpha_0}{\hat{\alpha}_1}$

$$\text{i.e.; } R = \left[\frac{2(n_1-1)\alpha_0}{b}, \frac{2(n_1-1)\alpha_0}{a}\right] \quad \text{...(10)}$$

Where $a (= X^2_{1-\Delta/2, 2n_1})$ and $b (= X^2_{\Delta/2, 2n_1})$ ... (11) are the lower and upper 100(Δ/2) percentile point of chi-square distribution with degree of freedom ($2n_1$) respectively.

The expression for Bias of DSSE ($\tilde{\alpha}$) is defined as below

$$\mathrm{Bias}(\tilde{\alpha}/\alpha,R)=E(\tilde{\alpha}-\alpha)$$
$$=\int_{\hat{\alpha}_2=0}^{\infty}\int_{\hat{\alpha}_1\in R}\left[k(\hat{\alpha}_1-\alpha_0)+(\alpha_0-\alpha)\right]f(\hat{\alpha}_1)f(\hat{\alpha}_2)d\hat{\alpha}_1 d\hat{\alpha}_2+$$
$$\int_{\hat{\alpha}_2=0}^{\infty}\int_{\hat{\alpha}_1\notin \bar{R}}\left[\hat{\alpha}_p-\alpha\right]f(\hat{\alpha}_1)f(\hat{\alpha}_2)d\hat{\alpha}_1 d\hat{\alpha}_2$$

Where $\bar{R}$ is the complement region of R in real space and

$$f(\hat{\alpha}_i)=\begin{cases}\dfrac{\left[\dfrac{(n_i-1)\alpha}{\hat{\alpha}_i}\right]^{n_i+1}e^{-\frac{(n_i-1)\alpha}{\hat{\alpha}_i}}}{\Gamma(n_i)\cdot(n_i-1)\alpha} & \text{for } \hat{\alpha}_i>0,\alpha>0 \\ 0 & \text{o.w.}\end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

We conclude,

$$\mathrm{Bias}(\tilde{\alpha}/\alpha,R)=\alpha\left\{(K+1)(\zeta-1)J_0(a^*,b^*)+\frac{1}{1+u}[(n_1-1)J_1(a^*,b^*)-J_0(a^*,b^*)]\right\} \ldots (13)$$

Where $J_\ell(a^*,b^*)=\int_{a^*}^{b^*} y^{-\ell}\frac{y^{n_1-1}e^{-y}}{\Gamma(n_1)}dy;\ \ell=0,1,2$, ………………………………,,(14)

Also $y=\dfrac{(n_1-1)\alpha}{\hat{\alpha}_1}$ ……………………………………………………..(15)

The Bias ratio [B (·)] of DSSE ($\tilde{\alpha}$) is defined as:

$$B(\tilde{\alpha})=\frac{\mathrm{Bias}(\tilde{\alpha}/\alpha,R)}{\alpha} \ldots\ldots\ldots\ldots\ldots(16)$$

The expression of Mean squared error [MSE (·)] of $\tilde{\alpha}$ derived as below:-

$$\mathrm{MSE}(\tilde{\alpha}/\alpha,R)=E(\tilde{\alpha}-\alpha)^2$$
$$=\int_{\hat{\alpha}_2=0}^{\infty}\int_{\hat{\alpha}_1\in R}\left[k(\hat{\alpha}_1-\alpha)+(\alpha_0-\alpha)\right]^2 f(\hat{\alpha}_1)f(\hat{\alpha}_2)d\hat{\alpha}_1 d\hat{\alpha}_2+$$
$$\int_{\hat{\alpha}_2=0}^{\infty}\int_{\hat{\alpha}_1\notin \bar{R}}\left[\hat{\alpha}_p-\alpha\right]^2 f(\hat{\alpha}_1)f(\hat{\alpha}_2)d\hat{\alpha}_1 d\hat{\alpha}_2$$

And by simple computations, one can get:

$$MSE(\tilde{\alpha}/\alpha,R)=\alpha^2\begin{cases} k^2\left[(n_1-1)^2J_2(a^*,b^*)-2\zeta(n_1-1)J_1(a^*,b^*)+\zeta^2J_0(a^*,b^*)\right] \\ +2k(\zeta-1)\left[(n_1-1)J_1(a^*,b^*)-\zeta J_0(a^*,b^*)\right]+ \\ (\zeta-1)^2J_0(a^*,b^*)+ \\ \left(\frac{1}{1+u}\right)^2\left[\left(\frac{1}{n_1-2}\right)-[(n_1-1)^2J_2(a^*,b^*)+2(n_1-1)J_1(a^*,b^*)-J_0(a^*,b^*)\right]+ \\ \left(\frac{u}{1+u}\right)^2\left(\frac{1}{n_1u-2}\right)\left[1-J_0(a^*,b^*)\right] \end{cases} \quad \ldots\ldots\ldots(17)$$

Now, the Efficiency of $\tilde{\alpha}$ relative to $\hat{\alpha}$ denote by R.Eff ($\tilde{\alpha}/\alpha$, R) is defined by:

$$R.Eff(\tilde{\alpha}/\alpha,R)=\frac{MSE(\hat{\alpha})}{MSE(\tilde{\alpha}/\alpha,R)\cdot[E(n/\alpha,R/n]} \quad \ldots\ldots(18)$$

Where $E(n/\alpha,R)$ is the Expected sample size, which is defined as:

$$E(n/\alpha,R)=n\left[1-\frac{u}{1+u}J_0(a^*,b^*)\right] \quad \ldots\ldots(19)$$

See for example [10] and [11].

As well as the Expected sample size proportion E (n/α, R)/n equal to

$$1-\frac{u}{1+u}J_0(a^*,b^*) \quad \ldots\ldots(20)$$

Also, we have to define the percentage of the overall sample saved (p.o.s.s.) of $\tilde{\alpha}$ as:

$$p.o.s.s.=\frac{n_2}{n}J_0(a^*,b^*)*100 \quad \ldots\ldots(21)$$

And, finally, $P(\hat{\alpha}\in R)$ represent the probability of a voiding the second sample (stage).

4. Conclusions and Numerical Results

The computations of Relative Efficiency [R.Eff(·)] and Bias Ratio [B(·)], Expected sample size [$E(n/\alpha,R)$], Expected sample size proportion [$E(n/\alpha,R)/n$], Percentage of the overall sample saved (p.o.s.s.) and probability of a voiding the second sample $[P(\hat{\alpha}\in R)]$ were used for the estimator $\tilde{\alpha}$. These computations were performed for n_1 = 4, 8, 16, 20 u (= n_2/n_1) = 2, 6, 10, 12, ζ (= α_0/α) = 0.25(0.25)2, Δ = 0.01, 0.05, and $k=\psi(\hat{\alpha})=e^{\frac{-10}{n1}}$

Some of these computations are given in the tables (1)-(4).

The observation mentioned in the tables lead to the following results:

i. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ are adversely proportional with small value of Δ especially when $\zeta = 1$, i.e. $\Delta = 0.01$ yield highest efficiency

ii. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ has maximum value when $\alpha=\alpha_0(\zeta=1)$, for each n_1, Δ, and decreasing otherwise ($\zeta\neq1$). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy.

iii. Bias ratio [B (·)] of $\tilde{\alpha}$ are reasonably small when $\alpha=\alpha_0$ for each n_1, Δ, and increases otherwise. This property shown that the proposed estimator $\tilde{\alpha}$ is very closely to unbiasedness property especially when $\alpha=\alpha_0$.

iv. The Effictive interval of $\tilde{\alpha}$ [the value of $\tilde{\alpha}$ which makes R.Eff(·) of $\tilde{\alpha}$ greater than one] is [0.5, 1.5].

v. Bias ratio [B (·)] of $\tilde{\alpha}$ are increases with small value of u.

vi. R.Eff($\tilde{\alpha}$) is decreasing function with increasing of the first sample size n_1, for each Δ andζ.

vii. The expected values of sample size of $\tilde{\alpha}$ are close to n_1, especially when $\zeta \geq 1$ and start far-away otherwise.

viii. Percentage of the overall sample saved $\left[\frac{n_2}{n} J_0(a^*, b^*) * 100\right]$ is increasing value with increasing value of u ($u = n_2 / n_1$) andζ.

ix. R.Eff($\tilde{\alpha}$) is an increasing function with respect to u. This property shown the effective of proposed estimator using small n_1 relative to n_2 (or large n_2) which given higher efficiency and reduce the observation cost.

x. The considered estimator $\tilde{\alpha}$ is better than the classical estimator especially when $\alpha\approx\alpha_0$, this will given the effective of $\tilde{\alpha}$ relative to $\hat{\alpha}$ and also given an important weight of prior knowledge, and the augmentation of efficiency may be reach to tens times.

xi. The considered estimator $\tilde{\alpha}$ is more efficient than the estimators introduced by [8], in the sense of higher efficiency.

References

[1] D. Kundu and R.D.Gupta, Absolute continuous bivariate generalized exponential distribution, AStA Advances in Statistical Analysis, 2(95) (2011), 169–185.

[2] R.D.Gupta and D. Kundu, Generalized exponential distributions, Australian and New Zealand Journal of Statistics, 2(41) (1999), 173-188.

[3] R.D.Gupta and D. Kundu, Exponentiated exponential distribution: alternative to gamma and Weibull distributions, Biometrical Journal, (43) (2001), 117-130.

[4] R.D.Gupta and D. Kundu, Generalized exponential distributions: different methods of estimations, Statistical Computations and Simulations, 4(69) (2001), 315-338.
[5] R.D.Gupta and D. Kundu, Generalized exponential distributions: statistical inferences, Statistical Theory and Applications, 2(1) (2002), 101-118.
[6] R.D.Gupta and D. Kundu, Discriminating between Weibull and generalized exponential distributions, Computational Statistics and Data Analysis, 43(2003), 179-196.
[7] M.Z.Raqab, Inference for generalized exponential distribution based on record statistics, Statistical Planning and Inference, 2(104) (2002), 339-350.
[8] J.R., Thompson, Some shrinkage techniques for estimating the mean, American Statistical Association, 63(1968), 113-122.
[9] S.K. Katti, Use of some a prior knowledge in the estimation of means from Double Samples, Biometrics, 18(1962), 139-147.
[10] V.B Waikar, F.J. Schuurmann and T.E Raghunathan, On a Two-Stage shrinkage testimator of the mean of a normal distribution , Communications in Statistics –Theory and Methods, 13 (1984), 1901-1913.
[11]A.N. Al-Joboori ,.Pre-Test single and double stage shrunken estimators for the mean of normal distribution with known variance, Baghdad Journal for Science, 17(4)(2010),1432-1442.

Table (1)

Shown Bias ratio [B (·)] and R.E.ff of $\tilde{\alpha}$ w.r.t Δ, n_1 and ζ when u = 2

			ζ					
Δ	**n_1**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
0.01	4	R.Eff(·) B(·)	0.525 -0.215	2.386 -0.295	6.722 -0.035	1.257 0.517	0.601 0.793	0.345 1.064
	8	R.Eff(·) B(·)	0.9 -6.319*10^-4	1.21 -0.249	4.882 -0.037	0.978 0.616	0.456 0.947	0.259 1.271
	16	R.Eff(·) B(·)	0.956 -3.902*10^-1	0.761 -0.073	1.806 -0.034	1.01 0.716	0.478 1.127	0.277 1.519
	20	R.Eff(·) B(·)	0.965 0	0.811 -0.029	1.288 -0.028	1.076 0.735	0.509 1.177	0.297 1.59
0.05	4	R.Eff(·) B(·)	0.723 -0.027	1.591 -0.255	3.851 -0.065	1.147 0.468	0.569 0.744	0.329 1.01
	8	R.Eff(·) B(·)	0.904 -3.918*10^-6	0.94 -0.134	2.616 -0.05	0.977 0.552	0.451 0.894	0.252 1.215
	16	R.Eff(·) B(·)	0.956 0	0.878 -0.015	1.157 -0.025	1.168 0.612	0.503 1.056	0.273 1.455
	20	R.Eff(·) B(·)	0.965 0	0.93 -3.98*10^-3	0.975 0.016	1.324 0.611	0.549 1.097	0.295 1.523

Table (2)

Shown Bias ratio [B (·)] and R.E.ff of $\tilde{\alpha}$ w.r.t Δ, n_1 and ζ when u = 6

			ζ					
Δ	**n_1**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
0.01	4	R.Eff(·) B(·)	0.312 -0.193	2.514 -0.264	22.294 -0.015	1.18 0.527	0.54 0.799	0.304 1.068
	8	R.Eff(·) B(·)	0.943 -5.631*10^-4	0.832 -0.221	7.295 -0.016	0.889 0.623	0.42 0.951	0.239 1.272
	16	R.Eff(·) B(·)	0.98 -3.501*10^-1	0.501 -0.064	1.462 -0.015	0.872 0.723	0.444 1.13	0.261 1.519
	20	R.Eff(·) B(·)	0.985 0	0.606 -0.025	0.93 -0.012	0.902 0.743	0.471 1.18	0.281 1.59
0.05	4	R.Eff(·) B(·)	0.692 -0.024	1.417 -0.213	8.917 -0.028	0.984 0.49	0.464 0.76	0.259 1.02
	8	R.Eff(·) B(·)	0.956 -3.453*10^-6	0.652 -0.112	3.074 -0.021	0.77 0.569	0.369 0.904	0.206 1.219
	16	R.Eff(·) B(·)	0.98 0	0.748 -0.013	0.889 -0.011	0.858 0.626	0.405 1.064	0.228 1.457
	20	R.Eff(·) B(·)	0.985 0	0.88 -3.365*10^-3	0.731 -6.765*10^-3	0.96 0.625	0.437 1.104	0.248 1.525

Table (3)

Shown Bias ratio [B (·)] and R.E.ff of $\tilde{\alpha}$ w.r.t Δ, n_1 and ζ when u = 10

			ζ					
Δ	**n_1**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
0.01	4	R.Eff(·) B(·)	0.216 -0.186	2.129 -0.256	36.212 -9.59*10^-3	1.108 0.529	0.51 0.801	0.285 1.069
	8	R.Eff(·) B(·)	0.95 -5.44*10^-4	0.605 -0.213	7.064 -0.01	0.815 0.624	0.398 0.952	0.227 1.273
	16	R.Eff(·) B(·)	0.987 -3.392*10^-1	0.37 -0.061	1.143 -9.307*10^-3	0.754 0.724	0.417 1.131	0.25 1.52
	20	R.Eff(·) B(·)	0.99 0	0.482 -0.024	0.705 -7.706*10^-3	0.759 0.745	0.439 1.18	0.27 1.59
0.05	4	R.Eff(·) B(·)	0.616 -0.023	1.106 -0.201	12.111 -0.018	0.83 0.496	0.396 0.765	0.219 1.023
	8	R.Eff(·) B(·)	0.971 -3.327*10^-6	0.485 -0.107	2.856 -0.014	0.62 0.573	0.313 0.907	0.176 1.22
	16	R.Eff(·) B(·)	0.987 0	0.645 -0.012	0.705 -6.792*10^-3	0.655 0.63	0.337 1.066	0.197 1.458
	20	R.Eff(·) B(·)	0.99 0	0.826 -3.19*10^-3	0.58 -4.305*10^-3	0.722 0.629	0.36 1.106	0.214 1.525

Table (4)

Shown Bias ratio [B (·)] and R.E.ff of $\tilde{\alpha}$ w.r.t Δ, n_1 and ζ when u = 12

			ζ					
Δ	**n_1**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
0.01	4	R.Eff(·) B(·)	0.187 -0.185	1.956 -0.254	41.524 -8.117*10^-3	1.078 0.53	0.498 0.801	0.278 1.07
	8	R.Eff(·) B(·)	0.95 -5.393*10^-4	0.531 -0.211	6.731 -8.48*10^-3	0.783 0.625	0.388 0.952	0.222 1.273
	16	R.Eff(·) B(·)	0.989 -3.362*10^-1	0.327 -0.06	1.025 -7.875*10^-3	0.706 0.725	0.405 1.131	0.245 1.52
	20	R.Eff(·) B(·)	0.992 0	0.437 -0.024	0.627 -6.52*10^-3	0.703 0.745	0.425 1.181	0.264 1.59
0.05	4	R.Eff(·) B(·)	0.58 -0.022	0.988 -0.198	13.181 -0.015	0.769 0.498	0.37 0.766	0.204 1.024
	8	R.Eff(·) B(·)	0.975 -3.292*10^-6	0.429 -0.105	2.712 -0.011	0.565 0.574	0.292 0.907	0.165 1.22
	16	R.Eff(·) B(·)	0.989 0	0.603 -0.012	0.638 -5.747*10^-3	0.584 0.631	0.311 1.067	0.184 1.458
	20	R.Eff(·) B(·)	0.992 0	0.801 -3.151*10^-3	0.525 -3.643*10^-3	0.641 0.63	0.331 1.107	0.2 1.526

Table (5)

Shown Probability of a Voiding Second Sample w.r.t Δ, u, n_1 and ζ

			ζ					
u	**n_1**	**Δ**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
2	4	0.01	0.216	0.893	0.952	0.986	0.991	0.99
		0.05	0.026	0.668	0.823	0.937	0.952	0.95
6	8	0.01	53×10^{-4}	0.62	0.851	0.976	0.988	0.99
		0.05	322×10^{-6}	0.3	0.612	0.903	0.944	0.95
10	16	0.01	278×10^{-12}	0.147	0.544	0.948	0.983	0.99
		0.05	0	0.029	0.264	0.83	0.929	0.95
12	20	0.01	0	0.055	0.409	0.931	0.981	0.99
		0.05	0	722×10^{-3}	0.159	0.791	0.921	0.95

Table (6)

Shown Expected Sample Size of $\tilde{\alpha}$ w.r.t Δ, u, and ζ when n_1 = 4

		ζ					
u	**Δ**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
2	0.01	10.271	4.859	4.381	4.108	4.075	4.080
	0.05	11.793	6.655	5.414	4.503	4.384	4.40
6	0.01	22.812	6.578	5.143	4.325	4.226	4.240
	0.05	27.38	11.965	8.242	5.508	5.153	5.20

Table (7)

Shown Expected Sample Size Proportion w.r.t Δ, u, n_1 and ζ

			ζ					
u	**n_1**	**Δ**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
2	4	0.01	0.856	0.405	0.365	0.342	0.340	0.340
		0.05	0.983	0.555	0.451	0.375	0.365	0.367
6	8	0.01	1	0.468	0.27	0.164	0.153	0.151
		0.05	1	0.743	0.475	0.226	0.190	0.186
10	16	0.01	1	0.866	0.496	0.138	0.106	0.1
		0.05	1	0.974	0.760	0.246	0.155	0.136
12	20	0.01	1	0.949	0.623	0.14	0.095	0.086
		0.05	1	0.993	0.853	0.27	0.150	0.123

Table (8)

Shown Percentage of Overall Sample Saved w.r.t Δ, u, n_1 and ζ

			ζ					
u	**n_1**	**Δ**	**0.25**	**0.75**	**1**	**1.5**	**1.75**	**2**
2	4	0.01	14.410	59.505	63.492	65.726	66.039	66.000
		0.05	1.723	44.542	54.883	62.4604	63.463	63.333
6	8	0.01	0.045	53.157	72.976	83.657	84.715	84.857
		0.05	276×10^{-4}	25.698	52.493	77.396	80.954	81.428
10	16	0.01	253×10^{-10}	13.408	50.397	86.1818	89.404	89.999
		0.05	135×10^{-14}	2.644	24.032	75.447	84.456	86.363
12	20	0.01	292×10^{-15}	5.3097	37.734	85.938	90.537	91.385
		0.05	218×10^{-20}	0.667	14.655	73.0148	85.028	87.692

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