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Doubly Asymptotic Trajectories of Lagrangian Systems in Homogeneous Force Fields (*).

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Abstract. – *We study Lagrangian systems with symmetry under the action of a constant generalized force in the direction of the symmetry field. After Routh's reduction, such systems become nonautonomous with Lagrangian quadratic in time. We prove the existence of solutions tending to an orbit of the symmetry group as $t \to \pm\infty$. As an example, we study doubly asymptotic solutions for the Kirchhoff problem of a heavy rigid body in an infinite volume of incompressible ideal fluid performing a potential motion.*

1. – Introduction.

Before going into the mathematical formulation of the problem, we briefly describe a general situation where systems of the type we study arise. In § 3 we will give a concrete physical example.

Consider a Lagrangian system with the configuration manifold N and kinetic energy $T = T(q, \dot{q})$ of class C^2 which is a positive definite homogeneous quadratic form in velocity $\dot{q} \in T_q N$. We assume also the presence of a generalized force Q of class C^1, which is a covector field on N. Thus $Q(q) \in T_q^* N$ for any $q \in N$. The equations of motion take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q\,. \tag{1.1}$$

Recall that although this formula is written in local coordinates, the left hand side is a well defined covector in $T_q^* N$ independent of the choice of local coordinates.

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Suppose that the kinetic energy is invariant under a one-parameter transformation group $\mathbb{R}$. Thus there exists a smooth free group action of $\mathbb{R}$ on N conserving T. The transformation group is called a symmetry group and the corresponding vector field v on N is called a symmetry field. We assume that the force field $Q = \nabla\phi$ is potential and invariant under the action of the symmetry group. Moreover, the projection $F = \langle Q, v\rangle$ of the force field on v is a nonzero constant. Hence the potential ϕ isn't invariant. We call such force fields Q homogeneous. An example is provided by the homogeneous gravitational force field.

Under the assumptions above, the fibration of N to the orbits of the group action turns out to be trivial. Namely, N is diffeomorphic to $M\{x\} \times \mathbb{R}\{y\}$, where M is a smooth manifold, and the group action corresponds to the translation $(x, y) \in M \times \mathbb{R} \to$ $\to (x, y+s) \in M \times \mathbb{R}$, $s \in \mathbb{R}$. From now on we identify N with $M \times \mathbb{R}$. Then $\phi = V(x) + Fy$ and the force field Q takes the form

$$Q(x, y) = (\nabla V(x), F) \in T_x^* M \times \mathbb{R},$$

where V is a C^2 function on M. Thus we have a constant generalized force F in the direction of the coordinate y. Due to this fact and since the kinetic energy doesn't depend on y, the coordinate y can be actually ignored in determining x along the solutions. For that reason, we will refer to y as to a cyclic coordinate.

Specifically, T is a C^2 function on $TM \times \mathbb{R}$:

$$T = T(x, \dot{x}, \dot{y}) = \frac{1}{2}\langle A(x)\dot{x}, \dot{x}\rangle + \langle b(x), \dot{x}\rangle \dot{y} + \frac{1}{2} c(x)\dot{y}^2, \tag{1.2}$$

where for all $x \in M$, $A(x)\colon T_x M \to T_x^* M$ is a symmetric positive definite operator, b is a covector field on M, $b(x) \in T_x^* M$ and c is a positive function on M. By $\langle \cdot, \cdot\rangle$ we denote the formal scalar product $T_x^* M \times T_x M \to \mathbb{R}$.

Let p_y be the generalized momentum corresponding to the coordinate y or, equivalently, the Noether integral corresponding to the symmetry group:

$$p_y = \langle T_{\dot{q}}, v\rangle = \frac{\partial T}{\partial \dot{y}} = \langle b(x), \dot{x}\rangle + c(x)\dot{y}.$$

Then Lagrange's equations (1.1) take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \frac{\partial V}{\partial x}, \qquad \dot{p}_y = F. \tag{1.3}$$

The second equation (1.3) yields $p_y(t) = Ft + p_y(0)$. Performing a time shift, without loss of generality, we can put $p_y(t) = Ft$.

An important role will be played by the function U on M defined by the formula

$$U(x) = -\frac{F^2}{2c(x)}, \qquad x \in M. \tag{1.4}$$

Note that $-t^2 U(x)$ is the kinetic energy of the motion along the orbit $\Gamma_x = \{x\} \times \mathbb{R} \subset N$ of the symmetry group with the momentum $p_y(t) = Ft$.

Let M be a compact manifold. Then U has a minimum on M. Suppose that there exists a unique minimum point $x_0 \in M$ and it is nondegenerate. Without loss of generality, we can assume that $b(x_0) = 0$ and $\nabla w(x_0)$ is an antisymmetric operator, where $w(x) = b(x)/c(x)$. Indeed, if this isn't so, it is sufficient to perform the transformation of the cyclic coordinate $y \to y - f(x)$, where $f \in C^3(M)$ is a function such that $\nabla f(x_0) = w(x_0)$ and the operator $\nabla^2 f(x_0) - \nabla w(x_0)$ is antisymmetric. Denote

$$W(x) = -\frac{F^2}{2(c(x) - \langle A^{-1}(x)\, b(x), b(x)\rangle)}, \qquad x \in M .$$

The goal of this paper is to prove the following

THEOREM 1.1. – *Suppose that x_0 is a point of strict nondegenerate minimum of the function W on M. Then there exists an infinite number of trajectories of the system* (1.3) *such that $x(t) \to x_0$ and $\dot{x}(t) \to 0$ as $t \to \pm\infty$. Thus these trajectories tend to Γ_{x_0} as $t \to \pm\infty$.*

Of course, we identify trajectories obtained by a time shift or a translation of the coordinate y. Note that in general Γ_{x_0} isn't a trajectory, unless some additional assumptions below are satisfied. If Γ_{x_0} is a trajectory, then, in classical mechanics, it is called a stationary solution [1]. Physically, Γ_{x_0} is the orbit of the symmetry group such that the kinetic energy of the motion with $\dot{y} = 1$ is minimal.

To understand the meaning of the function W, it is convenient to use Routh's reduction.

2. – Routh's reduction and reformulation of the main result.

We will apply the classical Routh method of ignoring a cyclic coordinate y to reduce the system to a time dependent Lagrangian system with the configuration space M. Since T is positive definite in the velocity, the equation $p_y = Ft$ can be solved for $\dot{y}$ in terms of x, $\dot{x}$ and t:

$$\dot{y} = g(x, \dot{x}, t) = (Ft - \langle b(x), \dot{x}\rangle)/c(x).$$

Define the Routh function $L = L(x, \dot{x}, t)$ on $TM \times \mathbb{R}$ by the standard formula

$$L(x, \dot{x}, t) = \min_{\dot{y}}\{T - p_y(t)\,\dot{y}\} + V = (T(x, \dot{x}, \dot{y}) - Ft\dot{y})\,|_{\dot{y} = g(x, \dot{x}, t)} + V(x). \tag{2.1}$$

By the Routh theorem [1], the trajectories of system (1.3) such that $p_y(t) = Ft$ satisfy the Routh equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 . \tag{2.2}$$

Thus we obtain a nonautonomous Lagrangian system with the Lagrangian L and configuration space M. This system is called the reduced Lagrangian system.

REMARK. – It is well known that in general the Routh reduction isn't an invariantly defined procedure: it depends on the choice of the diffeomorphism of N and $M \times \mathbb{R}$. Moreover, when the symmetry group is a circle S^1 and the fibration of N to the orbits of the group is topologically nontrivial, it is impossible to define the Routh function globally [1]. Thus, in this case, the Routh equations are only locally Lagrangian and the Routh function has no physical meaning. However, for our system $N = M \times \mathbb{R}$ and so the Routh function L is well defined. However, since the cyclic coordinate y is defined up to a shift $y \to y - f(x)$, the Routh function isn't unique.

To obtain an explicit expression for the Routh function, we use the formula (1.2) for the kinetic energy T. Then L takes the form

$$L(x, \dot{x}, t) = \frac{1}{2}\left(\langle A(x)\dot{x}, \dot{x}\rangle - \frac{\langle b(x), \dot{x}\rangle^2}{c(x)}\right) + \frac{\langle b(x), \dot{x}\rangle}{c(x)} Ft - \frac{F^2}{2c(x)} t^2 + V(x).$$

Denote $w = Fb/c$ and let U be the function (1.4). Then we obtain

$$L(x, \dot{x}, t) = K(x, \dot{x}) + t\langle w(x), \dot{x}\rangle + t^2 U(x) + V(x), \tag{2.3}$$

where $K(x, \dot{x}) = \langle B(x)\dot{x}, \dot{x}\rangle/2$ is a positive definite quadratic form in $\dot{x} \in T_x M$.

From now on we forget about the origin of the Lagrangian (2.3). Thus we consider a reversible Lagrangian system (2.2) with compact configuration manifold M and the Lagrangian $L \in C^2(TM \times \mathbb{R})$ of the form (2.3). Explicitly, system (2.2) takes the form

$$B(x)\, D_t \dot{x} + tG(x)\dot{x} + w(x) - t^2 \nabla U(x) - \nabla V(x) = 0, \tag{2.4}$$

where D_t is the covariant derivative with respect to the metric K and $G = \nabla w - (\nabla w)^T$. A trajectory will always mean a solution of the Lagrangian system (2.2) or (2.4).

System (2.4) has no equilibria unless the equations

$$\nabla U(x) = 0, \qquad \nabla V(x) = w(x)$$

have a common solution. Even if this is not the case, there exist solutions similar to equilibria. For example, the following result hold.

LEMMA 2.1. – *Let x_0 be a nondegenerate critical point of U. Suppose for simplicity that $L \in C^\infty$. Then there exists a solution $x(t)$ such that $x(t) \to x_0$ and $\dot{x}(t) \to 0$ as $t \to \infty$. By reversibility, $x(-t)$ is a solution asymptotic to x_0 as $t \to -\infty$.*

PROOF [11]. – There exists a unique formal solution of the equation (2.4) of the form

$$x(t) = x_0 + \sum_{k=2}^{\infty} \frac{a_k}{t^k}, \qquad a_2 = \left(\nabla^2 U(x_0)\right)^{-1} (w(x_0) - \nabla V(x_0)), \tag{2.5}$$

where we use a local chart around x_0 [11]. The coefficients a_k are obtained recursively. In general, the series is divergent, even if the system is analytic. However, by the

Kuznetsov theorem [12], there exists a C^∞ solution with the asymptotic expansion (2.5), tending to x_0 as $t \to \infty$. Note that in general this solution isn't unique. ∎

A similar result holds for $L \in C^2$. However, we will not prove this, since for the points x_0 we study the existence of asymptotic solutions can be obtained easier by variational methods.

Note that if, following [10], we perform a transformation of time $\tau = t^2/2$, then equation (2.4) takes the form

$$B(x)\, D_\tau x' + G(x)\, x' - \nabla U(x) + \frac{1}{2\tau}(w(x) - \nabla V(x) + B(x)\, x') = 0 .$$

Hence in the limit $\tau \to \infty$ we obtain an autonomous Lagrangian system with the Lagrangian

$$L_\infty(x, x') = K(x, x') + \langle w(x), x' \rangle + U(x).$$

Thus any critical point x_0 of U may be thought of as an equilibrium at infinity and it makes sense to look for homoclinic solutions. Under the condition on the function W in the next theorem, there always exist homoclinic solutions of the system with the Lagrangian L_∞ (see, for example, [3], [4]). However, in general the existence of an infinite number of homoclinics isn't proved, since the *PS* sequences are divergent in general [7]. For our system (2.4), this difficulty disappears.

Now we reformulate Theorem 1.1. Let x_0 be the unique minimum point of U. Suppose that it is nondegenerate. Recall that the Lagrangian is defined up to addition of a time derivative. This makes it possible to calibrate L in such a way that $w(x_0) = 0$ and $\nabla w(x_0)$ is an antisymmetric operator. This calibration is equivalent to a shift of the cyclic coordinate y in the previous section. Let $f \in C^3(M)$ be any function such that $\nabla f(x_0) = w(x_0)$. Then

$$t\langle w(x), \dot{x} \rangle = \frac{d}{dt}(tf(x)) - f(x) + t\langle w(x) - \nabla f(x), \dot{x} \rangle .$$

Being a total time derivative, the first term on the right hand side doesn't change Lagrange's equations. The second term can be added to $V(x)$ and treated together. Hence $w - \nabla f$ plays the role of the original field w. Choosing f appropriately, we can also kill the symmetric part in $\nabla w(x_0)$.

Define a function W on M by the formula

$$W(x) = U(x) - \frac{1}{2}\langle w(x), B(x)^{-1} w(x) \rangle . \tag{2.6}$$

This is an analog of the Hagedorn function [8], which appears in the sufficient conditions for instability of an equilibrium of a system with the Lagrangian L_∞. Of course, W depends on the choice of the function f. A similar nonuniqueness arises for the Hagedorn function. The point x_0 is a critical point of W, but not necessarily a minimum.

THEOREM 2.1. – *Let $x_0 \in M$ be a nondegenerate unique minimum point of W. Then there exist infinitely many trajectories of the system* (2. 2) *such that $x(t) \to x_0$ and $\dot{x}(t) \to 0$ as $t \to \pm\infty$.*

Obviously, this theorem implies Theorem 1.1. Theorem 2.1 is proved in §§ 4-5 by variational methods. In the next section we discuss an application to a classical mechanical system.

REMARKS. – 1) If w is a gradient, we can put $W = U$. For example, this is so if the coefficient b in the kinetic energy (1.2) is zero. In classical mechanics, such systems are called gyroscopically disconnected.

2) If x_0 is a point of nondegenerate minimum of the function U, but not of W, then it is possible that there exist no homoclinic solutions. Indeed, in the example below, x_0 is a point of nondegenerate minimum of U and it is an equilibrium. Moreover, for almost all solutions $x(t)$ close to x_0 we have $x(t) \to x_0$ as $t \to \pm\infty$. However, for all these solutions except the trivial one, $\limsup |\dot{x}(t)| = \infty$ as $t \to \pm\infty$. Thus there are no doubly asymptotic solutions. This example also shows that in general it isn't true that if $x(t) \to x_0$ as $t \to \infty$, then $\dot{x}(t) \to 0$. However, if W has a nondegenerate minimum at x_0, then this is so. The proof is contained in § 4.

3) If x_0 is a point of nondegenerate maximum for U and $w = 0$, then under weak additional assumptions most of the trajectories $x(t)$ tend to x_0 as $t \to \pm\infty$. The proof of this result involves Lyapunov type arguments [11] and is much simpler than the proof of Theorem 2.1. However, $\limsup_{t \to \pm\infty} |\dot{x}(t)| = \infty$ and so these solutions can't be called asymptotic.

4) Suppose that there exist a finite number of minimum points x_i of U. We calibrate w so that $w(x_i) = 0$ for all i and $\nabla w(x_i)$ is antisymmetric. Suppose that the points x_i are nondegenerate minimum points for the function W in (2. 6) and that the function V takes the same value at all points x_i. Then there exist heteroclinics joining any two points x_i and x_j. For $i \neq j$, the proof is simpler than that of Theorem 2.1 since it involves only minimization of the action functional. Moreover, for each pair x_i, x_j (also with $i = j$), there exist an infinite number of heteroclinic trajectories connecting x_i with x_j. The proof of this result is similar to the proof of Theorem 2.1.

5) We will show that the number of doubly asymptotic trajectories is at least $\operatorname{cat} \Omega(M)$, where $\Omega(M)$ is the loop space of M. By Serre's theorem [16], $\operatorname{cat} \Omega(M)$ is infinite for a compact manifold M.

The existence of $\operatorname{cat} \Omega(M)$ trajectories with the properties claimed in Theorem 2.1 can be obtained also for noncompact M if certain additional completeness assumptions are satisfied. For example, it is sufficient that the distance d on M defined by the Riemannian metric K is complete, V is bounded from below and $W(q) d^2(q, x_0) \geq c > 0$ for all q outside a small neighborhood of x_0.

Next we give an example showing that in general, if x_0 is a nondegenerate minimum of U, but not of W, then Theorem 2.1 doesn't hold.

EXAMPLE. – Suppose that in the local coordinates $x \in \mathbb{R}^2$ in a neighborhood of x_0 the Lagrange function has rotational symmetry, for example

$$L = \frac{1}{2} |\dot{x}|^2 + at\langle Jx, \dot{x}\rangle + \frac{t^2}{2} |x|^2 ,$$

where J is the standard symplectic matrix. We can always assume that L is defined on $TM \times \mathbb{R}$ with compact M. For example, we can explicitly extend L to $TS^2 \times \mathbb{R}$ conserving rotational symmetry. We have

$$U = \frac{1}{2} |x|^2 , \qquad W = \frac{1}{2}(1 - a^2) |x|^2 .$$

Thus $x = 0$ is a nondegenerate minimum for U, and it is a nondegenerate minimum for W if $|a| < 1$. It is easy to show that for $a > 1$ and any solution $x(t) \neq 0$, we have $\lim x(t) = 0$ and $\lim\sup |\dot{x}(t)| = \infty$ as $t \to \infty$. Thus for $a > 1$ there exist no asymptotic solutions to the equilibrium $x = 0$.

Indeed, perform the transformation of variables $x = \exp(-(at^2/2) J) y$. Then

$$L = \frac{1}{2} |\dot{y}|^2 + (1 - a^2) \frac{t^2}{2} |y|^2 ,$$

and the result follows from the WKB approximation [11].

3. – Kirchhoff problem.

This paper was motivated by the following classical problem of the rigid body dynamics. Consider a rigid body moving in an infinite volume of incompressible ideal fluid. We assume that the fluid is at rest at infinity and has zero vorticity, so that the flow is potential. Then the motion of the fluid is completely determined by the motion of the body. Thus the system of the body and the fluid has six degrees of freedom and the configuration space N is diffeomorphic to the Euclidean group $E(3)$, which is a semidirect product $SO(3) \times \mathbb{R}^3$. Since the system of the body and the fluid is Lagrangian, the motion of the body is described by a Lagrangian system with the configuration space N.

Let $-P\gamma$, where γ is the unit vertical vector, be the sum of the weight of the body and the Archimedus force. Depending of the density of the body, P can be positive or negative. Denote by O the point of the body where the force $P\gamma$ is applied. Let $\omega, v \in \mathbb{R}^3$ be the angular velocity of the body and the velocity of the point O represented in some coordinate frame e_1, e_2, e_3 connected to the body. Since the kinetic energy of the system body-fluid is a positive definite quadratic form, invariant under the action of the group $E(3)$ on itself by left translations, we obtain [9], [13]

$$T = \frac{1}{2}\langle A\omega, \omega\rangle + \langle B\omega, v\rangle + \frac{1}{2}\langle Cv, v\rangle, \tag{3.1}$$

where the matrices A, B, C are constant and the symmetric matrices A, C and

$$D = A - B^T C^{-1} B \tag{3.2}$$

are positive definite. The matrix A is a sum of the inertia tensor of the body and an additional inertia tensor corresponding to the liquid. The eigenvalues of the matrix C are sums of the mass of the body and the so called additional masses describing inertial properties of the fluid [13].

Introducing Euler angles and the Cartesian coordinates of the point O in some fixed coordinate frame, it is possible to write down the equations of motion in the ordinary Lagrangian form. However, they turn out to be quite complicated. It is simpler to rewrite these equations in a moving coordinate frame e_1, e_2, e_3. Then we obtain Kirchhoff's equations [9]

$$\begin{cases} \dot{p} + \omega \times p = -P\gamma\,, & p = T_v \\ \dot{J} + \omega \times J + v \times p = 0\,, & J = T_\omega\,, \end{cases} \tag{3.3}$$

where the unit vertical vector $\gamma \in S^2$ is represented in the coordinate frame e_1, e_2, e_3. Equations (3.3) follow, for example, from the theorem on momentum and angular momentum respectively in a moving coordinate frame. Let α and β be two horizontal fixed vectors, again written in the coordinate frame e_1, e_2, e_3 connected with the body. They satisfy the Poisson equations

$$\dot{\alpha} + \omega \times \alpha = 0\,, \qquad \dot{\beta} + \omega \times \beta = 0\,, \qquad \dot{\gamma} + \omega \times \gamma = 0\,. \tag{3.4}$$

Equations (3.3) and (3.4) form a complete system of equations of motion of the body.

REMARK. – Equations (3.3) are a particular case of the general Poincaré equations of a Lagrangian system obtained by projecting the ordinary Lagrange equations to some basis vector fields on the configuration space N. In our case, these fields are the left invariant vector fields on the group $N = E(3)$. They correspond to the translations of the body with unit velocity in the directions of the basis vectors e_1, e_2, e_3 and the rotations of the body with unit angular velocity about the same vectors.

The kinetic energy T admits a symmetry group $SO(3) \times \mathbb{R}^3$ acting on the configuration space by rotations and translations. Obviously, the force field $-P\gamma$ is invariant under the action of the group $S^1 \times \mathbb{R}^3$, where $S^1 \subset SO(3)$ corresponds to rotations about the vertical. Thus we are in the situation of § 1. The force of magnitude P plays the role of the homogeneous generalized component F of the force in § 1 and the coordinate y is now the height of the point O. The only difference is that the symmetry group is $S^1 \times \mathbb{R}^3$ and so there are additional integrals of motion. Let

$$p_\alpha = \langle p, \alpha \rangle, \qquad p_\beta = \langle p, \beta \rangle, \qquad p_\gamma = \langle p, \gamma \rangle$$

be the components of momentum. Then p_α, p_β and $p_\gamma + Pt$ are integrals of motion. For simplicity we assume that the horizontal momentum is zero: $p_\alpha = p_\beta = 0$. Without loss of generality we can put $p_\gamma = -Pt$. We also assume that the integral of vertical angular momentum $J_\gamma = \langle J, \gamma \rangle$ is zero.

By using the last Poisson equation (3.4), equations

$$p_\alpha = p_\beta = J_\gamma = 0\,, \qquad p_\gamma = -Pt$$

can be resolved in v, ω:

$$v = -C^{-1}(Pt\gamma + B\omega)\,, \qquad \omega = \frac{\dot\gamma \times D\gamma + Pt\gamma\langle B^T C^{-1}\gamma, \gamma\rangle}{\langle D\gamma, \gamma\rangle}\,. \tag{3.5}$$

The reduced configuration space

$$M = (SO(3)\times \mathbb{R}^3)/(S^1 \times \mathbb{R}^3) \cong S^2 = \{\gamma \in \mathbb{R}^3 \colon |\gamma| = 1\}$$

is the Poisson sphere. Substituting equations (3.5) in the kinetic energy, we obtain the Routh function (2.1) on $TS^2 \times \mathbb{R}$. After a simple vector algebra calculation, L takes the form

$$\begin{aligned} L(\gamma, \dot\gamma, t) = T + Pt\langle v, \gamma\rangle &= \frac{1}{2}\langle D\omega, \omega\rangle - \frac{P^2t^2}{2}\langle C^{-1}\gamma, \gamma\rangle - Pt\langle C^{-1}B\omega, \gamma\rangle = \\ &= \frac{\det D}{2}\frac{\langle D^{-1}\dot\gamma, \dot\gamma\rangle}{\langle D\gamma, \gamma\rangle} - Pt\frac{\langle D\gamma \times B^T C^{-1}\gamma, \dot\gamma\rangle}{\langle D\gamma, \gamma\rangle} + t^2 U(\gamma)\,, \end{aligned} \tag{3.6}$$

where D is the matrix (3.2) and we denoted

$$U(\gamma) = -\frac{P^2}{2}\left(\langle C^{-1}\gamma, \gamma\rangle + \frac{\langle C^{-1}B\gamma, \gamma\rangle^2}{\langle D\gamma, \gamma\rangle}\right). \tag{3.7}$$

The function U coincides with the function (1.4).

REMARK. – The subsequent results can be generalized to the case of nonzero p_α and p_β, only the Routh function becomes more complicated. However, if $J_\gamma = c \neq 0$, then the variational methods don't work. Indeed, since the projection

$$SO(3) \to SO(3)/S^1 = S^2$$

is a nontrivial Hopf fibration, the Routh function isn't a well defined function on $TS^2 \times \mathbb{R}$ for $c \neq 0$ [1]. The reduced system is a system with gyroscopic forces and the differential 2-form Ω of gyroscopic forces is nonexact on S^2. Indeed, it is well known that $\iint_{S^2} \Omega = 4\pi c$ [1]. Thus the reduced system is only locally Lagrangian and the variational methods don't work.

The function (3.7) on the sphere S^2 is even and so it is natural to regard it as a function on the projective plane $\mathbb{R}P^2$. On S^2, the function (3.7) has at least $\operatorname{cat}(\mathbb{R}P^2) = 3$ pairs of critical points. Let $\gamma_\pm \in S^2$, $\gamma_- = -\gamma_+$ be a pair of minimum points of U. We assume that it is unique. The doubly asymptotic trajectories are divided into two classes: doubly asymptotic to the same point γ_+ or γ_- as $t \to \pm\infty$, or connecting different points γ_+ and γ_-. The latter solutions may be called heteroclinics.

The doubly asymptotic solutions correspond to the following motions of the rigid body [10]. As $t \to \infty$, the body is falling down in the fluid (or floating up, depending on whether P is positive or negative), its mass center moves asymptotically along a straight line and the solid doesn't rotate. The same holds for $t \to -\infty$, with reversed time direction. In general, these straight lines for $t \to \infty$ and $t \to -\infty$ are different. The homoclinic and heteroclinic trajectories differ in the following way: for homoclinic solutions the orientation of the body is the same, up to a rotation about the vertical, as $t \to \pm\infty$ and for heteroclinic ones the body turns upside down as $-\infty < t < \infty$.

Since the function W in (2.6) turns out to be rather complicated and it is also not easy to describe the critical points of the function (3.7) explicitly, for simplicity we assume that $B = 0$. Then the Routh function (3.6) takes the simple form

$$(3.8) \qquad L(\gamma, \dot{\gamma}, t) = \frac{\det A}{2} \frac{\langle A^{-1}\dot{\gamma}, \dot{\gamma}\rangle}{\langle A\gamma, \gamma\rangle} + t^2 U(\gamma), \qquad U(\gamma) = -\frac{P^2}{2}\langle C^{-1}\gamma, \gamma\rangle.$$

Thus, $L(-\gamma, -\dot{\gamma}, t) = L(\gamma, \dot{\gamma}, t)$ and we obtain a Lagrangian system on $\mathbb{R}P^2$.

In this case, there exist six equilibrium points which are the critical points of the quadratic form U on S^2. Let c_i be the eigenvalues of the matrix C and e_i the corresponding unitary eigenvectors. If c_1 is the smallest eigenvalue and $c_1 < c_{2,3}$, then $\gamma_{\pm} = \pm e_1$ are the equilibrium points corresponding to the minimum of U on S^2. We obtain

PROPOSITION 3.1. – *If $B = 0$ and $c_1 < c_{2,3}$, there exists an infinite number of homoclinic motions of the body such that $\gamma(t) \to e_1$ and $\dot{\gamma} \to 0$ as $t \to \pm\infty$ and also an infinite number of heteroclinic motions such that $\gamma(t) \to \pm e_1$ and $\dot{\gamma} \to 0$ as $t \to \pm\infty$.*

Physically, the direction of e_1 and $-e_1$ is those, in which the resistance of the fluid is minimal.

PROOF. – The existence of heteroclinic trajectories from γ_- to γ_+ is practically obvious: they are obtained by minimizing the action functional on the set of curves connecting γ_- with γ_+ on S^2 and so the proof doesn't even need the manifold structure on the space of curves. On $\mathbb{R}P^2$, heteroclinic trajectories correspond to homotopically nontrivial homoclinic trajectories.

The existence of an infinite number of heteroclinic trajectories from γ_- to γ_+ and an infinite number of homoclinic trajectories to one and the same point γ_+ or γ_- follows from Theorem 2.1, applied to the system on $\mathbb{R}P^2$. Indeed, both connected components of the loop space of $\mathbb{R}P^2$ are isomorphic to the loop space of S^2 and so their category is infinite. ∎

EXAMPLE. – Suppose that the body has three orthogonal symmetry planes and the distribution of masses is also symmetric. Then $B = 0$ and the Routh function takes the form (3.8), where the matrices A and C can be diagonalized simultaneously:

$$A = \mathrm{diag}(a_1, a_2, a_3), \qquad C = \mathrm{diag}(c_1, c_2, c_3).$$

Hence Lagrange's equations with the Lagrangian (3.8) have three invariant submanifolds

$$N_i = \{(\gamma, \dot{\gamma}) \in TS^2 \colon \gamma_i = \dot{\gamma}_i = 0\}, \qquad i = 1, 2, 3$$

in the phase space TS^2. For example, on N_3 we can use the generalized coordinate $\phi \bmod 2\pi$ defined by

$$\gamma_1 = \cos\phi\,, \qquad \gamma_2 = \sin\phi\,.$$

Passing to the projective plane means performing the transformation $\theta = 2\phi$. Then, up to a function of t and a constant multiplier, the Lagrangian $L_3 = L|_{N_3}$ takes the form

$$L_3 = \frac{1}{2}\dot{\theta}^2 + kt^2(1-\cos\theta)\,, \qquad k = P^2\,\frac{(c_2-c_1)}{c_1 c_2 a_3}\,.$$

The equation of motion is as follows:

$$\ddot{\theta} = kt^2 \sin\theta\,. \tag{3.9}$$

This equation was first obtained and studied by Chaplygin [5]. The qualitative properties of the solutions were analyzed by Kozlov [11].

In this case, Theorem 2.1 yields the following

PROPOSITION 3.2. – *For any $m \in \mathbb{Z}$ there exists a solution $\theta(t)$ of equation* (3.9) *such that $\lim_{t\to-\infty}\theta(t) = 0$ and $\lim_{t\to\infty}\theta(t) = 2\pi m$. Thus the body performs $m/2$ full rotations around a horizontal axis. For even m these solutions are homoclinic trajectories and for odd m heteroclinic ones.*

In fact, since the configuration space is a circle, Proposition 3.2 has an elementary proof based on minimizing of the Hamilton action. For $m = 1$, Proposition 3.2 was proved in [10].

4. – The variational problem.

In this section we reformulate the statement of Theorem 2.1 in a variational form. Let $x_0 \in M$ be the unique minimum point of U on M and let it be nondegenerate. Without loss of generality, we can assume that

$$w(x_0) = \nabla U(x_0) = 0\,, \qquad U(x_0) = V(x_0) = 0\,, \qquad \nabla w(x_0) = -(\nabla w(x_0))^T\,. \tag{4.1}$$

Indeed, adding a constant to U or V doesn't change Lagrange's equations.

To prove Theorem 2.1, we represent doubly asymptotic solutions as critical points of the action functional on a suitable function space. The manifold M can be smoothly embedded in $\mathbb{R}^N$ for $N = 2n+1$ and, up to a translation, x_0 can be assumed to coincide with the origin of $\mathbb{R}^N$. Denote by $\langle\cdot,\cdot\rangle = |\cdot|^2$ the the Euclidean metric in $\mathbb{R}^N$, and also its restriction to M. Then we can identify T_xM with T_x^*M for any $x \in M$.

REMARK. – It is standard [14] to embed M into some $\mathbb{R}^N$ isometrically by using Nash's theorem. Then $K(x,\dot{x}) = |\dot{x}|^2/2$, which simplifies the notations. However, this approach requires that $K \in C^5$ and also it isn't natural from the physical point of view.

Denote

$$\mathcal{H} = \{v \in AC(\mathbb{R}, \mathbb{R}^N) : \|v\| < \infty\},$$

where $AC(\mathbb{R}, \mathbb{R}^N)$ is the set of absolutely continuous curves in $\mathbb{R}^N$ and

$$\|v\|^2 = \int_{-\infty}^{\infty} \left(|\dot{v}(t)|^2 + t^2 |v(t)|^2 \right) dt .$$

Denote by $\mathcal{E}$ the Sobolev space $W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ with the norm

$$\|u\|_1^2 = \int_{-\infty}^{\infty} (|\dot{u}(t)|^2 + |u(t)|^2)\, dt .$$

It is easy to see that $\mathcal{H} \subset \mathcal{E}$ and

$$\|u\|_1 \leqslant \sqrt{2}\, \|u\| \quad \text{for all } u \in \mathcal{H} . \tag{4.2}$$

Recall that $\mathcal{E}$ is continuously embedded into $C^0(\mathbb{R}, \mathbb{R}^N)$ and

$$\|u\|_\infty = \sup_{t \in \mathbb{R}} |u(t)| \leqslant \|u\|_1 \quad \text{for all } u \in \mathcal{E} . \tag{4.3}$$

LEMMA 4.1. – *The set $\mathcal{H}$ is a Hilbert space.*

PROOF. – Only the completeness needs to be proved. Let $\{q_n\}$ be a Cauchy sequence in $\mathcal{H}$. By (4.2), $\{q_n\}$ is also a Cauchy sequence in $\mathcal{E}$. Since $\mathcal{E}$ is complete, there is $q \in \mathcal{E}$ such that, up to a subsequence, $\|q_n - q\|_1 \to 0$ as $n \to \infty$. We can then extract a subsequence, always denoted $\{q_n\}$, such that $\|q_n - q\|_\infty \to 0$ and $\dot{q}_n \to \dot{q}$ almost everywhere. By the assumption and the Fatou lemma, for any $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that if $n > m_\varepsilon$, then

$$\int_{-\infty}^{\infty} \lim_{m \to \infty} \left(|\dot{q}_n(t) - \dot{q}_m(t)|^2 + t^2 |q_n(t) - q_m(t)|^2 \right) dt \leqslant \liminf_{m \to \infty} \|q_n - q_m\|^2 \leqslant \varepsilon^2 .$$

Hence $\|q_n - q\| \leqslant \varepsilon$. And this means that $\|q_n - q\| \to 0$ as $n \to \infty$. ■

PROPOSITION 4.1. – *The set*

$$\mathcal{M} = \{q \in \mathcal{H} : q(t) \in M \text{ for all } t \in \mathbb{R}\}$$

is a complete Hilbert submanifold in $\mathcal{H}$ and its tangent space at any point $q \in \mathcal{M}$ is given by

$$T_q \mathcal{M} = \{v \in \mathcal{H} : v(t) \in T_{q(t)} M \text{ for all } t \in \mathbb{R}\} .$$

PROOF. – From (4.2)-(4.3) it follows that $\mathcal{M}$ is a closed subset in $\mathcal{H}$. The proof of the fact that $\mathcal{M}$ is a manifold is standard. We skip the definition of the manifold structure here since later we will use another representation of the Hilbert manifold structure on $\mathcal{M}$ which follows from the decomposition in Lemma 4.3 below. ■

The Hamilton action I is defined on $\mathcal{M}$ as

$$I(q) = \int_{-\infty}^{\infty} L(q(t), \dot{q}(t), t)\, dt\,. \tag{4.4}$$

Since $K(x, \dot{x})$ is a homogeneous quadratic form in $\dot{x}$, by (4.1) there exist constants $\alpha, \beta > 0$ such that

$$|K(x, \dot{x}) + t\langle w(x), \dot{x}\rangle + t^2 U(x)| \leq \alpha(|\dot{x}|^2 + t^2|x|^2), \qquad |V(x)| \leq \beta |x|$$

for all $(x, \dot{x}) \in TM$. Since

$$\int_{|t| \geq 1} |q(t)|\, dt \leq \left(\int_{|t| \geq 1} \frac{1}{t^2}\, dt\right)^{1/2} \left(\int_{|t| \geq 1} t^2 |q(t)|^2 dt\right)^{1/2} \leq 2\|q\|\,,$$

the integral (4.4) is convergent for any $q \in \mathcal{M}$.

LEMMA 4.2. – *For any $c > 0$ there exists $c_1 > 0$ such that*

$$\|q\|^2 \leq c_1\,, \qquad \sup_{t \in \mathbb{R}} t|q(t)|^2 \leq 4c_1 \tag{4.5}$$

for all $q \in \mathcal{M}^c = \{q \in \mathcal{M} : I(q) < c\}$.

Here and henceforth by c_i we denote positive constants depending only on c.

PROOF. – Take sufficiently small $\delta > 0$ and let $C(x)\colon T_x M \to T_x^* M$ be the symmetric positive definite operator such that

$$K(x, \dot{x}) = \frac{1}{2}\langle C(x)\, \dot{x}, \dot{x}\rangle + \delta|\dot{x}|^2\,. \tag{4.6}$$

Since the function (2.6) has a strict nondegenerate minimum at x_0, for sufficiently small $\delta > 0$ we have

$$F(x) = U(x) - \frac{1}{2}\langle w(x), C(x)^{-1} w(x)\rangle \geq \delta|x|^2 \quad \text{for all } x \in M\,. \tag{4.7}$$

Thus, denoting for simplicity $y = \dot{x} + tC(x)^{-1} w(x)$, we have

$$L(x, \dot{x}, t) = \delta|\dot{x}|^2 + \frac{1}{2}\langle C(x) y, y\rangle + t^2 F(x) + V(x) \geq \delta(|\dot{x}|^2 + t^2|x|^2) + V(x)\,.$$

In turn, this gives

$$\int_{|t|\leqslant 1} L(q,\dot q,t)\,dt \geqslant \delta \int_{|t|\leqslant 1}\left(|\dot q|^2+t^2|q|^2\right)dt+2\min V,$$

$$\int_{|t|>1} L(q,\dot q,t)\,dt \geqslant \int_{|t|>1}\left(\delta|\dot q|^2+\frac{\delta}{2}t^2|q|^2+\frac{\delta}{2}t^2|q|^2-\beta|q|\right)dt \geqslant$$

$$\geqslant \int_{|t|>1}\left(\delta|\dot q|^2+\frac{\delta}{2}t^2|q|^2-\frac{\beta^2}{2\delta t^2}\right)dt \geqslant \frac{\delta}{2}\int_{|t|>1}\left(|\dot q|^2+t^2|q|^2\right)dt-\frac{\beta^2}{\delta}\,.$$

Those estimates, together with the assumption $I(q)\leqslant c$, yield the first inequality (4.5). Now, using the Schwartz inequality and (4.2), we have

$$t|q(t)|^2\leqslant\left|\int_0^t\left(|q(s)|^2+2s\langle\dot q(s),q(s)\rangle\right)ds\right|\leqslant\int_{-\infty}^{\infty}|q(s)|^2\,ds+$$

$$+2\left(\int_{-\infty}^{\infty}|\dot q(s)|^2ds\right)^{1/2}\left(\int_{-\infty}^{\infty}s^2|q(s)|^2ds\right)^{1/2}\leqslant 4c_1. \qquad\blacksquare$$

Next we are going to perform computations in a coordinate chart around x_0. In order to simplify the notations, we embed M to $\mathbb{R}^N$ in such a way that some neighborhood of x_0 in M is contained in the linear subspace $\mathbb{R}^n\subset\mathbb{R}^N$. We may also assume that this neighborhood coincides with the ball

$$B_2=\{x\in\mathbb{R}^n:\ |x|<2\}.$$

Such an embedding can be easily constructed by a partition of unity. Thus we have $x\in\mathbb{R}^n$ for any $x\in M\subset\mathbb{R}^N$ such that $|x|<2$.

Fix a constant $c>0$. By Lemma 4.2, there exists $T=T(c)>0$ such that $q(t)\in B_1\subset B_2$ for all $q\in\mathcal{M}^c$ and $|t|>T$. Set $\Lambda=W^{1,2}([-T,T],M)$ and

$$\Lambda_-=\{q\in AC((-\infty,-T],B_1):\ \|q\|_-^2<\infty\},$$

$$\Lambda_+=\{q\in AC([T,\infty),B_1):\ \|q\|_+^2<\infty\},$$

where

$$\|q\|_-^2=\int_{-\infty}^{-T}\left(t^2|q(t)|^2+|\dot q(t)|^2\right)dt,$$

$$\|q\|_+^2=\int_T^{\infty}\left(t^2|q(t)|^2+|\dot q(t)|^2\right)dt$$

and $|\cdot|$ is the Euclidean metric in $\mathbb{R}^n\subset\mathbb{R}^N$. It is well known that Λ is a complete Hilbert manifold [14], while $\Lambda_\pm$ are open sets in the Hilbert spaces

$$\Lambda_- \subset \mathcal{H}_- = \{v \in AC((-\infty, -T], \mathbb{R}^n) \colon \|v\|_-^2 < \infty\},$$

$$\Lambda_+ \subset \mathcal{H}_+ = \{v \in AC([T, \infty), \mathbb{R}^n) \colon \|v\|_+^2 < \infty\}.$$

Define the mappings $g_\pm \colon \mathcal{M} \to \Lambda_\pm$ and $g \colon \mathcal{M} \to \Lambda$ as

$$g_-(q) = q|_{(-\infty, -T]}, \qquad g(q) = q|_{[-T, T]}, \qquad g_+(q) = q|_{[T, \infty)}.$$

LEMMA 4.3. – *The mappings $g_\pm \colon \mathcal{M} \to \Lambda_\pm$ and $g \colon \mathcal{M} \to \Lambda$ are of class C^∞.*

Since we skipped the definition of the smooth structure on $\mathcal{M}$, this lemma can be regarded also as a definition of the structure of a Hilbert manifold on $\mathcal{M}$. Indeed, the map

$$g_+ \times g \times g_- \colon \mathcal{M}^c \to \Lambda_+ \times \Lambda \times \Lambda_-$$

yields an identification of $\mathcal{M}^c$ with an open set in the Hilbert submanifold

$$\{(q_+, \gamma, q_-) \in \Lambda_+ \times \Lambda \times \Lambda_- \colon q_\pm(\pm T) = \gamma(\pm T)\}$$

of codimension $2n$ in $\Lambda_+ \times \Lambda \times \Lambda_-$.

Define the functionals $J_\pm \colon \Lambda_\pm \to \mathbb{R}$ and $J \colon \Lambda \to \mathbb{R}$ as

$$J_-(q) = \int_{-\infty}^{-T} L(q(t), \dot q(t), t)\, dt,$$

$$J(q) = \int_{-T}^{T} L(q(t), \dot q(t), t)\, dt,$$

$$J_+(q) = \int_{T}^{\infty} L(q(t), \dot q(t), t)\, dt.$$

It is convenient to split the restriction $I|_{\mathcal{M}^c}$ of the action functional in the following way:

$$I = J_- \circ g_- + J \circ g + J_+ \circ g_+.$$

PROPOSITION 4.2. – *The functional $I \in C^1(\mathcal{M})$ and the derivative I' is locally Lipschitz.*

Actually $I \in C^2(\mathcal{M})$, but we won't use this.

PROOF. – Take arbitrary $c > 0$, choose $T = T(c) > 0$ and define the sets $\Lambda_\pm$ and Λ, the mappings $g_\pm$ and g and the functionals $J_\pm$ and J as above. In view of Lemma 4.3, Proposition 4.2 follows if we prove that J_-, J and J_+ are of class C^1 with locally Lipschitz derivative.

We start showing that J_+ is differentiable on Λ_+. Fix $q \in \Lambda_+$ and take $v \in T_q \Lambda_+ = \mathcal{H}_+$ such that $\|v\|_+^2 < 1/2$. By (4.2) and (4.3), $\|v\|_\infty < 1$. Hence $q(t) + v(t) \in B_2$ for all $t > T$, and it is possible working in $\mathbb{R}^n$. By Taylor's formula,

$$L(q(t) + v(t), \dot{q}(t) + \dot{v}(t), t) =$$

$$= L(q(t), \dot{q}(t), t) + \langle L_q(q(t), \dot{q}(t), t), v(t)\rangle + \langle L_{\dot{q}}(q(t), \dot{q}(t), t), \dot{v}(t)\rangle +$$

$$+ \frac{1}{2}\langle L_{qq} v(t), v(t)\rangle + \langle L_{q\dot{q}} \dot{v}(t), v(t)\rangle + \frac{1}{2}\langle L_{\dot{q}\dot{q}} \dot{v}(t), \dot{v}(t)\rangle,$$

where the second derivatives of L are evaluated at a point $(q(t) + \theta(t)\, v(t), \dot{q}(t) + \theta(t)\, \dot{v}(t), t)$ with $0 \leq \theta(t) \leq 1$ a measurable function. In view of (2.3), for any $(q, \dot{q}, t) \in B_2 \times \mathbb{R}^n \times \mathbb{R}$, we have

$$\|L_{qq}\| \leq c_2(1 + t^2 + |\dot{q}|^2), \qquad \|L_{q\dot{q}}\| \leq c_2(t + |\dot{q}|), \qquad \|L_{\dot{q}\dot{q}}\| \leq c_2, \tag{4.8}$$

where $\|\cdot\|$ denotes the operator norm of a matrix. Since $q(t) + \theta(t)\, v(t) \in B_2$ for all $t > T$, by using (4.8) and the Schwartz inequality, we get

$$\left|\int_T^\infty \langle L_{qq} v(t), v(t)\rangle\, dt\right| \leq \int_T^\infty c_2\big(1 + t^2 + |\dot{q}(t) + \theta(t)\dot{v}(t)|^2\big)\, |v(t)|^2\, dt \leq$$

$$\leq c_2(3\|v\|_+^2 + 2\|v\|_\infty^2(\|q\|_+^2 + \|v\|_+^2)) \leq c_2(3 + 4(\|q\|_+^2 + \|v\|_+^2))\|v\|_+^2 \leq c_3\|v\|_+^2\,,$$

$$\left|\int_T^\infty \langle L_{q\dot{q}} \dot{v}(t), v(t)\rangle\, dt\right| \leq \int_T^\infty c_2(t + |\dot{q}(t) + \theta(t)\dot{v}(t)|)\,|\dot{v}(t)\|v(t)|\, dt \leq$$

$$\leq c_2(\|v\|_+^2 + \|v\|_\infty(\|q\|_+\|v\|_+ + \|v\|_+^2)) \leq c_2(1 + \sqrt{2}(\|q\|_+ + \|v\|_+))\|v\|_+^2 \leq c_4\|v\|_+^2\,,$$

$$\left|\int_T^\infty \langle L_{\dot{q}\dot{q}} \dot{v}(t), \dot{v}(t)\rangle\, dt\right| \leq \int_T^\infty c_2\, |\dot{v}(t)|^2\, dt \leq c_2\|v\|_+^2\,.$$

These inequalities yield the differentiability of J_+ on Λ_+, providing at the same time an expression for $J_+'(q)\, v$. Indeed,

$$\left|\int_T^\infty \big(L(q(t) + v(t), \dot{q}(t) + \dot{v}(t), t) - L(q(t), \dot{q}(t), t) - \right.$$

$$\left. - \langle L_q(q(t), \dot{q}(t), t), v(t)\rangle - \langle L_{\dot{q}}(q(t), \dot{q}(t), t), \dot{v}(t)\rangle\big)\, dt\right| \leq c_5\|v\|_+^2\,.$$

Now we prove that J_+' is locally Lipschitz. Let $q_1, q_2 \in \Lambda_+$ and $v \in \mathcal{H}_+$. Since q_1 and

q_2 take values in $B_1 \subset \mathbb{R}^n$, we can write, denoting $u = q_2 - q_1$

$$(4.9) \quad \langle J'_+(q_2), v\rangle - \langle J'_+(q_1), v\rangle = \int_T^\infty (\langle L_q(q_2(t), \dot q_2(t), t) - L_q(q_1(t), \dot q_1(t), t), v(t)\rangle -$$

$$-\langle L_{\dot q}(q_2(t), \dot q_2(t), t) - L_{\dot q}(q_1(t), \dot q_1(t), t), \dot v(t)\rangle)\, dt$$

$$= \int_T^\infty \Big(\langle L_{qq} u(t), v(t)\rangle + \langle L_{q\dot q}\dot u(t), v(t)\rangle + \langle L_{\dot q q} u(t), \dot v(t)\rangle + \langle L_{\dot q\dot q}\dot u(t), \dot v(t)\rangle\Big)\, dt\,,$$

where the second derivatives of L are evaluated at a point $(q_1(t) + s(t)u(t), \dot q_1(t) + s(t)\,\dot u(t), t)$ with $0 \leqslant s(t) \leqslant 1$ a measurable function. By using (4.8) and the Schwartz inequality, we see that the right hand side in (4.9) is majorized by $c_6\|u\|_+\|v\|_+$, where c_6 is a constant depending on $\|q_1\|_+$ and $\|q_2\|_+$. Consequently,

$$\|J'_+(q_2) - J'_+(q_1)\|_+ \leqslant c_6\|q_2 - q_1\|_+$$

and we conclude that J'_+ is locally Lipschitz.

The functional J_- can be studied in a similar way. Finally, notice that the functional J acts on curves which are defined on a compact interval $[-T, T]$. Regularity of such functionals is well studied in the literature. See, for example, Eells [6], Palais and Smale [15] or Benci [2] for the fixed boundary value problem or the periodic boundary problem. The only difference in our case is that the boundary points are free. Exactly the same proof yields that $J \in C^1(\Lambda)$ and has locally Lipschitz derivative. Proposition 4.2 is proved. ■

PROPOSITION 4.3. – *Critical points of I correspond to the doubly asymptotic solutions such that $q(t) \to x_0$ and $\dot q(t) \to 0$ as $t \to \pm\infty$.*

Obviously, if $I'(q) = 0$ for some $q \in \mathcal{M}$, then $q(t)$ satisfies Lagrange's equations. Thus Proposition 4.3 follows from

LEMMA 4.4. – *If $q: [T, \infty) \to M$ is a trajectory of the system such that $q(t) \to x_0$ as $t \to \infty$, then $\dot q(t) \to 0$ and*

$$(4.10) \qquad \|q\|_+^2 = \int_T^\infty (|\dot q|^2 + t^2|q|^2)\, dt < \infty\,.$$

PROOF. – Changing T if necessary, we can assume that $q(t) \in B_1 \subset \mathbb{R}^n$ for $t \geqslant T$. Then Lagrange's equations (2.4) take the form

$$B(q)\ddot q + \Gamma(q, \dot q) + tG(q)\,\dot q + w(q) - t^2\nabla U(q) - \nabla V(q) = 0\,,$$

where Γ is quadratic in $\dot q$. Denote $B = B(0)$, $E = \nabla w(0) = G(0)/2$ and $D = \nabla^2 U(0)$. Then

(4.6) and (4.7) yield the inequality

$$\text{(4.11)}\quad \frac{1}{2}\langle Bv, v\rangle + \langle Eu, v\rangle + \frac{1}{2}\langle Du, u\rangle \geq \delta(|u|^2 + |v|^2) \quad \text{for all} \quad u, v \in \mathbb{R}^n .$$

First we prove (4.10). Let $f(t) = \langle Bq(t), q(t)\rangle/2$. Then $\dot{f} = \langle q, B\dot{q}\rangle$. Using (4.11), for sufficiently large t we obtain a version of the well known Lagrange inequality

$$\text{(4.12)}\quad \ddot{f} \geq \langle B\dot{q}, \dot{q}\rangle - 2t\langle q, E\dot{q}\rangle + t^2\langle Dq, q\rangle - \alpha|q|(|\dot{q}|^2 + t^2|q|^2) - \beta|q| \geq$$
$$\geq \delta(|\dot{q}|^2 + t^2|q|^2) - \beta|q| \geq \frac{1}{2}\delta(|\dot{q}|^2 + t^2|q|^2) - \frac{\beta^2}{2\delta t^2},$$

where α and β are positive constants. Since $f(t) \to 0$, we have $\liminf_{\infty} |\dot{f}(t)| = 0$. Integrating inequality (4.12), we get (4.10) as in the proof of (4.5). We also obtain

$$\text{(4.13)}\qquad \ddot{f} \geq \frac{t^2\delta f}{\lambda} - \frac{\beta^2}{2\delta t^2}, \qquad \lambda = \|B\| .$$

Now we show that $t^2 f(t) \to 0$ as $t \to \infty$. Fix $\varepsilon > 0$ and suppose that there exists a sequence $t_k \to \infty$ with $t_k^2 f(t_k) > \varepsilon$. Put $s = t_k$. Without loss of generality, we can assume that

$$\text{(4.14)}\qquad s > \frac{\beta\sqrt{\lambda}}{\delta\sqrt{2\varepsilon}} + 3 .$$

Suppose, for example, that $\dot{f}(s) \leq 0$. The case $\dot{f}(s) \geq 0$ is similar. Inequalities (4.13) and (4.14) yield $\ddot{f}(t) \geq 0$ and $\dot{f}(t) \leq 0$ for all $t \in [s-1, s]$. Hence $f(t) \geq f(s) > \varepsilon s^{-2}$ for all $t \in [s-1, s]$. Thus

$$\int_{s-1}^{s} t^2 |q(t)|^2 dt \geq \frac{2\varepsilon}{s^2\lambda} \int_{s-1}^{s} t^2\, dt \geq \frac{\varepsilon}{\lambda} .$$

Taking a sum over infinitely many $s = t_k$, we obtain a contradiction to (4.10).

To show that $\dot{q}(t) \to 0$ as $t \to \infty$, we use the energy theorem. Let $H = K - t^2 U - V$. Then

$$\dot{H} = -L_t = -\langle w(q), \dot{q}\rangle - 2tU(q) .$$

Thus for any $T < t_0 < t_1$ we have

$$|H|_{t_1} - H|_{t_0}| \leq c_7 \int_{t_0}^{t_1} (|q||\dot{q}| + t|q|^2)\, dt \leq \frac{2c_7}{t_0} \|q\|_+^2 ,$$

where c_7 is some positive constant. Thus there exists the limit of H as $t \to \infty$. Since $t^2|q|^2 \to 0$ as $t \to \infty$, we obtain $\dot{q}(t) \to 0$. ■

In next section we'll prove that the functional I satisfies the Palais-Smale condition, which implies Theorem 2.1.

5. – The Palais-Smale condition.

PROPOSITION 5.1. – *The action functional I satisfies the Palais-Smale condition.*

This means [15] that any PS sequence, i.e. any sequence $\{q_n\} \subset \mathcal{M}$, for which $I(q_n)$ is bounded and $I'(q_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

PROOF. – Let $\{q_n\} \subset \mathcal{M}$ be a PS sequence. We know that $q_n \in \mathcal{M}^c$ for some $c > 0$ and so, by Lemma 4.2, $\|q_n\|$ is uniformly bounded. Also, up to a subsequence, q_n converges uniformly to some curve $q_\infty \in L^\infty(\mathbb{R}, M)$:

$$\|q_n - q_\infty\|_\infty \to 0 \quad \text{as } n \to \infty . \tag{5.1}$$

To the constant c, we can associate some $T > 0$, the sets $\Lambda_\pm$ and Λ, the mappings $g_\pm$ and g and the functionals $J_\pm$ and J introduced in § 4. In fact, for any $\varepsilon > 0$, we can take $T = T_\varepsilon > 0$ so large that $q_n(t) \in \mathbb{R}^n$ and $|q_n(t)| < \varepsilon$ for all $|t| > T$ and all $n \in \mathbb{N}$. Plainly, this still holds true if a larger T is chosen. Later on, we'll take $\varepsilon > 0$ sufficiently small and $T > 0$ sufficiently large.

Denote

$$\sigma_n^- = g_-(q_n) : (-\infty, -T] \to B_\varepsilon ,$$

$$\gamma_n = g(q_n) : (-T, T] \to M ,$$

$$\sigma_n^+ = g_+(q_n) : (T, \infty) \to B_\varepsilon .$$

Thus $\gamma_n \in \Lambda$ and $\sigma_n^\pm \in \Lambda_\pm$. Since $I'(q_n) \to 0$ as $n \to \infty$, we have,

$$|\langle I'(q_n), v\rangle| \leq \varepsilon_n \|v\| \qquad \text{for all } v \in T_{q_n}\mathcal{M}$$

with $\varepsilon_n \to 0$ as $n \to \infty$. For any $v \in T_{q_n}\mathcal{M}$ such that $v|_{\mathbb{R}\setminus[-T, T]} \equiv 0$, we have

$$\langle I'(q_n), v\rangle = \langle J'(\gamma_n), v|_{[-T, T]}\rangle .$$

Similarly, by taking $v \in T_{q_n}\mathcal{M}$ such that $v(t) \equiv 0$ for all $t \leq T$, we have

$$\langle I'(q_n), v\rangle = \langle J'_+(\sigma_n^+), v|_{[T, \infty)}\rangle$$

and an analogous statement holds true for the functional J_-. Thus

$$\langle J'(\gamma_n), v\rangle \leq \varepsilon_n \|v\| \tag{5.2}$$

for all $v \in T_{\gamma_n}\Lambda$ such that $v(\pm T) = 0$ with $\varepsilon_n \to 0$ as $n \to \infty$. Here $\|v\|$ means the norm in $T_{\gamma_n}\Lambda$. Similarly

$$\langle J'_\pm(\sigma_n^\pm), v\rangle \leq \varepsilon_n \|v\|_\pm \tag{5.3}$$

for all $v \in \mathcal{H}_\pm$ such that $v(\pm T) = 0$.

Let γ_∞, σ_∞^- and σ_∞^+ be the restrictions of q_∞ to $[-T, T]$, $(-\infty, -T]$ and $[T, \infty)$ respectively. From (5.1) it follows that $\gamma_n \to \gamma_\infty$ and $\sigma_n^\pm \to \sigma_\infty^\pm$ uniformly.

LEMMA 5.1. – *If $J(\gamma_n)$ is uniformly bounded, $\gamma_n \to \gamma_\infty$ uniformly and* (5.2) *holds true, then $\{\gamma_n\}$ has a subsequence converging in Λ to γ_∞.*

PROOF. – Denote $x_\pm = \gamma_\infty(\pm T)$. If the boundary points $\gamma_n(\pm T)$ of the curves γ_n were fixed, then Lemma 5.1 would coincide with the well known fact that the Palais-Smale condition holds for the Hamilton functional J on

$$\Omega = \{\gamma \in W^{1,2}([-T, T], M) \colon \gamma(\pm T) = x_\pm\}.$$

In our case, the proof goes with some small modifications. However, also the reduction to the case $\gamma_n(\pm T) = x_\pm$ is possible. For the convenience of the reader we outline the proof.

There exists a sequence of smooth families of maps $f_{n,t} \colon M \to M$, depending on $t \in [-T, T]$, such that $f_{n, \pm T}(\gamma_n(\pm T)) = x_\pm$ and the sequence $f_{n,t}$ tends to id_M in the C^2 topology. For example, the map $f_{n,t}$ can be defined as follows. Let η_n be the affine map from $[-T, T]$ to $\mathbb{R}^N$ such that

$$\eta_n(\pm T) = \gamma_n(\pm T) - x_\pm .$$

Then $\|\eta_n\|_{C^2} \to 0$. Let $f_{n,t}(x)$ be the point in M closest to the point $x - \eta_n(t)$. If n is sufficiently large, the map $f_{n,t}$ is well defined and satisfies the stated conditions.

Now define the sequence $\omega_n \in \Omega$ by the formula $\omega_n(t) = f_{n,t}(\gamma_n(t))$. Since $\|\omega_n - \gamma_n\| \to 0$ and the map $x \to df_{n,t}(x) \colon T_x M \to T_{f_{n,t}(x)} M$ tends to the map $x \to id_{T_x M}$ in the C^1 topology, we obtain

$$|\langle J'(\omega_n), u\rangle - \langle J'(\gamma_n), v\rangle| \leq \delta_n \|u\|, \qquad \delta_n \to 0 ,$$

for any $u \in T_{\omega_n}\Omega$. Here $v(t) = df_{n,t}(\gamma_n(t))^{-1} u(t)$. Since $v(\pm T) = 0$, by (5.2) $\omega_n \in \Omega$ is a PS sequence and hence it converges to γ_∞. Hence γ_n is also converging to γ_∞ in Λ. ∎

Since the behaviour of the sequences $\{\sigma_n^+\}$ and $\{\sigma_n^-\}$ can be studied in a similar way, below we carry out the details for $\{\sigma_n^+\}$ and write $\sigma_n = \sigma_n^+$ and $\sigma_\infty = \sigma_\infty^+$.

LEMMA 5.2. – *Along a subsequence, $\|\sigma_n - \sigma_\infty\|_+ \to 0$ as $n \to \infty$.*

PROOF. – It is very similar to the proof of the PS condition in [14]. It is sufficent to show that $\{\sigma_n\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_+$.

For $t \in [T, \infty)$ put

$$\xi_{nm}(t) = \sigma_m(t) - \sigma_n(t) \text{ and } \zeta_{nm}(t) = (T^2/t^2)\, \xi_{nm}(T).$$

We point out that $\xi_{nm}(T) - \zeta_{nm}(T) = 0$ and $\|\zeta_{nm}\|_+ \to 0$ as $n, m \to \infty$. By (5.3), $\langle J'_+(\sigma_n), \xi_{nm} - \zeta_{nm}\rangle \to 0$. Since $\|J'_+(\sigma_n)\|$ is bounded, also $\langle J'_+(\sigma_n), \xi_{nm}\rangle \to 0$ as $n, m \to \infty$. We can extract a subsequence such that the sequence $J_+(\sigma_n)$ is converging.

Hence

$$\Psi_{nm} = J_+(\sigma_m) - J_+(\sigma_n) - \langle J'_+(\sigma_n), \xi_{nm}\rangle \to 0 \qquad \text{as } n, m \to \infty .$$

That $\{\sigma_n\}$ is a Cauchy sequence will follow from an estimate on Ψ_{nm} which we are going to establish next. We estimate separately the terms in the integral Ψ_{nm} corresponding to the terms in the Lagrangian (2.3).

We assume that $\varepsilon > 0$ is so small that

$$\|B(x) - B\| \leqslant \delta/2 , \qquad \|\nabla w(x) - E\| \leqslant \delta/2 , \qquad \|\nabla^2 U(x) - D\| \leqslant \delta/2 \tag{5.4}$$

for all x such that $|x| < \varepsilon$.

First we estimate the terms in Ψ_{nm} involving K by using (5.4) and the fact that $|\sigma_n(t)| < \varepsilon$ for all $t \in [T, \infty)$:

$$\begin{aligned} K(\sigma_m, \dot\sigma_m) - K(\sigma_n, \dot\sigma_n) - \langle K_q(\sigma_n, \dot\sigma_n), \xi_{nm}\rangle - \langle K_{\dot q}(\sigma_n, \dot\sigma_n), \dot\xi_{nm}\rangle = \\ = K(\sigma_n, \dot\xi_{nm}) + \langle K_q(\tau_{nm}, \dot\sigma_m), \xi_{nm}\rangle - \langle K_q(\sigma_n, \dot\sigma_n), \xi_{nm}\rangle \geqslant \\ \geqslant \frac{1}{2}\langle B\dot\xi_{nm}, \dot\xi_{nm}\rangle - \frac{\delta}{4}|\dot\xi_{nm}|^2 - c_1(|\dot\sigma_m|^2 + |\dot\sigma_n|^2)|\xi_{nm}| .\end{aligned} \tag{5.5}$$

Here and in (5.6)-(5.8) we denote by $\tau_{nm}(t)$ some intermediate point between $\sigma_n(t)$ and $\sigma_m(t)$. By c_i we denote some fixed positive constants independent of ε.

For the linear in velocity term in L, we have by (5.4)

$$\begin{aligned} \langle w(\sigma_m), \dot\sigma_m\rangle - \langle w(\sigma_n), \dot\sigma_n\rangle - \langle \nabla w(\sigma_n)\xi_{nm}, \dot\sigma_n\rangle - \langle w(\sigma_n), \dot\xi_{nm}\rangle = \\ = \langle \nabla w(\tau_{nm})\xi_{nm}, \dot\sigma_m\rangle - \langle \nabla w(\sigma_n)\xi_{nm}, \dot\sigma_n\rangle \geqslant \langle \nabla w(\sigma_n)\xi_{nm}, \dot\xi_{nm}\rangle - c_2|\tau_{nm} - \sigma_n||\xi_{nm}||\dot\sigma_m| \geqslant \\ \geqslant \langle E\xi_{nm}, \dot\xi_{nm}\rangle - \frac{\delta}{2}|\xi_{nm}||\dot\xi_{nm}| - c_2|\xi_{nm}|^2|\dot\sigma_m| .\end{aligned} \tag{5.6}$$

For the term involving U, we have

$$\begin{aligned} U(\sigma_m) - U(\sigma_n) - \langle \nabla U(\sigma_n), \xi_{nm}\rangle = \\ = \frac{1}{2}\langle \nabla^2 U(\tau_{nm})\xi_{nm}, \xi_{nm}\rangle \geqslant \frac{1}{2}\langle D\xi_{nm}, \xi_{nm}\rangle - \frac{\delta}{4}|\xi_{nm}|^2 .\end{aligned} \tag{5.7}$$

Finally,

$$V(\sigma_m) - V(\sigma_n) - \langle \nabla V(\sigma_n), \xi_{nm}\rangle = \frac{1}{2}\langle \nabla^2 V(\tau_{nm})\xi_{nm}, \xi_{nm}\rangle \geqslant -c_3|\xi_{nm}|^2 . \tag{5.8}$$

Putting together (5.5)-(5.8) and using (4.11), we obtain

$$\Psi_{nm} \geqslant \int_T^\infty \left(\frac{1}{2} \langle B\dot{\xi}_{nm}, \dot{\xi}_{nm} \rangle + t \langle E\xi_{nm}, \dot{\xi}_{nm} \rangle + \frac{t^2}{2} \langle D\xi_{nm}, \xi_{nm} \rangle - \right.$$

$$\left. - \frac{\delta}{2} (|\dot{\xi}_{nm}|^2 + t^2 |\xi_{nm}|^2) - c_3 |\xi_{nm}|^2 \right) dt - \Theta_{nm} \geqslant$$

$$\geqslant \int_T^\infty \left(\frac{\delta}{2} |\dot{\xi}_{nm}|^2 + \delta t^2 \left(\frac{1}{2} - \frac{c_3}{T^2 \delta} \right) |\xi_{nm}|^2 \right) dt - \Theta_{nm},$$

where we denoted

$$\Theta_{nm} = c_1 (\|\sigma_m\|_+^2 + \|\sigma_n\|_+^2) \|\xi_{nm}\|_\infty + c_2 (\|\sigma_m\|_+ + \|\sigma_n\|_+) \|\sigma_m\|_+ \|\xi_{nm}\|_\infty .$$

Since $\|\xi_{nm}\|_\infty \to 0$ as $n, m \to \infty$ and $\|\sigma_n\|_+$ is bounded, we have $\Theta_{nm} \to 0$.

As already noticed, we can choose T as large as we like. Let $T > 2\sqrt{c_3/\delta}$. Then

$$\frac{\delta}{4} \|\xi_{nm}\|_+^2 \leqslant \Psi_{nm} - \Theta_{nm} \to 0$$

as $n, m \to \infty$. Hence σ_n is a Cauchy sequence. ∎

Lemmas 5.1-5.2 imply that, up to a subsequence, $q_n \to q_\infty$ in $\mathcal{H}$. Proposition 5.1 is proved. ∎

PROOF OF THEOREM 2.1. – The manifold $\mathcal{M}$ is homotopically equivalent to the loop space

$$\Omega(M) = \{ q \in C^0([0, 1], M) \colon q(0) = x_0 = q(1) \}.$$

We recall that since M is a compact manifold, $\operatorname{cat} \Omega(M) = \infty$. Indeed, if $\pi_1(M)$ is finite, then the universal covering $\widetilde{M}$ is a compact simply connected manifold. By Serre's theorem [16], $\operatorname{cat} \Omega(\widetilde{M}) = \infty$. A simple argument then shows that also $\operatorname{cat} \Omega(M) = \infty$. If $\pi_1(M)$ is infinite, then $\Omega(M)$ has infinitely many connected components and so again $\operatorname{cat} \Omega(M) = \infty$.

This fact together with Propositions 4.1, 4.2 and 5.1 implies that the action functional I defined on $\mathcal{M}$ has infinitely many critical points. By Proposition 4.3, each of them corresponds to a solution $q(t)$ of system (2.2) such that $q(t) \to x_0$ and $\dot{q}(t) \to 0$ as $t \to \pm \infty$. ∎

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