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Doubly coprime factorizations, reduced-order observers, and dynamic state estimate feedback

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Doubly coprime factorizations of the transfer function of a lumped linear time-invariant system are a starting point for many of the results in the factorization approach to multivariable control system analysis and synthesis. In work by Nett *et al.* (1984), explicit state-space realizations of these factorizations are derived using results from state estimation/state feedback theory. Here new doubly coprime factorizations are developed based on minimal-order observers. Following on from this, various extensions are noted, and it is proved that the class of all proper stabilizing controllers for a given plant can be generated by dynamic feedback of the reduced-order state estimate.

1. Introduction

A doubly coprime factorization of the transfer function of a lumped linear time-invariant system is the starting point for many of the powerful results in the factorization approach to multivariable control system analysis and synthesis (Vidyasagar 1985). In an important work by Nett *et al.* (1984), explicit formulae are given for state-space realizations of the Bezout identity elements. The results of Nett *et al.* (1984) are based on ideas from the theory of state feedback and state estimation, and use existing computational algorithms, namely pole-placement algorithms.

Recently, Hippe (1989) has derived modified factorizations which are related to compensators based on reduced-order observers, rather than full-order state observers. One problem with these factorizations is that some of the Bezout identity elements are non-proper, and consequently are not suitable for use with the factorization approach. In § 2 of the present work we derive doubly coprime factorizations related to minimal-order observers, with all Bezout identity elements stable and proper.

In the work by Moore *et al.* (1988), the factorizations of Nett *et al.* (1984) have been generalized to allow for the possibility of dynamic state estimate feedback gains, as well as dynamic state estimator gains. Section 3 of the present work generalizes the factorizations of § 2 in a similar manner. To give an example of the utility of the results, it is then proved that all stabilizing controllers for a given plant can be structured as a minimal-order observer, with dynamic state estimate feedback gains. Finally, some dual results are summarized in § 4.

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2. Factorizations related to minimal-order observers

2.1. *Preliminaries*

Consider the ring of real rational functions defined on the complex plane that are stable and proper. Here a function $F(s)$ is stable if there are no poles in the closed right half-plane, and proper if $|F(\infty)|$ is finite. (In this work, we deal exclusively with continuous-time systems.) The class of matrix valued functions with entries in this ring will be denoted by RH^∞. The symbol I will be used to represent a real identity matrix of appropriate dimensions.

The following convenient notation for the state space realization of a transfer function matrix will be used:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_T \triangleq C(sI-A)^{-1}B+D$$

We shall make use of the following identities:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_T \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_T = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]_T \tag{2.1}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_T^{-1} = \left[\begin{array}{c|c} A-BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]_T \tag{2.2}$$

A plant/controller pair $G(s)$, $K(s)$, as depicted in Fig. 1 will, be said to be well-posed and internally stable if and only if

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \text{ exists and belongs to } RH^\infty \tag{2.3}$$

This condition corresponds to the transfer functions from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ being stable and proper.

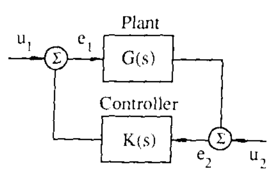

Figure 1. Closed-loop system with positive feedback.

The minimal-order observer for the m-input, p-output plant $G(s)$, with n state controllable and observable state-space realization $C(sI-A)^{-1}B$, will now be briefly reviewed. The treatment is similar to that found in the work by O'Reilly (1983). The observer equations are

$$\dot{z} = Rz + Sy + TBu \tag{2.4}$$

$$\hat{x} = [\Psi \quad \Theta]\begin{bmatrix} y \\ z \end{bmatrix} \tag{2.5}$$

where

$$C \text{ is full rank} \tag{2.6 a}$$

$$[\Psi \quad \Theta]\begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}[\Psi \quad \Theta] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.6 b}$$

$$R = TA\Theta, \quad S = TA\Psi \tag{2.6 c}$$

A suitable selection of a full row rank matrix T results in $(sI - R)^{-1} \in RH^{\infty}$, i.e. R is a matrix with all eigenvalues in the open left half-plane Re $[s] < 0$. For such selections, the error in the state estimate $x - \hat{x}$ due to an incorrect initial value of z will approach zero asymptotically.

Figure 2 shows the block diagram for an observer-based controller which uses feedback of the state estimate $\hat{x}$ through a constant, real matrix F. The transfer function matrix $K(s)$ of an equivalent controller in the simple positive feedback configuration of Fig. 1 is,

$$K(s) = \left[\begin{array}{c|c} R + TBF\Theta & S + TBF\Psi \\ \hline F\Theta & F\Psi \end{array}\right]_T \tag{2.7}$$

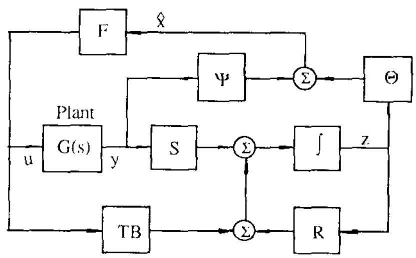

Figure 2. Minimal-order observer based control loop.

2.2. *Factorizations*

The main factorization result will now be stated.

Theorem 1

Consider the plant $G(s) = C(sI - A)^{-1}B$, with (A, B) controllable and (A, C) observable. Choose F and T such that $(sI - A - BF)^{-1}, (sI - R)^{-1} \in RH^{\infty}$, where R and T are described by the observer equations (2.4)–(2.6). With arbitrary Λ such that $(sI - \Lambda)^{-1} \in RH^{\infty}$, define

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & (A + BF - \Psi\Lambda C)\Psi \\ \hline F & I & F\Psi \\ C & 0 & I \end{array}\right]_T \tag{2.8}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A\Theta T + \Psi\Lambda C & -B & -(A - \Psi\Lambda C)\Psi \\ \hline F\Theta T & I & -F\Psi \\ C & 0 & I \end{array}\right]_T \tag{2.9}$$

Then the following hold:

(a) all transfer function matrices described by (2.8), (2.9) are stable and proper;

(b) M, $\tilde{M}$, V, $\tilde{V}$ have proper inverses;

(c) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$

(d) $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ where K is the observer-based controller given by (2.7);

(e)
$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & M \end{bmatrix}\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.10}$$

Proof

Considering (a), the transfer function matrices (2.8) and (2.9) are inherently proper. Since F is chosen such that $(sI - A - BF)^{-1} \in RH^{\infty}$, (2.8) is stable, and furthermore, to see that (2.9) is stable, apply a similarity transformation and use (2.6 b),

$$A\Theta T + \Psi\Lambda C = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}\begin{bmatrix} \Lambda & CA\Theta \\ 0 & R \end{bmatrix}\begin{bmatrix} C \\ T \end{bmatrix} \tag{2.11}$$

Since the similarity transformation leaves the eigenvalues unchanged, the eigenvalues of $A\Theta T + \Psi\Lambda C$ are simply equal to the eigenvalues of Λ, a matrix chosen such that its eigenvalues lie in the left half-plane, together with the eigenvalues of R, which lie in the left half-plane by virtue of the T selection.

It can be decuded from (2.2) that a square proper transfer function matrix has a proper inverse if its direct-feedthrough term D is non-singular. Considering (b), it follows that M, $\tilde{M}$, V, $\tilde{V}$ have proper inverses, because they have unity direct-feedthrough matrices. Application of (2.1) and (2.2) shows that (c), (d), and (e) hold. As an example of the proof technique, observe that

$$\tilde{M}^{-1}\tilde{N} = \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda C & -(A - \Psi\Lambda C)\Psi \\ \hline C & I \end{array}\right]_T^{-1}\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda C & B \\ \hline C & 0 \end{array}\right]_T$$

$$= \left[\begin{array}{c|c} A & (A - \Psi\Lambda C)\Psi \\ \hline C & I \end{array}\right]_T\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda C & B \\ \hline C & 0 \end{array}\right]_T \quad \text{by (2.2)}$$

$$= \left[\begin{array}{cc|c} A & A\Psi C - \Psi\Lambda C & 0 \\ 0 & A\Theta T + \Psi\Lambda C & B \\ \hline C & C & 0 \end{array}\right]_T \quad \text{by (2.1)}$$

$$= \left[\begin{array}{cc|c} A\Theta T + \Psi\Lambda C & A\Psi C - \Psi\Lambda C & 0 \\ 0 & A & B \\ \hline 0 & C & 0 \end{array}\right]_T \quad \text{(by change of basis)}$$

$$= \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]_T = G \quad \text{(by the removal of unobservable modes)}$$ □

Note that these factorizations, like those of Nett *et al.* (1984), are still nth order even though they are based on a minimal-order observer design. In this sense, they are non-minimal, as are the factorizations of Hippe (1989). We will now attempt to gain more intuition about the results, by comparing their properties with known properties of the full-order factorizations (Nett *et al.* 1984, Moore *et al.* 1988).

2.3. *The class of all stabilizing controllers*

Once doubly coprime factorizations for the plant $G(s)$ have been found, it is possible to parameterize the class of all proper stabilizing controllers in terms of an arbitrary $Q(s) \in RH^\infty$ (Vidyasagar 1985). Such a class $\{K(Q) | Q \in RH^\infty\}$ can be written in terms of linear fractional transformations as

$$\begin{aligned} K(Q) &= (U + MQ)(V + NQ)^{-1} = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) \\ &= UV^{-1} + \tilde{V}^{-1}Q(I + V^{-1}NQ)^{-1}V^{-1} \end{aligned} \tag{2.12}$$

or diagrammatically as in Fig. 3, where based on the third equality in (2.12),

$$J = \begin{bmatrix} K & \tilde{V}^{-1} \\ V^{-1} & -V^{-1}N \end{bmatrix} \tag{2.13}$$

With the factorizations (2.8) and (2.9),

$$J = \left[\begin{array}{cc|cc} \Lambda & C(A+BF)\Theta & C(A+BF-\Psi\Lambda C)\Psi & CB \\ 0 & T(A+BF)\Theta & T(A+BF)\Psi & TB \\ \hline 0 & F\Theta & F\Psi & I \\ -I & 0 & I & 0 \end{array}\right]_T \tag{2.14}$$

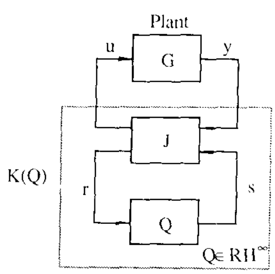

Figure 3. Class of all stabilizing controllers for G.

The scheme of Fig. 3 with J given by (2.14) has an interesting interpretation. To lead us into this, recall that if J is formed according to (2.13), and the doubly coprime factorizations of Nett *et al.* (1984) are used, then the scheme of Fig. 3 can be interpreted as in Fig. 4. That is, the class of all stabilizing controllers $\{K(Q) | Q \in RH^\infty\}$ for $G(s)$ can be generated by the use of an observer-based controller, with an additional internal feedback loop involving stable dynamics $Q(s)$—see the work by Doyle (1984). The residuals $r = (y - \hat{y})$ are filtered by Q to form s, which is added to $F\hat{x}$ to give the control signal u.

A reasonable question to ask is whether, analogously to the full state estimator based scheme of Fig. 4, the class of all stabilizing controllers can be obtained with a *minimal-order* observer-based compensator with added stable dynamics. There cannot be a direct analogue since the residuals $(y - \hat{y})$, obtained by defining $\hat{y} = C\hat{x}$, are equal to zero, as follows:

$$r = y - \hat{y} = y - C\hat{x} = y - C[\Psi \quad \Theta]\begin{bmatrix} y \\ z \end{bmatrix}$$

$$= y - [I \quad 0]\begin{bmatrix} y \\ z \end{bmatrix} = 0 \quad \text{by (2.6 } b)$$

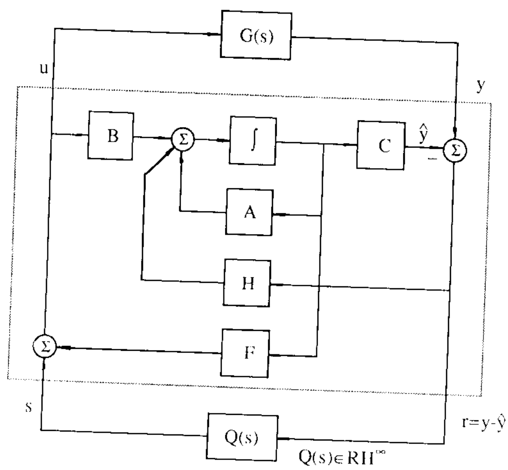

Figure 4. Controller class $\{K(Q) | Q \in RH^\infty\}$ based on full-order observer.

Consider instead residuals $r \triangleq (y - y_e)$, where the estimate y_e of y is, for the case $\Lambda = 0$, the integration of an estimate of the derivative $\dot{y}$. More generally, when Λ is chosen such that its eigenvalues lie in the closed left half-plane, y_e is the solution of

$$\dot{y}_e - \Lambda y_e = C(A\hat{x} + Bu) - \Lambda y = C(A + BF)\hat{x} - \Lambda y \tag{2.15}$$

Here $C(A\hat{x} + Bu)$ is an estimate of $\dot{y}$. From (2.15), we can obtain y_e explicitly by filtering a linear combination of the minimal-order state estimate $\hat{x}$ and the plant

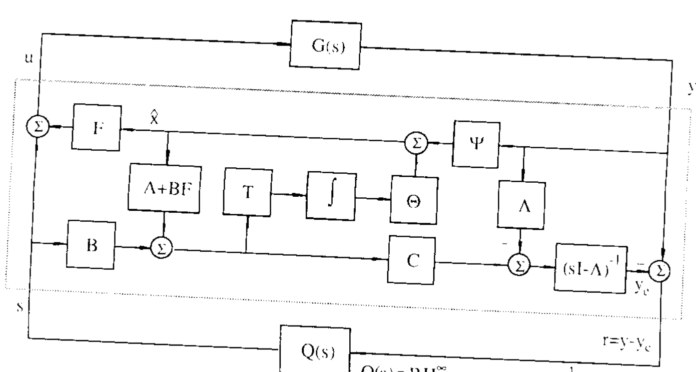

Figure 5. Controller class $\{K(Q) | Q \in RH^\infty\}$ based on minimal-order observer.

output y

$$y_e = (sI - \Lambda)^{-1}[C(A + BF)\hat{x} - \Lambda y] \tag{2.16}$$

With the residuals $r = y - y_e$, and referring to Fig. 3, it is reasonable to propose a minimal-order observer-based scheme as in Fig. 5. Evaluating the transfer function of the J block defined according to Fig. 5 gives precisely the J of (2.14).

In summary, we have found a minimal-order observer based compensator, with added stable dynamics, that generates the class of all stabilizing controllers for G as $Q(s)$ varies over RH^∞. Notice that the McMillan degree of J in (2.14) is $(n-p)+p=n$, which is the same as for the J in the full-order scheme of Fig. 4.

3. Dynamic state estimate feedback

From this point onwards, we will generalize the state estimate feedback gain F to be a proper, rational transfer function matrix, which may possibly be unstable. Assume that a left coprime factorization $F = \tilde{V}_F^{-1}\tilde{U}_F$ has been found: state-space realizations for such factorizations are readily available with the use of the doubly coprime factorizations given in § 2. It will be necessary to generalize the notation for a state-space realization so that, for example,

$$\left[\begin{array}{c|c} A + BF(s) & B\tilde{V}_F^{-1}(s) \\ \hline F(s) & \tilde{V}_F^{-1}(s) \end{array}\right]_T = F(s)[sI - A - BF(s)]^{-1}B\tilde{V}_F^{-1}(s) + \tilde{V}_F^{-1}(s) \tag{3.1}$$

To take account of the dynamic state estimate feedback, new doubly coprime factorizations will be defined.

In § 2, a constant F is chosen so that $(sI - A - BF)^{-1} \in RH^\infty$, or equivalently, so that F is a stabilizing controller for the system $(sI - A)^{-1}B$. Generalizing to the case when F is a transfer function matrix, we require $F(s)$ to be a stabilizing controller for the system $G_F \triangleq (sI - A)^{-1}B$.

3.1. *Factorizations*

Theorem 2: *Doubly coprime factorizations for* G_F

Given a plant $G_F \triangleq (sI - A)^{-1}B$ with (A, B) controllable, a proper stabilizing controller $F(s)$ with a left coprime factorization $\tilde{V}_F^{-1}\tilde{U}_F$, arbitrary Λ_1 such that $(sI_n - \Lambda_1)^{-1} \in RH^\infty$, and defining,

$$\begin{bmatrix} M_F & U_F \\ N_F & V_F \end{bmatrix} = \left[\begin{array}{c|cc} A + BF(s) & B\tilde{V}_F^{-1}(s) & A + BF(s) - \Lambda_1 \\ \hline F(s) & \tilde{V}_F^{-1}(s) & F(s) \\ I & 0 & I \end{array}\right]_T \tag{3.2}$$

$$\begin{bmatrix} \tilde{V}_F & -\tilde{U}_F \\ -\tilde{N}_F & \tilde{M}_F \end{bmatrix} = \left[\begin{array}{c|cc} \Lambda_1 & -B & -(A - \Psi\Lambda_1 C) \\ \hline 0 & \tilde{V}_F(s) & -\tilde{U}_F(s) \\ I & 0 & I \end{array}\right]_T \tag{3.3}$$

Then the following apply:

(a) the transfer matrices defined by (3.2) and (3.3) are stable and proper;

(b) M, $\tilde{M}$, V, $\tilde{V}$ have proper inverses;

(*c*) $G_F = N_F M_F^{-1} = \tilde{M}_F^{-1} \tilde{N}_F, \quad F = U_F V_F^{-1}$

(*d*)

$$\begin{bmatrix} \tilde{V}_F & -\tilde{U}_F \\ -\tilde{N}_F & \tilde{M}_F \end{bmatrix} \begin{bmatrix} M_F & U_F \\ N_F & V_F \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.4}$$

Proof

Statements (*b*), (*c*) and (*d*) can be proved by simple manipulations using (2.1) and (2.2). It remains to show that the transfer function matrices (3.2) and (3.3) are stable, since together with (*d*) this implies that the factorizations are coprime in RH^∞. First note that $\tilde{M}_F$ and $\tilde{N}_F$ are stable, since $(sI - \Lambda_1)^{-1} \in RH^\infty$. Consider then arbitrary stable proper stable factorizations $G_F = \mathcal{N}_F \mathcal{M}_F^{-1}$, $F = \mathcal{U}_F \mathcal{V}_F^{-1}$. Since F stabilizes G_F, then the standard arguments (Vidyasagar 1985) give that

$$(\tilde{V}_F \mathcal{M}_F - \tilde{U}_F \mathcal{N}_F)^{-1}, (\tilde{M}_F \mathcal{V}_F - \tilde{N}_F \mathcal{U}_F)^{-1} \in RH^\infty$$

Also from (*d*) we have $(\tilde{V}_F - \tilde{U}_F G_F) M_F = I$, $(\tilde{M}_F - \tilde{N}_F F) V_F = I$ so that

$$\begin{bmatrix} M_F \\ N_F \end{bmatrix} = \begin{bmatrix} I \\ G_F \end{bmatrix} (\tilde{V}_F - \tilde{U}_F G_F)^{-1} = \begin{bmatrix} \mathcal{M}_F \\ \mathcal{N}_F \end{bmatrix} (\tilde{V}_F \mathcal{M}_F - \tilde{U}_F \mathcal{N}_F)^{-1} \in RH^\infty \tag{3.5}$$

$$\begin{bmatrix} V_F \\ U_F \end{bmatrix} = \begin{bmatrix} I \\ F \end{bmatrix} (\tilde{M}_F - \tilde{N}_F F)^{-1} = \begin{bmatrix} \mathcal{V}_F \\ \mathcal{U}_F \end{bmatrix} (\tilde{M}_F \mathcal{V}_F - \tilde{N}_F \mathcal{U}_F)^{-1} \in RH^\infty \tag{3.6}$$

□

A generalization of Theorem 1 follows.

Theorem 3: *Doubly coprime factorizations for* G

Consider a plant $G = C(sI - A)^{-1}B$ with (A, B) controllable and (A, C) observable. Choose T such that $(sI - R)^{-1} \in RH^\infty$, and proper $F(s)$, with arbitrary right coprime factorization $\tilde{V}_F^{-1}\tilde{U}_F$, such that $F(s)$ stabilizes $G_F = (sI - A)^{-1}B$—the matrices R and T are defined by the observer equations (2.4)–(2.6). With arbitrary Λ_2 such that $(sI_p - \Lambda_2)^{-1} \in RH^\infty$ define

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A + BF(s) & B\tilde{V}_F^{-1}(s) & (A + BF(s) - \Psi\Lambda_2 C)\Psi \\ \hline F(s) & \tilde{V}_F^{-1}(s) & F(s)\Psi \\ C & 0 & I \end{array}\right]_T \tag{3.7}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A\Theta T + \Psi\Lambda_2 C & -B & -(A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{U}_F(s)\Theta T & \tilde{V}_F(s) & -\tilde{U}_F(s)\Psi \\ C & 0 & I \end{array}\right]_T \tag{3.8}$$

Then the following apply:

(*a*) the transfer functions defined by (3.7) and (3.8) are stable and proper;

(*b*) M, $\tilde{M}$, V, $\tilde{V}$ have proper inverses;

(*c*) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$

(*d*) $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ where K is the observer based controller given by (2.7);

(*e*)

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.9}$$

Proof

As in previous theorems, (*c*)–(*e*) can be proved by application of (2.1) and (2.2). Evaluation of M^{-1}, $\tilde{V}^{-1}$ shows that M, $\tilde{V}$ have proper inverses, because $\tilde{V}_F$ is proper with a proper inverse. Similarly, since V, $\tilde{M}$ have unity direct-feedthrough matrices, V^{-1}, $\tilde{M}^{-1}$ are proper, completing the proof of (*b*). It remains only to prove that all of the transfer functions are proper and stable. Consider first $\tilde{V}$,

$$\tilde{V} = \tilde{U}_F \Theta T(sI - A\Theta T - \Psi\Lambda_2 C)^{-1}(-B) + \tilde{V}_F \tag{3.10}$$

Since $(sI - A\Theta T - \Psi\Lambda_2 C)^{-1} \in RH^{\infty}$, $\tilde{V}$ is formed from the sum and product of stable proper transfer functions. It follows that $\tilde{V}$ is also proper and stable. The same can be seen of $\tilde{U}$, $\tilde{N}$, $\tilde{M}$. From the previous theorem, we have stable proper transfer functions M_F, N_F, and

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} M_F \\ N_F \end{bmatrix} \in RH^{\infty} \tag{3.11}$$

To establish that $\begin{bmatrix} U \\ V \end{bmatrix}$ is stable requires some intermediate results. Since $F(s)$ stabilizes $G_F(s)$,

$$\begin{bmatrix} I & -F(s) \\ -G_F(s) & I \end{bmatrix}^{-1} = \left[\begin{array}{c|cc} A & -B & 0 \\ \hline 0 & I & -F(s) \\ I & 0 & I \end{array}\right]_T^{-1} = \left[\begin{array}{c|cc} A + BF(s) & B & BF(s) \\ \hline F(s) & I & F(s) \\ I & 0 & I \end{array}\right]_T \in RH^{\infty} \tag{3.12}$$

$$\Rightarrow s\begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}B \in RH^{\infty} \quad \text{(by differentiation)}$$

$$\Rightarrow \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}\{A + BF(s)\}B + \begin{bmatrix} F(s) \\ I \end{bmatrix}B \in RH^{\infty}$$

$$\Rightarrow \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}AB + \left[\begin{array}{c|c} A + BF(s) & BF(s) \\ \hline F(s) & F(s) \\ I & I \end{array}\right]_T B \in RH^{\infty}$$

$$\Rightarrow \begin{bmatrix} F(s) \\ I \end{bmatrix}\{sI - A - BF(s)\}^{-1}AB \in RH^{\infty} \quad \text{by (3.12)}$$

Repeated differentiation leads to

$$\begin{bmatrix} F(s) \\ I \end{bmatrix} \{sI - A - BF(s)\}^{-1}[B \quad AB \quad A^2B \ldots A^{n-1}B] \in RH^{\infty}$$

$$\Rightarrow \begin{bmatrix} F(s) \\ I \end{bmatrix} \{sI - A - BF(s)\}^{-1} \in RH^{\infty} \quad ([A, B] \text{ controllable}) \tag{3.13}$$

Finally, with the following decomposition:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \left[\begin{array}{c|c} A + BF(s) & BF(s) \\ \hline F(s) & F(s) \\ I & 0 \end{array}\right]_T \Psi$$

$$+ \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} F(s) \\ I \end{bmatrix} \{sI - A - BF(s)\}^{-1}(A - \Psi\Lambda_2 C)\Psi \tag{3.14}$$

the first and second terms are proper stable transfer functions by virtue of (3.12) and (3.13), respectively. Thus $U, V \in RH^{\infty}$, and the proof of (*a*) is complete. □

Observe that when $F(s)$ equals a constant F, then (3.7) and (3.8) are identical to (2.8) and (2.9), and Theorem 3 specializes to Theorem 1.

The previous two theorems lead to the following corollary.

Corollary 1

Consider a plant $G = C(sI - A)^{-1}B$ with (A, B) controllable and (A, C) observable. Choose T such that the corresponding R results in a stable observer design. The controller for G obtained by dynamic state estimate feedback, via a proper $F(s)$, will be stabilizing if $F(s)$ is a proper stabilizing controller for G_F.

Proof

Start with an arbitrary $F(s)$ which stabilizes G_F (see Fig. 6). Choose a left coprime factorization $\tilde{V}_F^{-1}\tilde{U}_F$ for F, and construct the doubly coprime factorizations (3.7) and (3.8). A standard result from factorization theory (Vidyasagar 1985) is that $\tilde{V}^{-1}\tilde{U}$ will be a stabilizing controller for $\tilde{M}^{-1}\tilde{N}$. Since $G = C(sI - A)^{-1}B = \tilde{M}^{-1}\tilde{N}$, and $K = \tilde{V}^{-1}\tilde{U}$ is the observer based controller given by (2.7)—see Fig. 7—the corollary is proved.

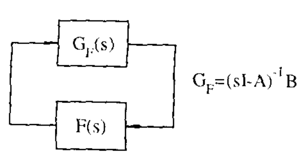

Figure 6. Controller $F(s)$ to stabilize plant $G_F(s)$.

A natural question to ask is the converse: Would the controller of Fig. 7 be destabilizing for G if F did not stabilize G_F? The answer to this question is not straight-forward, because the coprime factorizations of Theorem 3 rely on F to be stabilizing for G_F. The next section tackles this problem, and demonstrates the utility of the new factorizations at the same time.

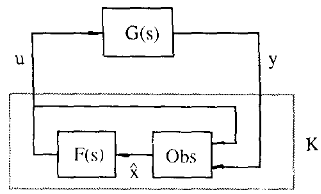

Figure 7. Observer-based controlller with dynamic state estimate feedback.

3.2. *All stabilizing controllers as minimal-order observer-based controllers*

The parameterization of the class of all proper stabilizing controllers for G will now be restated, thus

$$K(Q) = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}), \quad Q \in RH^\infty \tag{3.15}$$

Here $\tilde{M}$, $\tilde{N}$, $\tilde{U}$, $\tilde{V}$ now refer to the factorizations of Theorem 3, and can be thought of as functions of $F(s)$. At this point, it is convenient to introduce a new notation—instead of $K(Q)$ we will write $K[Q, F]$, to note explicitly the dependence of K on the choice of $F(s)$. The controller $K = \tilde{V}^{-1}\tilde{U}$ will be written as $K[0, F]$. Making use of the doubly coprime factorizations of Theorem 2, the class of all proper stabilizing controllers for G_F can be written as

$$F(Q_F) = (\tilde{V}_F + Q_F\tilde{N}_F)^{-1}(\tilde{U}_F + Q_F\tilde{M}_F) \quad Q_F \in RH^\infty \tag{3.16}$$

What we wish to show is that the class of all proper stabilizing controllers $\{K[Q, F] \mid Q \in RH^\infty\}$ is the same as the class of proper observer-based controllers $\{K[0, F(Q_F)] \mid Q_F \in RH^\infty\}$. The proof of this requires an alternative representation of $K[0, F(Q_F)]$.

Lemma 1

An observer-based controller $K[0, F(Q_F)]$ can be restructured as a linear fractional transformation,

$$K[0, F(Q_F)] = (\tilde{V} + Q_F\tilde{N}_0)^{-1}(\tilde{U} + Q_F\tilde{M}_0) \tag{3.17}$$

where

$$[-\tilde{N}_0 \quad \tilde{M}_0] = \left[\begin{array}{c|cc} A\Theta T - \Psi\Lambda_2 C & B & (A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{M}_F(s)\Theta T & -\tilde{N}_F(s) & \tilde{M}_F(s)\Psi \end{array}\right]_T \tag{3.18}$$

$$[0 \quad \Psi] = [-\tilde{N}_0 \quad \tilde{M}_0]\begin{bmatrix} M & U \\ N & V \end{bmatrix} \tag{3.19}$$

$$\tilde{M}_0\tilde{M}^{-1} = \Psi \tag{3.20}$$

Proof

$$K[0, F(Q_F)] = \tilde{V}^{-1}\tilde{U}|_{F=F(Q_F)=(\tilde{V}_F+Q_F\tilde{N}_F)^{-1}(\tilde{U}_F+Q_F\tilde{M}_F)}$$

$$= \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & -B \\ \hline (\tilde{U}_F(s) + Q_F(s)\tilde{M}_F(s))\Theta T & \tilde{V}_F(s) + Q_F(s)\tilde{N}_F(s) \end{array}\right]_T^{-1}$$

$$\times \left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & (A - \Psi\Lambda_2 C)\Psi \\ \hline (\tilde{U}_F(s) + Q_F(s)\tilde{M}_F(s))\Theta T & (\tilde{U}_F(s) + Q_F(s)\tilde{M}_F(s))\Psi \end{array}\right]_T$$

$$= \left\{\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & -B \\ \hline \tilde{U}_F(s)\Theta T & \tilde{V}_F(s) \end{array}\right]_T + Q_F(s)\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & -B \\ \hline \tilde{M}_F(s)\Theta T & \tilde{N}_F(s) \end{array}\right]_T\right\}^{-1}$$

$$\times \left\{\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & (A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{U}_F(s)\Theta T & \tilde{U}_F(s)\Psi \end{array}\right]_T + Q_F(s)\left[\begin{array}{c|c} A\Theta T + \Psi\Lambda_2 C & (A - \Psi\Lambda_2 C)\Psi \\ \hline \tilde{M}(s)_F\Theta T & \tilde{M}_F(s)\Psi \end{array}\right]_T\right\}$$

$$= (\tilde{V} + Q_F\tilde{N}_0)^{-1}(\tilde{U} + Q_F\tilde{M}_0)$$

where $\tilde{N}_0$ and $\tilde{M}_0$ are as defined above.

Finally, (3.19), (3.20) can be proved by application of (2.1) and (2.2). □

The main result is then as follows.

Theorem 4

The class of proper stabilizing observer-based controllers $\{K[0, F(Q_F)] \mid Q_F \in RH^\infty\}$ is the class of all proper stabilizing controllers $\{K[Q, F] \mid Q \in RH^\infty\}$ for G.

Proof

Let us consider $F(Q_F)$, with arbitrary $Q_F \in RH^\infty$. This is an arbitrary stabilizing controller for G_F. Define $Q = Q_F\Psi \in RH^\infty$, then

$$Q = Q_F\tilde{M}_0\tilde{M}^{-1} \quad \text{by (3.20)}$$
$$\Leftrightarrow Q\tilde{M} = Q_F\tilde{M}_0$$
$$\Leftrightarrow Q[-\tilde{N} \quad \tilde{M}] = Q_F[-\tilde{N}_0 \quad \tilde{M}_0] \quad \text{(multiplication by } [-G \quad I])$$
$$\Leftrightarrow (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) = (\tilde{V} + Q_F\tilde{N}_0)^{-1}(\tilde{U} + Q_F\tilde{M}_0) \quad \text{by (2.12) and (3.17)}$$
$$\Leftrightarrow K[Q, F] = K[0, F(Q_F)]$$

So the observer-based controller $K[0, F(Q_F)]$ is a stabilizing controller $K[Q, F]$ for G. Conversely, suppose we have an arbitrary stabilizing controller K for G, and we find $Q \in RH^\infty$ such that $K = K[Q, F]$. Then defining Q_F by

$$Q_F = Q(\Psi^T\Psi)^{-1}\Psi^T$$

it is clear that Q_F satisfies

$$Q = Q_F \Psi \in RH^\infty$$
$$\Leftrightarrow K[Q, F] = K[0, F(Q_F)] \quad \text{(as above)}$$

This completes the proof, by showing that the arbitrary stabilizing controller $K[Q, F]$ can be structured as an observer-based controller $K[0, F(Q_F)]$, where $F(Q_F)$ is stabilizing for G_F. □

4. The minimal-order dual observer

The reader may have noticed that Lemma 1 and Theorem 4 deal primarily with left coprime factorizations of $K[Q, F]$ and $F(Q_F)$. Are there dual results related to right coprime factorizations? In fact, we can exploit the dual minimal-order observer (O'Reilly 1983). Whereas the role of the observer is to make full use of the system information in the system outputs, the dual observer takes advantage of the fact that the system can be excited from more than one input. We claim that all of the results of this work can be derived in terms of the dual observer. To give an illustration of this, a dual version of Theorem 1 will be stated. The dual observer equations are

$$\dot{z} = Dz + \xi Hw, \quad w = y + CSz, \quad u = Lz + \eta Hw \tag{4.1}$$

where B is full rank and

$$[S \quad B]\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}[S \quad B] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.2}$$

$$D = \xi AS, \quad L = \eta AS \tag{4.3}$$

The transfer function matrix $K(s)$ of an equivalent controller for G is

$$K(s) = \left[\begin{array}{c|c} D + \xi HCS & \xi H \\ \hline A + \eta HCS & \eta H \end{array}\right]_T \tag{4.4}$$

Theorem 5

Consider the plant $G(s) = C(sI - A)^{-1}B$ with (A, B) controllable and (A, C) observable. Choose H and S such that $(sI - A - HC)^{-1}, (sI - D)^{-1} \in RH^\infty$ where H and S are described by the observer equations (4.1)–(4.3). With arbitrary Γ such that $(sI - \Gamma)^{-1} \in RH^\infty$, define

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} S\xi A + B\Gamma\eta & B & -S\xi H \\ \hline -\eta(A - B\Gamma\eta) & I & \eta H \\ C & 0 & I \end{array}\right]_T \tag{4.5}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + HC & -B & H \\ \hline -\eta(A + HC - B\Gamma\eta) & I & -\eta H \\ C & 0 & I \end{array}\right]_T \tag{4.6}$$

Then, the following apply:

(*a*) all transfer functions defined by (4.5) and (4.6) are stable and proper;

(*b*) M, $\tilde{M}$, V, $\tilde{V}$ have proper inverses;

(*c*) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$;

(*d*) $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ where K is the observer-based controller given by (2.7);

(*e*)

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & M \end{bmatrix}\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.7}$$

Close comparison of Theorem 1 and Theorem 5 shows that the corresponding factorizations are natural duals of each other. We can write down a dual to the controller-class of Fig. 5, with m integrators required to realize the transfer function $(sI - \Gamma)^{-1}$. The full-order observer-based class of Fig. 4 has no dual, as can be seen in the inherent symmetry of the block diagram.

5. Conclusions

For brevity, the results of the work have been obtained in terms of the minimal-order observer, which has McMillan degree $n - p$ $(p \geqslant 1)$. As shown in Fig. 2, the state estimate has an additive term Ψy involving direct-feedthrough of all plant outputs. The results can also be obtained in terms of a reduced-order observer of order $n - \chi$, with $\chi \leqslant p$. As shown in Fig. 8, the reduced-order observer-based controller has direct feedthrough to $\hat{x}$ of only χ plant outputs, the plant outputs being divided as follows:

$$y = Cx = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with a $(p - \chi) \times 1$ vector y_1 and a $\chi \times 1$ vector y_2.

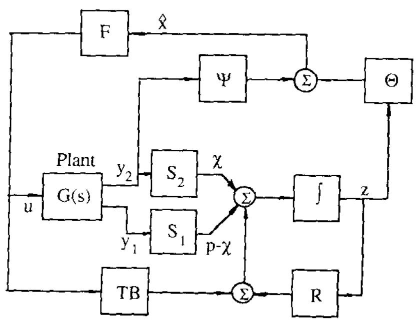

Figure 8. Reduced order observer-based control loop.

The case $\chi = p$ corresponds to the results of this work, while the $\chi = 0$ leads to the results of Nett *et al.* (1984), with F constant, and the results of Moore *et al.* (1988), with F dynamic. The requirement that C be full rank is not restrictive, since in practice C can always be made full rank by ignoring certain plant outputs, and deleting the corresponding rows of C.

Finally, it has been shown that an observer-based controller class $\{K[0, F(Q_F)] \mid Q_F \in RH^\infty\}$ is exactly the class of all proper stabilizing controllers $\{K[Q, F] \mid Q \in RH^\infty\}$ for G. Trivial extensions show that this is identical to the more general class $\{K[Q, F(Q_F)] \mid Q, Q_F \in RH^\infty\}$.

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References

Doyle, J. C., 1984, Lecture notes for ONR/Honeywell Workshop on Advances in Multivariable Control, Minneapolis.

Hippe, P., 1989, Modified doubly coprime fractional representations related to the reduced-order observer. *I.E.E.E. Transactions on Automatic Control*, **34**, 573–576.

Moore, J. B., Glover, K., and Telford, A. J., 1989, All stabilizing controllers as frequency shaped state estimate feedback. *I.E.E.E. Transactions on Automatic Control*, **34**, to be published.

Nett, C. N., Jacobson, C. A., and Balas, M. J., 1984, A connection between state-space and doubly coprime fractional representations. *I.E.E.E. Transactions on Automatic Control*, **29**, 831–832.

O'Reilly, J., 1983, *Observers for Linear Systems* (London: Academic Press).

Vidyasagar, M., 1985, *Control Systems Synthesis: A Factorization Approach* (Cambridge, MA: MIT Press).