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DOUBLY WARPED PRODUCTS WITH HARMONIC WEYL CONFORMAL CURVATURE TENSOR

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1. Introduction. An n-dimensional ($n \geq 4$) Riemannian manifold (whose metric g_{ij} need not be definite) is called *conformally symmetric* [4] if its Weyl conformal curvature tensor

$$(1)\qquad C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})$$

is parallel, i.e., if $C_{hijk,l} = 0$.

Here and in the sequel we denote by R_{hijk}, R_{ij} and R the curvature tensor, Ricci tensor and scalar curvature, respectively, while the comma stands for covariant differentiation with respect to g.

Clearly, the class of conformally symmetric manifolds contains all locally symmetric as well as all conformally flat manifolds of dimension $n \geq 4$.

It is easy to check that for every conformally symmetric manifold the condition

$$(2)\qquad R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik})$$

holds.

A Riemannian manifold is said to have *harmonic curvature* respectively *harmonic Weyl* (*conformal curvature*) *tensor* if the divergence of its curvature tensor (respectively Weyl tensor) is identically zero, i.e. $R^r{}_{ijk,r} = 0$, respectively $C^r{}_{ijk,r} = 0$ ([2], p. 440). The second Bianchi identity implies that $R^r{}_{ijk,r} = R_{ij,k} - R_{ik,j}$ and

$$C^r{}_{ijk,r} = \frac{n-3}{n-2}\left[R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik})\right].$$

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Including the case $n = 3$ in our considerations we say that M has *harmonic conformal curvature* if the tensor

$$R_{ij} - \frac{R}{2(n-1)} g_{ij}$$

is Codazzi on M, i.e. satisfies condition (2).

Now it is clear that any conformally symmetric manifold as well as manifolds with harmonic curvature have harmonic conformal curvature. Moreover, a Riemannian manifold has harmonic conformal curvature if and only if $C^r{}_{ijk,r} = 0$ in case $n \ge 4$ or if it is conformally flat in case $n = 3$.

In the paper [12] an n-dimensional ($n \ge 3$) Riemannian manifold satisfying condition (2) was called *nearly conformally symmetric*. (M^n, g) is *Einstein* if R_{ij} is proportional to g_{ij}. It is well known that every surface is Einstein and that an Einstein space satisfying $C^h{}_{ijk} = 0$ is a space of constant curvature.

The aim of this paper is to give a description of the local structure of doubly warped products with harmonic conformal curvature.

2. Definition and basic formulas. Let $(\overline{M}, \overline{g})$ and $(\overset{*}{M}, \overset{*}{g})$ ($\dim \overline{M} = r, \dim \overset{*}{M} = n - r, 1 \le r < n$) be Riemannian manifolds and let $f : \overline{M} \to \mathbb{R}^+$ and $h : \overset{*}{M} \to \mathbb{R}^+$ be positive C^∞-functions. The Riemannian manifold $M := \overline{M} \times \overset{*}{M}$ furnished with the metric tensor

$$g := (h \circ \sigma)^2 \pi^*(\overline{g}) + (f \circ \pi)^2 \sigma^*(\overset{*}{g}),$$

where $\pi : \overline{M} \times \overset{*}{M} \to \overline{M}$ and $\sigma : \overline{M} \times \overset{*}{M} \to \overset{*}{M}$ are the natural projections, will be called a *doubly warped product* and denoted by $\overline{M}\, {}_h\!\times_f \overset{*}{M}$ [6].

If in particular h or f is constant, then the doubly warped product reduces to a (singly) warped product [3] (or semi-decomposable space [10]).

The doubly warped product is the special case of the so-called conformal product ($g = h\overline{g} \oplus k\overset{*}{g}$, where h and k are functions defined on $\overline{M} \times \overset{*}{M}$) investigated by Yano [14] and Wong [13]. Some theorems on conformally symmetric, Ricci-symmetric and Ricci-pseudosymmetric doubly warped products can be found in [8] and [9]. Lorentzian doubly warped products have been studied by Beem and Powell [1].

By similar considerations to [11], §2, we obtain

THEOREM 1. *An n-dimensional doubly warped product $M = \overline{M}\, {}_h\!\times_f \overset{*}{M}$ ($\dim \overline{M} = r$, $1 \le r < n$) fibers into two mutually orthogonal $(n-r)$- and r-codimensional Riemannian foliations, the leaves of each being totally umbilical in M and pairwise (locally) homothetic.*

Let $\{\mathcal{U}\times\mathcal{V} : x^1,\dots,x^r,x^{r+1}=y^1,\dots,x^n=y^{n-r}\}$ be a product chart for $\overline{M}\times\overset{*}{M}$. The local components of the metric $g=\overline{g}\ {}_h\!\times_f \overset{*}{g}$ with respect to this chart are

$$g_{ij}=\begin{cases} h^2\overline{g}_{ab} & \text{if } i=a \text{ and } j=b,\\ f^2\overset{*}{g}_{\alpha\beta} & \text{if } i=\alpha \text{ and } j=\beta,\\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

where $a,b,c,d,e\in\{1,\dots,r\}$, $\alpha,\beta,\gamma,\delta,\varepsilon\in\{r+1,\dots,n\}$ and $i,j,k,l\in\{1,\dots,n\}$. We assume that each object marked with a dash is formed from $\overline{g}_{ab}$, and those marked with a star come from $\overset{*}{g}_{\alpha\beta}$. Moreover, the period and semicolon in subscripts denote covariant differentiation in $\overline{M}$ and $\overset{*}{M}$, respectively. The local components Γ^h_{ij} of the Levi-Civita connection on $\overline{M}\ {}_h\!\times_f \overset{*}{M}$ are

$$\begin{aligned} &\Gamma^a_{bc}=\overline{\Gamma}^a_{bc}, && \Gamma^a_{\alpha b}=\frac{1}{h}h_\alpha\delta^a_b, && \Gamma^\alpha_{\beta a}=\frac{1}{f}f_a\delta^\alpha_\beta,\\ &\Gamma^\alpha_{\beta\gamma}=\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}, && \Gamma^\alpha_{ab}=-\frac{1}{h}h^\alpha g_{ab}, &&\\ &\Gamma^a_{\alpha\beta}=-\frac{1}{f}f^a g_{\alpha\beta}, &&&& \end{aligned} \tag{4}$$

where $f_a=(\partial/\partial x^a)f$, $h_\alpha=(\partial/\partial x^\alpha)h$, $f^a=g^{ae}f_e$ and $h^\alpha=g^{\alpha\varepsilon}h_\varepsilon$.

The local components

$$R^l{}_{ijk}=\partial_k\Gamma^l_{ij}-\partial_j\Gamma^l_{ik}+\Gamma^r_{ij}\Gamma^l_{rk}-\Gamma^r_{ik}\Gamma^l_{rj},$$

where $\partial_j=\partial/\partial x^j$, of the curvature tensor of $\overline{M}\ {}_h\!\times_f \overset{*}{M}$, which in general do not vanish identically, are

$$\begin{aligned} R^a{}_{bcd}&=\overline{R}^a{}_{bcd}-\frac{(\Delta_1 h)}{h^2}(\delta^a_d g_{bc}-\delta^a_c g_{bd}),\\ R^\alpha{}_{bcd}&=\frac{1}{hf}(h^\alpha f_d g_{bc}-h^\alpha f_c g_{bd}),\\ R^a{}_{\beta\gamma\delta}&=\frac{1}{hf}(h_\delta f^a g_{\beta\gamma}-h_\gamma f^a g_{\beta\delta}),\\ R^\alpha{}_{bc\beta}&=-\frac{1}{f}\delta^\alpha_\beta f_{bc}-\frac{1}{h}h^\alpha_\beta g_{bc},\\ R^a{}_{\beta\gamma d}&=-\frac{1}{h}\delta^a_d h_{\beta\gamma}-\frac{1}{f}f^a_d g_{\beta\gamma},\\ R^\alpha{}_{\beta\gamma\delta}&=\overset{*}{R}{}^\alpha{}_{\beta\gamma\delta}-\frac{(\Delta_1 f)}{f^2}(\delta^\alpha_\delta g_{\beta\gamma}-\delta^\alpha_\gamma g_{\beta\delta}), \end{aligned} \tag{5}$$

where $(\Delta_1 h)=g^{\alpha\beta}h_\alpha h_\beta$, $(\Delta_1 f)=g^{ab}f_a f_b$, $h_{ij}=h_{i,j}$ and $f_{ij}=f_{i,j}$.

The local components $R_{ij} = R^r{}_{ijr}$ of the Ricci tensor of $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ are

$$
\begin{aligned}
R_{ab} &= \overline{R}_{ab} - \frac{n-r}{f} f_{ab} - (r-1)\frac{(\Delta_1 h)}{h^2} g_{ab} - \frac{1}{h}(h^{\varepsilon}_{.\varepsilon}) g_{ab},\\
R_{\alpha d} &= \frac{n-2}{hf} h_\alpha f_d, \qquad (6)\\
R_{\alpha\beta} &= \overset{*}{R}_{\alpha\beta} - \frac{r}{h} h_{\alpha\beta} - (n-r-1)\frac{(\Delta_1 f)}{f^2} g_{\alpha\beta} - \frac{1}{f}(f^{e}_{.e}) g_{\alpha\beta}
\end{aligned}
$$

where

$$(h^{\varepsilon}_{.\varepsilon}) = g^{\alpha\beta} h_{\alpha\beta} \quad \text{and} \quad (f^{e}_{.e}) = g^{ab} f_{ab}.$$

The scalar curvature R of the metric $\overline{g}\,{}_h\!\times_f \overset{*}{g}$ satisfies

$$
\begin{aligned}
R = \frac{\overline{R}}{h^2} + \frac{\overset{*}{R}}{f^2} - \frac{r(r-1)}{h^2}(\Delta_1 h) - \frac{(n-r)(n-r-1)}{f^2}(\Delta_1 f)\\
- \frac{2r}{h}(h^{\varepsilon}_{.\varepsilon}) - \frac{2(n-r)}{f}(f^{e}_{.e}).
\end{aligned} \tag{7}
$$

By straightforward computations we establish the following three lemmas.

LEMMA 1. *In a doubly warped product* $\overline{M}\,{}_h\!\times_f \overset{*}{M}$,

$$
\begin{aligned}
&h_{ab} = \frac{1}{h}(\Delta_1 h) g_{ab}, \quad h_{a\alpha} = -\frac{1}{f} f_a h_\alpha, \quad h_{\alpha\beta} = h_{\alpha;\beta},\\
&f_{ab} = f_{a.b}, \qquad f_{a\alpha} = -\frac{1}{h} f_a h_\alpha, \quad f_{\alpha\beta} = \frac{1}{f}(\Delta_1 f) g_{\alpha\beta}.
\end{aligned}
$$

LEMMA 2. *Let* $M = \overline{M}\,{}_h\!\times_f \overset{*}{M}$. *Then*

$$
\begin{aligned}
&\partial_c(\Delta_1 h) = -\frac{2}{f}(\Delta_1 h) f_c, \quad \partial_\gamma(\Delta_1 h) = 2h^\alpha h_{\alpha\gamma},\\
&\partial_\alpha(\Delta_1 f) = -\frac{2}{h}(\Delta_1 f) h_\alpha, \quad \partial_c(\Delta_1 f) = 2f^e f_{ec},\\
&\partial_c h_{\alpha\beta} = 0, \qquad \partial_c(h^{\varepsilon}_{.\varepsilon}) = -\frac{2}{f}(h^{\varepsilon}_{.\varepsilon}) f_c,\\
&\partial_\gamma f_{ab} = 0, \qquad \partial_\gamma(f^{e}_{.e}) = -\frac{2}{h}(f^{e}_{.e}) h_\gamma.
\end{aligned}
$$

LEMMA 3. *In* $\overline{M}\,{}_h\!\times_f \overset{*}{M}$,

$$
\begin{aligned}
h_{\alpha c\beta} &= -\frac{2}{f} f_c h_{\alpha\beta} + \frac{1}{hf}(\Delta_1 h) f_c g_{\alpha\beta} + \frac{1}{hf} f_c h_\alpha h_\beta,\\
f_{a\gamma b} &= -\frac{2}{h} h_\gamma f_{ab} + \frac{1}{hf}(\Delta_1 f) h_\gamma g_{ab} + \frac{1}{hf} f_a f_b h_\gamma,
\end{aligned}
$$

$$g^{ab}f_{abc} = \partial_c(f^e_{.e}) - \frac{2}{h^2}(\Delta_1 h)f_c,$$
$$g^{\alpha\beta}h_{\alpha\beta\gamma} = \partial_\gamma(h^\varepsilon_{.\varepsilon}) - \frac{2}{f^2}(\Delta_1 f)h_\gamma,$$

where $f_{ijk} = f_{i,jk}$ *and* $h_{ijk} = h_{i,jk}$.

On the other hand, the Ricci formula gives

$$\begin{aligned} g^{ab}f_{abc} &= g^{ab}f_{acb} - \frac{1}{h^2}f_e\overline{R}^e{}_c + \frac{r-1}{h^2}(\Delta_1 h)f_c,\\ g^{\alpha\beta}h_{\alpha\beta\gamma} &= g^{\alpha\beta}h_{\alpha\gamma\beta} - \frac{1}{f^2}h_\varepsilon\overset{*}{R}{}^\varepsilon{}_\gamma + \frac{n-r-1}{f^2}(\Delta_1 f)h_\gamma. \end{aligned} \tag{8}$$

Differentiating the local components of the Ricci tensor covariantly and taking into account (4), (6) and Lemmas 1 and 2, we get

$$\begin{aligned}
R_{ab,c} = {} & \overline{R}_{ab.c} - \frac{n-r}{f}f_{abc} + \frac{n-r}{f^2}f_cf_{ab}\\
& + \frac{2(r-1)}{h^2f}(\Delta_1 h)f_cg_{ab} + \frac{2}{hf}(h^\varepsilon_{.\varepsilon})f_cg_{ab}\\
& + \frac{r-2}{h^2f}(\Delta_1 h)(f_bg_{ac} + f_ag_{bc}),\\
R_{\alpha b,c} = {} & -\frac{1}{h}h_\alpha R_{bc} + \frac{n-2}{hf}h_\alpha f_{bc} - \frac{2(n-2)}{hf^2}h_\alpha f_bf_c\\
& + \frac{1}{h}h^\varepsilon R_{\alpha\varepsilon}g_{bc},\\
R_{ab,\gamma} = {} & -\frac{2}{h}h_\gamma\overline{R}_{ab} + \frac{2(n-r)}{hf}h_\gamma f_{ab}\\
& + \frac{2(r-1)}{h^3}(\Delta_1 h)h_\gamma g_{ab} + \frac{1}{h^2}(h^\varepsilon_{.\varepsilon})h_\gamma g_{ab}\\
& - \frac{2(n-2)}{hf^2}f_af_bh_\gamma - \frac{r-1}{h^2}g_{ab}\partial_\gamma(\Delta_1 h)\\
& - \frac{1}{h}g_{ab}\partial_\gamma(h^\varepsilon_{.\varepsilon}),\\
R_{\alpha\beta,c} = {} & -\frac{2}{f}f_c\overset{*}{R}_{\alpha\beta} + \frac{2r}{hf}f_ch_{\alpha\beta} + \frac{2(n-r-1)}{f^3}(\Delta_1 f)f_cg_{\alpha\beta}\\
& + \frac{1}{f^2}(f^e_{.e})f_cg_{\alpha\beta} - \frac{2(n-2)}{h^2f}h_\alpha h_\beta f_c\\
& - \frac{n-r-1}{f^2}g_{\alpha\beta}\partial_c(\Delta_1 f) - \frac{1}{f}g_{\alpha\beta}\partial_c(f^e_{.e}),
\end{aligned} \tag{9}$$

$$R_{\alpha\beta,\gamma} = -\frac{1}{f} f_a R_{\beta\gamma} + \frac{n-2}{hf} f_a h_{\beta\gamma} - \frac{2(n-2)}{h^2 f} f_a h_\beta h_\gamma + \frac{1}{f} f^e R_{ae} g_{\beta\gamma},$$

(9) [cont.]

$$R_{\alpha\beta,\gamma} = \overset{*}{R}_{\alpha\beta;\gamma} - \frac{r}{h} h_{\alpha\beta\gamma} + \frac{r}{h^2} h_\gamma h_{\alpha\beta} + \frac{2(n-r-1)}{hf^2} (\Delta_1 f) h_\gamma g_{\alpha\beta} + \frac{2}{hf} (f^e_{.e}) h_\gamma g_{\alpha\beta} + \frac{n-r-2}{hf^2} (\Delta_1 f)(h_\beta g_{\alpha\gamma} + h_\alpha g_{\beta\gamma}).$$

Moreover, from (7) we find

(10)

$$R_{,c} = \frac{1}{h^2} \overline{R}_{.c} - \frac{2}{f^3} \overset{*}{R} f_c + \frac{2r(r-1)}{h^2 f} (\Delta_1 h) f_c + \frac{2(n-r)(n-r-1)}{f^3} (\Delta_1 f) f_c - \frac{(n-r)(n-r-1)}{f^2} \partial_c (\Delta_1 f) + \frac{4r}{hf} (h^\varepsilon_{.\varepsilon}) f_c + \frac{2(n-r)}{f^2} (f^e_{.e}) f_c - \frac{2(n-r)}{f} \partial_c (f^e_{.e}),$$

$$R_{,\gamma} = -\frac{2}{h^3} \overline{R} h_\gamma + \frac{1}{f^2} \overset{*}{R}_{;\gamma} + \frac{2r(r-1)}{h^3} (\Delta_1 h) h_\gamma - \frac{r(r-1)}{h^2} \partial_\gamma (\Delta_1 h) + \frac{2(n-r)(n-r-1)}{hf^2} (\Delta_1 f) h_\gamma + \frac{2r}{h^2} (h^\varepsilon_{.\varepsilon}) h_\gamma - \frac{2r}{h} \partial_\gamma (h^\varepsilon_{.\varepsilon}) + \frac{4(n-r)}{hf} (f^e_{.e}) h_\gamma.$$

3. Doubly warped products with harmonic Weyl tensor and an r-dimensional factor $(1 < r < n-1)$**.** Let $M = \overline{M}\ {}_h\!\times_f \overset{*}{M}$ with $n = \dim M \geq 4$. We can now prove the following results.

LEMMA 4. *If $\overline{M}\ {}_h\!\times_f \overset{*}{M}$ has harmonic Weyl tensor and $r = \dim \overline{M} \geq 2$, $n - r = \dim \overset{*}{M} \geq 2$, then*

$$(\overline{11}) \qquad \overline{R}_{ab} + \frac{r-2}{f} f_{ab} - \frac{\overline{R}}{r} \overline{g}_{ab} - \frac{r-2}{rf} (\bar{f}^e_{.e}) \overline{g}_{ab} = 0$$

on $\overline{M}$, where $(\bar{f}^e_{.e}) = \overline{g}^{ab} f_{a.b}$, and

$$(\overset{*}{11}) \qquad \overset{*}{R}_{\alpha\beta} + \frac{n-r-2}{h} h_{\alpha\beta} - \frac{\overset{*}{R}}{n-r} \overset{*}{g}_{\alpha\beta} - \frac{n-r-2}{(n-r)h} (\overset{*}{h}{}^\varepsilon_{.\varepsilon}) \overset{*}{g}_{\alpha\beta} = 0$$

on $\overset{*}{M}$, where $(\overset{*}{h}{}^\varepsilon_{.\varepsilon}) = \overset{*}{g}{}^{\alpha\beta} h_{\alpha;\beta}$.

P r o o f. From (2), taking all the subscripts i, j, k from the same range $1, \ldots, r$ or $r+1, \ldots, n$ and substituting suitable formulae (9), we get

$$(\overline{12}) \quad \overline{R}_{ab.c} - \overline{R}_{ac.b} + \frac{n-r}{f} f_e \overline{R}^e{}_{abc} + \frac{n-r}{f^2}(f_c f_{ab} - f_b f_{ac}) + \frac{2}{hf}(h^{\varepsilon}_{.\varepsilon})(f_c g_{ab} - f_b g_{ac}) - \frac{n-2r}{h^2 f}(\Delta_1 h)(f_c g_{ab} - f_b g_{ac}) = \frac{1}{2(n-1)}(R_{,c} g_{ab} - R_{,b} g_{ac})$$

and

$$(\overset{*}{12}) \quad \overset{*}{R}_{\alpha\beta;\gamma} - \overset{*}{R}_{\alpha\gamma;\beta} + \frac{r}{h} h_\varepsilon \overset{*}{R}{}^{\varepsilon}{}_{\alpha\beta\gamma} + \frac{r}{h^2}(h_\gamma h_{\alpha\beta} - h_\beta h_{\alpha\gamma}) + \frac{2}{hf}(f^e_{.e})(h_\gamma g_{\alpha\beta} - h_\beta g_{\alpha\gamma}) + \frac{n-2r}{hf^2}(\Delta_1 f)(h_\gamma g_{\alpha\beta} - h_\beta g_{\alpha\gamma}) = \frac{1}{2(n-1)}(R_{,\gamma} g_{\alpha\beta} - R_{,\beta} g_{\alpha\gamma}).$$

Applying contractions to $(\overline{12})$ and $(\overset{*}{12})$, and making use of (10) and Lemma 2, we obtain

$$(\overline{13}) \quad \frac{1}{h^2}\overline{R}_{.c} = -\frac{2(r-1)}{n-r}\frac{\overset{*}{R}}{f^3} f_c - \frac{2(n-1)}{h^2 f} f_e \overline{R}^e{}_c + \frac{2n-nr+r^2-2}{f^2}\partial_c(\Delta_1 f) - \frac{2(r-1)}{f}\partial_c(f^e_{.e}) - \frac{2(n-r)}{f^2}(f^e_{.e}) f_c + \frac{2(r-1)(n-r-1)}{h^2 f}(\Delta_1 h) f_c - \frac{4(r-1)(n-r-1)}{(n-r)hf}(h^{\varepsilon}_{.\varepsilon}) f_c + \frac{2(r-1)(n-r-1)}{f^3}(\Delta_1 f) f_c$$

and

$$(\overset{*}{13}) \quad \frac{1}{f^2}\overset{*}{R}_{;\gamma} = -\frac{2(n-r-1)}{r}\frac{\overline{R}}{h^3} h_\gamma - \frac{2(n-1)}{hf^2} h_\varepsilon \overset{*}{R}{}^{\varepsilon}{}_\gamma + \frac{2n-nr+r^2-2}{h^2}\partial_\gamma(\Delta_1 h) - \frac{2(n-r-1)}{h}\partial_\gamma(h^{\varepsilon}_{.\varepsilon}) - \frac{2r}{h^2}(h^{\varepsilon}_{.\varepsilon}) h_\gamma + \frac{2(r-1)(n-r-1)}{hf^2}(\Delta_1 f) h_\gamma - \frac{4(r-1)(n-r-1)}{rhf}(f^e_{.e}) h_\gamma + \frac{2(r-1)(n-r-1)}{h^3}(\Delta_1 h) h_\gamma.$$

From (2), first taking $i = \alpha$, $j = \beta$, $k = c$ and remembering that the “mixed” components of the metric tensor are zero, we have

$$R_{\alpha\beta,c} - R_{\alpha c,\beta} = \frac{1}{2(n-1)} R_{,c} g_{\alpha\beta},$$

which, in view of (9), (10) and $(\overline{13})$, yields

$$\frac{f_c}{f}\left[\overset{*}{R}_{\alpha\beta} + \frac{n-r-2}{h} h_{\alpha\beta} - \frac{\overset{*}{R}}{n-r}\overset{*}{g}_{\alpha\beta} - \frac{n-r-2}{(n-r)h}(\overset{*}{h}{}^{\varepsilon}_{.\varepsilon})\overset{*}{g}_{\alpha\beta}\right] = 0.$$

The last result together with the non-constancy of f implies $(\overset{*}{11})$. In the same way, putting $i = a$, $j = b$, $k = \gamma$ in (2), we get

$$R_{ab,\gamma} - R_{a\gamma,b} = \frac{1}{2(n-1)} R_{,\gamma} g_{ab},$$

which, in view of (9), (10) and $(\overset{*}{13})$ yields similarly

$$\frac{h_\gamma}{h}\left[\overline{R}_{ab} + \frac{r-2}{f} f_{ab} - \frac{\overline{R}}{r}\overline{g}_{ab} - \frac{r-2}{rf}(\overline{f}{}^{e}_{.e})\overline{g}_{ab}\right] = 0,$$

and consequently $(\overline{11})$ by the assumption about h. This completes the proof.

R e m a r k 1. We observe that for $\overline{M}$ (resp. $\overset{*}{M}$) two-dimensional, the condition $(\overline{11})$ (resp. $(\overset{*}{11})$) is identically satisfied. For higher dimensions Lemma 4 states that in a doubly warped product $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ with harmonic Weyl tensor, the Hessian of f (resp. h) is proportional to the metric tensor $\overline{g}$ (resp. $\overset{*}{g}$) if and only if $\overline{M}$ (resp. $\overset{*}{M}$) is an Einstein manifold.

R e m a r k 2. If in particular $h = \text{const}$, then the doubly warped product reduces to a warped product and $(\overset{*}{11})$ shows that $\overset{*}{M}$ is an Einstein manifold (see also [7], Lemma 3).

We are now in a position to prove

THEOREM 2. *Let $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ be a doubly warped product with harmonic Weyl conformal curvature tensor. Then*

(i) *$\overline{M}$ has harmonic conformal curvature if and only if its curvature tensor and the function f satisfy*

$$(\overline{14})\qquad f_e\overline{R}^e{}_{abc} + \frac{1}{f}(f_c f_{ab} - f_b f_{ac}) = \frac{1}{r-1}\Big\{ f_e(\overline{R}^e{}_c\overline{g}_{ab} - \overline{R}^e{}_b\overline{g}_{ac}) + \frac{1}{f}[(\overline{f}{}^{e}_{.e})f_c - \overline{f}^e f_{ec}]\overline{g}_{ab} - \frac{1}{f}[(\overline{f}{}^{e}_{.e})f_b - \overline{f}^e f_{eb}]\overline{g}_{ac}\Big\},$$

where $\overline{f}^a = \overline{g}^{ae} f_e$, and

(ii) $\overset{*}{M}$ *has harmonic conformal curvature if and only if its curvature tensor and the function h satisfy*

$$(\overset{*}{14})\qquad h_\varepsilon \overset{*}{R}{}^{\varepsilon}{}_{\alpha\beta\gamma} + \frac{1}{h}(h_\gamma h_{\alpha\beta} - h_\beta h_{\alpha\gamma}) = \frac{1}{n-r-1}\Big\{ h_\varepsilon(\overset{*}{R}{}^{\varepsilon}{}_{\gamma}\overset{*}{g}_{\alpha\beta} - \overset{*}{R}{}^{\varepsilon}{}_{\beta}\overset{*}{g}_{\alpha\gamma}) + \frac{1}{h}[(\overset{*}{h}{}^{\varepsilon}_{.\varepsilon})h_\gamma - \overset{*}{h}{}^{\varepsilon}h_{\varepsilon\gamma}]\overset{*}{g}_{\alpha\beta} - \frac{1}{h}[(\overset{*}{h}{}^{\varepsilon}_{.\varepsilon})h_\beta - \overset{*}{h}{}^{\varepsilon}h_{\varepsilon\beta}]\overset{*}{g}_{\alpha\gamma}\Big\},$$

where $\overset{*}{h}{}^{\alpha} = \overset{*}{g}{}^{\alpha\varepsilon}h_\varepsilon$.

P r o o f. To prove (i), by calculating $\overset{*}{R}$ from $(\overline{13})$ and substituting the result to (10) one obtains

$$(\overline{15})\qquad R_{,c} = \frac{n-1}{(r-1)h^2}\overline{R}_{.c} + \frac{2(n-1)(n-r)}{(r-1)h^2 f} f_e \overline{R}{}^{e}{}_{c} - \frac{(n-1)(n-r)}{(r-1)f^2}\partial_c(\Delta_1 f) - \frac{2(n-1)(n-2r)}{h^2 f}(\Delta_1 h)f_c + \frac{4(n-1)}{hf}(h^{\varepsilon}_{.\varepsilon})f_c + \frac{2(n-1)(n-r)}{(r-1)f^2}(f^{e}_{.e})f_c,$$

which, in view of $(\overline{12})$, yields

$$\overline{R}_{ab.c} - \overline{R}_{ac.b} + \frac{n-r}{f}\Big[f_e\overline{R}{}^{e}{}_{abc} + \frac{1}{f}(f_c f_{ab} - f_b f_{ac})\Big] = \frac{1}{2(r-1)}(\overline{R}_{.c}\overline{g}_{ab} - \overline{R}_{.b}\overline{g}_{ac}) + \frac{n-r}{f}\cdot\frac{1}{r-1}\Big\{f_e(\overline{R}{}^{e}{}_{c}\overline{g}_{ab} - \overline{R}{}^{e}{}_{b}\overline{g}_{ac}) + \frac{1}{f}[(\overline{f}{}^{e}_{.e})f_c - \overline{f}{}^{e}f_{ec}]\overline{g}_{ab} - \frac{1}{f}[(\overline{f}{}^{e}_{.e})f_b - \overline{f}{}^{e}f_{eb}]\overline{g}_{ac}\Big\}.$$

Therefore

$$\overline{R}_{ab.c} - \overline{R}_{ac.b} = \frac{1}{2(r-1)}(\overline{R}_{.c}\overline{g}_{ab} - \overline{R}_{.b}\overline{g}_{ac})$$

is equivalent to $(\overline{14})$. Next, by the same argument as above, we conclude (ii) from the star formulas. This completes the proof.

R e m a r k 3. It is easy to verify that if the Hessian of f (resp. h) is proportional to the metric tensor $\overline{g}$ (resp. $\overset{*}{g}$), then the condition $(\overline{14})$ (resp. $(\overset{*}{14})$) is satisfied.

R e m a r k 4. By making use of Lemma 4 it is not difficult to show that for a doubly warped product with harmonic Weyl conformal curvature tensor the conditions ($\overline{14}$) and ($\overset{*}{14}$) are equivalent to

$$(\overline{16})\qquad f_e\overline{R}^e{}_{abc} + \frac{1}{f}(f_cf_{ab} - f_bf_{ac}) = \frac{\overline{R}}{r(r-1)}(f_c\overline{g}_{ab} - f_b\overline{g}_{ac}) + \frac{1}{f}\left[\frac{2}{r}(\overline{f}^e_{.e})f_c - \overline{f}^e f_{ec}\right]\overline{g}_{ab} - \frac{1}{f}\left[\frac{2}{r}(\overline{f}^e_{.e})f_b - \overline{f}^e f_{eb}\right]\overline{g}_{ac}$$

and

$$(\overset{*}{16})\qquad h_\varepsilon\overset{*}{R}{}^\varepsilon{}_{\alpha\beta\gamma} + \frac{1}{h}(h_\gamma h_{\alpha\beta} - h_\beta h_{\alpha\gamma}) = \frac{\overset{*}{R}}{(n-r)(n-r-1)}(h_\gamma\overset{*}{g}_{\alpha\beta} - h_\beta\overset{*}{g}_{\alpha\gamma}) + \frac{1}{h}\left[\frac{2}{n-r}(\overset{*}{h}{}^\varepsilon_{.\varepsilon})h_\gamma - \overset{*}{h}{}^\varepsilon h_{\varepsilon\gamma}\right]\overset{*}{g}_{\alpha\beta} - \frac{1}{h}\left[\frac{2}{n-r}(\overset{*}{h}{}^\varepsilon_{.\varepsilon})h_\beta - \overset{*}{h}{}^\varepsilon h_{\varepsilon\beta}\right]\overset{*}{g}_{\alpha\gamma},$$

respectively.

THEOREM 3. *A doubly warped product* $\overline{M}\ {}_h\!\times_f \overset{*}{M}$ *has harmonic Weyl conformal curvature tensor if and only if the conditions* ($\overline{12}$), ($\overline{11}$), ($\overset{*}{12}$) *and* ($\overset{*}{11}$) *are satisfied.*

P r o o f. The necessity follows from the harmonicity of the Weyl tensor and Lemma 4. To prove the sufficiency it suffices to check the condition (2) in the cases when i, j, k are

(a) a, b, c,
(b) α, β, γ,
(c) a, b, γ,
(d) α, β, c.

From ($\overline{12}$), ($\overset{*}{12}$), (9) and the Ricci formula, the cases (a) and (b) are evident. To verify the case (c), besides ($\overline{11}$) we need the condition ($\overline{13}$), which follows (as shown in the proof of Lemma 4) from ($\overline{12}$). Therefore, relations ($\overline{11}$), ($\overline{13}$), (9) and Lemmas 1–3 give (c). Analogously, we check the case (d). This completes the proof.

Differentiating ($\overline{11}$) covariantly in $\overline{M}$ and taking into account the Ricci formula, we get

$$(\overline{17})\qquad \overline{R}_{ab.c} - \overline{R}_{ac.b} - \frac{r-2}{f}f_e\overline{R}^e{}_{abc} - \frac{r-2}{f^2}(f_cf_{ab} - f_bf_{ac})$$

$$= \frac{1}{r}(\overline{R}_{.c}\overline{g}_{ab} - \overline{R}_{.b}\overline{g}_{ac}) - \frac{r-2}{rf^2}(\bar{f}^e_{.e})(f_c\overline{g}_{ab} - f_b\overline{g}_{ac})$$
$$+ \frac{r-2}{rf}[\overline{g}_{ab}\partial_c(\bar{f}^e_{.e}) - \overline{g}_{ac}\partial_b(\bar{f}^e_{.e})].$$

It is worth remarking that we may use $(\overline{17})$ and the analogous formula $(\overset{*}{17})$ to obtain some relations equivalent to $(\overline{12})$ and $(\overset{*}{12})$.

Lemma 5. *Suppose that $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ has harmonic Weyl conformal curvature tensor. Then*

(i) *for* $r = \dim \overline{M} > 2$,

$$(\overline{18}) \qquad \frac{1}{2}\overline{R}_{.c} + \frac{\overline{R}}{f}f_c = \frac{r(r-1)}{f^2}\left[\bar{f}^e f_{ec} - \frac{1}{r}(\bar{f}^e_{.e})f_c\right] - \frac{r-1}{f}\partial_c(\bar{f}^e_{.e}),$$

(ii) *for* $n - r = \dim \overset{*}{M} > 2$,

$$(\overset{*}{18}) \qquad \frac{1}{2}\overset{*}{R}_{;\gamma} + \frac{\overset{*}{R}}{h}h_\gamma = \frac{(n-r)(n-r-1)}{h^2}\left[\overset{*}{h}{}^\varepsilon h_{\varepsilon\gamma} - \frac{1}{n-r}(\overset{*}{h}{}^\varepsilon_{.\varepsilon})h_\gamma\right] - \frac{n-r-1}{h}\partial_\gamma(\overset{*}{h}{}^\varepsilon_{.\varepsilon}).$$

P r o o f. Contracting $(\overline{17})$ with $\overline{g}^{ab}$, we get, for $r > 2$,

$$(\overline{19}) \qquad \frac{1}{2r}\overline{R}_{.c} = -\frac{1}{f}f_e\overline{R}^e{}_c + \frac{1}{f^2}\bar{f}^e f_{ec} - \frac{1}{rf^2}(\bar{f}^e_{.e})f_c - \frac{r-1}{rf}\partial_c(\bar{f}^e_{.e}).$$

On the other hand, transvecting $(\overline{11})$ with $\bar{f}^a$, we obtain easily

$$(\overline{20}) \qquad f_e\overline{R}^e{}_c = \frac{\overline{R}}{r}f_c - \frac{r-2}{f}\bar{f}^e f_{ec} + \frac{r-2}{rf}(\bar{f}^e_{.e})f_c.$$

Together with $(\overline{19})$, this yields (i). The assertion (ii) follows from the analogous argument via the "star way" instead of the "dash way". This completes the proof.

Corollary 1. *Suppose that $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ has harmonic Weyl conformal curvature tensor. If the Hessian of f (resp. h) is proportional to the metric tensor $\overline{g}$ (resp. $\overset{*}{g}$) then*

(i) *for* $r = \dim \overline{M} > 2$,

$$\overline{R}f_c + (r-1)\partial_c(\bar{f}^e_{.e}) = 0,$$

(ii) *for* $n - r = \dim \overset{*}{M} > 2$,

$$\overset{*}{R}h_\gamma + (n-r-1)\partial_\gamma(\overset{*}{h}{}^\varepsilon_{.\varepsilon}) = 0.$$

LEMMA 6. *Suppose that* $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ *has harmonic Weyl conformal curvature tensor. Then for* $r > 2$,

$$\begin{aligned}\frac{n-2}{r-2}(\overline{R}_{ab.c}-\overline{R}_{ac.b}) &= \frac{2nr-r^2-2n}{2r(r-1)(r-2)}(\overline{R}_{.c}\overline{g}_{ab}-\overline{R}_{.b}\overline{g}_{ac})\\ &\quad+\frac{(n-r)\overline{R}}{r(r-1)f}(f_c\overline{g}_{ab}-f_b\overline{g}_{ac})+\frac{n-r}{rf^2}(\overline{f}^{e}_{.e})(f_c\overline{g}_{ab}-f_b\overline{g}_{ac})\\ &\quad-\frac{n-r}{f^2}(\overline{f}^e f_{ec}\overline{g}_{ab}-\overline{f}^e f_{eb}\overline{g}_{ac})+\frac{n-r}{rf}(\overline{g}_{ab}\partial_c(\overline{f}^{e}_{.e})-\overline{g}_{ac}\partial_b(\overline{f}^{e}_{.e}))\end{aligned}$$

on $\overline{M}$, *and the corresponding relation holds on* $\overset{*}{M}$.

Proof (for the dash formula). Calculating $f_e\overline{R}^e{}_{abc}+\frac{1}{f}(f_cf_{ab}-f_bf_{ac})$ from $(\overline{17})$, and substituting the result to $(\overline{12})$, together with $(\overline{15})$ and $(\overline{20})$, we get our assertion.

Since on a two-dimensional manifold the condition (2) is identically satisfied, we conclude from Lemmas 5 and 6 the following

THEOREM 4. *If a doubly warped product* $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ *has harmonic conformal curvature, then so do the factor manifolds* $\overline{M}$ *and* $\overset{*}{M}$.

4. Doubly warped products with harmonic Weyl tensor and a 1-dimensional factor. Let $\dim\overline{M}=1$. In a suitable product chart $t=x^1,x^2,\ldots,x^n$ for $\overline{M}\times\overset{*}{M}$, putting $g=\overline{g}\,{}_h\!\times_f\overset{*}{g}$, we have

$$\begin{gathered}g_{11}=h^2,\quad g_{1\alpha}=0,\quad g_{\alpha\beta}=f^2\overset{*}{g}_{\alpha\beta},\\ \Gamma^1_{11}=\overline{\Gamma}^1_{11}=0,\quad \Gamma^1_{\alpha 1}=\frac{1}{h}h_\alpha,\quad \Gamma^\alpha_{\beta 1}=\frac{1}{f}f'\delta^\alpha_\beta,\\ \Gamma^\alpha_{11}=-hh^\alpha,\quad \Gamma^1_{\alpha\beta}=-\frac{1}{h^2f}f'g_{\alpha\beta},\quad \Gamma^\alpha_{\beta\gamma}=\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}.\end{gathered}$$

Here and in the sequel α, β, γ run through $\{2,\ldots,n\}$, the prime stands for d/dt, while the Γ's (resp. $\overset{*}{\Gamma}$'s) are Christoffel symbols of g (resp. of $\overset{*}{g}$ with respect to the chart $x^2,\ldots,x^n$ of $\overset{*}{M}$).

Furthermore,

$$\begin{aligned}&R_{11}=-\frac{n-1}{f}f''-h(h^{\varepsilon}_{.\varepsilon}),\qquad R_{\alpha 1}=\frac{n-2}{hf}f'h_\alpha,\\ (21)\qquad &R_{\alpha\beta}=\overset{*}{R}_{\alpha\beta}-\frac{1}{h}h_{\alpha\beta}-\frac{1}{fh^2}\Big[f''+(n-2)\frac{(f')^2}{f}\Big]g_{\alpha\beta},\\ &R=\frac{\overset{*}{R}}{f^2}-\frac{2}{h}(h^{\varepsilon}_{.\varepsilon})-\frac{n-1}{fh^2}\Big[2f''+(n-2)\frac{(f')^2}{f}\Big].\end{aligned}$$

Hence in this case

$$
\begin{aligned}
R_{11,\gamma} = {} & \frac{2(n-1)}{hf} f'' h_\gamma + (h^{\varepsilon}_{.\varepsilon}) h_\gamma - \frac{2(n-2)}{hf^2}(f')^2 h_\gamma \\
& - h\partial_\gamma (h^{\varepsilon}_{.\varepsilon}), \\
R_{1\gamma,1} = {} & \frac{2(n-2)}{hf} f'' h_\gamma + (h^{\varepsilon}_{.\varepsilon}) h_\gamma - \frac{2(n-2)}{hf^2}(f')^2 h_\gamma \\
& + \frac{h}{f^2} h_\varepsilon \overset{*}{R}{}^{\varepsilon}{}_{\gamma} - \frac{1}{2}\partial_\gamma(\Delta_1 h) - \frac{n-2}{hf^2}(f')^2 h_\gamma, \\
R_{\alpha\beta,1} = {} & -\frac{2}{f} f' \overset{*}{R}_{\alpha\beta} + \frac{2}{hf} f' h_{\alpha\beta} + \frac{2(n-2)}{h^2 f^3}(f')^3 g_{\alpha\beta} \\
& - \frac{2(n-2)}{h^2 f} f' h_\alpha h_\beta - \frac{2n-5}{h^2 f^2} f'' f' g_{\alpha\beta} - \frac{1}{h^2 f} f''' g_{\alpha\beta}, \\
R_{\alpha 1,\beta} = {} & -\frac{1}{f} f' \overset{*}{R}_{\alpha\beta} + \frac{n-1}{hf} f' h_{\alpha\beta} + \frac{n-2}{h^2 f^3}(f')^3 g_{\alpha\beta} \\
& - \frac{1}{hf}(h^{\varepsilon}_{.\varepsilon}) f' g_{\alpha\beta} - \frac{2(n-2)}{h^2 f} f' h_\alpha h_\beta - \frac{n-2}{h^2 f^2} f'' f' g_{\alpha\beta}, \\
R_{,1} = {} & -\frac{2}{f^3}\overset{*}{R} f' + \frac{2(n-1)(n-2)}{h^2 f^3}(f')^3 \\
& - \frac{2(n-1)(n-3)}{h^2 f^2} f'' f' + \frac{4}{hf}(h^{\varepsilon}_{.\varepsilon}) f' - \frac{2(n-1)}{h^2 f} f''', \\
R_{,\gamma} = {} & \frac{1}{f^2}\partial_\gamma \overset{*}{R} + \frac{2(n-1)(n-2)}{h^3 f^2}(f')^2 h_\gamma + \frac{2}{h^2}(h^{\varepsilon}_{.\varepsilon}) h_\gamma \\
& - \frac{2}{h}\partial_\gamma (h^{\varepsilon}_{.\varepsilon}) + \frac{4(n-1)}{h^3 f} f'' h_\gamma.
\end{aligned}
\tag{22}
$$

LEMMA 7. *If* $\overline{M}\ {}_h\!\times_f \overset{*}{M}$ *has harmonic conformal curvature and* $r = \dim \overline{M} = 1$, $n-1 = \dim \overset{*}{M} \geq 2$, *then*

$$
\overset{*}{R}_{\alpha\beta} + \frac{n-3}{h} h_{\alpha\beta} - \frac{\overset{*}{R}}{n-1}\overset{*}{g}_{\alpha\beta} - \frac{n-3}{(n-1)h}(\overset{*}{h}{}^{\varepsilon}_{.\varepsilon})\overset{*}{g}_{\alpha\beta} = 0 \tag{23}
$$

on $\overset{*}{M}$. *Moreover, for* $n > 3$,

$$
\begin{aligned}
\frac{1}{2}\overset{*}{R}_{;\gamma} + \frac{\overset{*}{R}}{h} h_\gamma = {} & \frac{(n-1)(n-2)}{h^2}\left[\overset{*}{h}{}^{\varepsilon} h_{\varepsilon\gamma} - \frac{1}{n-1}(\overset{*}{h}{}^{\varepsilon}_{.\varepsilon}) h_\gamma\right] \\
& - \frac{n-2}{h}\partial_\gamma(\overset{*}{h}{}^{\varepsilon}_{.\varepsilon}).
\end{aligned}
\tag{24}
$$

Proof. Taking $i = j = 1$ and $k = \gamma$ in (2), we get

$$
R_{11,\gamma} - R_{1\gamma,1} = \frac{1}{2(n-1)} R_{,\gamma} g_{11},
$$

which, in view of (22), yields

$$\frac{1}{f^2} h_\varepsilon \overset{*}{R}{}^\varepsilon{}_\gamma = -\frac{1}{2(n-1)}\frac{h}{f^2}\partial_\gamma \overset{*}{R} + \frac{1}{2h}\partial_\gamma(\Delta_1 h) - \frac{n-2}{n-1}\partial_\gamma(h^\varepsilon_{.\varepsilon}) - \frac{1}{(n-1)h}(h^\varepsilon_{.\varepsilon})h_\gamma. \tag{25}$$

On the other hand, taking $i=\alpha$, $j=\beta$ and $k=1$ in (2), we have

$$R_{\alpha\beta,1} - R_{\alpha 1,\beta} = \frac{1}{2(n-1)} R_{,1} g_{\alpha\beta},$$

which, in view of (22), yields

$$\frac{f'}{f}\left[\overset{*}{R}_{\alpha\beta} + \frac{n-3}{h}h_{\alpha\beta} - \frac{\overset{*}{R}}{n-1}\overset{*}{g}_{\alpha\beta} - \frac{n-3}{(n-1)h}(\overset{*}{h}{}^\varepsilon_{.\varepsilon})\overset{*}{g}_{\alpha\beta}\right] = 0.$$

Together with the non-constancy of f, this implies (23). Now, from (23), (25) and Lemma 2, we get (24) by the same argument as in the proof of Lemma 5. This completes the proof.

By proceeding as in Section 3, in the case $\dim \overline{M} = 1$, from Lemma 7 one can easily obtain the following theorems.

THEOREM 5. *Let* $\dim \overline{M} = 1$ *and* $n \geq 3$. *A doubly warped product* $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ *has harmonic conformal curvature if and only if the conditions* (23) *and* $(\overset{*}{12})$ *are satisfied.*

THEOREM 6. *Let* $\dim \overline{M} = 1$ *and* $n \geq 3$. *If a doubly warped product* $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ *has harmonic conformal curvature, then so does* $\overset{*}{M}$.

5. Examples of doubly warped products with harmonic Weyl tensor. In this section we construct our examples using some ideas from the paper [5] (see Examples 1 and 3).

EXAMPLE 1. Assume that $\overline{M}$ ($\dim \overline{M} \geq 2$) is an open subset of $\mathbb{R}^r - \{(0,\ldots,0)\}$, $\overline{g}_{ab} = \delta_{ab}$, $a,b \in \{1,\ldots,r\}$,

$$f = f(x^1,\ldots,x^r) = \frac{B}{2}\Big(\sum_{a=1}^{r}(x^a)^2\Big),$$

and $\overset{*}{M}$ ($\dim \overset{*}{M} \geq 2$) is an open subset of $\mathbb{R}^{n-r} - \{(0,\ldots,0)\}$, $\overset{*}{g}_{\alpha\beta} = \delta_{\alpha\beta}$, $\alpha,\beta \in \{r+1,\ldots,n\}$, and

$$h = h(x^{r+1},\ldots,x^n) = \frac{A}{2}\Big(\sum_{\alpha=r+1}^{n}(x^\alpha)^2\Big),$$

with positive constants A and B. It is then easy to verify that

$$f_a = Bx^a, \qquad h_\alpha = Ax^\alpha, \quad (\varDelta_1 f) = \frac{2Bf}{h^2}, \quad (\varDelta_1 h) = \frac{2Ah}{f^2},$$

$$f_{ab} = B\overline{g}_{ab}, \quad h_{\alpha\beta} = A\overset{*}{g}_{\alpha\beta}, \quad (\bar{f}^{e}_{.e}) = rB, \qquad (\overset{*}{h}{}^{\varepsilon}_{.\varepsilon}) = (n-r)A$$

and

$$R = -\frac{2(n-1)r}{hf^2}A - \frac{2(n-1)(n-r)}{h^2 f}B \neq \text{const.}$$

Therefore, by Theorem 3, we conclude that the doubly warped product $\overline{M}\,{}_h\!\times_f \overset{*}{M}$ has harmonic Weyl conformal curvature tensor but it does not have harmonic curvature (since $R \neq$ const).

Example 2. Let $\overline{M} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with the metric tensor $\overline{g}$ defined by

$$\overline{g}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) = uv, \quad \overline{g}\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big) = \frac{1}{v}, \quad \overline{g}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = 0,$$

where $u = u(x)$, $v = v(y)$ are smooth functions not vanishing at any point on $\overline{M}$. Next let $f = f(y) = y$, and $\overset{*}{M} = \{(z,t) \in \mathbb{R}^2 : t > 0\}$ with $\overset{*}{g}$ defined by

$$\overset{*}{g}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) = pq, \quad \overset{*}{g}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = \frac{1}{q}, \quad \overset{*}{g}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}\Big) = 0,$$

where $p = p(z)$, $q = q(t)$ are smooth functions not vanishing at any point on $\overset{*}{M}$, and $h = h(t) = t$. Then the only non-zero components of the Christoffel symbols $\overline{\varGamma}^a_{bc}$ and curvature tensor $\overline{R}_{abcd}$ are

$$\overline{\varGamma}^1_{11} = \frac{1}{2u}\partial_x u, \quad \overline{\varGamma}^1_{12} = \frac{1}{2v}\partial_y v, \quad \overline{\varGamma}^2_{11} = -\frac{1}{2}uv\partial_y v,$$

$$\overline{\varGamma}^2_{22} = -\frac{1}{2v}\partial_y v, \qquad \overline{R}_{1212} = \frac{1}{2}u\partial_{yy}v.$$

Moreover,

$$f_1 = 0, \quad f_2 = 1, \quad (\varDelta_1 f) = \frac{v}{t^2}, \quad \overline{R} = -\partial_{yy}v,$$

$$f_{ab} = \frac{1}{2}(\partial_y v)\overline{g}_{ab}, \qquad (\bar{f}^{e}_{.e}) = \partial_y v$$

and we have analogous formulae on $\overset{*}{M}$. The conditions ($\overline{12}$) and ($\overset{*}{12}$) take the form

$$y^3\partial_{yyy}v + y^2\partial_{yy}v - 2y\partial_y v + 2v = -t^2\partial_{tt}q + 2t\partial_t q - 2q$$

and

$$t^3\partial_{ttt}q + t^2\partial_{tt}q - 2t\partial_t q + 2q = -y^2\partial_{yy}v + 2y\partial_y v - 2v,$$

respectively. Therefore, if we take the solutions $v(y) = c_1 y + c_2 y^2 - \varphi$, $q(t) = \gamma_1 t + \gamma_2 t^2 + \varphi$, where $c_1, c_2, \gamma_1, \gamma_2, \varphi \in \mathbb{R}$, then, by Theorem 3, the doubly warped product $\overline{M}\, {}_h\!\times_f \overset{*}{M}$ has harmonic Weyl conformal curvature. It is easily checked that

$$R = -12\left(\frac{c_2}{y^2} + \frac{\gamma_2}{t^2} + \frac{1}{2}\frac{c_1}{yt^2} + \frac{1}{2}\frac{\gamma_1}{y^2 t}\right) \neq \text{const},$$

so $\overline{M}\, {}_h\!\times_f \overset{*}{M}$ does not have harmonic curvature.

EXAMPLE 3. Let $\overline{M}$ ($\dim \overline{M} = 2$) and f be as in Example 2, and $\overset{*}{M}$ ($\dim \overset{*}{M} = n-2 \geq 2$) and h be as in Example 1. Then ($\overline{12}$) and ($\overset{*}{12}$) take the form

$$\partial_{yyy} v + \frac{n-3}{y}\partial_{yy} v - \frac{2(n-3)}{y^2}\partial_y v + \frac{2(n-3)}{y^3} v = 0$$

and

$$\frac{h_\gamma}{h^3}\left(\partial_{yy} v - \frac{2}{y}\partial_y v + \frac{2}{y^2} v\right) = 0,$$

respectively. Taking a solution $v(y) = c_1 y + c_2 y^2$, $c_1, c_2 \in \mathbb{R}$, we get a doubly warped product $\overline{M}\, {}_h\!\times_f \overset{*}{M}$ which has harmonic Weyl conformal curvature.

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