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DOUGALL'S BILATERAL ${}_2H_2$-SERIES AND RAMANUJAN-LIKE π-FORMULAE

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ABSTRACT. The modified Abel lemma on summation by parts is employed to investigate the partial sum of Dougall's bilateral ${}_2H_2$-series. Several unusual transformations into fast convergent series are established. They lead surprisingly to numerous infinite series expressions for π, including several formulae discovered by Ramanujan (1914) and recently by Guillera (2008).

Roughly speaking, hypergeometric series is defined to be a series $\sum C_n$ with term ratio C_{1+n}/C_n a rational function of n. In general, it can be explicitly written as

$$ {}_pF_q\left[\begin{matrix} a_1,\ a_2,\cdots,\ a_p \\ b_1,\ b_2,\cdots,\ b_q \end{matrix}\Big|\ z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!} $$

where the rising shifted factorial is given by

$$ (x)_0\equiv 1 \quad\text{and}\quad (x)_n=\Gamma(x+n)/\Gamma(x)=x(x+1)\cdots(x+n-1) \quad\text{for}\quad n=1,2,\cdots. $$

The Γ-function is defined by the Euler integral

$$ \Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u}\mathrm{d}u \quad\text{with}\quad \Re(x)>0 $$

which admits the well–known reciprocal formulae

$$ \Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x} \quad\text{and}\quad \Gamma(\tfrac12+x)\Gamma(\tfrac12-x)=\frac{\pi}{\cos\pi x}. $$

By means of Cauchy's residue theorem, Dougall [21, 1907] discovered the following bilateral series identity

$$ {}_2H_2\left[\begin{matrix} a,\ b \\ c,\ d \end{matrix}\Big|\ 1\right] = \sum_{k=-\infty}^{+\infty}\frac{(a)_k(b)_k}{(c)_k(d)_k} = \Gamma\left[\begin{matrix} 1-a,1-b,c,d,c+d-a-b-1 \\ c-a,\ d-a,\ c-b,\ d-b \end{matrix}\right] $$

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where $\Re(c+d-a-b) > 1$ for convergence. Different proofs can be found in Chu [19] and Slater [33, §6.1]. When $d = 1$, the last formula reduces to the well–known Gauss summation theorem (cf. Bailey [5, §1.3] and Slater [33, §1.7])

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{where} \quad \Re(c-a-b) > 0. $$

Throughout this paper, we shall adopt the notations of Slater [33] to denote the product and quotient forms of the shifted factorial and the Γ-function as follows:

$$ [A,\, B,\, \cdots,\, C]_n = (A)_n (B)_n \cdots (C)_n, \quad \Gamma[A,\, B,\, \cdots,\, C] = \Gamma(A)\Gamma(B)\cdots\Gamma(C); $$
$$ \begin{bmatrix} \alpha,\, \beta,\, \cdots,\, \gamma \\ A,\, B,\, \cdots,\, C \end{bmatrix}_n = \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}, \quad \Gamma\begin{bmatrix} \alpha,\, \beta,\, \cdots,\, \gamma \\ A,\, B,\, \cdots,\, C \end{bmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}. $$

By utilizing the modified Abel lemma on summation by parts, the author [18, 19] has recently reviewed several identities of classical hypergeometric series. The purpose of the present work is to explore this approach further to investigate acceleration of the partial sum of Dougall's ${}_2H_2$-series into fast convergent series. Five fundamental recurrence relations will be established. They will then be employed to establish several unusual transformation formulae which express the partial sum of Dougall's ${}_2H_2$-series in terms of fast convergent ones. Surprisingly, these transformations will further lead to numerous infinite series expressions for π, including three typical known ones due to Ramanujan [31, 1914]:

$$ \frac{4}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 1,\, 1,\, 1 \end{bmatrix}_k \frac{1+6k}{4^k}, \tag{1} $$

$$ \frac{8}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2},\, \frac{1}{4},\, \frac{3}{4} \\ 1,\, 1,\, 1 \end{bmatrix}_k \frac{3+20k}{(-4)^k}, \tag{2} $$

$$ \frac{16}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 1,\, 1,\, 1 \end{bmatrix}_k \frac{5+42k}{64^k}. \tag{3} $$

They are among the 17 formulae recorded by Ramanujan [31, 1914] (see Berndt [10, Chapter 29] also). However, the first rigorous mathematical proofs of Ramanujan's identities and generalizations of them were given by the Borwein brothers [11, §5] and the Chudnovsky brothers [20]. For the historical notes and introductive information on the formulae for π-series, there are three excellent survey papers by Bailey–Borwein [3], Baruah–Berndt–Chan [8] and Guillera [28]. Different proofs of Ramanujan's identities and further formulae for $1/\pi$ and $1/\pi^2$ can be found in Andrews–Berndt [2, §15.6], Baruah–Berndt [6, 7], Borweins [12, 13], Chan *et al.* [15]–[17], Guillera [24]–[29] Rogers [32] and Zudilin [35].

Two more spectacular examples from the iteration pattern **[5,5,10,10]** read as

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\left\{\frac{(10n)!}{(5n)!(4+5n)!}\right\}^3\frac{5}{2^{40+60n}} \quad \text{and} \quad \frac{\pi}{\sqrt{3}}=3^7 5^2\sum_{n=0}^{\infty}\frac{(15n)!(15+15n)!}{(30+30n)!}$$

$$\times\left\{\begin{array}{r} 967639691668545\\ +70481743821538938n\\ +1679601725341815240n^2\\ +20983953947697829200n^3\\ +162432411074663046000n^4\\ +844125894212126940000n^5\\ +3073307854429525600000n^6\\ +8009128408111512000000n^7\\ +15034125109741980000000n^8\\ +20170957811328600000000n^9\\ +188707547862600000000000n^{10}\\ +116921205291600000000000n^{11}\\ +43108745786000000000000n^{12}\\ +7158278820000000000000n^{13} \end{array}\right\} \times \left\{\begin{array}{r} 6729160970477568\\ +275897240208641792n\\ +4937369471226401216n^2\\ +51577093209056406400n^3\\ +353721530774616375400n^4\\ +1693516148807887876500n^5\\ +5861504657068749183750n^6\\ +14949937005553937671875n^7\\ +28315966186734060515625n^8\\ +39730036847597675859375n^9\\ +407347795569351128906 25n^{10}\\ +29647774656743291015625n^{11}\\ +14503229510183701171875n^{12}\\ +4274336559486181640625n^{13}\\ +573308927466064453125n^{14} \end{array}\right\}.$$

The rest of the paper will be organized as follows. Applying the modified Abel lemma on summation by parts, five recurrence relations for the partial sum of Dougall's ${}_2H_2$-series will be established in the next section. They will be employed in the second section to prove 60 infinite series identities for π and $1/\pi$, including several known formulae discovered by Ramanujan [31] and Guillera [29]. Finally, the more general iteration pattern will be examined in the third section, where, further, 40 Ramanujan–like formulae will be displayed. In order to assure the accuracy, all the formulae are checked by an appropriately devised *Mathematica* package.

1. Five recurrence relations for the Ω-sum

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla\tau_k=\tau_k-\tau_{k-1} \quad \text{and} \quad \Delta\tau_k=\tau_k-\tau_{k+1}$$

where Δ differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts may be reformulated as

$$\sum_{k=0}^{\infty}B_k\nabla A_k=[AB]_+-A_{-1}B_0+\sum_{k=0}^{\infty}A_k\Delta B_k \tag{4}$$

provided that the limit $[AB]_+:=\lim\limits_{n\to\infty}A_nB_{n+1}$ exists and one of both series just displayed is convergent.

Proof. According to the definition of the backward difference, we have

$$\sum_{k=0}^{m}B_k\nabla A_k=\sum_{k=0}^{m}B_k\{A_k-A_{k-1}\}=\sum_{k=0}^{m}A_kB_k-\sum_{k=0}^{m}A_{k-1}B_k.$$

Replacing k by $k+1$ for the last sum, we get the following expression:

$$\begin{aligned}\sum_{k=0}^{m} B_k \nabla A_k &= A_m B_{m+1} - A_{-1}B_0 + \sum_{k=0}^{m} A_k\{B_k - B_{k+1}\}\\ &= A_m B_{m+1} - A_{-1}B_0 + \sum_{k=0}^{m} A_k \Delta B_k.\end{aligned}$$

Letting $m \to \infty$ gives rise to the formula displayed in equation (4). □

According to Dougall's ${}_2H_2$-series, define its partial Ω-sum by

$$\Omega(a,b,c,d) := \sum_{k=0}^{\infty} \begin{bmatrix} a,\ b \\ c,\ d \end{bmatrix}_k \quad \text{where} \quad \Re(c+d-a-b) > 1.$$

For the four nonnegative integers p,q,r,s, a linear relation expressing $\Omega(a,b,c,d)$ in terms of $\Omega(a+p,b+q,c+r,d+s)$ will be called *iteration pattern* [**pqrs**]. Now we are ready to apply the modified Abel lemma on summation by parts to derive five recurrence relations for $\Omega(a,b,c,d)$, that will be employed to demonstrate infinite series identities for π and $1/\pi$ in the next section.

§1.1. **[1000].** For the two sequences A_k and B_k given respectively by

$$A_k = \begin{bmatrix} 1+a, c+d-a-1 \\ c,\ d \end{bmatrix}_k \quad \text{and} \quad B_k = \frac{(b)_k}{(c+d-a-2)_k},$$

it is almost trivial to compute the differences

$$\begin{aligned}\nabla A_k &= \begin{bmatrix} a, c+d-a-2 \\ c,\ d \end{bmatrix}_k \frac{(1+a-c)(1+a-d)}{a(2+a-c-d)},\\ \Delta B_k &= \frac{(b)_k}{(c+d-a-1)_k}\frac{2+a+b-c-d}{2+a-c-d},\end{aligned}$$

as well as two limiting relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1}B_0 = \frac{(c-1)(d-1)}{a(c+d-a-2)} \quad \text{with} \quad \Re(c+d-a-b) > 2.$$

By means of the modified Abel lemma on summation by parts, we can manipulate the Ω-series as follows:

$$\begin{aligned}\Omega(a,b,c,d) &= \frac{a(2+a-c-d)}{(1+a-c)(1+a-d)}\sum_k B_k \nabla A_k\\ &= \frac{a(2+a-c-d)}{(1+a-c)(1+a-d)}\Big\{[AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k\Big\}\\ &= \frac{(c-1)(d-1)}{(1+a-c)(1+a-d)} + \frac{a(2+a+b-c-d)}{(1+a-c)(1+a-d)}\sum_{k=0}^{\infty}\begin{bmatrix} 1+a,\ b \\ c,\ \ d \end{bmatrix}_k.\end{aligned}$$

This may be restated as the recurrence relation.

Lemma 1.1 (Recurrence relation **[1000]**)**.**

$$\Omega(a,b,c,d) = \Omega(1+a,b,c,d) \times \frac{a(2+a+b-c-d)}{(1+a-c)(1+a-d)} + \frac{(c-1)(d-1)}{(1+a-c)(1+a-d)}.$$

§1.2. **[0001].** Let A_k and B_k be the two sequences defined respectively by

$$A_k=\frac{(1+a+b-d)_k}{(c)_k}\quad\text{and}\quad B_k=\begin{bmatrix}a,\ b\\ a+b-d,d\end{bmatrix}_k.$$

We have no difficulty showing the differences

$$\begin{aligned}\nabla A_k&=\frac{(a+b-d)_k}{(c)_k}\frac{1+a+b-c-d}{a+b-d},\\ \Delta B_k&=\begin{bmatrix}a,\ b\\ 1+a+b-d,1+d\end{bmatrix}_k\frac{(d-a)(d-b)}{d(d-a-b)},\end{aligned}$$

as well as the two limiting relations

$$[AB]_+=0\quad\text{and}\quad A_{-1}B_0=\frac{c-1}{a+b-d}\quad\text{with}\quad\Re(c+d-a-b)>1.$$

According to the modified Abel lemma on summation by parts, the Ω-series can be reformulated as

$$\begin{aligned}\Omega(a,b,c,d)&=\frac{a+b-d}{1+a+b-c-d}\sum_k B_k\nabla A_k\\ &=\frac{a+b-d}{1+a+b-c-d}\Big\{[AB]_+-A_{-1}B_0+\sum_k A_k\Delta B_k\Big\}\\ &=\frac{c-1}{c+d-a-b-1}+\frac{(d-a)(d-b)}{d(c+d-a-b-1)}\sum_{k=0}^{\infty}\begin{bmatrix}a,\ \ b\\ c,\ 1+d\end{bmatrix}_k\end{aligned}$$

which yields the following recurrence relation.

Lemma 1.2 (Recurrence relation **[0001]**)**.**

$$\Omega(a,b,c,d)=\Omega(a,b,c,1+d)\times\frac{(d-a)(d-b)}{d(c+d-a-b-1)}+\frac{c-1}{c+d-a-b-1}.$$

§1.3. **[1001].** For the two sequences A_k and B_k given respectively by

$$A_k=\frac{(1+a)_k}{(c)_k}\quad\text{and}\quad B_k=\frac{(b)_k}{(d)_k}$$

it is not hard to verify the differences

$$\begin{aligned}\nabla A_k&=\frac{(a)_k}{(c)_k}\frac{1+a-c}{a},\\ \Delta B_k&=\frac{(b)_k}{(1+d)_k}\frac{d-b}{d},\end{aligned}$$

as well as two limiting relations

$$[AB]_+=0\quad\text{and}\quad A_{-1}B_0=\frac{c-1}{a}\quad\text{with}\quad\Re(c+d-a-b)>1.$$

In view of the modified Abel lemma on summation by parts, we can manipulate the Ω-series as follows:

$$\begin{aligned}\Omega(a,b,c,d) &= \frac{a}{1+a-c}\sum_k B_k \nabla A_k\\ &= \frac{a}{1+a-c}\Big\{[AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k\Big\}\\ &= \frac{1-c}{1+a-c} + \frac{a(d-b)}{d(1+a-c)}\sum_{k=0}^{\infty}\begin{bmatrix}1+a, & b\\ c, & 1+d\end{bmatrix}_k.\end{aligned}$$

This leads us to the following recurrence relation, which can also be derived by combining Lemma 1.1 with Lemma 1.2.

Lemma 1.3 (Recurrence relation [**1001**])**.**

$$\Omega(a,b,c,d) = \Omega(1+a,b,c,1+d)\times\frac{a(d-b)}{d(1+a-c)} + \frac{1-c}{1+a-c}.$$

Iterating m-times the equation displayed in Lemma 1.3, we get the relation

$$\Omega(a,b,c,d) = \Omega(a+m,b,c,d+m)\begin{bmatrix}a, d-b\\ d, 1+a-c\end{bmatrix}_m \tag{5a}$$

$$+ \frac{1-c}{1+a-c}\sum_{k=0}^{m-1}\begin{bmatrix}a, d-b\\ d, 2+a-c\end{bmatrix}_k. \tag{5b}$$

When $\Re(1+b-c)>0$, the limit of the last sum exists as $m\to\infty$. In order to evaluate the limit of (5a), recall the Kummer–Thomae–Whipple transformation (cf. Bailey [5, §3.2])

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a, & c, & e\\ & b, & d\end{matrix}\,\Big|\,1\right] &= {}_3F_2\left[\begin{matrix}b-a, d-a, b+d-a-c-e\\ b+d-a-c, b+d-a-e\end{matrix}\,\Big|\,1\right]\\ &\quad\times\Gamma\begin{bmatrix}b, & d, & b+d-a-c-e\\ a, & b+d-a-c, & b+d-a-e\end{bmatrix}.\end{aligned}$$

By rewriting $\Omega(a+m,b,c,d+m)$ in terms of the ${}_3F_2$-series, we may transform it into a convergent series

$$\begin{aligned}\Omega(a+m,b,c,d+m) &= {}_3F_2\left[\begin{matrix}1, & b, & a+m\\ & c, & d+m\end{matrix}\,\Big|\,1\right]\\ &= {}_3F_2\left[\begin{matrix}c-1, c+d-a-b-1, d-1+m\\ c+d-a-1, c+d-b-1+m\end{matrix}\,\Big|\,1\right]\\ &\quad\times\Gamma\begin{bmatrix}c, c+d-a-b-1, d+m\\ c+d-a-1, c+d-b-1+m\end{bmatrix}.\end{aligned}$$

Hence we can express the limit of (5a) as follows:

$$\begin{aligned}&\lim_{m\to\infty}\Omega(a+m,b,c,d+m)\begin{bmatrix}a, d-b\\ d, 1+a-c\end{bmatrix}_m = \Gamma\begin{bmatrix}c, c+d-a-b-1, d\\ c+d-a-1, c+d-b-1\end{bmatrix}\\ &\times\lim_{m\to\infty}{}_3F_2\left[\begin{matrix}c-1, c+d-a-b-1, d-1+m\\ c+d-a-1, c+d-b-1+m\end{matrix}\,\Big|\,1\right]\times\begin{bmatrix}a, & d-b\\ 1+a-c, & c+d-b-1\end{bmatrix}_m.\end{aligned}$$

First, it is not hard to check that

$$\lim_{m\to\infty}\begin{bmatrix}a, & d-b\\ 1+a-c, & c+d-b-1\end{bmatrix}_m=\Gamma\begin{bmatrix}1+a-c, & c+d-b-1\\ a, & d-b\end{bmatrix}.$$

Then the limit of the last $_3F_2$-series can be evaluated, under $\Re(1+b-c)>0$, by the Gauss summation theorem

$$\lim_{m\to\infty}{}_3F_2\left[\begin{matrix}c-1, c+d-a-b-1, d-1+m\\ c+d-a-1, c+d-b-1+m\end{matrix}\,\middle|\,1\right]={}_2F_1\left[\begin{matrix}c-1,\ c+d-a-b-1\\ c+d-a-1\end{matrix}\,\middle|\,1\right]$$
$$=\Gamma\begin{bmatrix}c+d-a-1, 1+b-c\\ b, \quad d-a\end{bmatrix}.$$

Summing up, we have proved the following limiting relation:

$$\lim_{m\to\infty}\Omega(a+m,b,c,d+m)\begin{bmatrix}a, d-b\\ d, 1+a-c\end{bmatrix}_m$$
$$=\Gamma\begin{bmatrix}c, d, 1+a-c, 1+b-c, c+d-a-b-1\\ a, \quad b, \quad d-a, \quad d-b\end{bmatrix}.$$

Finally, letting $m\to\infty$ in (5a–5b) and then relabeling the parameters by $a\to a+c$, $c\to c+1$, $d\to b+d$, we may reformulate the resulting equation as the following reciprocal relation.

Theorem 1.4 (Reciprocal relation)**.**

$$\sum_{k=0}^{\infty}\frac{(a+c)_k}{(b+d)_k}\left\{\frac{(b)_k}{(c)_{k+1}}+\frac{(d)_k}{(a)_{k+1}}\right\}=\Gamma\begin{bmatrix}a, c, b+d, b-c, d-a\\ a+c, b, d, b+d-a-c\end{bmatrix}.$$

This can be considered as a four–parameter extension of the Gauss summation theorem, because the latter can be deduced from it through multiplying across by c and then letting $c\to0$. Furthermore, letting $a\to\frac12-x$, $b\to\frac12$, $c\to x$, $d\to\frac12+x$ in Theorem 1.4, we recover the following interesting identity discovered by Guillera through the WZ–method.

Proposition 1.5 (Guillera [29, Identity 1: Equation 2])**.**

$$2x\sum_{k=0}^{\infty}\begin{bmatrix}\frac12, & \frac12\\ 1+x, & 1+x\end{bmatrix}_k=\frac{4^x\sqrt{\pi}}{\cos^2(\pi x)}\frac{\Gamma^3(1+x)}{\Gamma^3(\frac12+x)}+\frac{4x^2}{2x-1}\sum_{n=0}^{\infty}\begin{bmatrix}\frac12, & \frac12+x\\ 1+x, & \frac32-x\end{bmatrix}_n.$$

§1.4. **[1101].** Define the two sequences A_k and B_k, respectively, by

$$A_k=\begin{bmatrix}1+a, 1+b\\ c, 2+a+b-c\end{bmatrix}_k \quad\text{and}\quad B_k=\frac{(2+a+b-c)_k}{(d)_k}.$$

We can easily prove the differences

$$\nabla A_k = \begin{bmatrix}a, b\\ c, 2+a+b-c\end{bmatrix}_k\frac{(1+a-c)(1+b-c)}{ab},$$
$$\Delta B_k = \frac{(2+a+b-c)_k}{(1+d)_k}\frac{c+d-a-b-2}{d},$$

as well as two limiting relations

$$[AB]_+=0 \quad\text{and}\quad A_{-1}B_0=\frac{(c-1)(1+a+b-c)}{ab} \quad\text{with}\quad \Re(c+d-a-b)>2.$$

Taking into account the modified Abel lemma on summation by parts, we can reformulate the Ω-series as

$$\begin{aligned}\Omega(a,b,c,d) &= \frac{ab}{(1+a-c)(1+b-c)}\sum_k B_k \nabla A_k\\ &=\frac{ab}{(1+a-c)(1+b-c)}\Big\{[AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k\Big\}\\ &=\frac{(1-c)(1+a+b-c)}{(1+a-c)(1+b-c)} + \frac{ab(c+d-a-b-2)}{d(1+a-c)(1+b-c)}\sum_{k=0}^{\infty}\begin{bmatrix}1+a,\ 1+b\\ c,\quad 1+d\end{bmatrix}_k.\end{aligned}$$

This reads as the following recurrence relation, which can also be derived by combining Lemma 1.1 with Lemma 1.3.

Lemma 1.6 (Recurrence relation [**1101**]).

$$\Omega(a,b,c,d) = \Omega(1+a,1+b,c,1+d)\times\frac{ab(c+d-a-b-2)}{d(1+a-c)(1+b-c)}+\frac{(1-c)(1+a+b-c)}{(1+a-c)(1+b-c)}.$$

§1.5. **[1011].** For the two sequences A_k and B_k given, respectively, by

$$A_k = \frac{(1+a)_k}{(c+d-b)_k} \quad\text{and}\quad B_k = \begin{bmatrix}b, c+d-b\\ c, d\end{bmatrix}_k,$$

it is not difficult to check the differences

$$\begin{aligned}\nabla A_k &= \frac{(a)_k}{(c+d-b)_k}\frac{1+a+b-c-d}{a},\\ \Delta B_k &= \begin{bmatrix}b, c+d-b\\ 1+c, 1+d\end{bmatrix}_k \frac{(b-c)(b-d)}{cd},\end{aligned}$$

as well as two limiting relations

$$[AB]_+ = 0 \quad\text{and}\quad A_{-1}B_0 = \frac{c+d-b-1}{a} \quad\text{with}\quad \Re(c+d-a-b) > 1.$$

Then the modified Abel lemma on summation by parts allows us to manipulate the Ω-series as follows:

$$\begin{aligned}\Omega(a,b,c,d) &= \frac{a}{1+a+b-c-d}\sum_k B_k\nabla A_k\\ &=\frac{a}{1+a+b-c-d}\Big\{[AB]_+ - A_{-1}B_0 + \sum_k A_k\Delta B_k\Big\}\\ &=\frac{1+b-c-d}{1+a+b-c-d} + \frac{a(b-c)(b-d)}{cd(1+a+b-c-d)}\sum_{k=0}^{\infty}\begin{bmatrix}1+a,\quad b\\ 1+c,\ 1+d\end{bmatrix}_k.\end{aligned}$$

This is equivalent to the following recurrence relation, which can also be derived by combining Lemma 1.2 with Lemma 1.3.

Lemma 1.7 (Recurrence relation [**1011**]).

$$\Omega(a,b,c,d) = \Omega(1+a,b,1+c,1+d)\times\frac{a(b-c)(b-d)}{cd(1+a+b-c-d)}+\frac{1+b-c-d}{1+a+b-c-d}.$$

2. Fast convergent series and π-formulae

Applying the five recurrence relations established in the last section, we shall derive five infinite series expressions for $\Omega(a,b,c,d)$ with four of them being fast convergent ones. They will lead to numerous infinite series formulae for π.

§2.1. **[1011].** Iterating m-times the equation displayed in Lemma 1.7, we get

$$\begin{aligned}\Omega(a,b,c,d) &= \Omega(1+a,b,1+c,1+d)\times\frac{a(b-c)(b-d)}{cd(1+a+b-c-d)}+\frac{1+b-c-d}{1+a+b-c-d}\\ &= (-1)^m\Omega(a+m,b,c+m,d+m)\frac{(a)_m(c-b)_m(d-b)_m}{(c)_m(d)_m(c+d-a-b-1)_m}\\ &+\sum_{k=0}^{m-1}(-1)^k\frac{(a)_k(c-b)_k(d-b)_k}{(c)_k(d)_k(c+d-a-b-1)_k}\frac{c+d-b-1+2k}{c+d-a-b-1+k}.\end{aligned}$$

By means of the Weierstrass M-test on uniformly convergent series (cf. Stromberg [34, Page 141]), the limiting case $m\to\infty$ of the last equation subject to $\Re(c+d-2a+b)>1$ leads to the following transformation into well–poised series.

Theorem 2.1 (Transformation formula: $\Re(c+d)>1+\max\{\Re(a+b),\Re(2a-b)\}$)**.**

$$\Omega(a,b,c,d)=\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k(c-b)_k(d-b)_k}{(c)_k(d)_k(c+d-a-b)_k}\frac{c+d-b-1+2k}{c+d-a-b-1}.$$

When $d=1$, evaluating $\Omega(a,b,c,d)$ by the Gauss summation formula and then making the parameter replacements $a\to C$, $b\to 1-B$ and $c\to 1+A-B$, we derive from the last theorem the following expression.

Proposition 2.2 (Infinite series for Γ-function quotient: $\Re(1+A-2B-2C)>0$)**.**

$$\Gamma\left[\begin{matrix}1+A-B,1+A-C\\ A,\ 1+A-B-C\end{matrix}\right]=\sum_{k=0}^{\infty}(-1)^k\frac{(A+2k)(A)_k(B)_k(C)_k}{k!(1+A-B)_k(1+A-C)_k}.$$

This is equivalent to the limiting case $D\to-\infty$ of Dougall's well–known formula (cf. Bailey [5, §4.4] and Slater [33, §2.4])

Proposition 2.3 (Dougall [21, 1907]: $\Re(1+A-B-C-D)>0$)**.**

$$\begin{aligned}&{}_5F_4\left[\begin{matrix}A, & 1+A/2, & B, & C, & D\\ & A/2, & 1+A-B, & 1+A-C, & 1+A-D\end{matrix}\,\middle|\,1\right]\\ &=\Gamma\left[\begin{matrix}1+A-B,1+A-C,1+A-D,1+A-B-C-D\\ 1+A,1+A-B-C,1+A-B-D,1+A-C-D\end{matrix}\right].\end{aligned}$$

Corollary 2.4 ($A=B=C=x$ in Proposition 2.2)**.**

$$\frac{\sin\pi x}{\pi x}=\sum_{k=0}^{\infty}(-1)^k\left[\begin{matrix}x,\ x,\ x\\ 1,\ 1,\ 1\end{matrix}\right]_k\frac{x+2k}{x}.$$

Corollary 2.5 ($A=0$ and $B=-C=x$ in Proposition 2.2)**.**

$$\frac{\pi x}{\sin\pi x}=1+2\sum_{k=1}^{\infty}\frac{(-1)^kx^2}{x^2-k^2}.$$

Two particular identities may be worth mentioning. The first one was discovered by Bauer in 1859, which can also be found in Glaisher [22, §25] and Hardy [30, Equation 2].

Example 1 ($x = 1/2$ in Corollary 2.4: Bauer [9, §4])**.**

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{bmatrix}_k \big\{1+4k\big\}.$$

The next one was established by Glaisher in 1905 in a less well–known paper.

Example 2 ($A = B = C = D = -1/2$ in Proposition 2.3: Glaisher [22, §33])**.**

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty} \begin{bmatrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, \quad 1, \quad 1, \quad 1 \end{bmatrix}_k \big\{1-4k\big\}.$$

Further series for $1/\pi$ and $1/\pi^2$ with convergence rate equal to one can be found in Glaisher [22], who made a systematic investigation on π-formulae through elliptic functions before Ramanujan, even though all of the 30 principal series for $1/\pi$ and $1/\pi^2$ collected by him can be derived from Dougall's ${}_5F_4$-series published in 1907, two years after Glaisher's paper.

§2.2. **[0011].** Applying Lemma 1.2 to $\Omega(a, b, 1 + d, c)$, we have

$$\Omega(a, b, c, 1 + d) = \Omega(a, b, 1 + c, 1 + d)\frac{(c-a)(c-b)}{c(c+d-a-b)} + \frac{d}{c+d-a-b}.$$

Substituting this into the equation displayed in Lemma 1.2

$$\Omega(a, b, c, d) = \Omega(a, b, c, 1 + d)\frac{(d-a)(d-b)}{d(c+d-a-b-1)} + \frac{c-1}{c+d-a-b-1}$$

we can simplify the result to the following recurrence relation.

Lemma 2.6 (Recurrence relation **[0011]**)**.**

$$\begin{aligned}\Omega(a, b, c, d) =&\Omega(a, b, 1 + c, 1 + d)\frac{(c-a)(c-b)(d-a)(d-b)}{cd(c+d-a-b-1)_2}\\ &+\frac{c-1}{c+d-a-b-1} + \frac{(d-a)(d-b)}{(c+d-a-b-1)_2}.\end{aligned}$$

Iterating this relation m-times, we get

$$\begin{aligned}\Omega(a, b, c, d) =&\Omega(a, b, c + m, d + m)\frac{(c-a)_m(c-b)_m(d-a)_m(d-b)_m}{(c)_m(d)_m(c+d-a-b-1)_{2m}}\\ &+\sum_{k=0}^{m-1} \alpha_k(a, b, c, d)\frac{(c-a)_k(c-b)_k(d-a)_k(d-b)_k}{(c)_k(d)_k(c+d-a-b-1)_{2+2k}}\end{aligned}$$

where $\alpha_k(a, b, c, d)$ is the quadratic polynomial in k given by

$$\alpha_k(a, b, c, d) = (c-1+k)(c+d-a-b+2k) + (d-a+k)(d-b+k). \tag{6}$$

By means of the Weierstrass M-test on uniformly convergent series, the limiting case $m \to \infty$ of the last equation leads to the transformation.

Theorem 2.7 (Transformation formula **[0011]**)**.**

$$\Omega(a,b,c,d)=\sum_{k=0}^{\infty}\alpha_k(a,b,c,d)\frac{(c-a)_k(c-b)_k(d-a)_k(d-b)_k}{(c)_k(d)_k(c+d-a-b-1)_{2+2k}}.$$

When $a=b=\frac{1}{2}$ and $c=d=1+x$, the last theorem recovers the following interesting identity discovered by Guillera through the WZ–method.

Proposition 2.8 (Guillera [29, Identity 1: Equation 1])**.**

$$8x\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}\\ 1+x, & 1+x\end{bmatrix}_k=\sum_{n=0}^{\infty}\begin{bmatrix}\frac{1}{2}+x, & \frac{1}{2}+x, & \frac{1}{2}+x\\ 1+x, & 1+x, & 1+x\end{bmatrix}_n\frac{1+6x+6n}{4^n}.$$

Instead, for $d=1$, we can evaluate $\Omega(a,b,c,d)$ by the Gauss summation formula and derive from the last theorem the following expression.

Proposition 2.9 (Infinite series for Γ-function quotient)**.**

$$\begin{aligned}\Gamma\begin{bmatrix}1+c, 1+c-a-b\\ 1+c-a, 1+c-b\end{bmatrix}&=\sum_{k=0}^{\infty}\frac{[1-a,1-b,c-a,c-b]_k}{[1,c]_k(2+c-a-b)_{2k}}\\&\quad\times\frac{c\{3k^2+k(1+3c-2a-2b)+(c-a)(c-b)\}}{(c-a)(c-b)(1+c-a-b)}\\&=\frac{c}{1+c-a-b}+\sum_{k=1}^{\infty}\frac{[1-a,1-b,1+c-a,1+c-b]_k}{[1,c]_k(2+c-a-b)_{2k}}\\&\quad\times\frac{c\{3k^2+k(1+3c-2a-2b)+(c-a)(c-b)\}}{(c-a+k)(c-b+k)(1+c-a-b)}.\end{aligned}$$

Three infinite series with a free variable x can be derived from this proposition.

Corollary 2.10 ($a+b=c=1$ in Proposition 2.9)**.**

$$\frac{\sin\pi x}{\pi}=\sum_{k=0}^{\infty}\frac{[x,x,1-x,1-x]_k}{[1,1]_k(2)_{2k}}\Big\{3k^2+2k+x(1-x)\Big\}.$$

Corollary 2.11 ($a=b=x$ and $c=1+x$ in Proposition 2.9)**.**

$$\frac{\pi x}{\sin\pi x}=\sum_{k=0}^{\infty}\frac{[1,1-x,1-x]_k}{(1+x)_k(3-x)_{2k}}\frac{1+4k+3k^2-kx}{(1-x)(2-x)}.$$

Corollary 2.12 ($a=b=c=1-x$ in Proposition 2.9)**.**

$$\frac{\pi x}{\sin\pi x}=1+\sum_{k=1}^{\infty}\frac{[1,x,x]_k}{(1+x)_{2k}(1-x)_k}\frac{x+3k}{k}.$$

Letting $x=1/2$ in Corollary 2.10, we recover immediately the identity (1) discovered by Ramanujan [31, Equation 28] (cf. Berndt [10, Equation 20.1, Page 352] also). This is only the tip of the iceberg, which represents a large class of infinite series with convergence rate equal to 1/4. They are displayed as examples.

Example 3 ($x=1/2$ in Corollary 2.11)**.**

$$\frac{3\pi}{4}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{1}{2}\\ \frac{5}{4}, & \frac{7}{4}\end{bmatrix}_k\frac{2+3k}{4^k}.$$

Example 4 ($x = 3/4$ in Corollary 2.12).

$$\frac{\pi}{2\sqrt{2}} = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{4^k} \begin{bmatrix} 1, \frac{3}{4}, \frac{3}{4} \\ \frac{1}{4}, \frac{7}{8}, \frac{11}{8} \end{bmatrix}_k \frac{1+4k}{4k}.$$

Example 5 ($x = 1/2$ in Corollary 2.12).

$$\frac{\pi}{2} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \begin{bmatrix} 1, \frac{1}{2} \\ \frac{3}{4}, \frac{5}{4} \end{bmatrix}_k \frac{1+6k}{2k}.$$

Example 6 ($x = 1/3$ in Corollary 2.12).

$$\frac{2\pi}{3\sqrt{3}} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \begin{bmatrix} 1, \frac{1}{3}, \frac{1}{3} \\ \frac{2}{3}, \frac{2}{3}, \frac{7}{6} \end{bmatrix}_k \frac{1+9k}{3k}.$$

Example 7 ($x = 2/3$ in Corollary 2.12).

$$\frac{4\pi}{3\sqrt{3}} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \begin{bmatrix} 1, \frac{2}{3}, \frac{2}{3} \\ \frac{1}{3}, \frac{4}{3}, \frac{5}{6} \end{bmatrix}_k \frac{2+9k}{3k}.$$

Example 8 ($x = 1/4$ in Corollary 2.12).

$$\frac{\pi}{2\sqrt{2}} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \begin{bmatrix} 1, \frac{1}{4}, \frac{1}{4} \\ \frac{3}{4}, \frac{5}{8}, \frac{9}{8} \end{bmatrix}_k \frac{1+12k}{4k}.$$

Example 9 ($x = 1/3$ in Corollary 2.10).

$$\frac{9\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1, 1, \frac{3}{2} \end{bmatrix}_k \frac{2+18k+27k^2}{4^k}.$$

Example 10 ($x = 2/3$ in Corollary 2.11).

$$\frac{16\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{1}{3}, \frac{1}{3} \\ \frac{5}{3}, \frac{5}{3}, \frac{7}{6} \end{bmatrix}_k \frac{3+10k+9k^2}{4^k}.$$

Example 11 ($x = 1/3$ in Corollary 2.11).

$$\frac{20\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{2}{3}, \frac{2}{3} \\ \frac{4}{3}, \frac{4}{3}, \frac{11}{6} \end{bmatrix}_k \frac{3+11k+9k^2}{4^k}.$$

Example 12 ($x = 1/4$ in Corollary 2.10).

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, \frac{3}{2} \end{bmatrix}_k \frac{3+32k+48k^2}{4^k}.$$

Example 13 ($x = 3/4$ in Corollary 2.11).

$$\frac{15\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{1}{4}, \frac{1}{4} \\ \frac{7}{4}, \frac{9}{8}, \frac{13}{8} \end{bmatrix}_k \frac{4+13k+12k^2}{4^k}.$$

Example 14 ($x = 1/4$ in Corollary 2.11).

$$\frac{21\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{3}{4}, \frac{3}{4} \\ \frac{5}{4}, \frac{11}{8}, \frac{15}{8} \end{bmatrix}_k \frac{4+15k+12k^2}{4^k}.$$

Example 15 ($x = 1/6$ in Corollary 2.10)**.**

$$\frac{18}{\pi} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2} \end{matrix}\right]_k \frac{5 + 72k + 108k^2}{4^k}.$$

§2.3. **[1012].** Applying Lemma 1.2 to $\Omega(1+a, b, 1+c, 1+d)$ gives

$$\Omega(1+a, b, 1+c, 1+d) = \Omega(1+a, b, 1+c, 2+d)\frac{(d-a)(1+d-b)}{(1+d)(c+d-a-b)} + \frac{c}{c+d-a-b}.$$

Then substituting this into the equation displayed in Lemma 1.7, we have

$$\begin{aligned} \Omega(a, b, c, d) &= \Omega(1+a, b, 1+c, 1+d)\frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} + \frac{1+b-c-d}{1+a+b-c-d} \\ &= \Omega(1+a, b, 1+c, 2+d)\frac{a(b-c)(d-a)(d-b)_2}{c(d)_2(c+d-a-b-1)_2} \\ &\quad + \frac{c+d-b-1}{c+d-a-b-1} - \frac{a(c-b)(d-b)}{d(c+d-a-b-1)_2}. \end{aligned}$$

Iterating this relation m-times results in the equality

$$\begin{aligned} \Omega(a, b, c, d) &= (-1)^m \Omega(a+m, b, c+m, d+2m)\frac{(a)_m(c-b)_m(d-a)_m(d-b)_{2m}}{(c)_m(d)_{2m}(c+d-a-b-1)_{2m}} \\ &\quad + \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k(c-b)_k(d-a)_k(d-b)_{2k}}{(c)_k(d)_{1+2k}(c+d-a-b-1)_{2+2k}} \beta_k(a, b, c, d) \end{aligned}$$

where $\beta_k(a, b, c, d)$ is the cubic polynomial of k given by

(7a) $\quad \beta_k(a, b, c, d) \quad = \quad (c+d-b-1+3k)(c+d-a-b+2k)(d+2k)$

(7b) $\quad\quad - \quad (a+k)(c-b+k)(d-b+2k).$

By means of the Weierstrass M-test on uniformly convergent series, the limiting case $m \to \infty$ of the last equation subject to $\Re(c+d-a-b) > 1$ leads to the transformation.

Theorem 2.13 (Transformation formula **[1012]**)**.**

$$\Omega(a, b, c, d) = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(c-b)_k(d-a)_k(d-b)_{2k}}{(c)_k(d)_{1+2k}(c+d-a-b-1)_{2+2k}} \beta_k(a, b, c, d).$$

When $a = \frac{1}{2} + x$, $b = \frac{1}{2}$, $c = 1 + x$ and $d = 1 + 2x$, the last theorem recovers the following interesting identity discovered recently by Guillera.

Proposition 2.14 (Guillera [29, Identity 4: Equation 7])**.**

$$16x \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}+x \\ 1+x, & 1+2x \end{matrix}\right]_k = \sum_{n=0}^{\infty} \{3 + 20x + 20n\} \left[\begin{matrix} \frac{1}{2}+x, \frac{1}{4}+x, \frac{3}{4}+x \\ 1+x, 1+x, 1+x \end{matrix}\right]_n \left(\frac{-1}{4}\right)^n.$$

When $c = 1$, we can evaluate $\Omega(a, b, c, d)$ by the Gauss summation formula and obtain from Theorem 2.13 the following proposition.

Proposition 2.15 (Infinite series for Γ-function quotient)**.**

$$\Gamma\left[\begin{matrix} 1+d, 2+d-a-b \\ d-a, d-b \end{matrix}\right] = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-b)_k(d-a)_k(d-b)_{2k}}{k!(1+d)_{2k}(2+d-a-b)_{2k}} \beta_k(a, b, 1, d).$$

Corollary 2.16 ($a = b = x$ and $d = 1 + x$ in Proposition 2.15).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} 1, & \frac{1}{2}, & x, & 1-x \\ \frac{2+x}{2}, & \frac{3+x}{2}, & \frac{3-x}{2}, & \frac{4-x}{2} \end{bmatrix}_k \frac{2+11k+19k^2+10k^3+kx-kx^2}{(1+x)(1-x)(2-x)}.$$

In Proposition 2.15, splitting the sum with respect to k into two according to the two terms of $\beta_k(a,b,1,d)$, then keeping the first sum invariant and replacing the summation index by $k \to k-1$ for the second sum, we may reformulate the resulting equation as another proposition.

Proposition 2.17 (Infinite series for Γ-function quotient).

$$\Gamma\begin{bmatrix} d, 1+d-a-b \\ d-a,\ d-b \end{bmatrix} = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-b)_k(d-a)_k(d-b)_{2k}}{k!(d)_{2k}(1+d-a-b)_{2k}} \times \Big\{ (d-b+3k) + \frac{k(d-1+2k)(d-a-b+2k)}{(d-a-1+k)(d-b-1+2k)} \Big\}.$$

Corollary 2.18 ($a + b = d = 1$ in Proposition 2.17).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} x, & x, & -x, & \frac{x}{2}, & \frac{x-1}{2} \\ 1, & 1, & 1, & \frac{1}{2}, & \frac{1}{2} \end{bmatrix}_k \frac{10k^3-3k^2+2kx-k^2x+x^2-4kx^2-x^3}{x(1-x)}.$$

Corollary 2.19 ($a = b = x$ and $d = 1 + x$ in Proposition 2.17).

$$\frac{\pi x}{\sin \pi x} = \frac{1}{1-x} + \sum_{k=1}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} 1, & \frac{1}{2}, & x, & 1-x \\ \frac{1+x}{2}, & \frac{2+x}{2}, & \frac{2-x}{2}, & \frac{3-x}{2} \end{bmatrix}_k \frac{x-x^2+4k+10k^2}{2k(1-x)}.$$

Furthermore, for $d = 1$, we can evaluate $\Omega(a,b,c,d)$ by the Gauss summation formula and derive from Theorem 2.13 the following expression.

Proposition 2.20 (Infinite series for Γ-function quotient).

$$\Gamma\begin{bmatrix} c, 2+c-a-b \\ c-a, c-b \end{bmatrix} = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-a)_k(c-b)_k(1-b)_{2k}}{(c)_k(2)_{2k}(2+c-a-b)_{2k}} \beta_k(a,b,c,1).$$

Corollary 2.21 ($a + b = c = 1$ in Proposition 2.20).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} x, & x, & 1-x, & \frac{x}{2}, & \frac{1+x}{2} \\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2} \end{bmatrix}_k \Big\{ (1+2k)^2(x+3k) - (x+k)^2(x+2k) \Big\}.$$

Corollary 2.22 ($a = b = x$ and $c = 1 + x$ in Proposition 2.20).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} x, & 1-x, & \frac{1-x}{2}, & \frac{2-x}{2} \\ \frac{3}{2}, & 1+x, & \frac{3-x}{2}, & \frac{4-x}{2} \end{bmatrix}_k \frac{2+11k+19k^2+10k^3-2x-7kx-7k^2x+x^2+kx^2}{(1-x)(2-x)}.$$

When $x = 1/2$ in Corollary 2.21, we get the identity (2) found by Ramanujan [31, Equation 35] (cf. Berndt [10, Equation 21.5, Page 353] also). From the following examples, it can be seen that (2) is the simplest one among many infinite series with convergence rate equal to $-1/4$.

Example 16 ($x = 1/3$ in Corollary 2.16).

$$\frac{40\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{4}\Big)^k \begin{bmatrix} 1, & \frac{1}{2} \\ \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \big\{ 9 + 10k \big\}.$$

Example 17 ($x = 1/3$ in Corollary 2.19)**.**

$$\frac{2\pi}{3\sqrt{3}} = \frac{3}{2} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{2} \\ \frac{5}{6}, & \frac{7}{6} \end{bmatrix}_k \frac{1+15k}{6k}.$$

Example 18 ($x = 1/2$ in Corollary 2.19)**.**

$$\frac{\pi}{2} = 2 + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ \frac{3}{4}, & \frac{3}{4}, & \frac{5}{4}, & \frac{5}{4} \end{bmatrix}_k \frac{1+16k+40k^2}{4k}.$$

Example 19 ($x = 1/4$ in Corollary 2.19)**.**

$$\frac{\pi}{2\sqrt{2}} = \frac{4}{3} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{5}{8}, & \frac{7}{8}, & \frac{9}{8}, & \frac{11}{8} \end{bmatrix}_k \frac{3+64k+160k^2}{24k}.$$

Example 20 ($x = 1/6$ in Corollary 2.19)**.**

$$\frac{\pi}{3} = \frac{6}{5} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} 1, & \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ \frac{7}{12}, & \frac{11}{12}, & \frac{13}{12}, & \frac{17}{12} \end{bmatrix}_k \frac{5+144k+360k^2}{60k}.$$

Example 21 ($x = 2/3$ in Corollary 2.22)**.**

$$\frac{8\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{1}{6} \\ \frac{3}{2}, & \frac{5}{3}, & \frac{7}{6} \end{bmatrix}_k \left\{5+23k+30k^2\right\}.$$

Example 22 ($x = 1/3$ in Corollary 2.22)**.**

$$\frac{20\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \\ \frac{3}{2}, & \frac{4}{3}, & \frac{11}{6} \end{bmatrix}_k \left\{13+40k+30k^2\right\}.$$

Example 23 ($x = 1/2$ in Corollary 2.22)**.**

$$\frac{3\pi}{2} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{3}{2}, & \frac{5}{4}, & \frac{7}{4} \end{bmatrix}_k \left\{5+21k+20k^2\right\}.$$

This is equivalent to the series of BBP–type due to Adamchik and Wagon [1]

$$\pi = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left\{\frac{2}{1+4k} + \frac{2}{2+4k} + \frac{1}{3+4k}\right\}.$$

Further BBP–type series for π can be found, for example, in [1, 3, 4, 14, 23, 28].

Example 24 ($x = 1/2$ in Corollary 2.18)**.**

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{4}, & -\frac{1}{2}, & -\frac{1}{4} \\ 1, & 1, & 1 \end{bmatrix}_k \left\{1-28k^2+80k^3\right\}.$$

Example 25 ($x = 1/3$ in Corollary 2.18)**.**

$$\frac{3\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{3}, & \frac{1}{3}, & -\frac{1}{3}, & -\frac{1}{3}, & \frac{1}{6} \\ 1, & 1, & 1, & \frac{1}{2}, & \frac{1}{2} \end{bmatrix}_k \left\{1+3k-45k^2+135k^3\right\}.$$

Example 26 ($x = 2/3$ in Corollary 2.18)**.**

$$\frac{3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{2}{3}, & -\frac{2}{3}, & -\frac{1}{6} \\ 1, & 1, & 1, & \frac{1}{2}, & \frac{1}{2} \end{bmatrix}_k \left\{4-12k-99k^2+270k^3\right\}.$$

Example 27 ($x = 1/3$ in Corollary 2.21)**.**

$$\frac{27\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k \begin{bmatrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{6} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{bmatrix}_k \Big\{8 + 105k + 315k^2 + 270k^3\Big\}.$$

Example 28 ($x = 2/3$ in Corollary 2.21)**.**

$$\frac{27\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k \begin{bmatrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{bmatrix}_k \Big\{10 + 105k + 306k^2 + 270k^3\Big\}.$$

Example 29 ($x = 1/4$ in Corollary 2.21)**.**

$$\frac{32\sqrt{2}}{\pi} = \sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k \begin{bmatrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{bmatrix}_k \Big\{15 + 240k + 752k^2 + 640k^3\Big\}.$$

Example 30 ($x = 3/4$ in Corollary 2.21)**.**

$$\frac{32\sqrt{2}}{\pi} = \sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k \begin{bmatrix} \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{bmatrix}_k \Big\{21 + 240k + 720k^2 + 640k^3\Big\}.$$

§2.4. **[1022].** According to Lemma 1.7, the following equality holds:

$$\begin{aligned}\Omega(a,b,1+c,1+d) \quad &= \quad \Omega(1+a,b,2+c,2+d)\frac{a(1+c-b)(1+d-b)}{(1+c)(1+d)(a+b-c-d-1)} \\ &+ \quad \frac{1+c+d-b}{1+c+d-a-b}.\end{aligned}$$

Substituting this into the equation displayed in Lemma 2.6

$$\begin{aligned}\Omega(a,b,c,d) =&\Omega(a,b,1+c,1+d)\frac{(c-a)(c-b)(d-a)(d-b)}{cd(c+d-a-b-1)_2} \\ &+\frac{c-1}{c+d-a-b-1} + \frac{(d-a)(d-b)}{(c+d-a-b-1)_2}\end{aligned}$$

and then simplifying the result, we get the relation

$$\begin{aligned}\Omega(a,b,c,d) = \Omega(1+a,b,2+c,2+d)&\frac{a(c-a)(d-a)(c-b)_2(d-b)_2}{(c)_2(d)_2(a+b-c-d-1)_3} \\ + \frac{(c-a)(c-b)(d-a)(d-b)(1+c+d-b)}{cd(c+d-a-b-1)_3}& \\ + \frac{c-1}{c+d-a-b-1} + \frac{(d-a)(d-b)}{(c+d-a-b-1)_2}&.\end{aligned}$$

Iterating this relation m-times yields the expression

$$\begin{aligned}\Omega(a,b,c,d) =&(-1)^m\Omega(a+m,b,c+2m,d+2m)\tfrac{(a)_m(c-a)_m(d-a)_m(c-b)_{2m}(d-b)_{2m}}{(c)_{2m}(d)_{2m}(c+d-a-b-1)_{3m}} \\ &+\sum_{k=0}^{m-1}(-1)^k\frac{(a)_k(c-a)_k(d-a)_k(c-b)_{2k}(d-b)_{2k}}{(c)_{1+2k}(d)_{1+2k}(c+d-a-b-1)_{3+3k}}\gamma_k(a,b,c,d)\end{aligned}$$

where $\gamma_k(a,b,c,d)$ is the quintic polynomial of k given by

(8a) $\quad \gamma_k(a,b,c,d) = (c-a+k)(d-a+k)(c-b+2k)(d-b+2k)(1+c+d-b+4k)$

(8b) $\quad + (c+2k)(c-1+2k)(d+2k)(c+d-a-b+3k)(1+c+d-a-b+3k)$

(8c) $\quad + (c+2k)(d+2k)(d-a+k)(d-b+2k)(1+c+d-a-b+3k).$

By means of the Weierstrass M-test on uniformly convergent series, the limiting case $m \to \infty$ of the last equation leads to the transformation.

Theorem 2.23 (Transformation formula **[1022]**).

$$\Omega(a,b,c,d) = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(c-a)_k(d-a)_k(c-b)_{2k}(d-b)_{2k}}{(c)_{1+2k}(d)_{1+2k}(c+d-a-b-1)_{3+3k}} \gamma_k(a,b,c,d).$$

When $d = 1$, we can evaluate $\Omega(a,b,c,d)$ by the Gauss summation formula and derive from the last theorem the following further relations.

Proposition 2.24 (Infinite series for Γ-function quotient).

$$\Gamma\begin{bmatrix} 1+c, 3+c-a-b \\ c-a, c-b \end{bmatrix} = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-a)_k(c-a)_k(c-b)_{2k}(1-b)_{2k}}{(2)_{2k}(1+c)_{2k}(3+c-a-b)_{3k}} \gamma_k(a,b,c,1).$$

Corollary 2.25 ($a + b = c = 1$ in Proposition 2.24).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} x, 1-x, 1-x, \frac{x}{2}, \frac{x}{2}, \frac{1+x}{2}, \frac{1+x}{2} \\ 1, \quad 1, \quad 1, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3} \end{bmatrix}_k \gamma_k(x,1-x,1,1).$$

Corollary 2.26 ($a = b = x$ and $c = 1 + x$ in Proposition 2.24).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, \frac{1}{2}, x, 1-x, \frac{1-x}{2}, \frac{2-x}{2} \\ \frac{3}{2}, \frac{2+x}{2}, \frac{3+x}{2}, \frac{4-x}{3}, \frac{5-x}{3}, \frac{6-x}{3} \end{bmatrix}_k \frac{\gamma_k(x,x,1+x,1)}{(1-x^2)(2-x)(3-x)}.$$

Corollary 2.27 ($a = b = c = 1 - x$ in Proposition 2.24).

$$\frac{2(1-x^2)\pi x}{(1+x^2)\sin \pi x} = 2 + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, x, 1-x, \frac{x}{2}, \frac{1+x}{2} \\ \frac{2-x}{2}, \frac{3-x}{2}, \frac{2+x}{3}, \frac{3+x}{3}, \frac{4+x}{3} \end{bmatrix}_k \frac{\gamma_k(1-x,1-x,1-x,1)}{k^2(2k+1)(1+x^2)}.$$

Letting $c = 1$ in Theorem 2.23 and then making the replacement $k \to k + 1$ for the sum corresponding to (8b) in the factor $\gamma_k(a,b,c,d)$, we derive the following expressions.

Proposition 2.28 (Infinite series for Γ-Function quotient).

$$\Gamma\begin{bmatrix} 2+d, 4+d-a-b \\ 1+d-a, 1+d-b \end{bmatrix} = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-a)_k(1+d-a)_k(1-b)_{2k}(1+d-b)_{2k}}{(2)_{2k}(2+d)_{2k}(4+d-a-b)_{3k}} \gamma_k^\star(a,b,d)$$

where $\gamma_k^\star(a,b,d)$ is the quintic polynomial of k given by

$$\begin{aligned} \gamma_k^\star(a,b,d) = {} & (1+d+2k)(1-a+k)(1-b+2k)(2+d-b+4k)(3+d-a-b+3k) \\ & - (a+k)(1-a+k)(1-b+2k)(2-b+2k)(1+d-b+2k) \\ & + (1+2k)(d+2k)(1+d+2k)(2+d-a-b+3k)(3+d-a-b+3k). \end{aligned}$$

Corollary 2.29 ($a = b = d = 1 - x$ in Proposition 2.28).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, x, 1-x, \frac{x}{2}, \frac{1+x}{2} \\ \frac{3-x}{2}, \frac{4-x}{2}, \frac{3+x}{3}, \frac{4+x}{3}, \frac{5+x}{3} \end{bmatrix}_k \frac{\gamma_k^\star(1-x,1-x,1-x)}{(2k+1)(1-x^2)(4-x^2)}.$$

By specifying x in Corollaries 2.25, 2.26, 2.27 and 2.29, we can deduce the following infinite series expressions for π and $1/\pi$ with convergence rate equal to $-1/27$.

Example 31 ($x = 1/2$ in Corollary 2.29)**.**

$$\frac{15\pi}{8} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{1}{2} \\ \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \big\{6 + 7k\big\}.$$

Example 32 ($x = 1/2$ in Corollary 2.27)**.**

$$\frac{\pi}{2} = \frac{5}{3} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{1}{2} \\ \frac{5}{6}, & \frac{7}{6} \end{bmatrix}_k \frac{3 + 28k}{6k}.$$

Example 33 ($x = 1/3$ in Corollary 2.27)**.**

$$\frac{4\pi}{3\sqrt{3}} = \frac{5}{2} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{2}{3}, & \frac{2}{3}, & \frac{1}{6} \\ \frac{5}{6}, & \frac{7}{9}, & \frac{10}{9}, & \frac{13}{9} \end{bmatrix}_k \frac{2 + 30k + 63k^2}{3k}.$$

Example 34 ($x = 3/4$ in Corollary 2.27)**.**

$$\frac{\pi}{2\sqrt{2}} = \frac{25}{21} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{3}{4}, & \frac{3}{8}, & \frac{7}{8} \\ \frac{5}{8}, & \frac{9}{8}, & \frac{11}{12}, & \frac{19}{12} \end{bmatrix}_k \frac{21 + 304k + 448k^2}{84k}.$$

Example 35 ($x = 2/3$ in Corollary 2.27)**.**

$$\frac{2\pi}{3\sqrt{3}} = \frac{13}{10} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{1}{3}, & \frac{1}{3}, & \frac{5}{6} \\ \frac{7}{6}, & \frac{8}{9}, & \frac{11}{9}, & \frac{14}{9} \end{bmatrix}_k \frac{(5 + 21k)(1 + 12k + 18k^2)}{15k}.$$

Example 36 ($x = 1/4$ in Corollary 2.27)**.**

$$\frac{3\pi}{2\sqrt{2}} = \frac{17}{5} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{1}{4}, & \frac{1}{8}, & \frac{5}{8} \\ \frac{7}{8}, & \frac{11}{8}, & \frac{13}{12}, & \frac{17}{12} \end{bmatrix}_k \frac{15 + 324k + 1408k^2 + 1792k^3}{20k}.$$

Example 37 ($x = 1/3$ in Corollary 2.25)**.**

$$\frac{243\sqrt{3}}{8\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} \frac{2}{3}, & \frac{2}{3}, & \frac{2}{3}, & \frac{1}{6}, & \frac{1}{6} \\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2} \end{bmatrix}_k \Big\{17 + 279k + 864k^2 + 756k^3\Big\}.$$

Example 38 ($x = 1/2$ in Corollary 2.25)**.**

$$\frac{64}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4}, & \frac{3}{4} \\ 1, & 1, & 1, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \Big\{21 + 296k + 992k^2 + 896k^3\Big\}.$$

Example 39 ($x = 2/3$ in Corollary 2.29)**.**

$$\frac{160\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{1}{3}, & \frac{1}{3}, & \frac{5}{6} \\ \frac{7}{6}, & \frac{11}{9}, & \frac{14}{9}, & \frac{17}{9} \end{bmatrix}_k \Big\{33 + 227k + 438k^2 + 252k^3\Big\}.$$

Example 40 ($x = 2/3$ in Corollary 2.25)**.**

$$\frac{243\sqrt{3}}{4\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{5}{6}, & \frac{5}{6} \\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2} \end{bmatrix}_k \Big\{35 + 504k + 1620k^2 + 1512k^3\Big\}.$$

Example 41 ($x = 1/6$ in Corollary 2.27)**.**

$$70\pi = 222 + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \begin{bmatrix} 1, & \frac{1}{6}, & \frac{5}{6}, & \frac{1}{12}, & \frac{7}{12} \\ \frac{11}{12}, & \frac{17}{12}, & \frac{13}{18}, & \frac{19}{18}, & \frac{25}{18} \end{bmatrix}_k \frac{35 + 1026k + 4608k^2 + 6048k^3}{k}.$$

Example 42 ($x = 3/4$ in Corollary 2.26)**.**

$$\frac{315\pi}{16\sqrt{2}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \left[\begin{matrix} 1, & \frac{1}{4}, & \frac{1}{8}, & \frac{5}{8} \\ \frac{11}{8}, & \frac{15}{8}, & \frac{13}{12}, & \frac{17}{12} \end{matrix}\right]_k \Big\{44 + 305k + 688k^2 + 448k^3\Big\}.$$

Example 43 ($x = 5/6$ in Corollary 2.27)**.**

$$110\pi = 366 + \sum_{k=1}^{\infty} \Big(\frac{-1}{27}\Big)^k \left[\begin{matrix} 1, & \frac{1}{6}, & \frac{5}{6}, & \frac{5}{12}, & \frac{11}{12} \\ \frac{7}{12}, & \frac{13}{12}, & \frac{17}{18}, & \frac{23}{19}, & \frac{29}{18} \end{matrix}\right]_k \frac{55 + 1314k + 5760k^2 + 6048k^3}{k}.$$

Example 44 ($x = 1/3$ in Corollary 2.29)**.**

$$\frac{280\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \left[\begin{matrix} 1, & \frac{2}{3}, & \frac{2}{3}, & \frac{1}{6} \\ \frac{11}{6}, & \frac{10}{9}, & \frac{13}{9}, & \frac{16}{9} \end{matrix}\right]_k \Big\{57 + 308k + 498k^2 + 252k^3\Big\}.$$

Example 45 ($x = 1/4$ in Corollary 2.26)**.**

$$\frac{1155\pi}{16\sqrt{2}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \left[\begin{matrix} 1, & \frac{3}{4}, & \frac{3}{8}, & \frac{7}{8} \\ \frac{9}{8}, & \frac{13}{8}, & \frac{19}{12}, & \frac{23}{12} \end{matrix}\right]_k \Big\{164 + 697k + 976k^2 + 448k^3\Big\}.$$

§2.5. **[1122].** Applying Lemma 1.7 to $\Omega(b, 1+a, 1+c, 1+d)$, we get

$\Omega(1+a, b, 1+c, 1+d) = \Omega(1+a, 1+b, 2+c, 2+d)\frac{b(a-c)(a-d)}{(1+c)(1+d)(a+b-c-d)} + \frac{a-c-d}{a+b-c-d}.$

Substituting this into the equation displayed in Lemma 1.7, we may simplify the resulting equation as follows:

$$\begin{aligned}\Omega(a,b,c,d) &= \Omega(1+a, b, 1+c, 1+d)\frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} + \frac{1+b-c-d}{1+a+b-c-d} \\ &= \Omega(1+a, 1+b, 2+c, 2+d)\frac{ab(c-a)(c-b)(d-a)(d-b)}{(c)_2(d)_2(c+d-a-b-1)_2} \\ &\quad + \frac{a(c-b)(d-b)(a-c-d)}{cd(c+d-a-b-1)_2} + \frac{c+d-b-1}{c+d-a-b-1}.\end{aligned}$$

Iterating this relation m-times, we get

$$\begin{aligned}\Omega(a,b,c,d) &= \Omega(a+m, b+m, c+2m, d+2m)\tfrac{(a)_m(b)_m(c-a)_m(c-b)_m(d-a)_m(d-b)_m}{(c)_{2m}(d)_{2m}(c+d-a-b-1)_{2m}} \\ &\quad + \sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c-a)_k(c-b)_k(d-a)_k(d-b)_k}{(c)_{1+2k}(d)_{1+2k}(c+d-a-b-1)_{2+2k}}\rho_k(a,b,c,d)\end{aligned}$$

where $\rho_k(a,b,c,d)$ is the quartic polynomial of k given by

(9a) $\qquad \rho_k(a,b,c,d) = (a+k)(c-b+k)(d-b+k)(a-c-d-3k)$

(9b) $\qquad\qquad + (c+2k)(d+2k)(c+d-b-1+3k)(c+d-a-b+2k).$

By means of the Weierstrass M-test on uniformly convergent series, the limiting case $m \to \infty$ of the last equation yields the transformation.

Theorem 2.30 (Transformation formula [**1122**])**.**

$$\Omega(a,b,c,d) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c-a)_k(c-b)_k(d-a)_k(d-b)_k}{(c)_{1+2k}(d)_{1+2k}(c+d-a-b-1)_{2+2k}}\rho_k(a,b,c,d).$$

When $a = b = \frac{1}{2} + x$ and $c = d = 1 + 2x$, the last theorem recovers the following interesting identity discovered recently by Guillera.

Proposition 2.31 (Guillera [29, Identity 2: Equation 3]).

$$32x\sum_{k=0}^{\infty}\begin{bmatrix}\frac12+x,\ \frac12+x\\1+2x,\ 1+2x\end{bmatrix}_k=\sum_{n=0}^{\infty}\begin{bmatrix}\frac12+x,\ \frac12+x,\ \frac12+x\\1+x,\ 1+x,\ 1+x\end{bmatrix}_n\frac{5+42x+42n}{64^n}.$$

In particular, for $d=1$, we can evaluate $\Omega(a,b,c,d)$ by the Gauss summation formula and derive from the last theorem the following expression.

Proposition 2.32 (Infinite series for Γ-function quotient).

$$\Gamma\begin{bmatrix}1+c,2+c-a-b\\c-a,c-b\end{bmatrix}=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(1-a)_k(1-b)_k(c-a)_k(c-b)_k}{(2)_{2k}(1+c)_{2k}(2+c-a-b)_{2k}}\rho_k(a,b,c,1).$$

Corollary 2.33 ($a+b=c=1$ in Proposition 2.32).

$$\frac{\sin\pi x}{\pi x}=\sum_{k=0}^{\infty}\begin{bmatrix}x,\ 1-x\\1,\quad \frac32\end{bmatrix}_k^3\frac{k(1+3k)(3+9k+7k^2)+x(1-x)(1+6k+6k^2+x-x^2)}{64^k}.$$

Corollary 2.34 ($a=b=x$ and $c=1+x$ in Proposition 2.32).

$$\frac{\pi x}{\sin\pi x}=\sum_{k=0}^{\infty}\begin{bmatrix}1,\ x,\ x,\ 1-x,\ 1-x\\\frac32,\ \frac{2+x}{2},\ \frac{3+x}{2},\ \frac{3-x}{2},\ \frac{4-x}{2}\end{bmatrix}_k\frac{2+14k+39k^2+48k^3+21k^4-x+3k^2x+x^2-3k^2x^2}{64^k\ (1+x)(1-x)(2-x)}.$$

Corollary 2.35 ($a=b=c=x$ in Proposition 2.32).

$$\frac{\pi x}{\sin\pi x}=\frac{1+x^3}{1-x^2}+\sum_{k=1}^{\infty}\frac{[1,\ x,\ 1-x]_k^2}{[2,1+x,2-x]_{2k}}\frac{(2+3k)(1-x)x+k(5+20k+21k^2)}{k(1-x)}.$$

The identity (3) due to Ramanujan [31, Equation 29] (cf. Berndt [10, Equation 20.2, Page 352] also) can be shown to be the case $x=1/2$ of Corollary 2.33. Similarly, we can derive further infinite series formulae with convergence rate equal to $1/64$.

Example 46 ($x=1/2$ in Corollary 2.35).

$$\pi=3+\sum_{k=1}^{\infty}\begin{bmatrix}1,\ \frac12,\ \frac12,\ \frac12\\\frac34,\ \frac34,\ \frac54,\ \frac54\end{bmatrix}_k\frac{(1+6k)(2+7k)}{64^k\cdot k}.$$

Example 47 ($x=1/3$ in Corollary 2.35).

$$\frac{4\pi}{\sqrt3}=7+\sum_{k=1}^{\infty}\begin{bmatrix}1,\ \frac13,\ \frac23\\\frac32,\ \frac56,\ \frac76\end{bmatrix}_k\frac{4+39k+63k^2}{64^k\cdot k}.$$

Example 48 ($x=1/3$ in Corollary 2.34).

$$\frac{40\pi}{9\sqrt3}=\sum_{k=0}^{\infty}\begin{bmatrix}1,\ \frac13,\ \frac23\\\frac32,\ \frac76,\ \frac{11}{6}\end{bmatrix}_k\frac{8+27k+21k^2}{64^k}.$$

Example 49 ($x=1/4$ in Corollary 2.35).

$$3\pi\sqrt2=13+\sum_{k=1}^{\infty}\begin{bmatrix}1,\ \frac14,\ \frac14,\ \frac34,\ \frac34\\\frac32,\ \frac58,\ \frac78,\ \frac98,\ \frac{11}{8}\end{bmatrix}_k\frac{6+89k+320k^2+336k^3}{64^k\cdot k}.$$

Example 50 ($x = 1/2$ in Corollary 2.34)**.**

$$\frac{9\pi}{4} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \end{bmatrix}_k \frac{7+42k+75k^2+42k^3}{64^k}.$$

Example 51 ($x = 1/5$ in Corollary 2.35)**.**

$$\frac{8\pi\sqrt{2}}{\sqrt{5-\sqrt{5}}} = 21 + \sum_{k=1}^{\infty} \begin{bmatrix} 1, \frac{1}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5} \\ \frac{3}{2}, \frac{3}{5}, \frac{7}{5}, \frac{9}{10}, \frac{11}{10} \end{bmatrix}_k \frac{8+137k+500k^2+525k^3}{64^k \cdot k}.$$

Example 52 ($x = 1/6$ in Corollary 2.35)**.**

$$10\pi = 31 + \sum_{k=1}^{\infty} \begin{bmatrix} 1, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{7}{12}, \frac{11}{12}, \frac{13}{12}, \frac{17}{12} \end{bmatrix}_k \frac{10+195k+720k^2+756k^3}{64^k \cdot k}.$$

Example 53 ($x = 2/5$ in Corollary 2.35)**.**

$$\frac{12\pi\sqrt{2}}{\sqrt{5+\sqrt{5}}} = 19 + \sum_{k=1}^{\infty} \begin{bmatrix} 1, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5} \\ \frac{3}{2}, \frac{4}{5}, \frac{6}{5}, \frac{7}{10}, \frac{13}{10} \end{bmatrix}_k \frac{12+143k+500k^2+525k^3}{64^k \cdot k}.$$

Example 54 ($x = 1/8$ in Corollary 2.35)**.**

$$\frac{14\pi}{\sqrt{2-\sqrt{2}}} = 57 + \sum_{k=1}^{\infty} \begin{bmatrix} 1, \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8} \\ \frac{3}{2}, \frac{9}{16}, \frac{15}{16}, \frac{17}{16}, \frac{23}{16} \end{bmatrix}_k \frac{14+341k+1280k^2+1344k^3}{64^k \cdot k}.$$

Example 55 ($x = 3/8$ in Corollary 2.35)**.**

$$\frac{30\pi}{\sqrt{2+\sqrt{2}}} = 49 + \sum_{k=1}^{\infty} \begin{bmatrix} 1, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8} \\ \frac{3}{2}, \frac{11}{16}, \frac{13}{16}, \frac{19}{16}, \frac{21}{16} \end{bmatrix}_k \frac{30+365k+1280k^2+1344k^3}{64^k \cdot k}.$$

Example 56 ($x = 1/3$ in Corollary 2.33)**.**

$$\frac{81\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{3}, \frac{2}{3} \\ 1, \frac{3}{2} \end{bmatrix}_k^3 \frac{22+351k+1566k^2+2754k^3+1701k^4}{64^k}.$$

Example 57 ($x = 1/4$ in Corollary 2.34)**.**

$$\frac{105\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{bmatrix}_k \frac{29+224k+633k^2+768k^3+336k^4}{64^k}.$$

Example 58 ($x = 1/4$ in Corollary 2.33)**.**

$$\frac{128\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{4}, \frac{3}{4} \\ 1, \frac{3}{2} \end{bmatrix}_k^3 \frac{57+1056k+4896k^2+8704k^3+5376k^4}{64^k}.$$

Example 59 ($x = 1/6$ in Corollary 2.34)**.**

$$\frac{385\pi}{18} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{13}{12}, \frac{17}{12}, \frac{19}{12}, \frac{23}{12} \end{bmatrix}_k \frac{67+504k+1419k^2+1728k^3+756k^4}{64^k}.$$

Example 60 ($x = 1/6$ in Corollary 2.33)**.**

$$\frac{648}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{6}, \frac{5}{6} \\ 1, \frac{3}{2} \end{bmatrix}_k^3 \frac{205+4968k+24408k^2+44064k^3+27216k^4}{64^k}.$$

3. More general transformation and further π-formulae

In general, we can even iterate p and q times of Lemma 1.1 with respect to parameters a and b, then r and s times of Lemma 1.2 with respect to parameters c and d. The resulting relation reads as follows:

$$\begin{aligned}\Omega(a,b,c,d) &= (-1)^{pm+qm}\frac{(a)_{pm}(b)_{qm}}{(c)_{rm}(d)_{sm}}\Omega(a+pm,b+qm,c+rm,d+sm)\\ &\times \frac{(c-a)_{rm-pm}(c-b)_{rm-qm}(d-a)_{sm-pm}(d-b)_{sm-qm}}{(c+d-a-b-1)_{m(r+s-p-q)}}\\ &+ \sum_{n=0}^{m-1}(-1)^{pn+qn}\frac{(a)_{pn}(b)_{qn}}{(c)_{rn}(d)_{sn}}W_n(a,b,c,d;p,q,r,s)\\ &\times \frac{(c-a)_{rn-pn}(c-b)_{rn-qn}(d-a)_{sn-pn}(d-b)_{sn-qn}}{(c+d-a-b-1)_{n(r+s-p-q)}}\end{aligned}$$

where

$$W_n(a,b,c,d;p,q,r,s) = W(a+pn,b+qn,c+rn,d+sn;p,q,r,s)$$

and

$$\begin{aligned}W(a,b,c,d;p,q,r,s) &= \frac{(c-1)(d-1)}{(1+a-c)(1+a-d)}\sum_{i=0}^{p-1}\begin{bmatrix}a,2+a+b-c-d\\2+a-c,2+a-d\end{bmatrix}_i\\ &+\frac{(c-1)(d-1)}{(1+b-c)(1+b-d)}\begin{bmatrix}a,2+a+b-c-d\\1+a-c,1+a-d\end{bmatrix}_p\sum_{j=0}^{q-1}\begin{bmatrix}b,2+a+b-c-d+p\\2+b-c,2+b-d\end{bmatrix}_j\\ &+\frac{(2+a+b-c-d)_{p+q-1}(a)_p(b)_q}{[1+a-c,1+a-d]_p[1+b-c,1+b-d]_q}\Bigg\{\sum_{k=0}^{r-1}\frac{(d-1)[c-a-p,c-b-q]_k}{(c)_k(c+d-a-b-p-q)_k}\\ &-\frac{(c-1)[c-a-p,c-b-q]_r}{(c-1)_r(c+d-a-b-p-q)_r}\sum_{\ell=0}^{s-1}\begin{bmatrix}d-a-p,d-b-q\\d,c+d-a-b-p-q+r\end{bmatrix}_\ell\Bigg\}.\end{aligned}$$

By means of the Weierstrass M-test on uniformly convergent series, the limiting case $m\to\infty$ of the last equation gives rise to the transformation.

Theorem 3.1 (Transformation formula: $p+q<r+s$)**.**

$$\begin{aligned}\Omega(a,b,c,d) &= \sum_{n=0}^{\infty}(-1)^{pn+qn}\frac{(a)_{pn}(b)_{qn}}{(c)_{rn}(d)_{sn}}W_n(a,b,c,d;p,q,r,s)\\ &\times \frac{(c-a)_{rn-pn}(c-b)_{rn-qn}(d-a)_{sn-pn}(d-b)_{sn-qn}}{(c+d-a-b-1)_{n(r+s-p-q)}}.\end{aligned}$$

This general theorem can be utilized to derive many summation formulae involving π. For instance, the two infinite series anticipated in the introduction are obtained from the iteration pattern **[5,5,10,10]**, respectively, under the parameter settings "$a=b=1/2$, $c=d=1$" and "$a=b=1/3$, $c=4/3$, $d=1$". Further infinite series with different convergence rates are computed through *Mathematica*. They are exhibited as examples, where iteration patterns and parameter settings are highlighted in the headers.

Example 61 (**[0022]** $a = b = 1/2$ and $c = d = 1$)**.**

$$\frac{128}{\pi} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac14, \frac14, \frac14, \frac34, \frac34, \frac34 \\ 1, 1, 1, \frac32, \frac32, \frac32 \end{bmatrix}_n \frac{39 + 672n + 3168n^2 + 5888n^3 + 3840n^4}{16^n}.$$

Example 62 (**[0022]** $a = b = 1/2$, $c = 3/2$ and $d = 1$)**.**

$$\frac{105\pi}{8} = \sum_{n=0}^{\infty} \begin{bmatrix} 1, \frac12, \frac14, \frac34 \\ \frac98, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{bmatrix}_n \frac{40 + 237n + 428n^2 + 240n^3}{16^n}.$$

Example 63 (**[0022]** $a = b = c = 1/2$ and $d = 1$)**.**

$$30\pi = 88 + \sum_{n=1}^{\infty} \begin{bmatrix} 1, \frac12, \frac14, \frac34 \\ \frac78, \frac98, \frac{11}{8}, \frac{13}{8} \end{bmatrix}_n \frac{15 + 272n + 992n^2 + 960n^3}{16^n \cdot n}.$$

Example 64 (**[1123]** $a = 1/3$, $b = 2/3$ and $c = d = 1$)**.**

$$\frac{729\sqrt{3}}{4\pi} = \sum_{n=0}^{\infty} \left(\frac{2}{27}\right)^{2n} \begin{bmatrix} \frac13, \frac23, \frac16, \frac56 \\ 1, 1, 1, \frac32 \end{bmatrix}_n \Big\{100 + 1521n + 2610n^2\Big\}.$$

Example 65 (**[1123]** $a = b = 1/2$, $c = 3/2$ and $d = 1$)**.**

$$15\pi = \sum_{n=0}^{\infty} \left(\frac{2}{27}\right)^{2n} \begin{bmatrix} 1, \frac12, \frac14, \frac34 \\ \frac43, \frac53, \frac76, \frac{11}{6} \end{bmatrix}_n \Big\{47 + 286n + 517n^2 + 290n^3\Big\}.$$

Example 66 (**[1123]** $a = b = c = 1/2$ and $d = 1$)**.**

$$12\pi = 37 + \sum_{n=1}^{\infty} \left(\frac{2}{27}\right)^{2n} \begin{bmatrix} 1, \frac12, \frac14, \frac34 \\ \frac43, \frac53, \frac56, \frac76 \end{bmatrix}_n \frac{36 + 415n + 1314n^2 + 1160n^3}{n}.$$

Example 67 (**[0012]** $a = b = 1/2$ and $c = d = 1$)**.**

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \left(\frac{4}{27}\right)^{n} \begin{bmatrix} \frac12, \frac14, \frac14, \frac34, \frac34 \\ 1, 1, 1, \frac43, \frac53 \end{bmatrix}_n \Big\{9 + 118n + 400n^2 + 368n^3\Big\}.$$

Example 68 (**[0012]** $a = 1/3$, $b = 2/3$ and $c = d = 1$)**.**

$$\frac{81\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \left(\frac{4}{27}\right)^{n} \begin{bmatrix} \frac13, \frac23, \frac16, \frac56 \\ 1, 1, 1, \frac32 \end{bmatrix}_n \Big\{20 + 243n + 414n^2\Big\}.$$

Example 69 (**[0012]** $a = b = 1/3$, $c = 4/3$ and $d = 1$)**.**

$$\frac{160\pi}{9\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{4}{27}\right)^{n} \begin{bmatrix} 1, \frac13, \frac56, \frac56 \\ \frac32, \frac{11}{9}, \frac{14}{9}, \frac{17}{9} \end{bmatrix}_n \Big\{28 + 151n + 257n^2 + 138n^3\Big\}.$$

Example 70 (**[0012]** $a = b = 2/3$, $c = 5/3$ and $d = 1$)**.**

$$\frac{56\pi}{9\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{4}{27}\right)^{n} \begin{bmatrix} 1, \frac23, \frac16, \frac16 \\ \frac32, \frac{10}{9}, \frac{13}{9}, \frac{16}{9} \end{bmatrix}_n \Big\{11 + 74n + 181n^2 + 138n^3\Big\}.$$

Example 71 (**[0012]** $a = b = c = 1/2$ and $d = 1$)**.**

$$3\pi = 7 + \sum_{n=1}^{\infty} \left(\frac{4}{27}\right)^{n} \begin{bmatrix} 1, \frac14, \frac14, \frac34, \frac34 \\ \frac12, \frac32, \frac32, \frac56, \frac76 \end{bmatrix}_n \frac{6 + 67n + 200n^2 + 184n^3}{n}.$$

Example 72 (**[0012]** $a=b=c=1/3$ and $d=1$)**.**

$$\frac{20\pi}{\sqrt{3}}=19+\sum_{n=1}^{\infty}\Big(\frac{4}{27}\Big)^n\begin{bmatrix}1,\ \frac13,\ \frac56,\ \frac56\\ \frac32,\ \frac89,\ \frac{11}{9},\ \frac{14}{9}\end{bmatrix}_n\frac{20+185n+495n^2+414n^3}{n}.$$

Example 73 (**[0012]** $a=b=c=2/3$ and $d=1$)**.**

$$\frac{8\pi}{\sqrt{3}}=13+\sum_{n=1}^{\infty}\Big(\frac{4}{27}\Big)^n\begin{bmatrix}1,\ \frac23,\ \frac16,\ \frac16\\ \frac32,\ \frac79,\ \frac{10}{9},\ \frac{13}{9}\end{bmatrix}_n\frac{8+119n+405n^2+414n^3}{n}.$$

Example 74 (**[1013]** $a=1/3$, $b=2/3$ and $c=d=1$)**.**

$$\frac{486\sqrt{3}}{\pi}=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac13,\ \frac56,\ \frac19,\ \frac49,\ \frac79\\ 1,\ 1,\ 1,\ \frac53,\ \frac53\end{bmatrix}_n\Big\{280+4314n+10908n^2+7533n^3\Big\}.$$

Example 75 (**[1013]** $a=2/3$, $b=1/3$ and $c=d=1$)**.**

$$\frac{243\sqrt{3}}{2\pi}=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac23,\ \frac16,\ \frac29,\ \frac59,\ \frac89\\ 1,\ 1,\ 1,\ \frac43,\ \frac43\end{bmatrix}_n\Big\{80+1281n+5697n^2+7533n^3\Big\}.$$

Example 76 (**[1013]** $a=b=1/2$, $c=3/2$ and $d=1$)**.**

$$30\pi=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac12,\ \frac14,\ \frac34,\ \frac16,\ \frac56\\ \frac32,\ \frac43,\ \frac53,\ \frac76,\ \frac{11}{6}\end{bmatrix}_n\Big\{97+931n+3222n^2+4548n^3+2232n^4\Big\}.$$

Example 77 (**[1013]** $a=b=1/3$, $c=4/3$ and $d=1$)**.**

$$\frac{320\pi}{\sqrt{3}}=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac13,\ \frac56,\ \frac29,\ \frac59,\ \frac89\\ \frac43,\ \frac53,\ \frac{11}{9},\ \frac{14}{9},\ \frac{17}{9}\end{bmatrix}_n\Big\{604+5080n+14655n^2+17604n^3+7533n^4\Big\}.$$

Example 78 (**[1013]** $a=b=2/3$, $c=5/3$ and $d=1$)**.**

$$\frac{112\pi}{\sqrt{3}}=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac23,\ \frac16,\ \frac19,\ \frac49,\ \frac79\\ \frac43,\ \frac53,\ \frac{10}{9},\ \frac{13}{9},\ \frac{16}{9}\end{bmatrix}_n\Big\{206+2045n+7941n^2+13095n^3+7533n^4\Big\}.$$

Example 79 (**[1013]** $a=b=1/4$, $c=5/4$ and $d=1$)**.**

$$462\pi\sqrt{2}=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac14,\ \frac38,\ \frac78,\ \frac{7}{12},\ \frac{11}{12}\\ \frac43,\ \frac53,\ \frac54,\ \frac{19}{12},\ \frac{23}{12}\end{bmatrix}_n\times\Big\{2137+15405n+39972n^2+44400n^3+17856n^4\Big\}.$$

Example 80 (**[1013]** $a=b=3/4$, $c=7/4$ and $d=1$)**.**

$$90\pi\sqrt{2}=\sum_{n=0}^{\infty}\Big(\frac{-4}{27}\Big)^n\begin{bmatrix}\frac34,\ \frac18,\ \frac58,\ \frac{1}{12},\ \frac{5}{12}\\ \frac43,\ \frac53,\ \frac74,\ \frac{13}{12},\ \frac{17}{12}\end{bmatrix}_n\times\Big\{403+3975n+16100n^2+28368n^3+17856n^4\Big\}.$$

Example 81 (**[1023]** $a=b=1/2$ and $c=d=1$)**.**

$$\frac{512}{\pi}=\sum_{n=0}^{\infty}\Big(\frac{-1}{64}\Big)^n\begin{bmatrix}\frac12,\ \frac14,\ \frac34,\ \frac16,\ \frac56\\ 1,\ 1,\ 1,\ \frac43,\ \frac53\end{bmatrix}_n\Big\{165+2792n+10068n^2+9360n^3\Big\}.$$

Example 82 (**[1023]** $a = b = 1/2$, $c = 3/2$ and $d = 1$)**.**

$$\frac{105\pi}{2} = \sum_{n=0}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{bmatrix}_n \left(\frac{-1}{64}\right)^n \times \left\{166 + 1814n + 7421n^2 + 14347n^3 + 13224n^4 + 4680n^5\right\}.$$

Example 83 (**[1023]** $a = b = c = 1/2$ and $d = 1$)**.**

$$60\pi = 194 + \sum_{n=1}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8} \end{bmatrix}_n \left(\frac{-1}{64}\right)^n \times \frac{90 + 1703n + 10737n^2 + 30160n^3 + 38856n^4 + 18720n^5}{n}.$$

Example 84 (**[0013]** $a = 1/3$, $b = 2/3$ and $c = d = 1$)**.**

$$\frac{1287\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9} \\ 1, 1, 1, \frac{3}{2}, \frac{5}{4}, \frac{7}{4} \end{bmatrix}_n \left(\frac{27}{256}\right)^n \times \left\{1120 + 23202n + 130464n^2 + 260253n^3 + 166941n^4\right\}.$$

Example 85 (**[0013]** $a = b = 1/2$, $c = 3/2$ and $d = 1$)**.**

$$105\pi = \sum_{n=0}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{bmatrix}_n \left(\frac{27}{256}\right)^n \times \frac{317 + 3842n + 18247n^2 + 40606n^3 + 42192n^4 + 16488n^5}{n}.$$

Example 86 (**[0013]** $a = b = c = 1/2$ and $d = 1$)**.**

$$60\pi = 149 + \sum_{n=1}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8} \end{bmatrix}_n \left(\frac{27}{256}\right)^n \times \frac{180 + 3167n + 19156n^2 + 52844n^3 + 67896n^4 + 32976n^5}{n}.$$

Example 87 (**[1034]** $a = b = 1/2$ and $c = d = 1$)**.**

$$\frac{65536}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{432}\right)^n \begin{bmatrix} \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \\ 1, 1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3} \end{bmatrix}_n \times \left\{20895 + 564400n + 4058132n^2 + 12197472n^3 + 16285440n^4 + 7981056n^5\right\}.$$

Example 88 (**[1133]** $a = 1/3$, $b = 2/3$ and $c = d = 1$)**.**

$$\frac{19683\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2}, \frac{5}{4}, \frac{7}{4} \end{bmatrix}_n \times \frac{5420 + 129105n + 737874n^2 + 1474848n^3 + 943488n^4}{27^{2n}}.$$

Example 89 (**[1133]** $a = b = c = 1/5$ and $d = 1$)**.**

$$\frac{3024\pi\sqrt{2}}{\sqrt{5 - \sqrt{5}}} = 8070 + \sum_{n=1}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{9}{10}, \frac{9}{10} \\ \frac{4}{3}, \frac{5}{3}, \frac{17}{10}, \frac{11}{15}, \frac{16}{15}, \frac{19}{20}, \frac{29}{20} \end{bmatrix}_n \times \frac{1134 + 34323n + 232833n^2 + 645080n^3 + 796500n^4 + 364000n^5}{27^{2n} \cdot n}.$$

Example 90 (**[1133]** $a = b = c = 3/5$ and $d = 1$)**.**

$$\frac{2688\pi\sqrt{2}}{\sqrt{5+\sqrt{5}}} = 4430 + \sum_{n=1}^{\infty} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{3}{5}, \frac{7}{10}, \frac{7}{10} \\ \frac{4}{3}, \frac{5}{3}, \frac{11}{10}, \frac{13}{15}, \frac{23}{15}, \frac{17}{20}, \frac{27}{20} \end{bmatrix}_n \times \frac{1008+23816n+167281n^2+509295n^3+705500n^4+364000n^5}{27^{2n} \cdot n}.$$

Example 91 (**[1124]** $a = b = 1/2$ and $c = d = 1$)**.**

$$\frac{9216}{\pi} = \sum_{n=0}^{\infty} \Big(\frac{3}{8}\Big)^{6n} \begin{bmatrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \end{bmatrix}_n \times \Big\{2925 + 71802n + 554280n^2 + 1794832n^3 + 2464912n^4 + 1195040n^5\Big\}.$$

Example 92 (**[1124]** $a = 1/4$, $b = 3/4$ and $c = d = 1$)**.**

$$\frac{262144}{\pi\sqrt{2}} = \sum_{n=0}^{\infty} \Big(\frac{3}{8}\Big)^{6n} \begin{bmatrix} \frac{1}{4}, \frac{3}{4}, \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{bmatrix}_n \times \Big\{58905 + 1578944n + 8002656n^2 + 14939136n^3 + 9560320n^4\Big\}.$$

Example 93 (**[1124]** $a = b = c = 1/2$ and $d = 1$)**.**

$$360\pi = 1119 + \sum_{n=1}^{\infty} \Big(\frac{3}{8}\Big)^{6n} \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{4}, \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8} \end{bmatrix}_n \times \frac{1440+32223n+275358n^2+1162864n^3+2565376n^4+2806352n^5+1195040n^6}{n}.$$

Example 94 (**[1124]** $a = b = c = 3/4$ and $d = 1$)**.**

$$4320\pi\sqrt{2} = 19119 + \sum_{n=1}^{\infty} \Big(\frac{3}{8}\Big)^{6n} \begin{bmatrix} 1, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{5}{12}, \frac{5}{12} \\ \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \frac{11}{8}, \frac{13}{16}, \frac{17}{16}, \frac{21}{16}, \frac{25}{16} \end{bmatrix}_n \times \frac{17280+531375n+5322772n^2+25256544n^3+62390912n^4+77511424n^5+38241280n^6}{n}.$$

Example 95 (**[1134]** $a = 1/5$, $b = 4/5$ and $c = d = 1$)**.**

$$\frac{5^{10}}{2\pi\sqrt{2}}\sqrt{5-\sqrt{5}} = \sum_{n=0}^{\infty} \Big(\frac{27}{50000}\Big)^{n} \begin{bmatrix} \frac{2}{5}, \frac{3}{5}, \frac{1}{10}, \frac{9}{10}, \frac{1}{15}, \frac{4}{15}, \frac{11}{15}, \frac{14}{15} \\ 1, 1, 1, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4} \end{bmatrix}_n \times \left\{ \begin{matrix} 1826748 + 77513850n + 750064850n^2 + 3126564375n^3 \\ +6462824375n^4 + 6516000000n^5 + 2555437500n^6 \end{matrix} \right\}.$$

Example 96 (**[1134]** $a = 2/5$, $b = 3/5$ and $c = d = 1$)**.**

$$\frac{5^{10}}{2\pi\sqrt{2}}\sqrt{5+\sqrt{5}} = \sum_{n=0}^{\infty} \Big(\frac{27}{50000}\Big)^{n} \begin{bmatrix} \frac{1}{5}, \frac{4}{5}, \frac{3}{10}, \frac{7}{10}, \frac{2}{15}, \frac{7}{15}, \frac{8}{15}, \frac{13}{15} \\ 1, 1, 1, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4} \end{bmatrix}_n \times \left\{ \begin{matrix} 2954952 + 87237600n + 779867350n^2 + 3165466875n^3 \\ +6481206875n^4 + 6516000000n^5 + 2555437500n^6 \end{matrix} \right\}.$$

Example 97 ([**4488**] $a = b = 1/2$ and $c = d = 1$)**.**

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{ \frac{(8n)!}{(4n)!(3+4n)!} \right\}^3 \frac{1}{2^{31+48n}} \times \left\{ \begin{array}{r} 147650093325+8160777962064n+139098317327424n^2 \\ +1197446420123648n^3+6134230398836736n^4 \\ +20072164579737600n^5+43115678544691200n^6 \\ +60594339629236224n^7+53671368012595200n^8 \\ +27188396308824064n^9+6004799145246720n^{10} \end{array} \right\}.$$

Example 98 ([**4488**] $a = b = 1/3$, $c = 4/3$ and $d = 1$)**.**

$$\frac{\pi}{\sqrt{3}} = 2^7 3^6 \sum_{n=0}^{\infty} \frac{(12n)!(12+12n)!}{(24+24n)!} \left\{ \begin{array}{r} 25177954620+758453650825n \\ +9733209838197n^2+70930479893210n^3 \\ +328524057424620n^4+1021122326418600n^5 \\ +2183094932530176n^6+3222033941162880n^7 \\ +3226937631759360n^8+2094002610739200n^9 \\ +794122229809152n^{10}+133590654812160n^{11} \end{array} \right\}.$$

Example 99 ([**3588**] $a = b = 1/2$ and $c = d = 1$)**.**

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{ \frac{(6n)!(10n)!}{(3n)!(5n)!(7+8n)!} \right\}^3 \frac{(1+2n)^3}{2^{22+48n}} \times \left\{ \begin{array}{r} 170923431471384375+1529758516138947 6750n \\ +56547066148732887 2100n^2+11994998601076524911880n^3 \\ +1671826805587160147422 08n^4+1646467252861184903405952n^5 \\ +11975534932107848739688704n^6+66202254492960380950225408n^7 \\ +283488775041181863422083584n^8+951933287952149824105513984n^9 \\ +2524298122093567646814943232n^{10}+5299439637329702566400421888n^{11} \\ +8791167003335176149318615040n^{12}+11445656080075560220793733120n^{13} \\ +11545634384559874555665776640n^{14}+8832431201134981263461580800n^{15} \\ +4948699155399925991009615872n^{16}+1913934453266746935647010816n^{17} \\ +456352821126098194819973120n^{18}+50528158441688990812209152n^{19} \end{array} \right\}.$$

Example 100 ([**3588**] $a = b = 1/3$, $c = 4/3$ and $d = 1$)**.**

$$\frac{2\pi}{\sqrt{3}} = 3^9 \sum_{n=0}^{\infty} \left\{ \frac{(9n)!(15n)!}{(23+24n)!} \right\} (1+3n)(2+3n) \times \left\{ \begin{array}{r} 2382277599313920000+147373047270939033600n+414070345571385811 2000n^2 \\ +7053111893816120116 4288n^3+81960359948327089499 4016n^4 \\ +692684451817775518004 2304n^5+4429136330910895477868 5648n^6 \\ +21990769742708193073551 1784n^7+86279935710290627339902 2538n^8 \\ +270628062445320096471702 5178n^9+683441366631158851538004 5481n^{10} \\ +1394084710423528240776489 0810n^{11}+2295875327278476603270424 4865n^{12} \\ +3039689424760411723205260 0932n^{13}+3207319342809975439690187 7594n^{14} \\ +2658183789378902914077381 3276n^{15}+1691423817932535803945267 5692n^{16} \\ +797036450640343924082066 5986n^{17}+261862328800095840603362 9277n^{18} \\ +53522423207011058165358 7842n^{19}+5121752748567512315031 2889n^{20} \end{array} \right\}.$$

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