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DRESHER'S INEQUALITY

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Melvin Dresher has proved by an ingenious method, using moment-space theory[1], the inequality which I shall call Dresher's inequality:

If $p \geq 1 \geq r \geq 0$, $f(x) \geq 0$, $g(x) \geq 0$, *and* $\phi(x)$ *is a distribution function*[2], *then*[3]

(1) $$\left[\frac{\int [f(x)+g(x)]^p d\phi(x)}{\int [f(x)+g(x)]^r d\phi(x)}\right]^{\frac{1}{p-r}} \leq \left[\frac{\int f(x)^p d\phi(x)}{\int f(x)^r d\phi(x)}\right]^{\frac{1}{p-r}} + \left[\frac{\int g(x)^p d\phi(x)}{\int g(x)^r d\phi(x)}\right]^{\frac{1}{p-r}}$$

~~It is the purpose of~~ this note ~~to~~ gives an elementary proof of ~~this~~ The Dresher inequality, based on the Minkowski inequality and an inequality due to Radon[4]. The Radon inequality is gotten easily by transforming the Hölder

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1 Forthcoming.

2 In particular, we may take $\phi(x) = x$, and read each $d\phi(x)$ as dx.

3 We assume that neither f nor g vanishes identically.

4 J. Radon, Über die absolut additiven Mengenfunktionen, Wiener Sitzungsber. (II a), 122 (1913), p. 1351. See also Hardy, Littlewood, and Polya, Inequalities Cambridge (1934) p. 61 problem 65, and C. B. Morrey, A class of representations of manifolds, Amer. J. Math. 55 (1933), p. 602.

inequality[§], thus Dresher's inequality is seen to be a mélange of the Hölder and Minkowski inequalities. It is appropriate to note here that Dresher's inequality is a generalization of an inequality due to Beckenbach[•].

The Radon inequality is as follows:

<u>If</u> $\lambda > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $i = 1, \ldots, n$, <u>then</u>

$$(2) \qquad \frac{(\sum \alpha_i)^{\lambda+1}}{(\sum \beta_i)^{\lambda}} \leq \sum \frac{\alpha_i^{\lambda+1}}{\beta_i^{\lambda}}$$

In using this to prove Dresher's inequality, we shall need only two terms. The inequality is then stated as follows:

<u>If</u> $\lambda > 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\beta_1 \geq 0$, <u>and</u> $\beta_2 \geq 0$, <u>then</u>

$$(3) \qquad \frac{(\alpha_1 + \alpha_2)^{\lambda+1}}{(\beta_1 + \beta_2)^{\lambda}} \leq \frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}} .$$

[§] Hardy, Littlewood, Polya, loc. cit.

[•] E. F. Beckenbach, A class of mean value functions, Amer. Math. Monthly <u>57</u> (1950), pp. 1-6.

We make in (3) the substitutions

$$\alpha_1 = \left[\int f^p d\phi\right]^{\frac{1}{p}} \qquad \beta_1 = \left[\int f^r d\phi\right]^{\frac{1}{r}}$$

$$\alpha_2 = \left[\int g^p d\phi\right]^{\frac{1}{p}} \qquad \beta_2 = \left[\int g^r d\phi\right]^{\frac{1}{r}} \qquad \lambda = \frac{r}{p-r}.$$

Here, for the moment, we have assumed $p > r > 0$

Then the right side of (3) becomes the right side of (1), and the left side of (3) becomes

$$(4) \qquad \frac{\left\{\left[\int f^p d\phi\right]^{\frac{1}{p}} + \left[\int g^p d\phi\right]^{\frac{1}{p}}\right\}^{\frac{p}{p-r}}}{\left\{\left[\int f^r d\phi\right]^{\frac{1}{r}} + \left[\int g^r d\phi\right]^{\frac{1}{r}}\right\}^{\frac{r}{p-r}}}$$

But by Minkowski's inequality with $p \geq 1$ and $0 < r \leq 1$ respectively, both

$$(5) \qquad \left\{\left[\int f^p d\phi\right]^{\frac{1}{p}} + \left[\int g^p d\phi\right]^{\frac{1}{p}}\right\}^{p} \geq \int (f + g)^p d\phi$$

and

$$(6) \qquad \left\{ \left[\int f^r d\phi \right]^{\frac{1}{r}} + \left[\int g^r d\phi \right]^{\frac{1}{r}} \right\}^r \leq \int (f + g)^r d\phi \ ,$$

so that the quantity (4) is not less than the expression on the left side of (1), and Dresher's inequality is proved for $p \geq 1$ and $p > r > 0$. Extension to the remaining cases is trivial, and thus the proof is complete.

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