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Dry rigid block masonry: safe solutions in presence of Coulomb friction

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Abstract

Part of this paper refers to a research study developed by Jossa and the writer [1] now awaiting publication.
It is well known that the static and kinematic behaviour of ancient dry block masonry structures is predominantly regulated by two parameters: self-weight and friction. And it is also well known that any investigation on the safety levels of these structures implies difficulties of analysis, due to the non-associated flow rules imposed by friction. Possibilities of non-unique solutions are directly consequent. This is, therefore, one of the main problems in every programme for the conservation and repair of such structures.
The basic guideline of this research is to provide appropriate requirements to treat frictional materials within the framework of the standard limit analysis. This is possible, as shown in the paper, if a way to limit the space of statically admissible solutions can be defined in favour of safety.
Generally, in classic plastic theory, an equilibrated distribution of internal forces gives a safe solution if the relative virtual work done, with reference to the true collapse configuration of the structure, is not greater than the virtual work done by the true force distribution.
Therefore, in order to define a safe rigid-plastic model for frictional materials, a heuristic procedure to evaluate the minimum value of normal forces, with regard to the whole set of the statically admissible distributions, is herein proposed. Thus, a safe solution is obtainable with standard limit analysis by assuming the limiting frictional resistance to be that associated with this minimum normal force.
The heuristic aspect of this procedure is particularly highlighted in the two case studies herein analysed: the 2D masonry wall subject to in-plane traction forces and the voussoir arch subject to its own weight.

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1 Introduction

Investigation of the behaviour of masonry structures comprising rigid blocks continued for most of the last century, Kooharian and Heyman ([2], [3]), being the first to provide a description of vaulted block structures in terms of limit analysis. These authors treated the analysis of such structures under the assumption that friction between block interfaces was sufficiently high to prevent sliding. This leads, in the case of arches, to the well known hinging mechanisms discussed by Heyman [4].

However, especially for ancient buildings, the quality of the contact surfaces or of the binding materials might be so deteriorated as to reduce the original friction coefficient substantially. In particular, with reference to ancient dry block masonry structures we know that friction and self-weight predominantly regulate the structural behaviour. In addition, some particular shapes of masonry structures, e.g. flat arches, would never collapse unless sliding occurred.

Hence, it is necessary to study this group of problems under the more realistic assumptions of presence of sliding. It is also known that within these problems the bounding theorems of plastic limit analysis do not generally provide unique solutions for collapse loads, due to the non-associated flow rules imposed by friction.

Drucker [5] was perhaps the first to point out that whilst the exact solution to a problem involving Coulomb friction interfaces could be bounded from above and below, unfortunately such bounds will often be too wide to be of use in practice. In this class of problems, in fact, the bound from below is generally the condition that assumes no friction and cannot obviously be propounded for the analysis of most masonry structures.

Over the last forty years many researchers have attempted to study this issue further, with varying degrees of success ([6], [7], [8], [9], [10], [11], [12], [13]). However a review of existing limit analysis approaches for block masonry structures has revealed a lack of such consistent methods that assures the reliability of solutions for non-standard materials.

Jossa and the present writer have done extensive work on this theme ([14], [15], [16], [17]). Particularly their more recent studies have allowed them to define significant cases in which the solution is unique and easily attainable ([18], [19]), or cases in which certain procedures can be developed to achieve a solution and, at the same time, to test its limits of validity [20].

In this paper we are interested in providing appropriate requirements for the treatment of this class of block masonry as a standard unilateral rigid-plastic material, with associated flow rules. This goal can be achieved by defining an appropriate model through a few general rules chosen for their safety factors, as proposed in the following sections.

2 A strategy for the analysis

We begin by pointing out that, as a consequence of the lack of available compatibility conditions, we must work within admissible equilibrium states.

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We then look at a general model that could be easily used in static limit analysis. We assume that friction obeys a cohesionless linear Coulomb's law, for which the normality flow rule is not generally satisfied in plastic analysis. This means that admissible equilibrium conditions cannot certainly assure a safety state of the structure, as it would be required. Therefore, we must deal with this problem first.

Consider the Coulomb's cone in Figure 1(a). It is both known ([21], [22]) and obvious, that, if the normal force is given at some block interface, then it can be ignored in defining the limit surface. Hence, the cone reduces to a circle, with a constant value of the limiting friction force and with the normality rule once again guaranteed, as shown in Figure 1(b). Obviously, in 2D problems, such as those we are herein interested, the Coulomb's cone and the circle reduce to a bilateral yield line and to a line segment, respectively.

This statement is assumed to be the basic guideline of the present work.

The main goal of this research is to propose a heuristic procedure to construct such reference sub-models for frictional material in order to bound the range of all the statically admissible solutions from above. These sub-models, with regard to the statement previously described, can allow us to develop static analysis and to work within a safe class of statically admissible solutions.

In point of fact, let us suppose that, at every block interface of a masonry structure, we have succeeded in evaluating the minimum absolute value that the compressive normal force can assume, with regard to the whole set of the statically admissible solutions. Let N_{min} be this value and let φ be the limit friction angle at the interface. Then, according to the Coulomb's law and in favour of safety, we may assume that at that interface the masonry behaves essentially as a standard rigid-plastic material with the limiting value of shear forces defined by the product $N_{min}\tan\varphi$. This means that the internal virtual work done by these limiting shear forces, related to an equilibrated solution and with respect to the true collapse configuration of the structure, results not greater than the internal work done by the true shear forces. As a consequence, the load factor corresponding to any statically admissible solution is not greater than the true collapse factor and hence the solution is safer than the exact one. In other words the construction of such minimum yield conditions must guarantee high levels of reliability in the analysis, in observance of the normality rule.

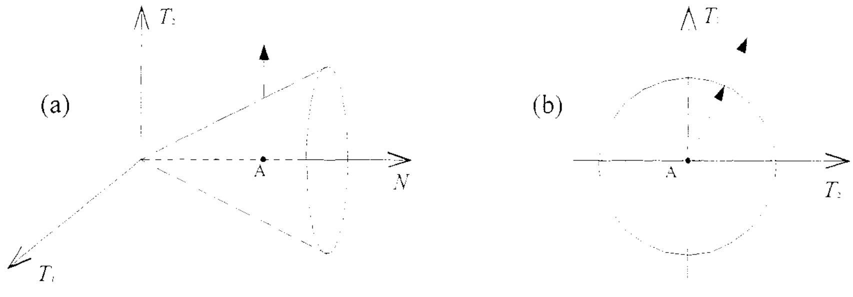

Figure 1: (a) Cohesionless Coulomb's cone
(b) Yield domain in the plane of the shear forces.

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Moreover this safety criterion allows us, as shown in later examples and better explained in [1], also to take into account irregularities of shape and layout that always characterise real masonry structures.
However, the theoretical simplicity of the described statement does not correspond to practical simplicity as well, because the local minimum normal force is not always easily attainable. This is due to difficulties in examining both every block interface, especially inside the complexity of a masonry wall, and all the statically admissible distributions of internal forces in a masonry structure.
This means that the formulation of a general procedure may not be obtainable and might lead us to abandon the attempt. Actually, we can still decide:

a) to define only certain general rules to follow in order to formulate a possible procedure;
b) to specify an appropriate method for particular cases.

Both these suggestions have been developed in the following sections, with reference to the two case studies of masonry walls subject to in-plane loads and voussoir arches under their own weight.

2.1 The model and a few general rules

Although it seems difficult to investigate every block interface in a masonry structure so as to determine the proposed N_{min}, our attention can easily be focused on a fracture line passing through a certain number of block interfaces where sliding between blocks is prevalently concentrated. It should be observed that this line, as the site of global sliding failure, occurs when the resultant frictional resistance, linearly dependent on the total weight resting on it, has been achieved. Therefore, according to the strategy described above, it is in our interest to identify such N_{min} as the minimum resultant of the normal forces acting on all block interfaces involved in the fracture line.
On the other hand the definition of the proposed reference sub-model can be useful only if easily practicable. To this end it was decided to search such integral values of N_{min}, along possible fracture lines, given by equilibrated distributions of internal forces, not necessarily admissible anywhere in the structure. This means that we have chosen to work inside a class of equilibrated distributions not less than the class of statically admissible ones. It is, once again, for the sake of safety.
For our purpose the triangular masonry scheme in Figure 2(a), with rigid blocks interacting only along the horizontal joints, has been appropriately chosen to analyse a limiting situation. It represents, for a masonry wall, the maximum distribution of the load Q acting on its top and is referred to the well known Tartaglia's triangle.
Due to the absence of shear forces at the interfaces the solution is statically determined and the distribution of Q is such that the generic normal force acting on the k^{th} support point and at the i^{th} row can be written in this form:

$$N_{i,k} = \frac{\binom{i-1}{k-1}}{2^{i-1}} Q = \frac{\frac{(i-1)(i-2)...(i-k+1)}{(k-1)(k-2)...1}}{2^{i-1}} Q \qquad (1)$$

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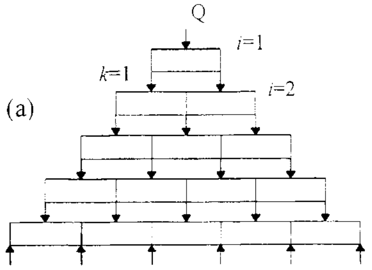

Figure 2: The Tartaglia's distribution

These values can be combined with a set of self-equilibrated reactions, as shear forces limited by friction. Indeed, as discussed elsewhere [1], although these forces contribute to maximise the spreading of Q, they do not affect the safety level of the analysis and are, therefore, not included in this law of spreading.
On the other hand, the Tartaglia's model is properly defined only to maximise the distribution of a single load. Obviously, its validity does not hold when referred to the whole structure, because it does not include such possible combinations of several loads as would otherwise allow greater spreading. The heuristic nature of this procedure is here quite evident.
However, by assuming that the chosen distribution corresponds approximately to a maximum possible spreading of loads inside block masonry, this procedure reveals its particular relevance. In fact, we shall later show that, in many situations, the required N_{min} on a fracture line does depend on this described condition of maximum distribution.
As a numerical example Figure 3 sketches the resultant normal forces acting on the fifth and tenth row of a scheme made by UNI blocks (6x12,5x25 cm) and the corresponding angles φ_1 of maximum spreading.
It should be noted that the angle φ_1 is defined by the line linking the application points of Q and the resultant normal force acting on a half row.

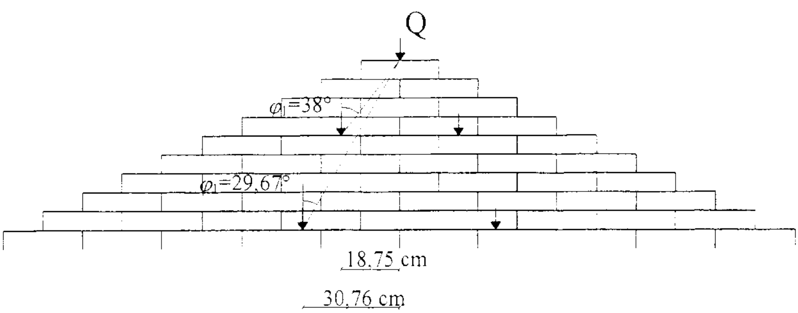

Figure 3: Representation of the angle φ_1.

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3 Case studies

3.1 Masonry wall subject to its own weight and to horizontal forces

The strategy proposed can be adopted, for example, in evaluating the frictional resistance for a retaining wall provided by a buttress wall connected to it (Fig. 4). In this case, if we assume that the dead load of the buttress wall is uniformly distributed, e.g. within the hypothesis of regularity of shape and layout of the wall, all the fracture lines shown in Figure 5 will have the same probability of occurring. That is to say that the resultant frictional resistance along a generic fracture line (e.g. line 2 in Figure 5) is equal to the resistance along the maximum possible slope (line 1 in Figure 5), defined by the dimension of the blocks. Then, with reference to the scheme in Figure 6(a), being:

$$\Omega = \frac{h^2 \tan\alpha}{2} \tag{2}$$

the area of the triangle of masonry bounded by the maximum slope (A-B = line 1) and the vertical from its lowest point (B-G), forming angle α, the said frictional resistance will be:

$$T = \gamma\, b\, \Omega \tan\varphi \tag{3}$$

where γ, b and $\tan\varphi$ are, respectively, the specific weight, the thickness and the friction coefficient between the blocks of the wall being analysed.
With reference now to a generic fracture line (C-B = line 2) inclined at $\underline{\alpha}$ in respect to the vertical, the resultant frictional resistance is still given by eqn (3) but the area of masonry now concerned will be:

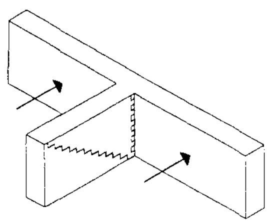

Figure 4: Retaining wall, with buttress, subject to horizontal forces.

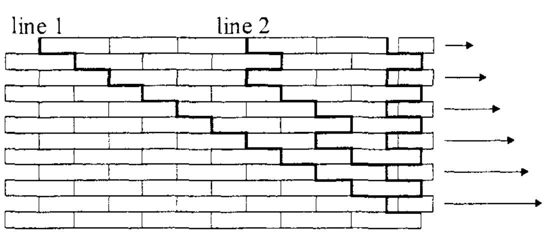

Figure 5: Possible cracks in the buttress wall.

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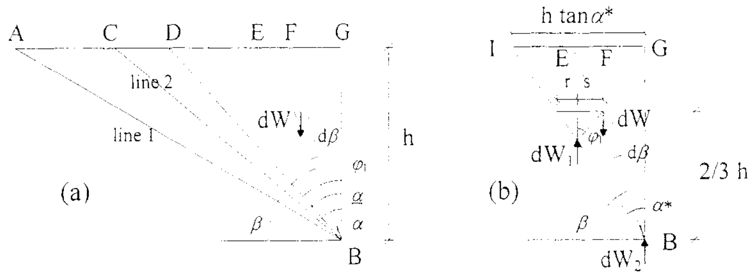

Figure 6: (a) Scheme of possible cracks.
(b) Distribution of the elementary weight dW according to angle φ_1.

$$\underline{\Omega} = \frac{h^2 \tan \underline{\alpha}}{2} \tag{4}$$

Therefore, to determine the frictional resistance as function of the area Ω (and the corresponding weight of masonry) we increase $\underline{\Omega}$ by the coefficient $\delta = \Omega/\underline{\Omega}$. This coefficient, since it expresses an equivalence justified only for geometrical reasons, is still valid when we take into account irregularities in the dead-load distribution. In this case, in fact, the search for N_{min} to define a safe resistance criterion does not alter the number of interfaces involved in the fracture line, but refers only to the reduction of weight resting on them.

Let us, then, imagine that collapse is caused by the activation of the fracture line C-B in Figure 6(a) (line 2). Having fixed angle φ_1 as reference for the spreading, in agreement with what has previously been said, we may distinguish two cases:

a) $\underline{\alpha} > \varphi_1$

In this case, although all the weight resting on the interfaces of the fracture line (region C-B-G), must be increased by the coefficient δ previously described, part of it (region D-B-G), defined by the spreading angle φ_1, may result in a different distribution, which we now wish to identify with the aim of minimising the normal force. Therefore, being:

$$dW = \frac{\gamma \, b \, h^2}{2 \sin^2 \beta} d\beta \tag{5}$$

the weight of the elementary cone of masonry E-B-F, it can easily be observed, in Figure 6(b), that the smallest value of dW_1 acting on the generic line I-B is found when the equilibrium condition maximises the distance s between the two forces dW and dW_1. Knowing that such maximisation is guaranteed, even if approximately, by the definition of reference angle φ_1 for the spreading, and that, in this case, the two distances r and s in Figure 6(b) are equal (since $\alpha^* = \varphi_1$), we simply obtain:

$$dW_1 = dW \frac{2}{1 + \tan\beta \tan\varphi_1} \tag{6}$$

In consequence, on the interfaces involved in the fracture line C-B in Figure 6(a), the minimum normal force will be:

$$N_{min} = \left[\gamma\, b\, h^2 \int_{\pi/2-\varphi_1}^{\pi/2} \frac{1}{\sin^2\beta\,(1+\tan\beta\,\tan\varphi_1)}\, d\beta + W_{(CBD)} \right] \delta \qquad (7)$$

where $W_{(CBD)}$ is the weight of the region C-B-D and δ is the amplifying coefficient (δ = Area$_{(ABG)}$/Area$_{(CBG)}$ = $\tan\alpha / \tan\underline{\alpha}$).

By integrating function (7) we get:

$$N_{min} = \left\lfloor \gamma\, b\, h^2 \left(1 + \ln(\cos\varphi_1) + \ln(\sin\varphi_1) - \ln(\sin(2\,\varphi_1))\right) \tan\varphi_1 + W_{(CBD)} \right\rfloor \delta \qquad (8)$$

b) $\underline{\alpha} < \varphi_1$

In this case the part of masonry resting on the interfaces of the fracture line is inside the slope of the spreading angle and the new value of dW_1, taking into account that $r = s\tan\underline{\alpha}\cot\varphi_1$ (since $\alpha^* = \underline{\alpha}$), becomes:

$$dW_1 = dW \frac{\tan\underline{\alpha} + \tan\varphi_1}{\tan\underline{\alpha}\,(1+\tan\beta\,\tan\varphi_1)} \qquad (9)$$

The integral of function (9), determined for the range ($\pi/2$-$\underline{\alpha}$) – ($\pi/2$), on the fracture line gives:

$$N_{min} = \frac{\gamma\, b\, h^2}{2} \left(\frac{\tan\underline{\alpha} + \tan\varphi_1}{\tan\underline{\alpha}} \right) \left[\tan\underline{\alpha} + \begin{pmatrix} \ln(\cos\underline{\alpha}) + \\ \ln(\sin\varphi_1) + \\ -\ln(\sin(\underline{\alpha} + \varphi_1)) \end{pmatrix} \tan\varphi_1 \right] \delta \qquad (10)$$

In conclusion, with reference to the possible fracture line C-B, one can safely assign the limiting shear force:

$$T = N_{min} \tan\varphi \qquad (11)$$

where φ is the friction angle and N_{min} is obtained from eqn (8) or from eqn (10) depending on the case.

By way of example for case a) let us consider a masonry wall consisting of UNI blocks (6x12,5x25 cm) and with specific weight γ = 18 KN/m^3, in which a crack inclined at angle $\underline{\alpha}$ = 40° in respect to the vertical, involving a height h = 0,54 m (equal to 9 rows), is generated.

Being φ_1 = 29,67° the relative spreading angle found in the example in Figure 3, and φ = 32° the friction angle, the limiting shear force along this line according to eqn (11) is T = 315 N, equal to about 73,8% of the resistance corresponding to the hypothesis of regular and uniform weight distribution of the wall.

3.2 Voussoir arch subject to its own weight

A circular masonry arch, with radial disposition of blocks, subject to its own weight, is now considered.

In this case, in determining N_{min} acting on the possible generic radial crack, one can choose to operate in a more or less precise manner.

An approximate calculation can be made, simply based on the conditions required for the equilibrium of the two parts of the arch divided by the fracture line and considered as rigid bodies. This means, the approximation of the choices not being changed, minimising in favour of safety the normal force, by operating

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on a range of solutions wider than that which covers the statically admissible solutions for all the sections.
Alternatively, one can decide to develop the analysis by constructing true and proper statically admissible solutions.
We wish now, by way of example, to investigate the minimum normal forces at the springing joints and in certain sections of the constant thickness arch in Figure 7, operating with reference to a conventional friction angle φ^*, chosen in function of the value of φ and of the spreading angle φ_1. The problem will then be reduced to that of minimising the normal forces in the sections chosen, in respect to the said angle, with reference to all possible three-hinge arches such as arch A-C-E in Figure 7. However, the minimisation of these forces, corresponding to the components of the vectors GF, GH and GI normal to the block interfaces, will have to respect the condition that these vectors do not fall outside the friction cone.
A simple computation programme has been worked out for the minimisation procedure, which developments are omitted here for the sake of brevity.

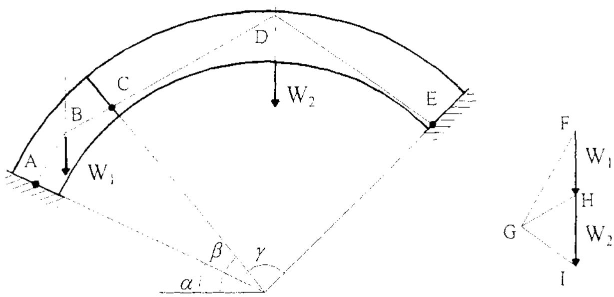

Figure 7. The voussoir arch of constant thickness

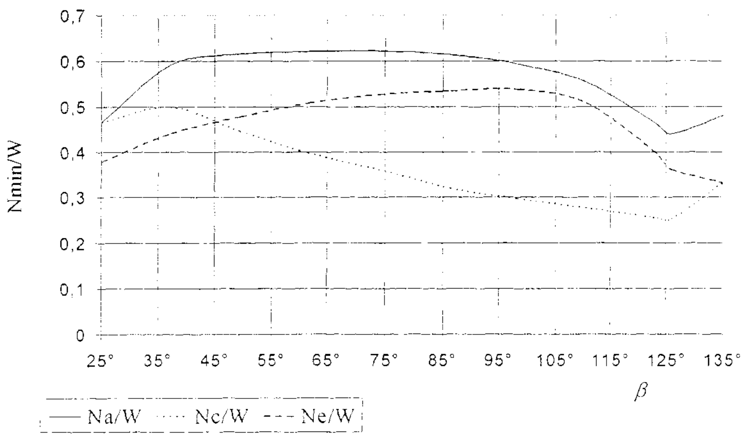

Figure 8. Minimum normal forces on the block interfaces identified by β.

Table 1. Limit shear forces on the block interfaces identified by β.

	T_a/W	T_c/W	T_e/W
$\beta = 25°$	0,326		
$\beta = 37,2°$		0,35	
$\beta = 61,65°$		0,28	
$\beta = 86,46°$		0,24	
$\beta = 110,56°$		0,21	
$\beta = 135°$			0,23

Figure 8 gives the results in terms of the adimensional ratios N_{min}/W (W is the total weight of the arch) in function of the position of the generic sections of the arch identified by β. The results refer to a constant thickness arch with a ratio thickness/average radius of 0,2, angles of embrace $\alpha = 25°$ and $\gamma = 135°$ and conventional friction angle $\varphi^* = 35°$.
The values obtained can now be used to construct a safe solution with limiting shear forces given in table 1.

4 Conclusions

A safe rigid-plastic model for frictional materials such as dry block masonry structures has been set out in this paper in order to treat them within the framework of the standard limit analysis. This provides such reduced resistance conditions for problems involving Coulomb friction that any statically admissible solution succeeds in assuring a safety state for them.
To this end a heuristic procedure has been outlined herein, accounting for both the irregularities of shape and layout of real masonry structures.
On the other hand the approximation that some choices have required so as to reduce the complexity of such problems, is rightly accomplished for the sake of reliability that sustains this research.
Further development of the model and the heuristic procedure are referred to [1].

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References

[1] Casapulla, C. & Jossa P. Heuristic safety methods in static analysis of block masonry. To be pubblished.
[2] Kooharian, A. Limit analysis of voussoir (segmental) and concrete arches. *Journ. Amer. Conc. Inst.*, **24(4)**, pp. 317-328, 1952.

[3] Heyman, J. The stone skeleton. *Int. Journ. of Solids and Structures*, **2**, pp. 249-279, 1966.
[4] Heyman, J. *The masonry arch*, Ellis Horwood: Chichester, 1982.
[5] Drucker, D. C. Coulomb friction, plasticity, and limit loads. *Journ. Appl. Mec. Trans. ASME*, **21(1)**, pp. 71-74, 1954.
[6] Livesley, R. K. Limit analysis of structures formed from rigid blocks. *Int. Journ. for Num. Meth. in Eng.*, **12**, pp. 1853-71, 1978.
[7] Lo Bianco, M. & Mazzarella, C. Limit load of masonry structures. *Final Report of the IABSE Symp.*, Venice, pp. 187-194, 1983.
[8] Baggio C., Masiani R. & Trovalusci P. Modelli discreti per lo studio della muratura a blocchi. *Proc. of the 5th Nat. Conf. on L'Ingegneria Sismica in Italia*, Palermo, vol. II, pp. 1205-18, 1991.
[9] D'Asdia, P. & D'Ayala, D. Limit analysis of block masonry shell structures. *Final Report of the IABSE Symposium*, Rome, pp. 353-360, 1993.
[10] Gilbert, M. & Melbourne, C. Rigid-block analysis of masonry structures. *The Struct. Eng.*, **72**, pp. 356-361, 1994.
[11] Boothby, T. E. Stability of masonry piers and arches including sliding. *Journ. Engin. Mech. ASCE*, **120(2)**, pp. 304-319, 1994.
[12] Begg, D. W. & Fishwick, R. J. Numerical analysis of rigid block structures including sliding. *Comp. Meth. Struct. Masonry*, **3**, pp. 177-183, 1995.
[13] Sinopoli A., Corradi M. & Foce F. Modern formulation for preelastic theories on masonry arches. *Journ. Engin. Mech. ASCE*, **2**, pp. 204-213, 1997.
[14] Casapulla, C., De Riggi T. & Jossa P. Il ruolo dell'attrito nella resistenza di pareti murarie. *Ing. Sism.*, **3**, pp. 32-40, 1996.
[15] Casapulla, C., De Riggi T. & Jossa P. Cinematica di blocchi affiancati con attrito in presenza di azioni sismiche. *Ing. Sism.*, **3**, pp. 21-34, 1998.
[16] Jossa, P. Reliability, a background of common rationality in Architecture. *Proc. of the World Congr. on The human being and the city*, Naples, 2000.
[17] Casapulla, C. Problematiche relative all'analisi limite di strutture murarie a blocchi in presenza di attrito. *Ph.D. Thesis*, Dept. of Construction, Univ. of Florence, 1999.
[18] Casapulla, C. & Lauro, F. A simple computation tool for the limit-state analysis of masonry arches. *Proc. of the 5th Int. Cong. on Restoration of Architectural Firenze Heritage 2000*, Florence, 2000.
[19] Casapulla, C. & D'Ayala D. Lower bound approach to the limit analysis of 3D vaulted block masonry structures. To be published in *Proc. of the 5th Int. Symp. on Computer Methods in Structural Masonry (STRUMAS V)*, Rome, 2001.
[20] Casapulla, C. Resistenze attritive in una parete muraria soggetta ad azioni normali al suo piano medio. *Proc. of the 9th Nat. Conf. on L'Ingegneria Sismica in Italia*, Turin, 1999.
[21] Michalowski R. & Mroz Z. Associated and non-associated sliding rules in contact friction problems. *Arch. of Mech.*, **30(3)**, pp. 259-276, 1978.
[22] Goyal S., Ruina A. & Papadopoulos J. Planar sliding with dry friction. Part 1. Limit surface and moment function. *Wear*, **143(2)**, pp. 307-330, 1991.