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DUAL FINITE ELEMENT ANALYSIS FOR CONTACT PROBLEM OF ELASTIC BODIES WITH AN ENLARGING CONTACT ZONE

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Summary. Dual finite element analysis of the contact problem of two elastic bodies with an enlarging contact zone is presented. Approximations of the solution are defined on two types of triangulations by piecewise constant stress fields. Convergence is proved in both cases.

Keywords: contact of elastic bodies, dual variational formulation, dual finite element method

AMS Subject class.: 65 N 30, 73 C 99

INTRODUCTION

Dual finite element analysis for the contact problem of two plane elastic bodies without friction was given in [1]−III by Haslinger and Hlaváček, but only for the case of a bounded contact zone. In the case of the contact zone enlarging during the deformation a primary finite element analysis was discussed in [1]−II. It is the aim of this paper to analyse the latter problem by a dual procedure. Two types of triangulations are used. The convergence of the dual finite approximations for each type of triangulations is proven.

1. THE DUAL VARIATIONAL FORMULATION

First of all we recall the primary variational formulation of a contact problem with an enlarging contact zone (see [1]−I, § 1 for details). We introduce the following notation: the norm and the semi-norm in the Sobolev space $H^k(\Omega)$ are denoted by $\|\cdot\|_k$, $|\cdot|_k$, respectively. $P_k(T)$ denotes the set of all polynomials of order k defined on T.

$$W = \{\mathbf{u} \mid \mathbf{u} = (\mathbf{u}', \mathbf{u}'') \in [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2\},$$

$$V = \{\mathbf{u} \in W \mid \mathbf{u}' = 0 \text{ on } \Gamma_u,\ u''_n = 0 \text{ on } \Gamma_0\},$$

$$K_\varepsilon = \{\mathbf{v} \in V \mid v''_\xi(\eta) - v'_\xi(\eta) \leqq \varepsilon(\eta) \text{ for a.e. } \eta \in \langle a, b\rangle\},$$

where $v^M_\xi(\eta)$ is the component of the displacement vector $\mathbf{v}^M$ in the direction ξ,

$\varepsilon(\eta) = f'(\eta) - f''(\eta)$; f', f'' are functions describing the arcs Γ'_K and Γ''_K, respectively (see Fig. 1), $f', f'' \in C^2(\langle a, b \rangle)$.

$$A(\mathbf{u}, \mathbf{v}) = \int_\Omega c_{ijkl} e_{ij}(\mathbf{u})\, e_{kl}(\mathbf{v})\, \mathrm{d}x\,, \quad \Omega = \Omega' \cup \Omega''\,,$$

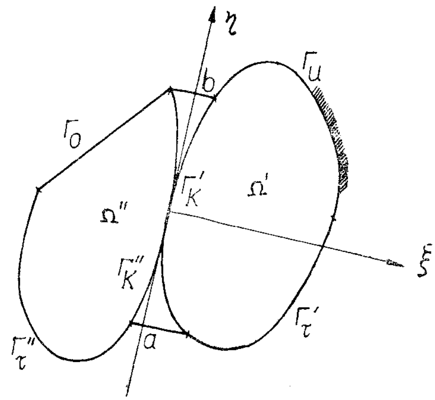

Fig. 1

where $e_{ij}(\mathbf{u}) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ are the components of the strain tensor with respect to the displacement vector $\mathbf{u}$, c_{ijkl} are coefficients defined by the generalized Hooke's law. Assume that there is a positive constant c_0 such that each symmetric tensor of the 2nd order $\tau = (\tau_{ij})$, $i, j = 1, 2$ satisfies

$$c_{ijkl}\tau_{ij}\tau_{kl} \geqq c_0 \tau_{ij}\tau_{ij} \quad \text{for a.e.} \quad x \in \Omega\,.$$

Throughout this paper we shall use the adding convention, i.e., sum from 1 to 2 for any twice repeated index.

$$L(\mathbf{v}) = \int_\Omega F_i v_i\, \mathrm{d}x + \int_{\Gamma_\tau' \cup \Gamma_\tau''} P_i v_i\, \mathrm{d}S\,,$$

where F_i, P_i are the components of the body forces and the surface loads, respectively.

$$\mathscr{L}(\mathbf{v}) = \tfrac{1}{2} A(\mathbf{v}, \mathbf{v},) - L(\mathbf{v})\,.$$

Then the primary problem is to find $\mathbf{u} \in K_\varepsilon$ such that

$$(1.1) \qquad \mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K_\varepsilon\,.$$

The existence and uniqueness of a solution were discussed in [1]–I.

We now consider the sets

$$S = \{\mathscr{N} = (\mathscr{N}_{ij}),\ i, j = 1, 2 \mid \mathscr{N}_{ij} \in L_2(\Omega),\ \mathscr{N}_{ij} = \mathscr{N}_{ji}\}\,,$$

$$\mathscr{W}_\varepsilon = S \times K_\varepsilon\,,$$

and the functional

$$\mathscr{H}([\mathcal{N}, \mathbf{v}]; \lambda) = \mathscr{L}_1(\mathcal{N}, \mathbf{v}) + \int_\Omega \lambda_{ij}(e_{ij}(\mathbf{v}) - \mathcal{N}_{ij})\, \mathrm{d}x\,; \quad [\mathcal{N}, \mathbf{v}] \in \mathscr{W}_\varepsilon\,, \quad \lambda \in S\,,$$

where

$$\mathscr{L}_1(\mathcal{N}, \mathbf{v}) = \frac{1}{2}\int_\Omega c_{ijkl}\mathcal{N}_{ij}\mathcal{N}_{kl}\, \mathrm{d}x - L(\mathbf{v})\,.$$

Lemma 1.1. 1. *If* $\{[\mathcal{N}^*, \mathbf{v}^*]; \lambda^*\}$ *is a saddle-point of the functional* $\mathscr{H}$ *on* $\mathscr{W}_\varepsilon \times S$, *i.e.*,

$$\mathscr{H}([\mathcal{N}^*, \mathbf{v}^*]; \mu) \leqq \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*]; \lambda^*) \leqq \mathscr{H}([\mathcal{N}, \mathbf{v}]; \lambda^*)$$

holds for all $[\mathcal{N}, \mathbf{v}] \in \mathscr{W}_\varepsilon$ *and* $\mu \in S$, *then a solution* $\mathbf{u}$ *of the primary problem* (1.1) *exists and*

$$\mathcal{N}^* = e(\mathbf{u})\,, \quad \mathbf{v}^* = \mathbf{u}\,, \quad \lambda^* = \tau(\mathbf{u})\,,$$

where $e(\mathbf{u})$ *and* $\tau(\mathbf{u})$ *are the strain and the stress tensor, respectively.*

2. *If* $\mathbf{u}$ *is a solution of the primary problem* (1.1), *then* $\{[e(\mathbf{u}), \mathbf{u}]; \tau(\mathbf{u})\}$ *is a saddle-point of* $\mathscr{H}$ *on* $\mathscr{W}_\varepsilon \times S$.

Proof. 1. From the properties of the saddle-point we deduce that

$$\delta_\lambda \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*]; \lambda^*) = 0 \Leftrightarrow \mathcal{N}^*_{ij} = e_{ij}(\mathbf{v})\,, \tag{1.2}$$

$$\delta_{\mathcal{N}} \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*]; \lambda^*) = 0 \Leftrightarrow \lambda^*_{ij} = c_{ijkl}\mathcal{N}^*_{kl}\,, \tag{1.3}$$

$$\delta_v \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*]; \lambda^*)(\mathbf{v} - \mathbf{v}^*) \geqq 0 \quad \forall \mathbf{v} \in K_\varepsilon\,, \tag{1.4}$$

where e.g. $\delta_\lambda \mathscr{H}$ denotes the partial Gâteaux differential of $\mathscr{H}$ with respect to λ.

From (1.4) it follows that

$$\int_\Omega \lambda^*_{ij} e_{ij}(\mathbf{v} - \mathbf{v}^*)\, \mathrm{d}x \geqq L(\mathbf{v} - \mathbf{v}^*) \quad \forall \mathbf{v} \in K_\varepsilon\,. \tag{1.5}$$

Making use of (1.5), (1.3) and (1.2), we obtain

$$\int_\Omega c_{ijkl} e_{ij}(\mathbf{v} - \mathbf{v}^*)\, e_{kl}(\mathbf{v})\, \mathrm{d}x \geqq L(\mathbf{v} - \mathbf{v}^*) \quad \forall \mathbf{v} \in K_\varepsilon\,,$$

i.e., $\mathbf{v}^* = \mathbf{u}$ is a solution of (1.1). Furthermore,

$$\mathcal{N}^* = e(\mathbf{u})\,, \quad \lambda^* = \tau(\mathbf{u})\,.$$

2. Let $\mathbf{u}$ be a solution of (1.1). We have to verify that

$$\mathscr{H}([e(\mathbf{u}), \mathbf{u}]; \mu) \leqq \mathscr{H}([e(\mathbf{u}), \mathbf{u}]; \tau(\mathbf{u})) \leqq \mathscr{H}([\mathcal{N}, \mathbf{v}]; \tau(\mathbf{u})) \tag{1.6}$$

$$\forall \mu \in S\,, \quad [\mathcal{N}, \mathbf{v}] \in \mathscr{W}_\varepsilon\,.$$

It is easy to show that the left inequality holds even with the equality sign.

We have

$$(1.7)\qquad \mathscr{H}([\mathscr{N}, \mathbf{v}]; \tau(\mathbf{u})) - \mathscr{H}([e(\mathbf{u}), \mathbf{u}]; \tau(\mathbf{u})) =$$

$$= \mathscr{L}_1(\mathscr{N}, \mathbf{v}) - \mathscr{L}_1(e(\mathbf{u}), \mathbf{u}) + \int_\Omega \tau_{ij}(\mathbf{u}) [e_{ij}(\mathbf{v}) - \mathscr{N}_{ij}] \, dx =$$

$$= \frac{1}{2} \int_\Omega c_{ijkl}[\mathscr{N}_{ij} - e_{ij}(\mathbf{u})] [\mathscr{N}_{kl} - e_{kl}(\mathbf{u})] \, dx + A(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) .$$

Since $\mathbf{u}$ is a solution of (1.1), we have

$$(1.8)\qquad A(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) \geqq 0 \quad \forall \mathbf{v} \in K_\varepsilon .$$

From (1.7), (1.8) and the positive definiteness of the coefficients c_{ijkl} the right inequality of (1.6) follows. Thus $\{[e(\mathbf{u}), \mathbf{u}]; \tau(\mathbf{u})\}$ is a saddle-point of $\mathscr{H}$ on $\mathscr{W}_\varepsilon \times S$. Q.E.D.

Using the definition of $\mathscr{H}$ and the relation

$$\operatorname*{Sup}_{\lambda \in S} \int_\Omega \lambda_{ij}[e_{ij}(\mathbf{v}) - \mathscr{N}_{ij}] \, dx = \begin{cases} 0 & \text{if } \mathscr{N} = e(\mathbf{v}) , \\ +\infty & \text{if } \mathscr{N} \neq e(\mathbf{v}) , \end{cases}$$

we arrive at

$$(1.9)\qquad \mathscr{L}(\mathbf{u}) = \operatorname*{Inf}_{\mathbf{v} \in K_\varepsilon} \mathscr{L}(\mathbf{v}) = \operatorname*{Inf}_{\substack{\mathbf{v} \in K_\varepsilon \\ \mathscr{N} \in S}} \operatorname*{Sup}_{\lambda \in S} \mathscr{H}([\mathscr{N}, \mathbf{v}]; \lambda) ,$$

where $\mathbf{u}$ is an arbitrary solution of (1.1).

The *problem*: to find

$$(1.10)\qquad \operatorname*{Sup}_{\lambda \in S} \operatorname*{Inf}_{[\mathscr{N}, \mathbf{v}] \in \mathscr{W}_\varepsilon} \mathscr{H}([\mathscr{N}, \mathbf{v}]; \lambda)$$

will be called *dual* to the primary problem (1.1).

We shall reformulate the dual problem (1.10) in a simpler form. To this end, we introduce the decomposition

$$\mathscr{H}([\mathscr{N}, \mathbf{v}]; \lambda) = \mathscr{H}_1(\mathscr{N}, \lambda) + \mathscr{H}_2(\lambda, \mathbf{v}) ,$$

where

$$\mathscr{H}_1(\mathscr{N}, \lambda) = \frac{1}{2} \int_\Omega c_{ijkl} \mathscr{N}_{ij} \mathscr{N}_{kl} \, dx - \int_\Omega \lambda_{ij} \mathscr{N}_{ij} \, dx ,$$

$$(1.11)\qquad \mathscr{H}_2(\lambda, \mathbf{v}) = \int_\Omega \lambda_{ij} e_{ij}(\mathbf{v}) \, dx - L(\mathbf{v}) .$$

The definitions of $\mathscr{H}_1$, $\mathscr{H}_2$ imply that

$$(1.12)\qquad \operatorname*{Inf}_{[\mathscr{N}, \mathbf{v}] \in \mathscr{W}_\varepsilon} \mathscr{H}([\mathscr{N}, \mathbf{v}]; \lambda) = \operatorname*{Inf}_{\mathscr{N} \in S} \mathscr{H}_1(\mathscr{N}, \lambda) + \operatorname*{Inf}_{v \in K_\varepsilon} \mathscr{H}_2(\lambda, \mathbf{v}) .$$

It is readily seen that

$$(1.13)\qquad \operatorname*{Inf}_{\mathscr{N} \in S} \mathscr{H}_1(\mathscr{N}, \lambda) = \mathscr{H}_1(\mathscr{N}^*, \lambda) = -\frac{1}{2} \int_\Omega a_{ijkl} \lambda_{ij} \lambda_{kl} \, dx ,$$

where a_{ijkl} are the coefficients of the inverse generalized Hooke's law, and $\mathscr{N}^*_{ij} = a_{ijkl} \lambda_{kl}$.

Since $K_0 \subset K_\varepsilon$ and K_0 is a convex cone (K_0 denotes K_ε with $\varepsilon = 0$), the infimum of $\mathscr{H}_2$ on K_ε is finite only if

$$\mathscr{H}_2(\lambda, \mathbf{v}) \geqq 0 \quad \forall \mathbf{v} \in K_0 . \tag{1.14}$$

We set

$$S^+_{F,P} = \{\lambda \in S \mid \mathscr{H}_2(\lambda, \mathbf{v}) \geqq 0 \quad \forall \mathbf{v} \in K_0\} .$$

It is possible to show that for any $\lambda \in S^+_{F,P}$ smooth enough we have

$$\frac{\partial \lambda_{ij}}{\partial x_j} + F_i = 0 \text{ in } \Omega , \quad i = 1, 2 , \tag{1.15}$$

$$\lambda_{ij} n_j = P_i \text{ on } \Gamma_\tau , \quad i = 1, 2 , \tag{1.16}$$

$$T''_t(\lambda'') = 0 \text{ on } \Gamma_0 , \tag{1.17}$$

$$T'_\eta(\lambda') = 0 \text{ on } \Gamma'_K, \ T''_\eta(\lambda'') = 0 \text{ on } \Gamma''_K , \tag{1.18}$$

$$-T'_\xi(\lambda')(\cos\alpha')^{-1} = T''_\xi(\lambda'')(\cos\alpha'')^{-1} \leqq 0 , \quad \eta \in \langle a, b \rangle \tag{1.19}$$

where α^M, $M = {}', {}''$ is the angle between the axis η and the tangent to Γ^M_K and $(\cos\alpha^M)^{-1} = [1 - (\partial f^M/\partial\eta)^2]^{1/2}$.

In fact, from the condition $\lambda \in S^+_{F,P}$ it follows that any $\mathbf{v} \in K_0$ satisfies

$$-\int_\Omega v_i \frac{\partial \lambda_{ij}}{\partial x_j} \, dx + \int_{\partial\Omega' \cup \partial\Omega''} v_i \lambda_{ij} n_j \, dS \geqq \int_\Omega F_i v_i \, dx + \int_{\Gamma_\tau} P_i v_i \, dS .$$

Inserting $v^M_i = \pm\varphi_i \in C^\infty_0(\Omega^M)$, $M = {}', {}''$ we obtain (1.15). Then we arrive at

$$\int_{\partial\Omega' \cup \partial\Omega''} v_i \lambda_{ij} n_j \, dS \geqq \int_{\Gamma_\tau} P_i v_i \, dS .$$

Choosing $v_i = \pm\psi_i$ such that the trace of ψ_i has its support in Γ_τ, we obtain (1.16). Consequently, we deduce

$$0 \leq \int_{\partial\Omega \div \Gamma_\tau} v_i \lambda_{ij} n_j \, dS = \int_{\Gamma_{K'}} [T'_\xi(\lambda') \, v'_\xi + T'_\eta(\lambda') \, v'_\eta] \, dS +$$

$$+ \int_{\Gamma_{K''}} [T''_\xi(\lambda'') \, v''_\xi + T''_\eta(\lambda'') \, v''_\eta] \, dS + \int_{\Gamma_0} T''_t(\lambda'') \, v''_t \, dS \quad \forall \mathbf{v} \in K_0 .$$

Choosing $\mathbf{v}' \equiv 0$, $\mathbf{v}''$ with $v''_n = 0$, $v''_t = \pm\psi$ on Γ_0, where the support of ψ is contained in Γ_0, we obtain (1.17). Thus we are left with the following inequality

$$\int_{\Gamma_{K'}} [T'_\xi v'_\xi + T'_\eta v'_\eta] \, dS + \int_{\Gamma_{K''}} [T''_\xi v''_\xi + T''_\eta v''_\eta] \, dS \geqq 0 \quad \forall \mathbf{v} \in K_0 . \tag{1.20}$$

Taking $\mathbf{v} \in V$ such that $v'_\eta = v''_\eta = 0$ and $v'_\xi = v''_\xi = \pm\varphi$ for points on $\Gamma'_K \cup \Gamma''_K$

having the same coordinates η and with $\varphi \in C_0^\infty(\langle a, b\rangle)$ we have

$$0 = \int_{\Gamma_{K'}} [T'_\xi(\lambda') \varphi] \, dS + \int_{\Gamma_{K''}} T''_\xi(\lambda'') \varphi \, dS =$$

$$= \int_a^b [T''_\xi(\lambda'') (\cos \alpha'')^{-1} + T'_\xi(\lambda') (\cos \alpha')^{-1}] \varphi \, d\eta .$$

Consequently, we deduce

$$(1.21) \qquad -T'_\xi(\lambda') (\cos \alpha')^{-1} = T''_\xi(\lambda'') (\cos \alpha'')^{-1} \quad \forall \eta \in \langle a, b\rangle .$$

Choosing $v'_\xi = v''_\xi = 0$, $v''_\eta = 0$, $v'_\eta = \pm\varphi$ on Γ'_K, we obtain from (1.20) that $T'_\eta(\lambda') = 0$ on Γ'_K. The condition $T'_\eta(\lambda'') = 0$ on Γ''_K can be deduced in the same way. Thus, (1.20) yields that

$$\int_a^b [T''_\xi(\lambda'') (\cos \alpha'')^{-1}] (v''_\xi - v'_\xi) \, d\eta \geqq 0 \quad \forall \mathbf{v} \in K_0 .$$

By virtue of the condition $v''_\xi - v'_\xi \leqq 0 \; \forall \mathbf{v} \in K_0$, (1.19) follows.

From (1.15) up to (1.18) and (1.21) it is readily seen that for any $v \in K_\varepsilon$, $\lambda \in S^+_{F,P}$ we have

$$(1.22) \qquad \mathscr{H}_2(\lambda, \mathbf{v}) = \int_a^b T''_\xi(\lambda'') (\cos \alpha'')^{-1} (v''_\xi - v'_\xi) \, d\eta .$$

We can show that for any $\lambda \in S^+_{F,P}$,

$$(1.23) \qquad \operatorname*{Inf}_{\mathbf{v} \in K_\varepsilon} \mathscr{H}_2(\lambda, \mathbf{v}) = \int_a^b T''_\xi(\lambda'') (\cos \alpha'')^{-1} \varepsilon(\eta) \, d\eta .$$

In fact, from (1.22), (1.19) and the definition of K_ε, it follows that

$$(1.24) \qquad \mathscr{H}_2(\lambda, \mathbf{v}) \geqq \int_a^b T''_\xi(\lambda'') (\cos \alpha'')^{-1} \varepsilon(\eta) \, d\eta \quad \forall \mathbf{v} \in K_\varepsilon , \quad \lambda \in S^+_{F,P} .$$

Let the functions f' and f'' describing Γ'_K and Γ''_K, respectively, be twice continuously differentiable on $\langle a, b\rangle$. There exists a function $\mathbf{u}''_0 \in [H^1(\Omega'')]^2$ such that $u''_{0n} = 0$ on Γ_0, $u''_{0\xi} = \varepsilon(\eta)$ on $\langle a, b\rangle$. Choosing $\mathbf{u}_0 = (\mathbf{u}'_0, \mathbf{u}''_0)$, where $\mathbf{u}'_0 \equiv 0$, we have $\mathbf{u}_0 \in K_\varepsilon$ and

$$(1.25) \qquad \mathscr{H}_2(\lambda, \mathbf{u}_0) = \int_a^b T''_\xi(\lambda'') (\cos \alpha'')^{-1} \varepsilon(\eta) \, d\eta .$$

Then (1.23) follows from (1.24) and (1.25).

Combining (1.23), (1.12) with (1.13) we arrive at

$$(1.26) \qquad \operatorname*{Sup}_{\lambda \in S} \operatorname*{Inf}_{[\mathscr{N}, \mathbf{v}] \in \mathscr{W}_\varepsilon} \mathscr{H}([\mathscr{N}, \mathbf{v}]; \lambda) = - \operatorname*{Inf}_{\lambda \in S^+_{F,P}} \mathscr{S}_1(\lambda) ,$$

where

$$\mathscr{S}_1(\lambda) = \frac{1}{2} \int_\Omega a_{ijkl} \lambda_{ij} \lambda_{kl} \, dx - \int_a^b T''_\xi(\lambda'') (\cos \alpha'')^{-1} \varepsilon(\eta) \, d\eta .$$

We introduce the following notation for $\sigma, \tau \in S$:

$$\langle \sigma, \tau \rangle = \int_\Omega \sigma_{ij}\tau_{ij}\,\mathrm{d}x\,, \quad (\sigma, \tau) = \langle c^{-1}\sigma, \tau \rangle\,,$$

$$\|\tau\|_0^2 = \langle \tau, \tau \rangle\,, \quad \|\tau\|^2 = (\tau, \tau)$$

where $c: S \to S$ is the isomorphism defined by the generalized Hooke's law:

$$\sigma = ce \Leftrightarrow \sigma_{ij} = c_{ijkl}e_{kl}\,.$$

Then, making use of (1.25) and (1.11), we may write

$$\mathscr{S}_1(\lambda) = = \tfrac{1}{2}\|\lambda\|^2 - \langle e(\boldsymbol{u}_0), \lambda \rangle + L(\boldsymbol{u}_0)$$

and the dual problem (1.10) is reformulated as follows: to find $\lambda^* \in S^+_{F,P}$ such that

$$\mathscr{S}(\lambda^*) \leqq \mathscr{S}(\lambda) \quad \forall \lambda \in S^+_{F,P}\,, \tag{1.27}$$

where $\mathscr{S}(\lambda) = \frac{1}{2}\|\lambda\|^2 - \langle e(\boldsymbol{u}_0), \lambda \rangle$.

Lemma 1.2. *Let* $\boldsymbol{u}$ *be a solution of the primary problem* (1.1). *Then* $\lambda^* = (\lambda^*_{ij}) = (c_{ijkl}e_{kl}(\boldsymbol{u}))$ *is the unique solution of the dual problem* (1.27).

Proof. It is easy to see that $\lambda^* \in S^+_{F,P}$ since $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \in K_\varepsilon$ for any $\boldsymbol{w} \in K_0$. For any $\lambda \in S^+_{F,P}$ we have

$$\mathscr{S}(\lambda) - \mathscr{S}(\lambda^*) = \frac{1}{2}\int_\Omega a_{ijkl}(\lambda_{ij} - \lambda^*_{ij})(\lambda_{kl} - \lambda^*_{kl})\,\mathrm{d}x +$$

$$+ \int_\Omega a_{ijkl}\lambda^*_{ij}(\lambda_{kl} - \lambda^*_{kl})\,\mathrm{d}x - \langle e(\boldsymbol{u}_0), \lambda - \lambda^* \rangle \geqq$$

$$\geqq \frac{a_0}{2}\int_\Omega (\lambda_{ij} - \lambda^*_{ij})(\lambda_{kl} - \lambda^*_{kl})\,\mathrm{d}x + \int_\Omega e_{ij}(\boldsymbol{u})(\lambda_{ij} - \lambda^*_{ij})\,\mathrm{d}x - \langle e(\boldsymbol{u}_0), \lambda - \lambda^* \rangle \geqq$$

$$\geqq \langle e(\boldsymbol{u} - \boldsymbol{u}_0), \lambda - \lambda^* \rangle\,,$$

where we have used the ellipticity of the coefficients a_{ijkl}. The solution $\boldsymbol{u}$ of the primary problem satisfies

$$\langle \lambda^*, e(\boldsymbol{v} - \boldsymbol{u}) \rangle \geqq L(\boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in K\,.$$

Since $\boldsymbol{u}_0 \in K$ we have

$$\langle e(\boldsymbol{u} - \boldsymbol{u}_0), -\lambda^* \rangle + L(\boldsymbol{u} - \boldsymbol{u}_0) \geqq 0\,.$$

From $\lambda \in S^+_{F,P}$ and $\boldsymbol{u} - \boldsymbol{u}_0 \in K_0$ it follows that

$$0 \leqq \mathscr{H}_2(\lambda, \boldsymbol{u} - \boldsymbol{u}_0) = \langle e(\boldsymbol{u} - \boldsymbol{u}_0), \lambda \rangle - L(\boldsymbol{u} - \boldsymbol{u}_0)\,.$$

Adding the two inequalities we arrive at

$$\langle e(\boldsymbol{u} - \boldsymbol{u}_0), \lambda - \lambda^* \rangle \geqq 0 \quad \forall \lambda \in S^+_{F,P}\,,$$

i.e.,

$$\mathscr{S}(\lambda) \geqq \mathscr{S}(\lambda^*) \quad \forall \lambda \in S^+_{F,P}\,.$$

This implies that λ^* solves the problem (1.27). Moreover, since the functional $\mathscr{S}(\lambda)$ is strictly convex on $S^+_{F,P}$, the uniqueness of λ^* follows. Q.E.D.

Remark 1.1. Let $\boldsymbol{u}$, λ^* be as in Lemma 1.2. Making use of (1.9), (1.25) and the relation between $\mathscr{S}_1(\lambda)$ and $\mathscr{S}(\lambda)$, it is possible to show that

$$\mathscr{L}(\boldsymbol{u}) + L(\boldsymbol{u}_0) + \mathscr{S}(\lambda^*) = 0\,.$$

Remark 1.2. A result analogous to that of Lemma 1.5 in the paper [1]–III can be obtained. Namely, if a solution $\boldsymbol{u}$ of the primary problem (1.1) exists, then the set $S^+_{F,P}$ is non-empty, convex and closed in S. Existence and uniqueness of a solution of (1.27) can be shown directly by using the strict convexity and the lower weak semicontinuity of the functional $\mathscr{S}(\lambda)$.

We here emphasize the fact that the dual problem (1.27) is uniquely solvable if the primary problem possesses at least one solution. The existence of the solution of the dual problem, however, can be proved directly, if in some way we show the non-emptiness of the set $S^+_{F,P}$. Thus, the dual problem may have a solution even in some cases when the primary problem has none.

2. APPROXIMATION OF THE DUAL PROBLEM

As in [2] it is possible to approximate the solution of the dual problem by means or piecewise constant stress tensors. Here we follow the procedure suggested by Haslinger and Hlaváček [2].

Let us consider a triangulation $\mathscr{T}_h^M$ of Ω^M, $M = {}'$, $''$ such that the triangles adjacent to the boundaries may have a curved side along the boundary, and the nodes on Γ_K^M are on lines parallel to the ξ axis. If a curved triangle $T \in \mathscr{T}_h = \mathscr{T}'_h \cup \mathscr{T}''_h$ adjacent to Γ_K^M is convex, it will be divided by the chord into the "straight" triangle T_0 and the segment T_s such that $T = T_0 \cup T_s$. If $T_c \in \mathscr{T}_h$ is not convex, one of its sides must be parallel to ξ axis. We define

$$V_h = \Big\{ \boldsymbol{v} \in V \mid \boldsymbol{v}|_{T_0} \in [P_1(T_0)]^2 \quad \forall T_0 \subset T \in \mathscr{T}_h \quad \text{adjacent to } \Gamma_K^M\,,$$

$$\left(\frac{\partial \boldsymbol{v}}{\partial \xi}\right)\bigg|_{T_s} = 0 \quad \forall T_s \subset T \in \mathscr{T}_h \quad \text{adjacent to } \Gamma_K^M\,,$$

$$\left(\frac{\partial \boldsymbol{v}}{\partial \xi}\right)\bigg|_{T_c} = 0 \quad \text{for each non-convex triangle } T_c \text{ adjacent to } \Gamma_K^M\,,$$

$$\boldsymbol{v}|_T \in [P_1(T)]^2 \quad \text{for the other triangles } T\Big\}\,.$$

We introduce the sets

$$S_h = \{\tau \in S \mid \tau_{ij} \in P_0(T^*)\,;\quad T^* = T, T_0, T_s, T_c\,;\quad i, j = 1, 2\}\,,$$

$$K_{0h} = \{\boldsymbol{v} \in V_h \mid v''_\xi - v'_\xi \leqq 0 \quad \forall \eta \in \langle a, b\rangle \quad \text{on} \quad \Gamma'_K \cup \Gamma''_K\}\,.$$

Then we define approximations $S^+_{F,P,h}$ of $S^+_{F,P}$ as follows:

$$S^+_{F,P,h} = \{\lambda \in S_h \mid \langle \lambda, e(\mathbf{v}_h)\rangle \geqq L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in K_{0h}\} .$$

It is readily seen that the condition $v''_\xi - v'_\xi \leqq 0$, $\mathbf{v} \in V_h$ holds everywhere on $\langle a, b\rangle$ if only it holds for the nodes lying on Γ^M_K.

Let the function $\mathbf{u}_0 = (0, \mathbf{u}''_0)$ be known (see (1.25)).

Then the approximate dual problem is formulated as follows: to find $\lambda_h \in S^+_{F,P,h}$ such that

$$\mathscr{S}(\lambda_h) \leqq \mathscr{S}(\lambda) \quad \forall \lambda \in S^+_{F,P,h} . \tag{2.1}$$

We introduce the projection mapping r_h: $S \to S_h$ defined by the relation

$$\langle \tau - r_h\tau, \chi_h\rangle = 0 \quad \forall \chi_h \in S_h . \tag{2.2}$$

Lemma 2.1. *Let $\tau \in S^+_{F,P}$. Then $r_h\tau \in S^+_{F,P,h}$.*

Proof. It is readily seen that

$$(r_h\tau)|_{T^*} = (\text{mes } T^*)^{-1} \int_{T^*} \tau \, dx , \quad \forall T^* = T, T_0, T_s, T_c \in \mathscr{T}_h , \tag{2.3}$$

$$\|r_h\tau - \tau\|_0 \to 0 , \quad h \to 0 .$$

Assume that $\mathbf{v}_h \in K_{0h}$. Then $e(\mathbf{v}_h) \in S_h$, and using (2.2) we obtain

$$\langle r_h\tau, e(\mathbf{v}_h)\rangle = \langle \tau, e(\mathbf{v}_h)\rangle \geqq L(\mathbf{v}_h)$$

because of $K_{0h} \subset K_0$, $\tau \in S^+_{F,P}$. This yields $r_h\tau \in S^+_{F,P,h}$. Q.E.D.

Next we recall some results of [2].

Lemma 2.2. *Let $\Gamma'_K \cap \bar{\Gamma}_u = \emptyset$, $\Gamma''_K \cap \bar{\Gamma}_0 = \emptyset$. Assume that the number of points of $\bar{\Gamma}_\tau \cap \bar{\Gamma}_u$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_0$ is finite, and $f^M \in C^m(\langle a - \delta, b + \delta\rangle)$, $\delta > 0$, $m \geqq 1$, $M = {}', {}''$. Then the set*

$$\mathscr{K}_m = K_0 \cap [C^m(\bar{\Omega}')]^2 \times [C^m(\bar{\Omega}'')]^2$$

is dense in K_0.

Lemma 2.3. *Let $\mathbf{v} \in [H^2(\Omega')]^2 \times [H^2(\Omega'')]^2$, $f^M \in C^2(\langle a, b\rangle)$. Define the Lagrange linear interpolation $\mathbf{v}_I \in V_h$ as follows: If A_i is a node of a curved non-convex triangle, $\mathbf{v}_I(A_i) = \mathbf{v}(D_i)$, where D_i is the projection of A_i onto Γ^M_K in the direction ξ; $\mathbf{v}_I(A_i) = \mathbf{v}(A_i)$ at the other nodes. We construct $\mathbf{v}_I$ such that $\mathbf{v}_I \in V_h$, i.e., $\mathbf{v}_I$ is a piecewise linear function which is extended continuously by constants on segments T_s and on non-convex curved triangles. Then*

$$\|\mathbf{v}_I - \mathbf{v}\|_{1,\Omega' \cup \Omega''} \to 0 \quad \textit{for} \quad h \to 0$$

for any regular system of triangulations.

We can prove the following result about the convergence of λ_h to λ^*.

Theorem 1. *Let $f^M \in C^2$, $M = {}', {}''$ in a neighbourhood of the interval $\langle a, b\rangle$*

and let the same assumptions on $\Gamma_K^M, \Gamma_u, \Gamma_0, \Gamma_\tau$ as in Lemma 2.2 hold. Assume that $S_{F,P}^+ \neq \emptyset$.

Then the approximate dual problem (2.1) *possesses a unique solution λ_h and*

$$(2.4) \qquad \|\lambda_h - \lambda^*\|_0 \to 0\,, \quad h \to 0$$

holds for any regular family of triangulations $\mathcal{T}_h$, where λ^* is the solution of the dual problem (1.27).

Proof. The set $S_{F,P,h}^+$ is convex and closed. Lemma 2.1 implies that it is non-empty. The functional $\mathcal{S}(\lambda)$ is strictly convex and lower weakly semicontinuous on $S_{F,P,h}^+$. The existence and uniqueness of the solution of (2.1) follows.

To show (2.4), we shall apply an abstract theorem about the convergence of the Ritz-Galerkin method (see e.g. [3], Chapt. 4). We only have to prove the following two conditions:

(i) $\exists\{\tau_h\}$, $\tau_h \in S_{F,P,h}^+$, $\tau_h \to \lambda^*$ in S for $h \to 0$,

(ii) $\tau_h \in S_{F,P,h}^+$, $\tau_h \rightharpoonup \tau$ (weakly) in S implies $\tau \in S_{F,P}^+$.

Choosing $\tau_h = r_h\lambda^*$, making use of Lemma 2.1 and (2.3) we deduce that (i) holds.

Now let $\mathbf{v}$ be an arbitrary element of K_0. By virtue of Lemma 2.2 we find $\mathbf{v}_\gamma \in \mathcal{K}_2$ such that for any $\gamma > 0$,

$$\|\mathbf{v}_\gamma - \mathbf{v}\|_{1,\Omega'\cup\Omega''} < \gamma\,.$$

Using Lemma 2.3 we arrive at

$$\|\mathbf{v}_{\gamma I} - \mathbf{v}_\gamma\|_{1,\Omega'\cup\Omega''} \to 0\,, \quad h \to 0\,.$$

It is clear that $\mathbf{v}_{\gamma I} \in K_{0h}$. Then we obtain

$$\|\mathbf{v}_{\gamma I} - \mathbf{v}\|_{1,\Omega'\cup\Omega''} \leqq \|\mathbf{v}_{\gamma I} - \mathbf{v}_\gamma\|_{1,\Omega'\cup\Omega''} + \|\mathbf{v}_\gamma - \mathbf{v}\|_{1,\Omega'\cup\Omega''} \to 0 \text{ for } \gamma, h \to 0.$$

If $\tau_h \in S_{F,P,h}^+$, we have

$$\langle\tau_h, e(\mathbf{v}_{\gamma I})\rangle \geqq L(\mathbf{v}_{\gamma I})\,.$$

Since $\tau_h \rightharpoonup \tau$ (weakly) in S and $e(\mathbf{v}_{\gamma I}) \to e(\mathbf{v})$ (strongly) in S, passing to the limit for $h, \gamma \to 0$ we conclude

$$\langle\tau, e(\mathbf{v})\rangle \geqq L(\mathbf{v}) \quad \forall \mathbf{v} \in K_0$$

i.e., $\tau \in S_{F,P}^+$. Q.E.D.

3. AN ALGORITHM FOR THE SOLUTION OF THE APPROXIMATE PROBLEM

As in [2] we can simplify the approximate problem (2.1) by eliminating the auxiliary functions $\mathbf{v}_h$, which appear in the definition of $S_{F,P,h}^+$.

We set

$$\mathbf{v}_h(\xi, \eta) = \sum_{i=1}^{N} q_i \varphi_i(\xi, \eta)\,,$$

where q_i are the values of the displacement components at the nodes of the triangulation $\mathscr{T}_h$.

If we describe the condition $v''_{h\xi} - v'_{h\xi} \leqq 0$ at the nodes lying on Γ_K, then precisely two components $\{q_{k_1}, q_{k_2}\}$ intervene at any pair of the nodes $(A_k, B_k) \in \bar{\Gamma}_K$ (see Fig. 2). Namely, the condition $v''_{h\xi} - v'_{h\xi} \leqq 0$ implies $q_{k_1} - q_{k_2} \leqq 0$. We now introduce that linear transformation

$$\mathbf{q} = F_K \mathbf{y}: R^2 \to R^2$$

defined by the relations

$$y_{k_1} = q_{k_1}, \quad y_{k_2} = q_{k_1} - q_{k_2}.$$

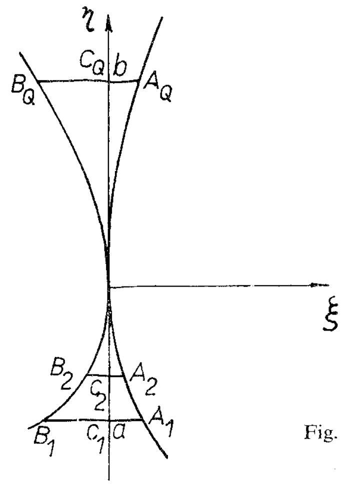

Fig. 2

It is readily seen that F_K is regular. We consider the same transformation for any pair $M_K = \{q_{k_1}, q_{k_2}\}$, $k = 1, \ldots, Q$ corresponding to the pair of nodes $(A_k, B_k) \in \bar{\Gamma}_K$. We also set $y_p = q_p$ for $q_p \notin \bigcup_{k=1}^{Q} M_k$, $1 \leqq p \leqq N$. Finally, we have

$$\mathbf{q} = F\mathbf{y}: R^N \to R^N, \quad \text{and} \tag{3.1}$$

$$\mathbf{v}_h \in K_{0h} \Leftrightarrow \mathbf{q} \in \mathscr{K}_q \Leftrightarrow \mathbf{y} \in \mathscr{K}_y = \{\mathbf{y} \in R^N,\ y_{k_2} \leqq 0,\ k = 1, \ldots, Q\}.$$

Let ψ_{T^*} be the characteristic function of the figure $T^* \in \mathscr{T}_h$, $T^* = T, T_0, T_s, T_c$. Then we have

$$\tau_h \in S_h \Leftrightarrow \tau_h(\xi, \eta) = \sum_{T^* \in \mathscr{T}_h} \tau(T^*)\, \psi_{T^*}(\xi, \eta). \tag{3.2}$$

Denoting by

$$\mathbf{t}^{\mathrm{T}} = \{\tau_{11}(T_1), \tau_{22}(T_1), \tau_{12}(T_1), \tau_{11}(T_2), \tau_{22}(T_2), \tau_{12}(T_2), \ldots\} \tag{3.3}$$

the corresponding vector in R^M, we obtain

$$\langle \tau, e(\mathbf{v}_h) \rangle = \sum_{T^* \in \mathscr{T}_h} \int_{T^*} \tau_{kj}(T^*) \sum_{i=1}^{N} q_i e_{kj}(\varphi_i)\, d\xi\, d\eta = (\mathbf{E}\mathbf{t}, \mathbf{q}),$$

where E is a matrix of order $N \times M$ $(N < M)$ and $(E\mathbf{t}, \mathbf{q}) = \mathbf{q}^{\mathrm{T}} E\mathbf{t}$. Since

$$L(\mathbf{v}_h) = \sum_{i=1}^{N} q_i L(\varphi_i) = (\mathbf{l}, \mathbf{q}),$$

where $\mathbf{l}$ is a fixed vector in R^N, the condition $\tau \in S^+_{F,P,h}$ can be written in the form

$$(\mathbf{l} - E\mathbf{t}, \mathbf{q}) \leqq 0 \quad \forall \mathbf{q} \in \mathscr{K}_q .$$

Applying the mapping F, we obtain an equivalent condition

$$(3.4) \qquad (\mathbf{l} - E\mathbf{t}, F\mathbf{y}) \leqq 0 \quad \forall \mathbf{y} \in \mathscr{K}_y .$$

Denote by J^- the set of all indexes k_2, $k = 1, \ldots, Q$, and $J^0 = \{1, \ldots, N\} \dot{-} J^-$. Since the cone $\mathscr{K}_y$ is defined by the vectors

$$\{\mathbf{e}_j, -\mathbf{e}_m;\ j \in J^0,\ m \in J^-\}$$

where $\{\mathbf{e}_j, \mathbf{e}_m\}$ create an orthonormal basis of R^N, (3.4) is equivalent to the system

$$(3.5) \qquad g_j(\mathbf{t}) = (\mathbf{l} - E\mathbf{t}, F\mathbf{e}_j) = 0, \quad j \in J^0,$$

$$(3.6) \qquad g_m(\mathbf{t}) = (\mathbf{l} - E\mathbf{t}, F\mathbf{e}_m) \leqq 0, \quad m \in J^- .$$

Inserting (3.2), (3.3) into the functional $\mathscr{S}(\tau_h)$, we arrive at the following problem: $\mathscr{S}_0(\mathbf{t}) = \min$ over the set of all $\mathbf{t}$ satisfying the conditions (3.5) and (3.6).

Remark 3.1. When choosing global coordinates (x_1, x_2) instead of (ξ, η) the procedure is analogous, but the situation is somewhat more complicated. In fact, if we describe the condition $v''_{h\xi} - v'_{h\xi} \leqq 0$ at the nodes on Γ_K, there exist precisely four components $\{q_{k_1}, q_{k_2}, q_{k_3}, q_{k_4}\}$ at any pair of nodes $(A_k, B_k) \in \bar{\Gamma}_K$. The condition $v''_{h\xi} - v'_{h\xi} \leqq 0$ implies $\sum_{j=1}^{4} b_j q_{k_j} \leqq 0$, where $b_1 = -b_3 = -\cos(x_1, \xi)$, $b_2 = -b_4 = \cos(x_2, \xi)$.

4. ANOTHER APPROXIMATION BY FINITE ELEMENT METHOD

In Section 2 the dual problem (1.27) was approximated by piecewise constant external approximations using the triangulation suggested by Haslinger and Hlaváček [2]. It is the aim of this section to approximate the same problem using another triangulation, analogous to that suggested by Hlaváček and Křížek in [4].

Assume that the parts Γ^M_K, $M = {}', {}''$ consist of a finite number of convex and concave arcs. Here an arc $\Gamma_0 \subset \partial\Omega$ is called convex (concave) if there exists a convex domain $\Omega_0 \subset \Omega$ $(\Omega_0 \subset R^2 \dot{-} \Omega)$ such that $\Gamma_0 \subset \partial\Omega_0$.

Next we define a triangulation of $\Omega = \Omega' \cup \Omega''$. We construct domains Ω^M_h, approximating Ω^M and such that $\bar{\Omega}^M_h \subset \bar{\Omega}^M$, $M = {}', {}''$. Each concave arc belonging to Γ^M_K is approximated by a polygonal curve consisting of a finite number of line segments whose lengths are not greater than h. Each of these line segments is tangential to the concave arc. Furthermore, each inflexion point of Γ^M_K must be a vertex of the polygonal curve. By Γ_{Kh}^M we denote the union of the convex arcs and the approximate polygonal curves. The part of Ω^M bounded by Γ_K^M and Γ_{Kh}^M will be denoted by D_h^M. Then the domain $\Omega_h^M = \Omega^M \dot{-} D_h^M$ will be triangulated in the standard way used in the finite element method. The triangles adjacent to the boundary may have a curved side along the boundary. We require each end point of Γ_K^M and each node of the approximate polygonal curves to be also a node of the triangulation. Moreover, the corresponding nodes A_i, B_i lying on Γ'_{Kh} and Γ''_{Kh} must lie on the lines A_iB_i parallel to ξ-axis (see Fig. 3). This triangulation will be denoted by $\mathscr{T}_h^M$ and $\mathscr{T}_h = \mathscr{T}'_h \cup \mathscr{T}''_h$.

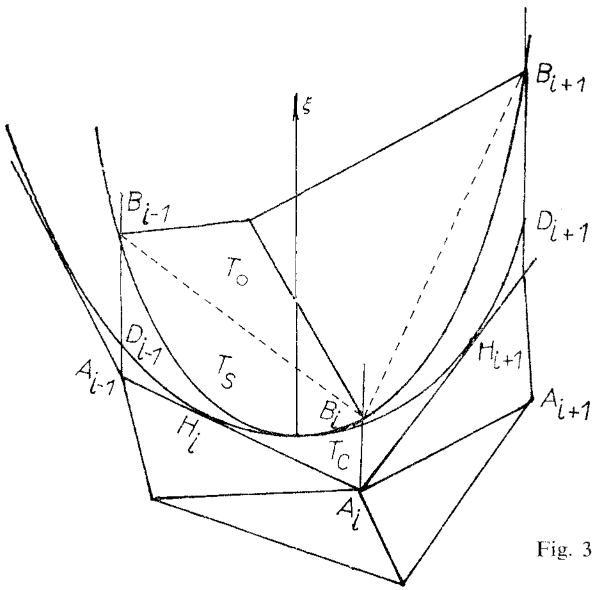

Fig. 3

We say that the system $\{\mathscr{T}_h\}$, $0 < h \leqq h_0$ is regular if the smallest internal angle of all triangles of $\mathscr{T}_h$, for any $0 < h \leqq h_0$, is not smaller than a constant $\theta > 0$, independent of h. Here if a triangle is curved, the internal angles are defined by the angles of the "straight" triangle with the same vertices.

If a curved triangle $T \in \mathscr{T}_h$ adjacent to Γ_K^M is convex, it is divided by a chord into a "straight" triangle T_0 and a segment T_s such that $T = T_0 \cup T_s$.

We introduce

$$V_h = \{\mathbf{v} \in V \mid \mathbf{v}|_{T_0} \in [P_1(T_0)]^2 \quad \forall T_0 \subset T \in \mathscr{T}_h \text{ adjacent to } \Gamma_K^M ,$$

$$\left(\frac{\partial}{\partial \xi} \mathbf{v}\right)\Bigg|_{T_s} = 0 \quad \forall T_s \subset T \in \mathscr{T}_h \text{ adjacent to } \Gamma_K^M ,$$

$$\left(\frac{\partial}{\partial \xi} \mathbf{v}\right)\Bigg|_{T_c} = 0 \quad \forall T_c \subset D_h , \quad \text{where } T_c \text{ is a curved triangle defined by } H_i, A_i, H_{i+1} \text{ (Fig. 3)},$$

$$\mathbf{v}|_T \in [P_1(T)]^2 \quad \text{for all the remaining triangles}\}.$$

In other words, V_h consists of piecewise linear vector-functions which are extended continuously on segments T_s and on non-convex curved triangles, constantly in the direction ξ.

Let $\varepsilon > 0$ be fixed. We define $G^M \subset \Omega^M$ to be the so called ε-skin of the curved part of Γ_K^M, i.e.,

$$G^M = \{y \in \Omega^M, \exists x \in \Gamma_K^M \dot{-} \Gamma_K^{*M} : \text{dist}(x, y) < \varepsilon\}, \quad M = ', ''$$

where $\Gamma_K^{*M} = \{x \in \Gamma_K^M; \text{there exists a line segment } S \subset \Gamma_K^M : x \in S\}$, and $G = G' \cup G''$ (Fig. 4).

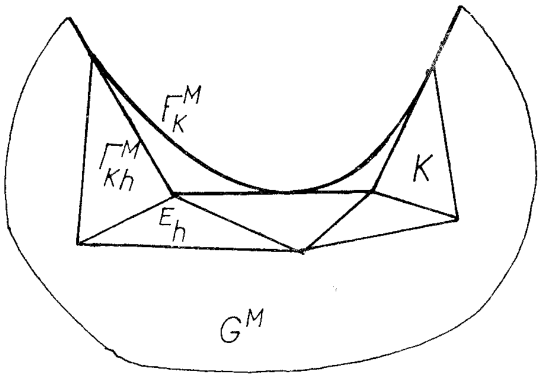

Fig. 4

The following assertion holds.

Lemma 4.1. *Let* $\mathbf{v} \in [H^2(\Omega') \cap W^{1,\infty}(G')]^2 \times [H^2(\Omega'') \cap W^{1,\infty}(\Omega'')]^2, f^M \in C^2(\langle a, b\rangle)$. *We define a Lagrange linear interpolation* $\mathbf{v}_I \in V_h$ *as follows: if* A_i *is a vertex of a side tangential to* Γ_K^M, $\mathbf{v}_I(A_i) = \mathbf{v}(D_i)$, *where* $D_i \in \Gamma_K^M$ *is the projection of* A_i *in the direction* ξ; $\mathbf{v}_I(a_i) = \mathbf{v}(a_i)$ *at the other nodes; then we construct* $\mathbf{v}_I$ *so that* $\mathbf{v}_I \in V_h$. *Then*

$$\|\mathbf{v}_I - \mathbf{v}\|_{1,\Omega' \cup \Omega''} \to 0 \quad \textit{for} \quad h \to 0$$

holds for any regular system of triangulations.

Proof. Let V_h^0 be the space of all piecewise linear functions on $\mathscr{T}_h \cup D_h$, continuous on $\bar{\Omega}^M$, $M = ', ''$, where each convex curved triangle $T = T_0 \cup T_s$ will remain undivided while each non-convex curved triangle $T_c \subset D_h$ is divided by the line segment A_iD_i into two triangles T_c^m, $m = 1, 2$, on which the functions from V_h^0 are linear and defined by the values at the nodes A_i, H_i, H_{i+1} (see Fig. 3). Let the functions $\mathbf{w}_h^0 \in V_h^0$ be constant in the direction ξ, i.e. $((\partial/\partial\xi)\, \mathbf{w}_h^0)|_{T_c^m} = 0$ for each $\mathbf{w}_h^0 \in V_h^0$, $m = 1, 2$. The interpolation $\mathbf{v}_h^0 \in V_h^0$ of $\mathbf{v}$ is defined so that $\mathbf{v}_h^0 = \mathbf{v}$ at all nodes of the triangulation $\mathscr{T}_h$, i.e., except the nodes D_i.

First of all, we shall show that

$$\|\mathbf{v}_h^0 - \mathbf{v}\|_{1,\Omega' \cup \Omega''} \to 0 \quad \text{for} \quad h \to 0\,. \tag{4.1}$$

In fact, there exists an extension $E\mathbf{v} \in [H^2(R^2)]^2$ of $\mathbf{v}$ such that

$$\|E\mathbf{v}\|_{2,R^2} \leqq C\|\mathbf{v}\|_{2,\Omega^M}$$

(see e.g. [5], Chapt. 2). For any convex curved triangle adjacent to Γ_K^M we define $\tilde{T} = \Delta\tilde{a}_i a_j \tilde{a}_k$ (i.e. twice extended T_0) (see Fig. 5). $\tilde{T} = T$ if T is a straight triangle. Let π denote the linear interpolation on $\tilde{T}$ with the nodes a_i, a_j, a_k. Using the affine equivalence and the regularity of the system $\{\mathscr{T}_h\}$ we deduce

$$(4.3) \qquad \|\pi E\mathbf{v} - E\mathbf{v}\|_{1,\tilde{T}} \leqq Ch|E\mathbf{v}|_{2,\tilde{T}},$$

where C is independent of h and $E\mathbf{v}$. Since $\pi E\mathbf{v} = \mathbf{v}_h^0$ on T^*, $\forall T^* \in \mathscr{T}_h$, making use of (4.2), (4.3) we arrive at

$$(4.4) \qquad \|\mathbf{v} - \mathbf{v}_h^0\|_{1,\Omega_h^M}^2 = \sum_{T^* \in \mathscr{T}_h^M} \|\mathbf{v} - \mathbf{v}_h^0\|_{1,T^*}^2 \leqq \sum_{T^* \in \mathscr{T}_h^M} \|\mathbf{v} - \mathbf{v}_h^0\|_{1,\tilde{T}^*}^2 \leqq$$
$$\leqq Ch^2 \sum_{T^* \in \mathscr{T}_h^M} \|E\mathbf{v}\|_{1,\tilde{T}^*}^2 \leqq 2Ch^2\|E\mathbf{v}\|_{2,R^2}^2 \leqq C_1 h^2 |\mathbf{v}|_{2,\Omega^M}^2 .$$

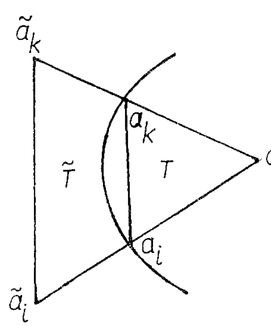

Fig. 5

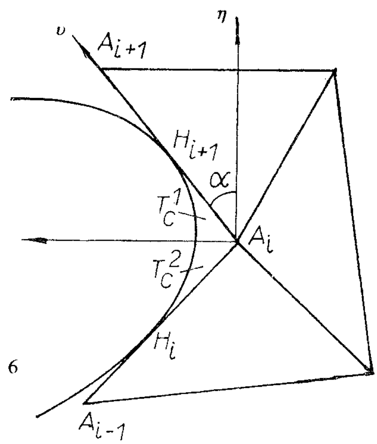

Fig. 6

Furthermore, it is readily seen that for each point $(\xi, \eta) \in T_c$ we have

$$|v_{hj}^0(\xi, \eta)| \leqq \max_{k=i-1,i,i+1} |v_j(A_k)| \leqq \|\mathbf{v}\|_{0,\infty,G}, \quad j = 1, 2,$$

because for h small enough $T_c \subset G$ and $\mathbf{v} \in [W^{1,\infty}(G)]^2$. Consequently, we have

$$|\mathbf{v}_h^0(\xi, \eta) - \mathbf{v}(\xi, \eta)| \leqq 2\|\mathbf{v}\|_{0,\infty,G} \quad \forall(\xi, \eta) \in T_c$$

and

$$(4.5) \qquad \|\mathbf{v}_h^0 - \mathbf{v}\|_{0,T_c}^2 \leqq 4\|\mathbf{v}\|_{0,\infty,G}^2 \text{ mes } T_c .$$

Let us consider $(\partial/\partial\eta)\, \mathbf{v}_h^0$ on T_c^1 (Fig. 6). It is readily seen that

$$\frac{\partial}{\partial \nu} \mathbf{v}_h^0 = \frac{\partial}{\partial \eta} \mathbf{v}_h^0 (\cos \alpha^M) + (\sin \alpha^M) \frac{\partial}{\partial \xi} \mathbf{v}_h^0 = (\cos \alpha^M) \frac{\partial}{\partial \eta} \mathbf{v}_h^0 ,$$

i.e.,

$$\frac{\partial}{\partial \eta} \mathbf{v}_h^0 = (\cos \alpha^M)^{-1} \frac{\partial}{\partial \nu} \mathbf{v}_h^0 .$$

But $\mathbf{v}_h^0$ is linear in the direction ν, therefore we have

$$\frac{\partial}{\partial \nu} \mathbf{v}_h^0 = \frac{\mathbf{v}_h^0(A_{i+1}) - \mathbf{v}_h^0(A_i)}{|A_i A_{i+1}|} = \frac{\mathbf{v}(A_{i+1}) - \mathbf{v}(A_i)}{|A_i A_{i+1}|} = \frac{\partial}{\partial \nu} \mathbf{v}(N_i) ,$$

where N_i is a point lying between A_i and A_{i+1}. Since

$$\frac{\partial}{\partial \nu} \mathbf{v}(N_i) = \frac{\partial}{\partial \eta} \mathbf{v}(N_i) (\sin \alpha^M) + (\cos \alpha^M) \frac{\partial}{\partial \xi} \mathbf{v}(N_i)$$

we have

$$\frac{\partial}{\partial \eta} \mathbf{v}_h^0 = \frac{\partial}{\partial \eta} \mathbf{v}(N_i) + (\operatorname{tg} \alpha^M) \frac{\partial}{\partial \xi} \mathbf{v}(N_i) ,$$

i.e.,

$$\left| \frac{\partial}{\partial \eta} \mathbf{v}_h^0 \right| \leqq C \|\mathbf{v}\|_{1,\infty,G} .$$

Here C is independent of T_c because of the fact that

$$|\operatorname{tg} \alpha^M| \leqq \|f^M\|_{C^1(\langle a,b \rangle)} .$$

A similar result holds for T_c^2.

Finally, we obtain

$$\left| \frac{\partial}{\partial \xi} (\mathbf{v}_h^0 - \mathbf{v}) \right| = \left| \frac{\partial}{\partial \xi} \mathbf{v} \right| \leqq \|\mathbf{v}\|_{1,\infty,G} ,$$

$$\left| \frac{\partial}{\partial \eta} (\mathbf{v}_h^0 - \mathbf{v}) \right| \leqq C \|\mathbf{v}\|_{1,\infty,G} .$$

Thus we deduce

(4.6) $$|\mathbf{v}_h^0 - \mathbf{v}|_{1,T_c}^2 \leqq C \|\mathbf{v}\|_{1,\infty,G}^2 \operatorname{mes} T_c .$$

Combining (4.5) with (4.6) we arrive at

$$\|\mathbf{v}_h^0 - \mathbf{v}\|_{1,T_c}^2 \leqq C \|\mathbf{v}\|_{1,\infty,G}^2 \operatorname{mes} T_c$$

and

(4.7) $$\|\mathbf{v}_h^0 - \mathbf{v}\|_{1,D_h^M}^2 \leqq \sum_{T_c \in D_h^M} \|\mathbf{v}_h^0 - \mathbf{v}\|_{1,T_c}^2 \leqq C \|\mathbf{v}\|_{1,\infty,G}^2 \operatorname{mes} D_h^M \to 0$$

for $h \to 0$ since $\operatorname{mes} D_h^M \to 0$ for $h \to 0$. Then (4.1) is a consequence of (4.4) and (4.7).

Next we show that

$$(4.8) \qquad \|\mathbf{v}_I - \mathbf{v}_h^0\|_{1,\Omega^M} \to 0 \quad \text{for} \quad h \to 0 .$$

In fact, it is clear that

$$\operatorname{Supp}(\mathbf{v}_I - \mathbf{v}_h^0) \subset \cup T_s \cup D_h \cup E_h ,$$

where E_h is the set of all "straight" triangles having at least one vertex at the point of intersection of two tangents to Γ_K^M.

In every T_s we may write

$$v_{hj}^0(\xi, \eta) - v_{Ij}(\xi, \eta) = v_{hj}^0(\xi, \eta) - v_{Ij}(\xi(s), \eta) = [\xi - \xi(s)] \frac{\partial}{\partial \xi} v_{hj}^0$$

for the j-th component, $j = 1, 2$, where $(\xi(s), \eta)$ is the projection of the point (ξ, η) on the chord in the direction ξ. Since $f^M \in C^2(\langle a, b \rangle)$ we have $|\xi - \xi(s)| \leqq h$. Thus we deduce

$$(4.9) \qquad \int_{T_s} (v_{hj}^0 - v_{Ij})^2 \, d\xi \, d\eta \leqq h^2 \int_{T_s} \left(\frac{\partial}{\partial \xi} v_{hj}^0 \right)^2 d\xi \, d\eta .$$

For T_s we also have

$$(4.10) \qquad \frac{\partial}{\partial \xi} (v_{hj}^0 - v_{Ij}) = \frac{\partial}{\partial \xi} v_{hj}^0 , \quad \frac{\partial}{\partial \eta} (v_{hj}^0 - v_{Ij}) = - \frac{\partial}{\partial \xi} v_{hj}^0 \frac{\partial}{\partial \eta} \xi(s) ,$$

$$\left| \frac{\partial}{\partial \eta} \xi(s) \right| = |\operatorname{tg} \alpha| \leqq \|f^M\|_{C^1(\langle a,b \rangle)} , \quad j = 1, 2 ,$$

where α is the angle between η-axis and the chord. From (4.9), (4.10) we obtain

$$(4.11) \qquad \|\mathbf{v}_h^0 - \mathbf{v}_I\|_{1,T_s}^2 \leqq C \|\mathbf{v}_h^0\|_{1,T_s}^2 .$$

Let us consider a $T \in E_h$. Then $T \subset G$ for h small enough. We know that T has either the position of T^* or T^{**} (see Fig. 7).

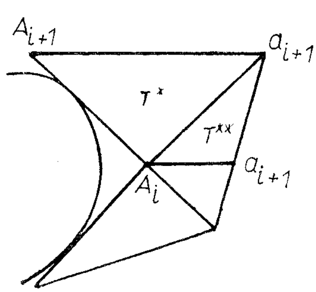

Fig. 7

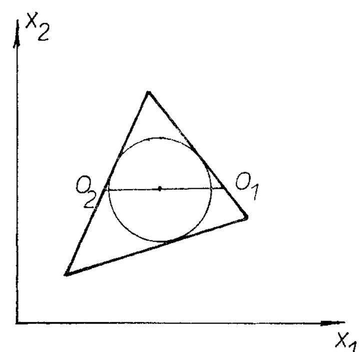

Fig. 8

If T has the position of T^*, then $v^0_{hj}(a_{i+1}) = v_{Ij}(a_{i+1})$, i.e.,

(4.12) $$|v^0_{hj} - v_{Ij}| \leqq \max_{k=i,i+1} |v^0_{hj}(A_k) - v_{Ij}(A_k)|\,, \quad j = 1, 2\,.$$

If T has the position of T^{**}, $v^0_{hj}(a_k) = v_{Ij}(a_k)$, $k = i, i + 1$, and for each $(\xi, \eta) \in T^{**}$ we have

(4.13) $$|v^0_{hj}(\xi, \eta) - v_{Ij}(\xi, \eta)| \leqq |v^0_{hj}(A_i) - v_{Ij}(A_i)|\,, \quad j = 1, 2\,.$$

Moreover, for the j-th component, $j = 1, 2$, we have

(4.14) $$|v^0_{hj}(A_i) - v_{Ij}(A_i)| = |v_j(A_i) - v_j(D_i)| = \left|\int_{A_i}^{D_i} \frac{\partial}{\partial \xi} v_j \,\mathrm{d}\xi\right| \leqq \int_{A_i}^{D_i} \left|\frac{\partial}{\partial \xi} v_j\right| \mathrm{d}\xi\,.$$

If h is sufficiently small, $\overline{A_iD_i} \subset G$, $|A_iD_i| \leqq Ch_T^2$, where C is independent of A_i and h_T. Furthermore, we have

$$\left|\frac{\partial}{\partial \xi} v_j\right| \leqq |\mathbf{v}|_{1,\infty,G}\,, \quad (\xi, \eta) \in \overline{A_iD_i}\,.$$

Thus, from (4.14) we get

(4.15) $$|v^0_{hj}(A_i) - v_{Ij}(A_i)| \leqq Ch_T^2 |\mathbf{v}|_{1,\infty,G}\,.$$

Combining (4.12)÷(4.15), for every $T \in E_h$ we obtain

(4.16) $$\|\mathbf{v}^0_h - \mathbf{v}_I\|_{0,\infty,T} \leqq Ch^2 |\mathbf{v}|_{1,\infty,G}\,,$$

(4.17) $$\|\mathbf{v}^0_h - \mathbf{v}_I\|^2_{0,T} \leqq Ch_T^4 |\mathbf{v}|_{1,\infty,G} \operatorname{mes} T\,.$$

Since $\mathbf{v}^0_h$ and $\mathbf{v}_I$ are linear on T and the system $\{\mathscr{T}_h\}$ is regular, setting $\mathbf{w} = \mathbf{v}^0_h - \mathbf{v}_I$ we have (see Fig. 8)

$$\frac{\partial w_j}{\partial x_1} = \frac{w_j(0_1) - w_j(0_2)}{|0_1 0_2|} \leqq \frac{2\|\mathbf{w}\|_{0,\infty,T}}{2\varrho_T} \leqq \frac{1}{\alpha h_T} \|\mathbf{w}\|_{0,\infty,T}\,, \quad j = 1, 2$$

where $\alpha > 0$ is the number from the definition of the regularity of $\{\mathscr{T}_h\}$. Making use of (4.16) we obtain

(4.18) $$\int_T \left|\frac{\partial w_j}{\partial x_1}\right|^2 \mathrm{d}x \leqq \frac{1}{\alpha^2}\, h_T^2 |\mathbf{v}|_{1,\infty,G} \operatorname{mes} T\,, \quad j = 1, 2\,,$$

and a similar estimate for $\int_T |\partial w_j/\partial x_2|^2 \,\mathrm{d}x$.

Finally, (4.17), (4.18) imply

(4.19) $$\|\mathbf{v}^0_h - \mathbf{v}_I\|^2_{1,T} \leqq Ch^2 |\mathbf{v}|^2_{1,\infty,G} \operatorname{mes} T\,.$$

Let us consider the last case, when $T_c \subset D_h$ is defined by sides A_iH_i, A_iH_{i+1} and the arc $\widehat{H_iH}_{i+1}$ (see Fig. 6). By definitions of $\mathbf{v}^0_h$, $\mathbf{v}_I$ we get

$$v_{hj}^0(A_i) = v_j(A_i), \quad v_{Ij}(A_i) = v_j(D_i),$$

i.e.,

$$v_{hj}^0(A_i) - v_{Ij}(A_i) = v_j(A_i) - v_j(D_i). \tag{4.20}$$

Furthermore, for each $(\xi, \eta) \in T_c$ we can show

$$|v_{hj}^0(\xi, \eta) - v_{Ij}(\xi, \eta)| \leqq \max_{k=i-1,i,i+1} |v_j(A_k) - v_j(D_k)| \tag{4.21}$$

because of the fact that

$$|v_{hj}^0(H_s) - v_{Ij}(H_s)| \leqq \max_{k=s,s-1} |v_j(A_k) - v_j(D_k)|, \quad s = i, i+1; \quad j = 1, 2$$

and the functions v_{hj}^0 are linear on T_c^m, $m = 1, 2$ (see Fig. 6).

From (4.14), (4.15) and (4.21) we deduce

$$\|\mathbf{v}_h^0 - \mathbf{v}_I\|_{0,T_c}^2 \leqq ch^4 |\mathbf{v}|_{1,\infty,G}^2 \text{ mes } T_c.$$

Moreoves, for each T_c we have

$$\frac{\partial}{\partial \xi}(\mathbf{v}_h^0 - \mathbf{v}_I) = 0, \quad \left|\frac{\partial}{\partial \eta}\mathbf{v}_h^0\right| \leqq C|\mathbf{v}|_{1,\infty,G}, \quad \left|\frac{\partial}{\partial \eta}\mathbf{v}_I\right| \leqq C|\mathbf{v}|_{1,\infty,G}.$$

Consequently,

$$|\mathbf{v}_h^0 - \mathbf{v}_I|_{1,T_c}^2 \leqq C|\mathbf{v}|_{1,\infty,G}^2 \text{ mes } T_c$$

holds, i.e.,

$$\|\mathbf{v}_h^0 - \mathbf{v}_I\|_{1,T_c}^2 \leqq (Ch^4 + C_1)\,|\mathbf{v}|_{1,\infty,G}^2 \text{ mes } T_c. \tag{4.22}$$

Using (4.11), (4.19) and (4.22) we arrive at

$$\|\mathbf{v}_h^0 - \mathbf{v}_I\|_{1,\Omega}^2 = \sum_{T_s} \|\mathbf{v}_h^0 - \mathbf{v}_I\|_{1,T_s}^2 + \sum_{T \in E_h} \|\mathbf{v}_h^0 - \mathbf{v}_I\|_{1,T}^2 + \tag{4.23}$$

$$+ \sum_{T_c \in D_h} \|\mathbf{v}_h^0 - \mathbf{v}_I\|_{1,T_c}^2 \leqq C\|\mathbf{v}_h^0\|_{1,\cup T_s}^2 + C_1\|\mathbf{v}\|_{1,\infty,G} \text{ mes}(E_h \cup D_h) \to 0, \quad h \to 0.$$

Here we have also used the fact that mes $(E_h \cup D_h) \to 0$ for $h \to 0$, (using (4.4)) mes $(\cup T_s) \to 0$ for $h \to 0$ and $\|\mathbf{v}_h^0\|_{1,\cup T_s} \leqq \|\mathbf{v}\|_{1,\cup T_s} + \|\mathbf{v}_h^0 - \mathbf{v}\|_{1,\cup T_s}$.

Combining (4.1), (4.23) with the triangle inequality we obtain the assertion of Lemma 4.1. Q.E.D.

Theorem 2. *Assume that $f^M \in C^2$, $M = {}', {}''$ in a neighbourhood of the interval $\langle a, b\rangle$. Let the assumptions on Γ_K^M, Γ_0, Γ_u, Γ_τ from Lemma 2.2 hold. Assume that $\mathbf{u}$ satisfies the conditions of Lemma 4.1 and $S_{F,P}^+ \neq \emptyset$.*

Then the approximate problem 2.1 possesses a unique solution and

$$\|\lambda_h - \lambda^*\| \to 0, \quad h \to 0$$

holds for any regular system of triangulations $\{\mathcal{T}_h\}$, where λ^* is the solution of the dual problem (1.27).

Proof is parallel to that of Theorem 1, but instead of Lemma 2.3 we use Lemma 4.1.

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Souhrn

DUÁLNÍ ANALÝZA KONTAKTU DVOU PRUŽNÝCH TĚLES S PROMĚNNÝM ROZSAHEM KONTAKTU METODOU KONEČNÝCH PRVKŮ

TRAN VAN BON

V práci je provedena duální analýza kontaktního problému dvou pružných těles s proměnným rozsahem kontaktu. Aproximace řešení metodou konečných prvků jsou definovány na dvou typech triangulací po částech konstantními poli napětí. Dokazuje se konvergence obou typů aproximací.

Резюме

ДВОЙСТВЕННЫЙ АНАЛИЗ КОНТАКТНОЙ ЗАДАЧИ УПРУГИХ ТЕЛ С РАСШИРЯЮЩЕЙСЯ ЗОНОЙ КОНТАКТА

TRAN VAN BON

Определяется двойственная вариационная формулировка контактной задачи двух упругих тел. Аппроксимации построены при помощи двух типов триангуляции и по частям постоянных полей напряжений. В обоих случаях доказывается сходимость аппроксимаций.

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