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DUAL FINITE ELEMENT ANALYSIS FOR ELLIPTIC PROBLEMS WITH OBSTACLES ON THE BOUNDARY, I

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INTRODUCTION

Recently, Mosco and Strang [5] have published an error analysis for a finite element procedure applied to unilateral problems with an obstacle in the domain. Using some ideas of their approach, the dual finite analysis has been accomplished in [3] for unilateral problems with conditions of Signorini's type on the boundary, i.e., with boundary obstacles given by a zero function.

In the present paper we extend the results of [3] to some problems with non-homogeneous obstacles on the boundary. The dual finite element procedures are proposed using piecewise linear polynomials on triangulations of the given domain and  $O(h)$  convergence in energy norm proved, provided the solution is sufficiently regular. Some a posteriori error estimates and two-sided bounds for the energy of the solution are also derived.

1. THE DUAL VARIATIONAL FORMULATIONS

Let us consider the following model problem

$$(1.1) \quad \begin{aligned} &-\Delta u + u = f \quad \text{in } \Omega \subset R^n, \\ &u - g \geq 0, \quad \partial u / \partial v \geq 0, \quad (u - g) \partial u / \partial v = 0 \quad \text{on } \partial \Omega \equiv \Gamma, \end{aligned}$$

where  $\partial u / \partial v$  denotes the derivative with respect to the outward normal  $v$  and  $f, g$  are given functions. Let  $\Omega$  be a bounded domain with Lipschitz boundary (cf. e.g. [1] for the definition). Henceforth we use the Sobolev spaces  $H^k(\Omega)$  with the usual norms  $\|u\|_k, H^0(\Omega) = L_2(\Omega), x = (x_1, x_2, \dots, x_n)$ ,

$$(u, v)_0 = \int_{\Omega} uv \, dx,$$

$$(u, v)_1 = (u, v)_0 + \sum_{i=1}^n (\partial u / \partial x_i, \partial v / \partial x_i)_0.$$

Assume that  $f \in L_2(\Omega)$  and that a function  $G \in H^2(\Omega)$  exists such that  $G = g$  on the boundary  $\Gamma$ .<sup>1)</sup>

The problem (1.1) can be recast as follows. Introduce the convex set

$$\mathcal{K} = \{v \mid v \in H^1(\Omega), \gamma v - g \geq 0 \text{ on } \Gamma\},$$

where  $\gamma v$  denotes the trace of  $v$  on the boundary, and the functional (potential energy)

$$\mathcal{L}(v) = \frac{1}{2} \|v\|_1^2 - (f, v)_0.$$

Then the problem to find  $u \in \mathcal{K}$  such that

$$(1.2) \quad \mathcal{L}(u) \leq \mathcal{L}(v) \quad \forall v \in \mathcal{K}$$

represents a variational formulation of the problem (1.1) and it will be called *primary*.

The problem can be reformulated in terms of the gradient-vector (cf. [3]). To this end we introduce the set

$$Q = \{\mathbf{q} \in [L_2(\Omega)]^n, \operatorname{div} \mathbf{q} \in L_2(\Omega)\},$$

where the operator

$$\operatorname{div} \mathbf{q} = \sum_{i=1}^n \partial q_i / \partial x_i$$

is defined in the sense of distributions. For  $\mathbf{q} \in Q$ , we may define the functional  $\mathbf{q} \cdot \mathbf{v} \in H^{-1/2}(\Gamma)$  by means of the relation<sup>2)</sup>

$$(1.3) \quad \langle \mathbf{q} \cdot \mathbf{v}, \gamma v \rangle = \int_{\Omega} (\mathbf{q} \cdot \operatorname{grad} v + v \operatorname{div} \mathbf{q}) \, dx \quad \forall v \in H^1(\Omega).$$

We write  $s \geq 0$  for an  $s \in H^{-1/2}(\Gamma)$  if

$$\langle s, \gamma v \rangle \geq 0 \quad \forall v \in \mathcal{C},$$

where

$$\mathcal{C} = \{v \in H^1(\Omega), \gamma v \geq 0 \text{ on } \Gamma\}.$$

Finally, introduce the set

$$(1.4) \quad \mathcal{U} = \{\boldsymbol{\lambda} \in [L_2(\Omega)]^{n+1}, \boldsymbol{\lambda} = [\boldsymbol{\lambda}', \lambda_{n+1}], \boldsymbol{\lambda}' \in Q, \\ \lambda_{n+1} = f + \operatorname{div} \boldsymbol{\lambda}', \boldsymbol{\lambda}' \cdot \mathbf{v} \geq 0 \text{ on } \Gamma\}$$

and the functional (complementary energy)

$$(1.5) \quad \mathcal{S}_g(\boldsymbol{\lambda}) = \frac{1}{2} \sum_{i=1}^{n+1} \|\lambda_i\|_0^2 - \langle \boldsymbol{\lambda} \cdot \mathbf{v}, g \rangle.$$

<sup>1)</sup> See e. g. [2], where some sufficient conditions for the existence of  $G$  are presented.

<sup>2)</sup> Henceforth  $\mathbf{a} \cdot \mathbf{b}$  denotes the scalar product  $\sum_{i=1}^n a_i b_i$ . See e.g. [1] for the definition of  $H^{-1/2}(\Gamma)$ .

The problem to find  $\lambda^0 \in \mathcal{U}$  such that

$$(1.6) \quad \mathcal{S}_g(\lambda^0) \leq \mathcal{S}_g(\lambda) \quad \forall \lambda \in \mathcal{U},$$

will be called *dual* to the primary problem (1.2).

It is easy to prove that both the primary and the dual problem possesses a unique solution. Moreover, there is an interpretation of the solution to the dual problem in terms of the solution to the primary problem.

**Theorem 1.1.** *If  $u$  is the solution to the primary problem (1.2) and  $\lambda^0$  the solution to the dual problem (1.6), then*

$$(1.7) \quad \lambda_i^0 = \partial u / \partial x_i, \quad i = 1, \dots, n, \quad \lambda_{n+1}^0 = u.$$

*Proof.* First we rewrite the dual problem into an equivalent one. Setting

$$(1.8) \quad \begin{aligned} \lambda_i &= p_i + \partial G / \partial x_i, \quad i = 1, \dots, n, \\ \lambda_{n+1} &= p_{n+1} + G, \end{aligned}$$

we may write for  $\lambda \in \mathcal{U}$

$$(1.9) \quad \mathcal{S}_g(\lambda) = \mathcal{S}(\mathbf{p}) + (G, f)_0 - \frac{1}{2} \|G\|_1^2,$$

where

$$\mathcal{S}(\mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n+1} \|p_i\|_0^2.$$

It is readily seen that  $\lambda \in \mathcal{U}$  if and only if  $\mathbf{p} \in \mathcal{U}_G$ , where

$$\begin{aligned} \mathcal{U}_G &= \{ \mathbf{p} = [\mathbf{p}', p_{n+1}], \quad \mathbf{p}' \in Q, \quad p_{n+1} = f + \Delta G - G + \operatorname{div} \mathbf{p}', \\ &\quad \mathbf{p}' \cdot \mathbf{v} + \partial G / \partial v = 0 \quad \text{on} \quad \Gamma \}. \end{aligned}$$

Consequently, the problem to find  $\mathbf{p}^0 \in \mathcal{U}_G$  such that

$$(1.10) \quad \mathcal{S}(\mathbf{p}^0) \leq \mathcal{S}(\mathbf{p}) \quad \forall \mathbf{p} \in \mathcal{U}_G$$

is equivalent with the dual problem (1.6).

We can prove the following

**Lemma 1.1.** *There exists  $w \in H_+^{1/2}(\Gamma)$  such that*

$$(1.11) \quad \begin{aligned} \mathcal{S}(\mathbf{p}^0) - \langle \mathbf{p}^0 \cdot \mathbf{v} + \partial G / \partial v, \mu \rangle &\leq \mathcal{S}(\mathbf{p}^0) - \langle \mathbf{p}^0 \cdot \mathbf{v} + \partial G / \partial v, w \rangle \leq \\ &\leq \mathcal{S}(\mathbf{p}) - \langle \mathbf{p} \cdot \mathbf{v} + \partial G / \partial v, w \rangle \end{aligned}$$

holds for any  $\mu \in H_+^{1/2}(\Gamma)$ ,  $\mathbf{p} \in Q_{fG}$ , where

$$H_+^{1/2} = \{ v \in H^{1/2}(\Gamma), v \geq 0 \},$$

$Q_{fG} = \{\mathbf{p} \in [L_2(\Omega)]^{n+1}, \mathbf{p} = [\mathbf{p}', p_{n+1}], \mathbf{p}' \in Q, p_{n+1} = f + \Delta G - G + \operatorname{div} \mathbf{p}'\}$ .  
 Moreover,

$$\langle \mathbf{p}^0 \cdot \mathbf{v} + \partial G / \partial v, w \rangle = 0.$$

Proof of this Lemma is based on a Corollary of Hahn-Banach theorem being parallel to that of Lemma 1.1 in [3].

Using Lemma 1.1 and following the proof of Theorem 1.1 in [3], we show that

$$(1.12) \quad \mathbf{p}^0 = [\mathbf{p}^{0'}, p_{n+1}^0], \quad \mathbf{p}^{0'} = \operatorname{grad} \tilde{u}, \quad p_{n+1}^0 = \tilde{u},$$

where  $\tilde{u}$  solves the problem

$$(1.13) \quad -\Delta \tilde{u} + \tilde{u} = f + \Delta G - G \quad \text{in } \Omega, \quad \gamma \tilde{u} = w \quad \text{on } \Gamma.$$

Finally, let us prove that  $\tilde{u} = u - G$ , which will complete the proof of Theorem 1.1, by virtue of (1.8), (1.10) and (1.12). Setting  $V = v - G$ ,  $U = u - G$ , we have  $V \in \mathcal{C}$ ,  $U \in \mathcal{C}$  (see (1.3)). The function  $u$  is a solution of (1.2), precisely if

$$(u, v - u)_1 \geq (f, v - u)_0 \quad \forall v \in \mathcal{X}.$$

Thus for  $U$  we obtain an equivalent version:

$$(1.14) \quad (U, V - U)_1 \geq (f, V - U)_0 - (G, V - U)_1 \quad \forall V \in \mathcal{C}.$$

Inserting  $V = 0$  and  $V = 2U$ , we derive

$$(1.15) \quad (U, U)_1 = (f, U)_0 - (G, U)_1.$$

Consequently, (1.15) and (1.14) result in

$$(1.16) \quad (U, V)_1 = (f, V)_0 - (G, V)_1 \quad \forall V \in \mathcal{C}.$$

$U$  is a solution of (1.14) if and only if it satisfies (1.15), (1.16). Let us verify (1.15), (1.16) for  $\tilde{u}$ . In fact, we have

$$0 \leq \left\langle \frac{\partial}{\partial v} (\tilde{u} + G), \gamma V \right\rangle = \int_{\Omega} [\operatorname{grad} (\tilde{u} + G) \cdot \operatorname{grad} V + V \operatorname{div} \operatorname{grad} (\tilde{u} + G)] dx \\ \forall V \in \mathcal{C},$$

where (1.12) and the definition of  $\mathcal{U}_G$  has been used.

On the other hand, from (1.13) we obtain that

$$\operatorname{div} \operatorname{grad} (\tilde{u} + G) = \tilde{u} + G - f.$$

Consequently,

$$0 \leq \int_{\Omega} [\operatorname{grad} (\tilde{u} + G) \cdot \operatorname{grad} V + V(\tilde{u} + G - f)] dx = (\tilde{u} + G, V)_1 - \\ -(V, f)_0 \quad \forall V \in \mathcal{C},$$

i.e., (1.16) is satisfied for  $U = \tilde{u}$ .

Making use of Lemma (1.1), we may write

$$0 = \left\langle \frac{\partial}{\partial v} (\tilde{u} + G), \gamma u \right\rangle = (\tilde{u} + G, \tilde{u})_1 - (f, \tilde{u})_0,$$

which is (1.15). Q.E.D.

## 2. FINITE ELEMENT APPROXIMATIONS OF THE PRIMARY PROBLEM

To propose a consistent dual finite element procedure, we restrict ourselves to plane polygonal domains (multiply connected, in general). Thus let  $\Omega$  be a polygonal bounded domain. We carve it into triangles  $T$ , generating a triangulation  $\mathcal{T}_h$ . Denote  $h$  the maximal side of all triangles in  $\mathcal{T}_h$  and  $S_h$  the space of continuous (in  $\Omega$ ) piecewise linear functions on  $\mathcal{T}_h$ . Henceforth we shall consider only  $\alpha$ - $\beta$ -regular families of triangulations  $\{\mathcal{T}_h\}$ ,  $0 < h \leq 1$ , i.e. such that positive parameters  $\alpha$ ,  $\beta$  exist, independent of  $h$ , and such that (i) no angle of all the triangles in  $\mathcal{T}_h$  is less than  $\alpha$ , (ii) the ratio of any two sides in  $\mathcal{T}_h$  is less than  $\beta$ .

Let us define  $g_h$  as the linear interpolate of  $g$  on  $\Gamma$  with the nodes determined by the vertices of the triangulation  $\mathcal{T}_h$ .

Introduce the following sets:

$$\begin{aligned} \mathcal{X}_h &= \{v \in S_h, \gamma v - g_h \geq 0 \text{ on } \Gamma\}, \\ \mathcal{C}_h &= \{v \in S_h, \gamma v \geq 0 \text{ on } \Gamma\} = \mathcal{C} \cap S_h. \end{aligned}$$

We say that  $u_h \in \mathcal{X}_h$  is a finite element approximation of the primary problem (1.2) if

$$(2.1) \quad \mathcal{L}(u_h) \leq \mathcal{L}(v) \quad \forall v \in \mathcal{X}_h.$$

Since  $\mathcal{X}_h$  is a closed convex subset of  $H^1(\Omega)$ , it is easy to see that (2.1) has a unique solution. To find it, we can apply e.g. the algorithm of Gauss-Seidel with constraints (cf. [4] Chpt., 4, § 1.4 or [3]-Section 2).

Next let us derive an error estimate for  $u - u_h$ . First we prove the following (cf. an analogous result of [5])

**Lemma 2.1.** *Let a function  $W_h \in \mathcal{C}_h$  exist such that  $2(u - G) - W_h \in \mathcal{C}$ . Then*

$$(2.2) \quad \|u - u_h\|_1 \leq \|u - G - W_h\|_1 + \|G - G_I\|_1,$$

where  $G_I$  denotes the linear interpolate of  $G$  on the triangulation  $\mathcal{T}_h$ .

*Proof.* Denote  $u = G + U$  and set  $v = G + W_h$ . Then  $v \in \mathcal{X}$  and  $2u - v = G + (2U - W_h) \in \mathcal{X}$ . We have

$$(2.3) \quad (u, w - u)_1 - (f, w - u)_0 \geq 0 \quad \forall w \in \mathcal{X}.$$

Consequently, inserting  $w = v$  and  $w = 2u - v$ , we derive the equation

$$(2.4) \quad (u, W_h - U)_1 = (f, W_h - U)_0.$$

Denoting  $U_h = u_h - G_I$  and setting  $v = G + U_h$ , we have  $\gamma U_h = \gamma u_h - g_h \geq 0$ , consequently  $v \in \mathcal{X}$ . If  $w = v$  is inserted in (2.3), it follows that

$$(2.5) \quad (u, U_h - U)_1 \geq (f, U_h - U)_0.$$

Third, choosing  $v_h = G_I + W_h$ , we have  $v_h \in \mathcal{X}_h$ . From (2.1) we obtain that

$$(2.6) \quad (u_h, v_h - u_h)_1 = (u_h, W_h - U_h)_1 \geq (f, W_h - U_h)_0.$$

Then using (2.4), (2.5) and (2.6), we may write

$$\begin{aligned} (u - u_h, U_h - W_h)_1 &= (u, U - W_h + U_h - U)_1 - (u_h, U_h - W_h)_1 \geq \\ &\geq (f, U - W_h)_0 + (f, U_h - U)_0 + (f, W_h - U_h)_0 = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \|u - u_h\|_1^2 &= (u - u_h, G - G_I + U - U_h)_1 \leq \\ &\leq (u - u_h, G - G_I)_1 + (u - u_h, U - U_h + U_h - W_h)_1 \leq \\ &\leq \|u - u_h\|_1 \{ \|G - G_I\|_1 + \|U - W_h\|_1 \}, \end{aligned} \quad \text{Q.E.D.}$$

According to Lemma 2.1, it remains to show the existence of a function  $W_h \in \mathcal{C}_h$ , sufficiently close to  $U = u - G$  and such that  $2U - W_h \in \mathcal{C}$ . The answer to this question is contained in the following

**Theorem 2.1.** *Assume that  $u \in H^2(\Omega)$  and  $u - g \in H^2(\Gamma_m)$ ,  $m = 1, \dots, M$ , where  $\Gamma_m$  denotes any side of the polygonal boundary  $\Gamma$ .*

*Then there exists  $W_h \in \mathcal{S}_h$  such that*

$$0 \leq W_h \leq u - g \quad \text{on } \Gamma$$

and

$$\|u - G - W_h\|_1 \leq Ch(\|u - G\|_2 + \sum_{m=1}^M \|u - g\|_{H^2(\Gamma_m)}),$$

where  $C$  is independent of  $h$ ,  $u$  and  $G$ .

For the proof – see [3] Section 2.

**Corollary 2.1.** *Let the assumption of Theorem 2.1 be satisfied. Then*

$$\|u - u_h\|_1 = O(h).$$

The proof follows from Lemma 2.1, Theorem 2.1 and the inequalities

$$2(u - g) - W_h \geq u - g - W_h \quad \text{on } \Gamma,$$

$$\|G - G_I\|_1 \leq Ch \|G\|_2.$$

### 3. FINITE ELEMENT APPROXIMATIONS OF THE DUAL PROBLEM

Making use of the definition (1.4), we can transform the dual problem (1.6) into an equivalent one: to find  $\mathbf{q}^0 \in \mathcal{U}_0$  such that

$$(3.1) \quad I(\mathbf{q}^0) \leq I(\mathbf{q}) \quad \forall \mathbf{q} \in \mathcal{U}_0,$$

where

$$\mathcal{U}_0 = \{\mathbf{q} \in \mathcal{Q}, \mathbf{q} \cdot \mathbf{v} \geq 0 \quad \text{on } \Gamma\},$$

$$(3.2) \quad I(\mathbf{q}) = \frac{1}{2} \left( \sum_{i=1}^n \|q_i\|_0^2 + \|\operatorname{div} \mathbf{q}\|_0^2 \right) + (f, \operatorname{div} \mathbf{q})_0 - \langle \mathbf{q} \cdot \mathbf{v}, g \rangle.$$

Then

$$\lambda_i^0 = q_i^0, \quad (i = 1, \dots, n), \quad \lambda_{n+1}^0 = f + \operatorname{div} \mathbf{q}^0.$$

Consider again the  $\alpha$ - $\beta$ -regular triangulations  $\mathcal{T}_h$  of  $\Omega \subset R^2$  and the spaces  $S_h$  of piecewise linear functions on  $\mathcal{T}_h$ . Introducing the subset

$$\mathcal{U}_{0h} = \mathcal{U}_0 \cap [S_h]^2,$$

we may define:

a vector  $\mathbf{q}^h \in \mathcal{U}_{0h}$  will be called a finite element approximation of the dual problem (3.1), if

$$(3.3) \quad I(\mathbf{q}^h) \leq I(\mathbf{q}) \quad \forall \mathbf{q} \in \mathcal{U}_{0h}.$$

The problem (3.3) has a unique solution (cf. an analogue in [3]-Section 3, where also some algorithm for solving (3.3) has been proposed). Note that the last term in (3.2) reduces to an integral, i.e.,

$$(3.4) \quad -\langle \mathbf{q} \cdot \mathbf{v}, g \rangle = - \int_{\Gamma} \mathbf{q} \cdot \mathbf{v} g \, ds \quad \forall \mathbf{q} \in S_h.$$

As far as the error estimate for  $\mathbf{q}^0 - \mathbf{q}^h$  is concerned, we may apply the approach of [3]-Section 3, completing only the functional  $I$  of [3] by the term (3.4). Thus we come to the following

**Theorem 3.1.** *Assume that  $\mathbf{q}^0 \in [H^2(\Omega)]^2$  and  $\mathbf{q}^0 \cdot \mathbf{v} \in H^2(\Gamma_m)$ , where  $\Gamma_m$  is any side of the polygonal boundary  $\Gamma$ . Then*

$$\sum_{i=1}^2 \|q_i^0 - q_i^h\|_0 + \|\operatorname{div}(\mathbf{q}^0 - \mathbf{q}^h)\|_0 = O(h).$$



Remark 3.1. If  $\mathbf{q}^h$  is a solution of (3.3), then

$$\lambda^h = \{q_1^h, q_2^h, f + \operatorname{div} \mathbf{q}^h\} \in \mathcal{U}$$

is an approximation to the solution  $\lambda^0$  of (1.6). By virtue of Theorem 1.1 and 3.1, it holds

$$\sum_{i=1}^2 \|q_i^h - \partial u / \partial x_i\|_0 = O(h), \quad \|\operatorname{div} \mathbf{q}^h + f - u\|_0 = O(h).$$

#### 4. A POSTERIORI ERROR ESTIMATES AND TWO-SIDED BOUNDS OF ENERGY

In this section, we derive some a posteriori error estimates, utilising the dual finite element analysis.

Since  $u$  satisfies (2.3), we may write for any  $w \in \mathcal{X}$

$$(4.1) \quad \begin{aligned} 2[\mathcal{L}(w) - \mathcal{L}(u)] &= \|w\|_1^2 - \|u\|_1^2 - 2(f, w - u)_0 \geq \\ &\geq \|w\|_1^2 - \|u\|_1^2 - 2(u, w - u)_1 = \|w - u\|_1^2. \end{aligned}$$

Let us search an upper bound for  $-\mathcal{L}(u)$ . From the duality theory (cf. e.g. [4] chpt. 5, § 3 for an analogous problem) it follows

$$(4.2) \quad \mathcal{L}(u) = \operatorname{Max}_{\mu \in H_+^{-1/2}(\Gamma)} \operatorname{Min}_{v \in H^1(\Omega)} \{\mathcal{L}(v) - \langle \mu, \gamma v - g \rangle\},$$

where

$$H_+^{-1/2}(\Gamma) = \{s \in H^{-1/2}(\Gamma), \quad s \geq 0\}.$$

Setting

$$V = v - G,$$

we can write

$$\begin{aligned} \mathcal{L}(v) &= \langle \mu, \gamma v - g \rangle = \frac{1}{2} \|V + G\|_1^2 - (f, V + G)_0 - \langle \mu, \gamma V \rangle = \\ &= \frac{1}{2} \|V\|_1^2 + (V, G)_1 - (f, V)_0 - \langle \mu, \gamma V \rangle + \frac{1}{2} \|G\|_1^2 - (f, G)_0. \end{aligned}$$

Then obviously

$$(4.3) \quad \operatorname{Min}_{v \in H^1(\Omega)} \{\mathcal{L}(v) - \langle \mu, \gamma v - g \rangle\} = \frac{1}{2} \|G\|_1^2 - (f, G)_0 + \operatorname{Min}_{V \in H^1(\Omega)} \mathcal{L}_1(V),$$

where

$$\mathcal{L}_1(V) = \frac{1}{2} \|V\|_1^2 + (V, G)_1 - (f, V)_0 - \langle \mu, \gamma V \rangle.$$

It is well-known that if  $V_\mu$  minimizes  $\mathcal{L}_1(V)$  over  $H^1(\Omega)$ , then

$$\lambda(\mu) = \{\partial V_\mu / \partial x_1, \partial V_\mu / \partial x_2, V_\mu\}$$

minimizes the functional (of complementary energy – cf. e.g. [6])

$$\mathcal{S}(\mathbf{p}) = \frac{1}{2} \sum_{i=1}^3 \|p_i\|_0^2$$

over the set  $A_\mu$  and

$$\text{Min}_{V \in H^1(\Omega)} \mathcal{L}_1(V) = - \text{Min}_{\mathbf{p} \in A_\mu} \mathcal{S}(\mathbf{p}),$$

where

$$\begin{aligned} A_\mu &= \{ \mathbf{p} \in [L_2(\Omega)]^3, \sum_{i=1}^2 (p_i, \partial V / \partial x_i)_0 + (p_3, V)_0 = \\ &= (f, V)_0 - (V, G)_1 + \langle \mu, \gamma V \rangle \quad \forall V \in H^1(\Omega) \}. \end{aligned}$$

The latter relation, however, can be rewritten as follows

$$\langle \mu, \gamma V \rangle = \int_{\Omega} \left[ \sum_{i=1}^2 (p_i + \partial G / \partial x_i) \partial V / \partial x_i + (p_3 + G - f) V \right] dx.$$

Inserting  $V = \varphi \in C_0^\infty(\Omega)$ , we obtain that

$$\mathbf{p}' + \text{grad } G \in Q,$$

$$\text{div}(\mathbf{p}' + \text{grad } G) = p_3 + G - f,$$

consequently

$$(4.4) \quad p_3 = f + \text{div } \mathbf{p}' + \Delta G - G.$$

Using also (1.3), we may write

$$\langle \mu, \gamma V \rangle = \langle \mathbf{p}' \cdot \mathbf{v} + \partial G / \partial v, \gamma V \rangle \quad \forall V \in H^1(\Omega).$$

It means that

$$(4.5) \quad \mathbf{p}' \cdot \mathbf{v} + \partial G / \partial v = \mu \geq 0.$$

Now (4.4), (4.5) imply that (cf. the proof of Theorem 1.1)

$$(4.6) \quad \bigcup_{\mu \in H_+^{-1/2}(\Gamma)} A_\mu = \mathcal{U}_G.$$

Inserting (4.3) into (4.2), we obtain

$$\mathcal{L}(u) = \text{Max}_{\mu \in H_+^{-1/2}(\Gamma)} \left\{ \frac{1}{2} \|G\|_1^2 - (f, G)_0 - \text{Min}_{\mathbf{p} \in A_\mu} \mathcal{S}(\mathbf{p}) \right\}.$$

We have, by virtue of (4.6),

$$\text{Max}_{\mu \in H_+^{-1/2}(\Gamma)} \left( - \text{Min}_{\mathbf{p} \in A_\mu} \mathcal{S}(\mathbf{p}) \right) = - \text{Min}_{\mu} \text{Min}_{\mathbf{p}} \mathcal{S}(\mathbf{p}) = - \text{Min}_{\mathbf{p} \in U_G} \mathcal{S}(\mathbf{p}),$$

which results in the desired bound

$$(4.7) \quad \begin{aligned} -\mathcal{L}(u) &= -\frac{1}{2}\|G\|_1^2 + (f, G)_0 + \mathcal{L}(\mathbf{p}^0) \leq \\ &\leq \mathcal{L}(G) + \mathcal{L}(\mathbf{p}) = \mathcal{L}_g(\lambda) \quad \forall \lambda \in \mathcal{U}, \end{aligned}$$

where (1.9) has been used. Thus we are led to the following

**Theorem 4.1.** *Let  $\tilde{u}_h$  be any approximation of the primary problem (1.2) such that  $\tilde{u}_h \in \mathcal{X}^1$ . Let  $\mathbf{q}^h \in \mathcal{U}_{0h}$  be a finite element approximation of the dual problem (3.1). Then*

$$(4.8) \quad \begin{aligned} \|\tilde{u}_h - u\|_1^2 &\leq \sum_{i=1}^2 \|q_i^h - \partial u_h / \partial x_i\|_0^2 + \|f + \operatorname{div} \mathbf{q}^h - \tilde{u}_h\|_0^2 + \\ &+ 2 \int_{\Gamma} \mathbf{q}^h \cdot \mathbf{v}(\tilde{u}_h - g) \, ds \equiv E(\mathbf{q}^h, \tilde{u}_h), \end{aligned}$$

Proof is parallel to that of Theorem 6.1 in [3].

Remark 4.1. Suppose that  $G$  is known explicitly. Then

$$\tilde{u}_h = u_h + G - G_I$$

can be substituted in (4.8), where  $u_h \in \mathcal{X}_h$  is the finite element approximation (or any iterative solution of the problem (2.1), obtained by means of the Gauss-Seidel algorithm with constraints). Instead of  $\mathbf{q}^h$  we may insert any  $\mathbf{q}^{hm} \in \mathcal{U}_{0h}$ .

Note that all terms in the right-hand side of (4.8) are non-negative.

**Theorem 4.2.** *(Two-sided bounds for the energy). Let  $\tilde{u}_h$  and  $\mathbf{q}^h$  be the same as in Theorem 4.1. Then for  $U = u - G$  it holds*

$$(4.9) \quad \begin{aligned} 2\mathcal{L}(G) - 2\mathcal{L}(\tilde{u}_h) &\leq \|U\|_1^2 \leq \|q_i^h - \partial G / \partial x_i\|_0^2 + \\ &+ \|f - G + \operatorname{div} \mathbf{q}^h\|_0^2 \equiv F(\mathbf{q}^h), \end{aligned}$$

$$(4.10) \quad 2\mathcal{L}(G) - 2\mathcal{L}(\tilde{u}_h) \leq (f, U)_0 - (G, U)_1 \leq F(\mathbf{q}^h).$$

Proof. From (1.15) we know that

$$\|U\|_1^2 = (f, U)_0 - (G, U)_1.$$

Then we have

$$\begin{aligned} \mathcal{L}(u) &= \frac{1}{2}\|U + G\|_1^2 - (f, U + G)_0 = \frac{1}{2}\|U\|_1^2 - (f, U)_0 + (G, U)_1 + \\ &+ \frac{1}{2}\|G\|_1^2 - (f, G)_0 = -\frac{1}{2}\|U\|_1^2 + \mathcal{L}(G), \end{aligned}$$

<sup>1)</sup> Note that  $\mathcal{X}_h \not\subset \mathcal{X}$  unless  $g_h \geq g$ ! Therefore, the finite element approximations  $u_h \in \mathcal{X}_h$  cannot be used, in general.

consequently,

$$\|U\|_1^2 = 2\mathcal{L}(G) - 2\mathcal{L}(u) \geq 2\mathcal{L}(G) - 2\mathcal{L}(\tilde{u}_h).$$

Using (4.7) and (1.9), we obtain for any  $\mathbf{p} \in U_G$

$$\|U\|_1^2 = 2[\mathcal{L}(G) - \mathcal{L}(u)] \geq 2[\mathcal{L}(G) + \mathcal{S}_g(\lambda)] = 2\mathcal{L}(\mathbf{p}).$$

Finally, if  $\mathbf{q}^h \in \mathcal{U}_{oh}$ , then  $\mathbf{p} = [\mathbf{p}', p_3] \in \mathcal{U}_G$  with

$$(4.11) \quad \mathbf{p}' = \mathbf{q}^h - \text{grad } G, \quad p_3 = f + \text{div } \mathbf{q}^h - G. \quad \text{Q.E.D.}$$

**Remark 4.2** If  $f = 0$ , a two-sided estimate for  $\|u^2\|$  follows easily from (4.9) and (4.10), as

$$\|u\|_1^2 = \|U + G\|_1^2 = \|U\|_1^2 + \|G\|_1^2 + 2(G, U)_1.$$

**Theorem 4.3.** Let  $\tilde{u}_h$ ,  $\mathbf{q}^h$  and  $E(\mathbf{q}^h, \tilde{u}_h)$  be the same as in Theorem 4.1. Then it holds

$$(4.12) \quad \sum_{i=1}^2 \|q_i^h - \partial u / \partial x_i\|_0^2 + \|f + \text{div } \mathbf{q}^h - u\|_0^2 \leq E(\mathbf{q}^h, \tilde{u}_h).$$

**Proof.** The solution  $\mathbf{p}^0$  of (1.10) satisfies the inequality

$$(\mathbf{p}^0, \mathbf{p} - \mathbf{p}^0) \geq 0 \quad \forall \mathbf{p} \in \mathcal{U}_G,$$

where

$$(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^3 (p_i, q_i)_0.$$

Consequently, we may write for any  $\mathbf{p} \in \mathcal{U}_G$

$$(4.13) \quad \begin{aligned} 2\mathcal{L}(\mathbf{p}) - 2\mathcal{L}(\mathbf{p}^0) &= \|\mathbf{p}\|^2 - \|\mathbf{p}^0\|^2 \geq \|\mathbf{p}\|^2 - (\mathbf{p}^0, \mathbf{p}) = \\ &= (\mathbf{p}, \mathbf{p} - \mathbf{p}^0) - (\mathbf{p}^0, \mathbf{p} - \mathbf{p}^0) + (\mathbf{p}^0, \mathbf{p} - \mathbf{p}^0) \geq \|\mathbf{p} - \mathbf{p}^0\|^2. \end{aligned}$$

From (1.9) and (4.7) it follows that

$$(4.14) \quad \begin{aligned} \mathcal{S}(\mathbf{p}) - \mathcal{S}(\mathbf{p}^0) &= \mathcal{S}_g(\lambda) - \mathcal{S}_g(\lambda^0) = \mathcal{S}_g(\lambda) + \mathcal{L}(u) \leq \\ &\leq \mathcal{S}_g(\lambda) + \mathcal{L}(v) \quad \forall \lambda \in \mathcal{U}, \quad v \in \mathcal{H}, \end{aligned}$$

if  $\lambda = [\lambda', \lambda_3]$ , where  $\lambda' = \mathbf{p}' + \text{grad } G$ ,  $\lambda_3 = p_3 + G$ . We have  $\mathbf{p} - \mathbf{p}^0 = \lambda - \lambda^0$  and substituting

$$\lambda = \lambda^h = \{q_1^h, q_2^h, f + \text{div } \mathbf{q}^h\}, \quad \lambda^0 = \{\partial u / \partial x_1, \partial x / \partial x_2, u\},$$

from (4.13) and (4.14) we obtain that the left-hand side of (4.12) is bounded above

by  $2\mathcal{L}_g(\lambda^h) + 2\mathcal{L}(\tilde{u}_h)$ . The latter sum, however, can be rearranged to  $E(\mathbf{q}^h, \tilde{u}_h)$  (cf. the proof of Theorem 6.1 in [3]).

Remark 4.3. Using Corollary 2.1 and Theorem 3.1, it is easy to prove that  $E(\mathbf{q}^h, \tilde{u}_h)$ , where  $\tilde{u}_h = u_h + G - G_I$  (cf. Remark 4.1), tends to zero with  $h \rightarrow 0$ .

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#### Souhrn

### DUÁLNÍ ROZBOR ELIPTICKÝCH ÚLOH S PŘEKÁŽKAMI NA HRANICI METODOU KONEČNÝCH PRVKŮ, I

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V nedávné práci [3] předložil autor duální analýzu eliptických úloh druhého řádu s okrajovými podmínkami Signoriniho typu, tj. s překážkami na hranici danými nulovou funkcí. V tomto článku se rozšiřují výsledky z [3] na jednu třídu podobných úloh, ale s nehomogenními překážkami na hranici.

Pomocí po částech lineárních polynomů na triangulaci dané oblasti jsou navrženy duální metody konečných prvků a dokazuje se jejich  $O(h)$ -konvergence v energetické normě, za předpokladu, že řešení je dostatečně hladké. Dále se odvozují též některé aposteriori odhady chyb obou duálních metod a oboustranné odhady energie řešení.

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