Progress In Electromagnetics Research B, Vol. 16, 107–126, 2009

DUAL INTEGRAL EQUATIONS TECHNIQUE IN ELECTROMAGNETIC WAVE SCATTERING BY A THIN DISK

M. V. Balaban

Institute of Radiophysics and Electronics NASU
Ul. Proskury, 12, Kharkov 61085, Ukraine

R. Sauleau

Institut d'Electronique et de Télécommunications de Rennes
Université de Rennes 1
Rennes cedex, France

T. M. Benson

George Green Institute for Electromagnetics Research
University of Nottingham
University Park, Nottingham NG7 2RD, UK

A. I. Nosich

Institute of Radiophysics and Electronics NASU
Ul. Proskury, 12, Kharkov 61085, Ukraine

Abstract—The scattering of an arbitrary electromagnetic wave by a thin disk located in free space is formulated rigorously in terms of coupled dual integral equations (CDIEs) for the unknown images of the jumps and average values of the normal to the disk scattered-field components. Considered are three cases of the disk: (1) Zero-thickness perfectly electrically conducting (PEC) disk, (2) thin electrically resistive (ER) disk and (3) dielectric disk. Disk thickness is assumed much smaller than the disk radius and the free space wavelength, in ER and dielectric disk cases, and also much smaller than the skin-layer depth, in the ER disk case. The set of CDIEs are "decoupled" by introduction of the coupling constants. Each set of DIEs are reduced to a Fredholm second kind integral equation by using the semi-inversion of DIE integral operators. The set of "coupling" equations for finding

Corresponding author: M. V. Balaban (mikhail.balaban@gmail.com).

the coupling constants is obtained additionally from the edge behavior condition. Thus, each problem is reduced to a set of coupled Fredholm second kind integral equations. It is shown that each set can be reduced to a block-type three-diagonal matrix equation, which can be effectively solved numerically by iterative inversions of the two diagonal blocks and 2×2 matrix.

1. INTRODUCTION

The problem of electromagnetic wave scattering by a thin disk has been attracting the interest of researchers since long ago. This is explained by many applications of this canonical shape. Besides of traditional applications in the printed disk antennas with PEC [1, 2] or ER disks [3], thin dielectric disk is met as a simplified model of the tree leave [4]. Moreover, thin few-micron radius disks are used as resonators of semiconductor lasers with ultralow thresholds [5–7]. Many approximations and computational electromagnetics methods have been used in their analysis. High frequency approximation techniques (for the PEC disk scattering problem) are approximate methods commonly used when the size of the disk is much larger then the free-space wavelength. They are physical optics technique and physical theory of diffraction method; geometrical theory of diffraction; uniform theory of diffraction and uniform asymptotic theory method; equivalent current method. Boundary element method (BEM) is a numerical method using the Method of Moments (MoM) to solve an electric field (in PEC and ER disk case) or coupled field (electric field and magnetic field) integral equations for electric and magnetic currents on the disk. As it requires calculating only the boundary values, it is significantly more efficient (in terms of computational resources) for not large (versus the free-space wavelength) disk, otherwise using this method leads to the inversion of the large matrix. Moreover, in both cases (small and large disk) sometimes using of this method leads to the computational errors associated with ill-conditioned matrices. Finite element method (FEM) is also a numerical method. Every FEM code divides the entire problem domain into small elements. The domain must be finite and bounded. Modeling an unbounded (e.g., radiation) problem requires that the problem domain be bounded with special boundary that absorbs all incident energy. The boundary condition models have been well developed for 2D problems but have not for FEM 3D codes yet. The Finite Difference Time Domain (FDTD) method builds a direct solution of Maxwell's time-dependent curl equations. It uses simple approximations to evaluate the space and time derivatives and absorbing boundary condition to truncate the computational domain for an open 3D region problems. FDTD-based code divides space domain into the small elements as a FEM code and uses space and time recurrent formulas to find electrical and magnetic fields. Time stepping is continued until a steady state solution is obtained. Although FDTD is known as powerful numerical method, it has disadvantages such as numerical dispersion, backreflection from the borders of computational window, etc [8].

Here we develop the method of spectral domain integral equations combined with analytical regularization [9–14]. In contrast to high-frequency approximation techniques, FEM and FDTD methods, it is a rigorous method. It enables us to reduce each problem to the sets of coupled Fredholm second kind integral equations. These sets of equations can be discretized to the matrix equations of the same kind by applying any reasonable descritization scheme. Favorable features of this-kind equations guarantee that the obtained matrices are well conditioned.

2. PROBLEM STATEMENT

We consider the problem of diffraction of harmonic electromagnetic field by a zero-thickness disk of radius a. Assume that the center of the disk is located at the point $(\rho = 0, z = 0)$. Denote the total field as a sum of the scattered and incident fields:

$$E = E_{in} + E_{sc} \quad H = H_{in} + H_{sc} \tag{1}$$

Introduce dimensionless cylindrical coordinates $(\rho = r/a, \varphi, \zeta = z/a)$ with the origin at the disk center point. Demand the incident and scattered fields to satisfy the set of homogeneous Maxwell equations outside of the sources and outside the disk.

$$\operatorname{curl} E = ikaZ_0 H, \quad Z_0 \operatorname{curl} H = -ikaE. \tag{2}$$

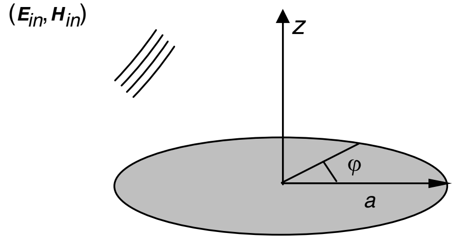

Figure 1. Problem geometry.

Demand the field components to satisfy the following generalized boundary conditions on the disk [15]:

$$\begin{aligned} \left[E_{\rm tg}^{+} + E_{\rm tg}^{-}\right] &= 2Z_0 R \cdot \overrightarrow{n} \times \left[H_{\rm tg}^{+} - H_{\rm tg}^{-}\right], \\ Z_0 \left[H_{\rm tg}^{+} + H_{\rm tg}^{-}\right] &= -2Q \cdot \overrightarrow{n} \times \left[E_{\rm tg}^{+} - E_{\rm tg}^{-}\right]. \end{aligned} \tag{3}$$

Here, Z_0 is the free-space impedance and R and Q are the electric and magnetic resistivities. For the dielectric disk they are given by

$$R = \frac{iZ}{2} \cot\left(\frac{\sqrt{\varepsilon_r \mu_r} k\tau}{2}\right), \quad Q = R/Z^2, \quad |\varepsilon_r \mu_r| \gg 1,\, \tau \ll \lambda_0. \tag{4}$$

Here, Z is the relative impedance of the disk material, $k = \omega/c$ is the wavenumber, ε_r is the relative permittivity, μ_r is the relative conductivity, λ_0 is the wavelength in free space, and τ is the thickness of the disk. For the ER disk they are given by

$$R = \frac{1}{Z_0 \sigma \tau}, \quad Q = \infty, \quad \sigma/\omega \gg 1,\, \tau \ll \lambda_0. \tag{5}$$

For a zero-thickness PEC disk they are given by

$$R = 0, \quad Q = \infty \tag{6}$$

On the rest part of the plane ($z = 0$) the components of the field are continuous. Besides, the components of the scattered field must satisfy the 3-D radiation condition and the edge condition [16] (condition of local integrability of power).

3. FIELDS COMPONENTS REPRESENTATION

To obtain convenient field component representations, assume that they are continuous everywhere in the free space except the plane ($\zeta = \xi$) and satisfy the homogeneous Maxwell Equations (2) in doubly-connected domain $\Omega = \Omega^+ \cup \Omega^- = \mathbb{R}^3 \backslash \{\zeta = \xi\}$. Then each component satisfies Helmholtz equation $\Delta u + (ka)^2 u = 0$ in Ω and can be presented in terms of Fourier-Bessel transform:

$$u(\rho, \varphi, \zeta) = \sum_{m=-\infty}^{\infty} e^{im\varphi} u_m(\rho, \zeta) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta - \xi|} u_m^{\pm}(\kappa) J_{|m|}(\kappa\rho) \kappa \, d\kappa \tag{7}$$

Note that the function $u^{\pm}(\rho, \varphi, \zeta)$ is presented in terms of Fourier series in φ where each coefficient is the m-th order scalar Hankel transformation of $e^{i\gamma(\kappa)|\zeta-\xi|} u_m^{\pm}(\kappa)$. Here, $\gamma(\kappa) = \sqrt{(ka)^2 - \kappa^2}$ is the complex valued function with the chosen branch $\mathrm{Re}(\gamma(\kappa)) \geq 0$, $\mathrm{Im}(\gamma(\kappa)) \geq 0$, and $u_m^{\pm}(\kappa)$ are the images of $u_m(\rho,\zeta)$ in Ω^+ and Ω^-, respectively. Introduce normal to the plane ($\zeta=\xi$) field components in terms of Fourier-Bessel transforms like (7) and use Maxwell's Equations (2) to find tangential field components. Finally we obtain:

$$\begin{pmatrix} E_z^{\mathrm{sgn}(\zeta-\xi)} \\ Z_0 H_z^{\mathrm{sgn}(\zeta-\xi)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta-\xi|} J_{|m|}(\kappa\rho) \begin{pmatrix} \kappa e_{m,z}^{\mathrm{sgn}(\zeta-\xi)}(\kappa) \\ \kappa h_{m,z}^{\mathrm{sgn}(\zeta-\xi)}(\kappa) \end{pmatrix} d\kappa \quad (8)$$

$$\begin{pmatrix} E_r^{\mathrm{sgn}(\zeta-\xi)} \\ -iE_\varphi^{\mathrm{sgn}(\zeta-\xi)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta-\xi|} \bar{H}_m(\kappa\rho) \begin{pmatrix} \mathrm{sgn}(\zeta-\xi)\, i\gamma(\kappa) e_{m,z}^{\mathrm{sgn}(\zeta-\xi)}(\kappa) \\ -ka\, h_{m,z}^{\mathrm{sgn}(\zeta-\xi)}(\kappa) \end{pmatrix} d\kappa \quad (9)$$

$$\begin{pmatrix} Z_0 H_r^{\mathrm{sgn}(\zeta-\xi)} \\ -iZ_0 H_\varphi^{\mathrm{sgn}(\zeta-\xi)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta-\xi|} \bar{H}_m(\kappa\rho) \begin{pmatrix} \mathrm{sgn}(\zeta-\xi)\, i\gamma(\kappa) h_{m,z}^{\mathrm{sgn}(\zeta-\xi)}(\kappa) \\ ka\, e_{m,z}^{\mathrm{sgn}(\zeta-\xi)}(\kappa) \end{pmatrix} d\kappa \quad (10)$$

here

$$\bar{H}_m(\kappa\rho) = \begin{pmatrix} J'_{|m|}(\kappa\rho) & mJ_{|m|}(\kappa\rho)/(\kappa\rho) \\ mJ_{|m|}(\kappa\rho)/(\kappa\rho) & J'_{|m|}(\kappa\rho) \end{pmatrix} \quad (11)$$

is vector Hankel transform [17, 18].

Note that thus presented fields components satisfy radiation condition of Silver-Muller automatically.

4. SET OF DUAL INTEGRAL EQUATIONS

Our goal is to find integral equations for the following unknown functions:

$$u_m^{\pm}(\kappa) = \left(e_{m,z}^{+}(\kappa) \pm e_{m,z}^{-}(\kappa)\right)/2, \quad v_m^{\pm}(\kappa) = \left(h_{m,z}^{+}(\kappa) \pm h_{m,z}^{-}(\kappa)\right)/2.$$

To this end, we substitute the incident and scattered field expressions into the generalized boundary conditions (3) and satisfy field continuity condition outside the disk. The following set of CDIEs are obtained:

$$\begin{cases} \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} \gamma(\kappa)\left(u_m^-(\kappa) + u_m^{0,-}(\kappa)\right) + 2Rkau_m^-(\kappa) \\ ika\left(v_m^+(\kappa) + v_m^{0,+}(\kappa)\right) + 2Ri\gamma(\kappa)v_m^+(\kappa) \end{pmatrix} d\kappa = \bar{0} & (\rho<1) \\ \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} ikau_m^-(\kappa) \\ -\gamma(\kappa)v_m^+(\kappa) \end{pmatrix} d\kappa = \bar{0} & (\rho>1) \end{cases} \quad (12)$$

$$\begin{cases} \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} \gamma(\kappa)\left(v_m^-(\kappa) + v_m^{0,-}(\kappa)\right) + 2Qkav_m^-(\kappa) \\ -\left(ika\left(u_m^+(\kappa) + u_m^{0,+}(\kappa)\right) + 2Qi\gamma(\kappa)u_m^+(\kappa)\right) \end{pmatrix} d\kappa = \bar{0} & (\rho<1) \\ \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} ikav_m^-(\kappa) \\ \gamma(\kappa)u_m^+(\kappa) \end{pmatrix} d\kappa = \bar{0} & (\rho>1) \end{cases} \quad (13)$$

Here $u_m^{0,\pm}(\kappa)$ and $v_m^{0,\pm}(\kappa)$ are certain given functions (generated by the incident field).

Thus, our problem is reduced to the search of $u_m^{\pm}(\kappa)$ and $v_m^{\pm}(\kappa)$, which are unknown functions considered in the following functional spaces:

1. For the zero-thickness PEC disk case: $\gamma^{1/2-\varepsilon}(\kappa)u_m^-(\kappa)$, $\gamma^{1/2-\varepsilon}(\kappa)v_m^+(\kappa) \in L_2(\mathbb{R}_+)$
2. For the thin ER disk case: $\gamma^{1/2-\varepsilon}(\kappa)u_m^-(\kappa) \in L_2(\mathbb{R}_+)$, $\gamma^{1/2-\varepsilon/2}(\kappa)\kappa^{1/2-\varepsilon/2}v_m^+(\kappa) \in L_2(\mathbb{R}_+)$
3. For the thin dielectric disk case: $\gamma^{1/2-\varepsilon}(\kappa)u_m^-(\kappa) \in L_2(\mathbb{R}_+)$, $\gamma^{1/2-\varepsilon/2}(\kappa)\kappa^{1/2-\varepsilon/2}v_m^+(\kappa) \in L_2(\mathbb{R}_+)$ and $\gamma^{1/2-\varepsilon}(\kappa)v_m^-(\kappa) \in L_2(\mathbb{R}_+)$, $\gamma^{1/2-\varepsilon/2}(\kappa)\kappa^{1/2-\varepsilon}u_m^+(\kappa) \in L_2(\mathbb{R}_+)$,

where $0 < \varepsilon \ll 1/2$ (this is a parameter which enable us to show existence and uniqueness of solution in $L_2(\mathbb{R}_+)$ and does not effect numerical scheme, see Discussion). Note that this choice of spaces is not arbitrary, but follows from the edge behavior condition. Also note that, in the case $m = 0$, DIEs (20), (21) decouple and can be reduced to two independent integral equations of the Fredholm second kind [19, 20]. Hereinafter we will consider the case $m \neq 0$ and reduce each of the coupled DIEs (12), (13) to four "quasi-coupled" DIEs. To do this, we consider the following functions:

$$f_m^-(\rho)=\left\{\begin{matrix} f_m^{-,l}(\rho), \ \rho<1 \\ f_m^{-,r}(\rho), \rho>1\end{matrix}\right\}=\int_0^\infty\left(\gamma(\kappa)\left(u_m^-(\kappa)+u_m^{0,-}(\kappa)\right)+2Rkau_m^-(\kappa)\right)J_{|m|}(\kappa\rho)d\kappa \quad (14)$$

$$g_m^+(\rho)=\left\{\begin{matrix} g_m^{+,l}(\rho), \ \rho<1 \\ g_m^{+,r}(\rho), \rho>1\end{matrix}\right\}=\int_0^\infty\left(ika\left(v_m^+(\kappa)+v_m^{0,+}(\kappa)\right)+2Ri\gamma(\kappa)v_m^+(\kappa)\right)J_{|m|}(\kappa\rho)d\kappa \quad (15)$$

$$g_m^-(\rho)=\left\{\begin{matrix} g_m^{-,l}(\rho), \ \rho<1 \\ g_m^{-,r}(\rho), \rho>1\end{matrix}\right\}=\int_0^\infty\left(\gamma(\kappa)\left(v_m^-(\kappa)+v_m^{0,-}(\kappa)\right)+2Qkav_m^-(\kappa)\right)J_{|m|}(\kappa\rho)d\kappa \quad (16)$$

$$f_m^+(\rho)=\left\{\begin{matrix} f_m^{+,l}(\rho), \ \rho<1 \\ f_m^{+,r}(\rho), \rho>1\end{matrix}\right\}=-\int_0^\infty\left(ika\left(u_m^+(\kappa)+u_m^{0,+}(\kappa)\right)+2Qi\gamma(\kappa)u_m^+(\kappa)\right)J_{|m|}(\kappa\rho)d\kappa \quad (17)$$

$$p_m^-(\rho)=\left\{\begin{matrix} p_m^{-,l}(\rho), & \rho<1 \\ p_m^{-,r}(\rho), & \rho>1\end{matrix}\right\}=\int_0^\infty ikau_m^-(\kappa)J_{|m|}(\kappa\rho)d\kappa \quad (18)$$

$$q_m^+(\rho)=\left\{\begin{matrix} q_m^{+,l}(\rho), & \rho<1 \\ q_m^{+,r}(\rho), & \rho>1\end{matrix}\right\}=-\int_0^\infty \gamma(\kappa)v_m^+(\kappa)J_{|m|}(\kappa\rho)d\kappa \quad (19)$$

$$q_m^-(\rho)=\left\{\begin{matrix} q_m^{-,l}(\rho), & \rho<1 \\ q_m^{-,r}(\rho), & \rho>1\end{matrix}\right\}=\int_0^\infty ikav_m^-(\kappa)J_{|m|}(\kappa\rho)d\kappa \quad (20)$$

$$p_m^+(\rho)=\left\{\begin{array}{ll} p_m^{+,l}(\rho), & \rho<1 \\ p_m^{+,r}(\rho), & \rho>1 \end{array}\right\}=\int_0^\infty \gamma(\kappa)u_m^+(\kappa)J_{|m|}(\kappa\rho)d\kappa \tag{21}$$

In terms of these functions, DIEs (12) can be written as follows:

$$\begin{array}{l}\left\{\begin{array}{l} \frac{\partial}{\partial\rho}f_m^{-,l}(\rho)+\mathrm{sgn}(m)\frac{|m|}{\rho}g_m^{+,l}(\rho)=0 \\ \mathrm{sgn}(m)\frac{|m|}{\rho}f_m^{-,l}(\rho)+\frac{\partial}{\partial\rho}g_m^{+,l}(\rho)=0 \end{array}\right. (\rho<1) \\ \left\{\begin{array}{l} \frac{\partial}{\partial\rho}p_m^{-,r}(\rho)+\mathrm{sgn}(m)\frac{|m|}{\rho}q_m^{+,r}(\rho)=0 \\ \mathrm{sgn}(m)\frac{|m|}{\rho}p_m^{-,r}(\rho)+\frac{\partial}{\partial\rho}q_m^{+,r}(\rho)=0 \end{array}\right. (\rho>1)\end{array} \tag{22}$$

The solutions of Equations (22) are the following functions:

$$f_m^{-,l}(\rho)=A_m^l\rho^{|m|}+B_m^l\rho^{-|m|} \qquad (\rho<1) \tag{23}$$

$$g_m^{+,l}(\rho)=-\mathrm{sgn}(m)A_m^l\rho^{|m|}+\mathrm{sgn}(m)B_m^l\rho^{-|m|} \qquad (\rho<1) \tag{24}$$

$$p_m^{-,r}(\rho)=C_m^r\rho^{|m|}+D_m^r\rho^{-|m|} \qquad (\rho>1) \tag{25}$$

$$q_m^{+,r}(\rho)=-\mathrm{sgn}(m)C_m^r\rho^{|m|}+\mathrm{sgn}(m)D_m^r\rho^{-|m|} \qquad (\rho>1) \tag{26}$$

Demand solutions (23), (24) to be in $L_2(0,1)$ and (25), (26) in $L_2(1,\infty)$. Then the constants of integration B_m^l, C_m^r must be zero. The constants of integration A_m^l and D_m^r (hereinafter the coupling constants) are unknowns constants to be found. Thus, coupled DIEs (12) are equivalent to the following "quasi-coupled" DIEs:

$$\left\{\begin{array}{ll} \int_0^\infty\left(\gamma(\kappa)(u_m^-(\kappa)+u_m^{0,-}(\kappa))+2Rkau_m^-(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=A_m^l\rho^{|m|} & (\rho<1) \\ \int_0^\infty ikau_m^-(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=D_m^r\rho^{-|m|} & (\rho>1) \end{array}\right. \tag{27}$$

$$\left\{\begin{array}{ll} \int_0^\infty\left(ika(v_m^+(\kappa)+v_m^{0,+}(\kappa))+2Ri\gamma(\kappa)v_m^+(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=-\mathrm{sgn}(m)A_m^l\rho^{|m|} & (\rho<1) \\ \int_0^\infty \gamma(\kappa)v_m^+(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=-\mathrm{sgn}(m)D_m^r\rho^{-|m|} & (\rho>1) \end{array}\right. \tag{28}$$

Similarly, coupled DIEs (13) are equivalent to the following "quasi-coupled" DIEs:

$$\left\{\begin{array}{ll} \int_0^\infty\left(\gamma(\kappa)(v_m^-(\kappa)+v_m^{0,-}(\kappa))+2Qkav_m^-(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=M_m^l\rho^{|m|} & (\rho<1) \\ \int_0^\infty ikav_m^-(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=P_m^r\rho^{-|m|} & (\rho>1) \end{array}\right. \tag{29}$$

$$\begin{cases} \int\limits_0^\infty \left(ika(u_m^+(\kappa)+u_m^{0,+}(\kappa))+2Qi\gamma(\kappa)u_m^+(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=\mathrm{sgn}(m)M_m^l\rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty \gamma(\kappa)u_m^+(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=\mathrm{sgn}(m)P_m^r\rho^{-|m|} & (\rho>1) \end{cases} \quad (30)$$

There are many of works that address the way of solving each of obtained DIE, (e.g., [9, 10, 19, 21, 22]). Typically, in these works such equations are reduced to the solving of the operator equation $(I+K)\,X=B$ i.e., the equation of the Fredholm second kind. Here, K is a compact operator (matrix or integral, it depends on the method), I is a unit operator, X is a vector of unknown functions, and B is a given right-hand vector. Further we will consider the method of the reduction of DIEs to the equivalent Fredholm second kind integral equations.

5. FREDHOLM INTEGRAL EQUATIONS FOR THE PEC DISK CASE

Consider the case of PEC disk ($R=0, Q=\infty$). Then we have the following DIEs:

$$\begin{cases} \int\limits_0^\infty \gamma(\kappa)(u_m^-(\kappa)+u_m^{0,-}(\kappa))\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=A_m^l\rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty ikau_m^-(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=D_m^r\rho^{-|m|} & (\rho>1) \end{cases} \quad (31)$$

$$\begin{cases} \int\limits_0^\infty ika(v_m^+(\kappa)+v_m^{0,+}(\kappa))\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=-\mathrm{sgn}(m)A_m^l\rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty \gamma(\kappa)v_m^+(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa=-\mathrm{sgn}(m)D_m^r\rho^{-|m|} & (\rho>1) \end{cases} \quad (32)$$

and $v_m^-(\kappa)\equiv 0$, $u_m^+(\kappa)\equiv 0$. Consider two pairs of direct and inverse Abel integral transforms (Erdeyi-Kober transform or fractional (order 1/2) integration and differentiation operators).

$$X^{(0)}(v) = \int\limits_0^v \frac{x^{(0)}(\rho)}{\sqrt{v^2-\rho^2}}\rho d\rho \quad (33)$$

$$Y^{(0)}(\rho) = \frac{2}{\pi}\frac{d}{d\rho}\int\limits_0^\rho \frac{y^{(0)}(v)}{\sqrt{\rho^2-v^2}}v dv \quad (34)$$

$$X^{(\infty)}(v) = \int\limits_v^\infty \frac{x^{(\infty)}(\rho)}{\sqrt{\rho^2-v^2}}\rho d\rho \quad (35)$$

$$Y^{(\infty)}(\rho) = -\frac{2}{\pi}\frac{d}{d\rho}\int_{\rho}^{\infty}\frac{y^{(\infty)}(v)}{\sqrt{v^2-\rho^2}}vdv \tag{36}$$

Multiply the first equation of (31) with $\rho^{|m|}$ and apply Abel integral transform (33). Multiply the result with $\sqrt{2/\pi}\,v^{-|m|-1/2}$. Multiply the second equation of (31) with $\rho^{-|m|}$ and apply inverse Abel integral transform (36). Multiply the result with $\sqrt{\pi/2}\,v^{|m|-1/2}$ and obtain:

$$\begin{cases} \int_0^\infty \gamma(\kappa)\,\kappa^{-3/2}\left(u_m^-(\kappa)+u_m^{0,-}(\kappa)\right)J_{|m|+1/2}(\kappa v)d\kappa = A_m^l\frac{1}{\sqrt{2}}\frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)}v^{|m|+1/2} & (v<1) \\ \int_0^\infty ika\,\kappa^{-1/2}u_m^-(\kappa)\,J_{|m|+1/2}(\kappa v)d\kappa = D_m^r\sqrt{2}\frac{\Gamma(|m|+1/2)}{\Gamma(|m|)}v^{-|m|-1/2} & (v>1) \end{cases} \tag{37}$$

Now, multiply the first equation of (32) with $\rho^{|m|}$ and apply inverse Abel integral transform (34). Multiply the result with $\sqrt{2/\pi}\,v^{-|m|-1/2}$. Multiply the second equation of (31) with $\rho^{-|m|}$ and apply Abel integral transform (35). Multiply the result by $\sqrt{\pi/2}\,v^{|m|-1/2}$ and obtain:

$$\begin{cases} \int_0^\infty ika\,\kappa^{-1/2}\left(v_m^+(\kappa)+v_m^{0,+}(\kappa)\right)J_{|m|-1/2}(\kappa v)d\kappa = -\mathrm{sgn}(m)A_m^l\sqrt{2}\frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)}v^{|m|-1/2} & (v<1) \\ \int_0^\infty \gamma(\kappa)\,\kappa^{-3/2}v_m^+(\kappa)\,J_{|m|-1/2}(\kappa v)d\kappa = -\mathrm{sgn}(m)\,D_m^r\frac{1}{\sqrt{2}}\frac{\Gamma(|m|-1/2)}{\Gamma(|m|)}v^{-|m|+1/2} & (v>1) \end{cases} \tag{38}$$

Extract the most singular parts of integrals in the neighborhood of $\kappa=\infty$ and transform equations to "quasi-canonic" type:

$$\int_0^\infty i\kappa^{-1/2}u_m^-(\kappa)J_{|m|+1/2}(\kappa v)d\kappa =$$
$$\begin{cases} -\int_0^\infty \kappa^{-3/2}\left(w(\kappa)u_m^-(\kappa)+\gamma(\kappa)u_m^{0,-}(\kappa)\right)J_{|m|+1/2}(\kappa v)d\kappa + A_m^l\frac{1}{\sqrt{2}}\frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)}v^{|m|+1/2} & (v<1) \\ (ka)^{-1}D_m^r\sqrt{2}\frac{\Gamma(|m|+1/2)}{\Gamma(|m|)}v^{-|m|-1/2} & (v>1) \end{cases} \tag{39}$$

$$\int_0^\infty i\kappa^{-1/2}v_m^+(\kappa)J_{|m|-1/2}(\kappa v)d\kappa =$$
$$\begin{cases} -\int_0^\infty i\kappa^{-1/2}v_m^{0,+}J_{|m|-1/2}(\kappa v)d\kappa - (ka)^{-1}\mathrm{sgn}(m)A_m^l\sqrt{2}\frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)}v^{|m|-1/2} & (v<1) \\ -\int_0^\infty w(\kappa)\kappa^{-3/2}v_m^+(\kappa)J_{|m|-1/2}(\kappa v)d\kappa - \mathrm{sgn}(m)D_m^r\frac{1}{\sqrt{2}}\frac{\Gamma(|m|-1/2)}{\Gamma(|m|)}v^{-|m|+1/2} & (v>1) \end{cases} \tag{40}$$

where $w(\kappa) = \gamma(\kappa) - i\kappa = \sqrt{(ka)^2 - \kappa^2} - i\kappa$. Note that $w(\kappa) = \underline{\mathrm{O}}\left((ka)^2/\kappa\right)$ if $ka/\kappa \to 0$. One can show that $\int_0^\infty i\kappa^{-1/2} u_m^-(\kappa) J_{|m|+1/2}(\kappa v)\, d\kappa$ and $\int_0^\infty i\kappa^{-1/2} v_m^+(\kappa) J_{|m|-1/2}(\kappa v)\, d\kappa$ are uniformly convergent integrals around the point $v = 1$ if $\gamma^{1/2-\varepsilon}(\kappa) u_m^-(\kappa), \gamma^{1/2-\varepsilon}(\kappa) v_m^+(\kappa) \in L_2(\mathbb{R}_+)$. Hence, they are continuous at the point $v = 1$. That is,

$$A_m^l \frac{1}{\sqrt{\pi}} \frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)} - (ka)^{-1} D_m^r \frac{2}{\sqrt{\pi}} \frac{\Gamma(|m|+1/2)}{\Gamma(|m|)}$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \kappa^{-3/2}(w(\kappa) u_m^-(\kappa) + \gamma(\kappa) u_m^{0,-}(\kappa)) J_{|m|+1/2}(\kappa) d\kappa \tag{41}$$

$$(ka)^{-1}\mathrm{sgn}(m) A_m^l \frac{2}{\sqrt{\pi}} \frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)} - \mathrm{sgn}(m) D_m^r \frac{1}{\sqrt{\pi}} \frac{\Gamma(|m|-1/2)}{\Gamma(|m|)}$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \kappa^{-1/2}(w(\kappa)\kappa^{-1} v_m^+(\kappa) - i v_m^{0,+}(\kappa)) J_{|m|-1/2}(\kappa) d\kappa \tag{42}$$

Hereinafter, we will consider (41) and (42) as additional "coupling" equations to find (exclude from consideration) the coupling constants.

To reduce DIEs (39), (40) to the integral equations of the Fredholm second kind, we multiply (39) with $vJ_{|m|+1/2}(\lambda v)$, and (40) by $vJ_{|m|-1/2}(\lambda v)$ and integrate in v from 0 to ∞. Using the fact that

$$\begin{aligned} x(l) &= \int_0^\infty \int_0^\infty \kappa x(\kappa) J_m(\kappa v) d\kappa J_m(lv) v dv \\ &= \int_0^1 \int_0^\infty \kappa x(\kappa) J_m(\kappa v) d\kappa J_m(lv) v dv + \int_1^\infty \int_0^\infty \kappa x(\kappa) J_m(\kappa v) d\kappa J_m(lv) v dv \end{aligned} \tag{43}$$

and

$$\int_0^x v^{\nu+1} J_\nu(\lambda v) dv = x^{\nu+1} \lambda^{-1} J_{\nu+1}(\lambda x) \tag{44}$$

$$\int_x^\infty v^{-\nu+1} J_\nu(\lambda v) dv = x^{-\nu+1} \lambda^{-1} J_{\nu-1}(\lambda x) \tag{45}$$

we obtain the integral equations, which can be reduced to the following form:

$$iu_m^-(\lambda) = -\lambda \int_0^\infty \kappa^{-2} \left(w(\kappa) u_m^-(\kappa) + \gamma(\kappa) u_m^{0,-}(\kappa) \right) S_{|m|+1/2}(\kappa, \lambda) d\kappa$$
$$+A_m^l \frac{1}{\sqrt{2}} \frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)} \lambda^{1/2} J_{|m|+3/2}(\lambda) + (ka)^{-1} D_m^r \sqrt{2} \frac{\Gamma(|m|+1/2)}{\Gamma(|m|)} \lambda^{1/2} J_{|m|-1/2}(\lambda) \quad (46)$$

$$\gamma(\lambda) v_m^+(\lambda) = \lambda^2 \int_0^\infty \kappa^{-1} \left(w(\kappa) \kappa^{-1} v_m^+(\kappa) - i v_m^{0,+}(\kappa) \right) S_{|m|-1/2}(\kappa, \lambda) d\kappa$$
$$-(ka)^{-1} \text{sgn}(m) A_m^l \sqrt{2} \frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)} \lambda^{3/2} J_{|m|+1/2}(\lambda) - \text{sgn}(m) D_m^r \frac{1}{\sqrt{2}} \frac{\Gamma(|m|-1/2)}{\Gamma(|m|)} \lambda^{3/2} J_{|m|-3/2}(\lambda) \quad (47)$$

Here,

$$\begin{aligned} S_\mu(\kappa, \lambda) &= \kappa^{1/2} \lambda^{1/2} \int_0^1 J_\mu(\kappa v) J_\mu(\lambda v) v dv \\ &= \frac{\kappa^{1/2} \lambda^{1/2}}{\kappa^2 - \lambda^2} \left(\lambda J_{\mu-1}(\lambda) J_\mu(\kappa) - \kappa J_{\mu-1}(\kappa) J_\mu(\lambda) \right) \end{aligned} \quad (48)$$

One can show that

$$\begin{aligned} u_m^-(\lambda) &= U_1^-(\lambda) + \frac{U_2^-(\lambda)}{\lambda} + \ldots (\lambda \to \infty), \\ \gamma(\lambda) v_m^+(\lambda) &= \lambda \cdot V_1^+(\lambda) + V_2^+(\lambda) + \frac{V_3^+(\lambda)}{\lambda} + \ldots (\lambda \to \infty), \end{aligned}$$

where $U_n^-(\lambda), V_n^+(\lambda) \underset{\lambda\to\infty}{\cong} \underline{O}(1)$ and

$$U_1^-(\lambda) \underset{\lambda\to\infty}{=} \left(\sqrt{\frac{2}{\pi}} \int_0^\infty \kappa^{-3/2} \left(w(\kappa) u_m^-(\kappa) + \gamma(\kappa) u_m^{0,-}(\kappa) \right) J_{|m|+1/2}(\kappa) d\kappa \right.$$
$$\left. -A_m^l \frac{1}{\sqrt{\pi}} \frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)} + (ka)^{-1} D_m^r \frac{2}{\sqrt{\pi}} \frac{\Gamma(|m|+1/2)}{\Gamma(|m|)} \right) \lambda^{1/2} J_{|m|-1/2}(\lambda) \quad (49)$$

$$V_1^+(\lambda) \underset{\lambda\to\infty}{=} \left(\sqrt{\frac{2}{\pi}} \int_0^\infty \kappa^{-1/2} \left(w(\kappa) \kappa^{-1} v_m^+(\kappa) - i v_m^{0,+}(\kappa) \right) J_{|m|-1/2}(\kappa) d\kappa \right.$$
$$\left. -(ka)^{-1} \text{sgn}(m) A_m^l \frac{2}{\sqrt{\pi}} \frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)} + \text{sgn}(m) D_m^r \frac{1}{\sqrt{\pi}} \frac{\Gamma(|m|-1/2)}{\Gamma(|m|)} \right) \lambda^{1/2} J_{|m|+1/2}(\lambda) \quad (50)$$

As we require (41) and (42), then $U_1^-(\lambda) = 0$ and $V_1^+(\lambda) = 0$. Multiply (41) with $\sqrt{\pi/2}\,\lambda^{1/2} J_{|m|-1/2}(\lambda)$ and add to (46), and multiply (42) with $\sqrt{\pi/2}\,\lambda^{3/2}J_{|m|+1/2}(\lambda)$ and add to (47), to obtain:

$$\begin{aligned} iu_m^-(\lambda) = -\lambda \int_0^\infty \kappa^{-2}\left(w(\kappa)u_m^-(\kappa)+\gamma(\kappa)u_m^{0,-}(\kappa)\right)S_{|m|+1/2}(\kappa,\lambda)d\kappa \\ -\lambda^{1/2}J_{|m|-1/2}(\lambda)\int_0^\infty \kappa^{-3/2}\left(w(\kappa)u_m^-(\kappa)+\gamma(\kappa)u_m^{0,-}(\kappa)\right)J_{|m|+1/2}(\kappa)d\kappa \\ +A_m^l\frac{1}{\sqrt{2}}\frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)}\left(\lambda^{1/2}J_{|m|+3/2}(\lambda)+\lambda^{1/2}J_{|m|-1/2}(\lambda)\right) \end{aligned} \tag{51}$$

$$\begin{aligned} \gamma(\lambda)v_m^+(\lambda) = \lambda^2 \int_0^\infty \kappa^{-1}\left(w(\kappa)\kappa^{-1}v_m^+(\kappa)-iv_m^{0,+}(\kappa)\right)S_{|m|-1/2}(\kappa,\lambda)d\kappa \\ -\lambda^{3/2}J_{|m|+1/2}(\lambda)\int_0^\infty \kappa^{-1/2}\left(w(\kappa)\kappa^{-1}v_m^+(\kappa)-iv_m^{0,+}(\kappa)\right)J_{|m|-1/2}(\kappa)d\kappa \\ -\operatorname{sgn}(m)D_m^r\frac{1}{\sqrt{2}}\frac{\Gamma(|m|-1/2)}{\Gamma(|m|)}\left(\lambda^{3/2}J_{|m|-3/2}(\lambda)+\lambda^{3/2}J_{|m|+1/2}(\lambda)\right) \end{aligned} \tag{52}$$

Use the fact that

$$\begin{aligned} S_\mu(\kappa,\lambda) &= \frac{\kappa^{1/2}\lambda^{1/2}}{\kappa^2-\lambda^2}\left(\lambda J_{\mu-1}(\lambda)J_\mu(\kappa)-\kappa J_{\mu-1}(\kappa)J_\mu(\lambda)\right) \\ &= \frac{\kappa^{1/2}\lambda^{1/2}}{\kappa^2-\lambda^2}\left(\kappa J_\mu(\lambda)J_{\mu+1}(\kappa)-\lambda J_\mu(\kappa)J_{\mu+1}(\lambda)\right) \end{aligned} \tag{53}$$

and

$$J_{\mu+1}(\lambda)+J_{\mu-1}(\lambda) = 2\mu\lambda^{-1}J_\mu(\lambda) \tag{54}$$

Finally we obtain the following integral equations:

$$\begin{aligned} \gamma^{1/2-\varepsilon}(\lambda)u_m^-(\lambda) = i\gamma^{1/2-\varepsilon}(\lambda)\int_0^\infty \kappa^{-1}\left(w(\kappa)u_m^-(\kappa)+\gamma(\kappa)u_m^{0,-}(\kappa)\right)S_{|m|-1/2}(\kappa,\lambda)d\kappa \\ -iA_m^l\sqrt{2}\frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)}\frac{\gamma^{1/2-\varepsilon}(\lambda)}{\lambda^{1/2}}J_{|m|+1/2}(\lambda) \end{aligned} \tag{55}$$

$$\begin{aligned} \gamma^{1/2-\varepsilon}(\lambda)v_m^+(\lambda) = \frac{\lambda}{\gamma^{1/2+\varepsilon}(\lambda)}\int_0^\infty \left(w(\kappa)\kappa^{-1}v_m^+(\kappa)-iv_m^{0,+}(\kappa)\right)S_{|m|+1/2}(\kappa,\lambda)d\kappa \\ -\operatorname{sgn}(m)D_m^r\sqrt{2}\frac{\Gamma(|m|+1/2)}{\Gamma(|m|)}\frac{\lambda^{1/2}}{\gamma^{1/2+\varepsilon}(\lambda)}J_{|m|-1/2}(\lambda) \end{aligned} \tag{56}$$

Equations (55), (56) together with the "coupling" equations (41), (42) form a set of coupled Fredholm second kind integral equations in $L_2(\mathbb{R}_+)$ for the unknown functions $\gamma^{1/2-\varepsilon}(\lambda)u_m^-(\lambda)$ and $\gamma^{1/2-\varepsilon}(\lambda)\,v_m^+(\lambda)$. Therefore, from the Fredholm alternative, there exists the unique solution of these equations.

6. FREDHOLM INTEGRAL EQUATIONS FOR THE ER DISK CASE

Consider the ER disk case ($0 < R < \infty$, $Q = 0$). In this case, as shown above, we have the following DIEs:

$$\begin{cases} \int_0^\infty \left(\gamma(\kappa)(u_m^-(\kappa)+u_m^{0,-}(\kappa)) + 2Rkau_m^-(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = A_m^l\rho^{|m|} & (\rho<1) \\ \int_0^\infty ikau_m^-(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = D_m^r\rho^{-|m|} & (\rho>1) \end{cases} \tag{57}$$

$$\begin{cases} \int_0^\infty \left(ika(v_m^+(\kappa)+v_m^{0,+}(\kappa))+2Ri\gamma(\kappa)v_m^+(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = -\mathrm{sgn}(m)A_m^l\rho^{|m|} & (\rho<1) \\ \int_0^\infty \gamma(\kappa)v_m^+(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = -\mathrm{sgn}(m)D_m^r\rho^{-|m|} & (\rho>1) \end{cases} \tag{58}$$

and $v_m^-(\kappa) \equiv 0$, $u_m^+(\kappa) \equiv 0$. The DIE (57), similarly to the DIE (31) (in the case of PEC disk), can be reduced to the following integral equations of the Fredholm second kind in $L_2(\mathbb{R}_+)$

$$\begin{aligned} &\gamma^{1/2-\varepsilon}(\lambda)u_m^-(\lambda) \\ &= i\gamma^{1/2-\varepsilon}(\lambda)\int_0^\infty \kappa^{-1}\left(w(\kappa)u_m^-(\kappa) + 2Rkau_m^-(\kappa) + \gamma(\kappa)u_m^{0,-}(\kappa)\right)S_{|m|-1/2}(\kappa,\lambda)d\kappa \\ &\qquad -iA_m^l\sqrt{2}\frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)}\frac{\gamma^{1/2-\varepsilon}(\lambda)}{\lambda^{1/2}}J_{|m|+1/2}(\lambda) \end{aligned} \tag{59}$$

with the additional "coupling" equation to find (exclude) the coupling constants:

$$\begin{aligned} &A_m^l\frac{1}{\sqrt{\pi}}\frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)} - (ka)^{-1}D_m^r\frac{2}{\sqrt{\pi}}\frac{\Gamma(|m|+1/2)}{\Gamma(|m|)} \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \kappa^{-3/2}\left(w(\kappa)u_m^-(\kappa)+2Rkau_m^-(\kappa)+\gamma(\kappa)u_m^{0,-}(\kappa)\right)J_{|m|+1/2}(\kappa)d\kappa \end{aligned} \tag{60}$$

One can see that (59) and (60) differ from the Equations (41) and (55) only by the presence of additional term $2R\,ka\,u_m^-(\kappa)$, which is zero in the case of PEC disk. Write the second DIE (58) as follows:

$$2Ri\int_0^\infty \gamma(\kappa)\kappa^{-1}v_m^+(\kappa)J_{|m|}(\kappa\rho)d\kappa =$$

$$\begin{cases} -\int_0^\infty \kappa^{-1}(ika(v_m^+(\kappa)+v_m^{0,+}(\kappa)))J_{|m|}(\kappa\rho)d\kappa - \operatorname{sgn}(m)A_m^l\rho^{|m|} & (\rho<1) \\ -2Ri\operatorname{sgn}(m)D_m^r\rho^{-|m|} & (\rho>1) \end{cases} \quad (61)$$

Similarly to the case of PEC disk one can show that $\int_0^\infty \gamma(\kappa)\kappa^{-1}v_m^+(\kappa)J_{|m|}(\kappa\rho)d\kappa$ are uniformly convergent integrals around the point $v = 1$ if $\gamma^{1/2-\varepsilon/2}(\kappa)\kappa^{1/2-\varepsilon/2}v_m^+(\kappa) \in L_2(\mathbb{R}_+)$. Hence, they are continuous at the point $v = 1$. That is,

$$\frac{i}{2R}\operatorname{sgn}(m)A_m^l + \operatorname{sgn}(m)D_m^r = \frac{ka}{2R}\int_0^\infty \kappa^{-1}\left(v_m^+(\kappa)+v_m^{0,+}(\kappa)\right)J_{|m|}(\kappa)d\kappa \quad (62)$$

Apply inverse Hankel transform to the Equation (61). We obtain:

$$v_m^+(\lambda) = -\frac{ka}{2R}\frac{\lambda^{3/2}}{\gamma(\lambda)}\int_0^\infty \kappa^{-3/2}\left(v_m^+(\kappa)+v_m^{0,+}(\kappa)\right)S_{|m|}(\kappa,\lambda)d\kappa$$
$$+\frac{i}{2R}\operatorname{sgn}(m)A_m^l\frac{\lambda}{\gamma(\lambda)}J_{|m|+1}(\lambda) - \operatorname{sgn}(m)D_m^r\frac{\lambda}{\gamma(\lambda)}J_{|m|-1}(\lambda) \quad (63)$$

Finally, as already done in the case of PEC disk, multiply (62) with $\lambda\gamma^{-1}(\lambda)J_{|m|+1}(\lambda)$ and add to the (63). Then we obtain the following integral equation:

$$\gamma^{1/2-\varepsilon/2}(\lambda)\lambda^{1/2-\varepsilon/2}v_m^+(\lambda) =$$
$$-\frac{ka}{2R}\frac{\lambda^{1-\varepsilon/2}}{\gamma^{1/2+\varepsilon/2}(\lambda)}\int_0^\infty \kappa^{-1/2}(v_m^+(\kappa)+v_m^{0,+}(\kappa))S_{|m|+1}(\kappa,\lambda)d\kappa - 2mD_m^r\frac{\lambda^{1/2-\varepsilon/2}}{\gamma^{1/2+\varepsilon/2}(\lambda)}J_{|m|}(\lambda)$$
$$(64)$$

Equations (59), (64) together with the "coupling" equations (60), (62) form a set of coupled Fredholm second kind integral equations in $L_2(\mathbb{R}_+)$ for the unknown functions $\gamma^{1/2-\varepsilon}(\lambda)u_m^-(\lambda)$ and $\gamma^{1/2-\varepsilon/2}(\lambda)\lambda^{1/2-\varepsilon/2}v_m^+(\lambda)$. Therefore, from the Fredholm alternative, there exists the unique solution of these equations.

7. FREDHOLM INTEGRAL EQUATIONS FOR THE DIELECTRIC DISK CASE

Consider the dielectric disk case $(0 < R < \infty, 0 < Q < \infty)$. Then, as shown above, we have the following DIEs:

$$\begin{cases} \int\limits_0^\infty \left(\gamma(\kappa)(u_m^-(\kappa)+u_m^{0,-}(\kappa)) + 2Rkau_m^-(\kappa)\right) \kappa^{-1} J_{|m|}(\kappa\rho)d\kappa = A_m^l \rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty ikau_m^-(\kappa)\kappa^{-1} J_{|m|}(\kappa\rho)d\kappa = D_m^r \rho^{-|m|} & (\rho>1) \end{cases} \quad (65)$$

$$\begin{cases} \int\limits_0^\infty \left(ika(v_m^+(\kappa)+v_m^{0,+}(\kappa))+2Ri\gamma(\kappa)v_m^+(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = -\mathrm{sgn}(m)A_m^l\rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty \gamma(\kappa)v_m^+(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = -\mathrm{sgn}(m)D_m^r\rho^{-|m|} & (\rho>1) \end{cases} \quad (66)$$

$$\begin{cases} \int\limits_0^\infty \left(\gamma(\kappa)(v_m^-(\kappa)+v_m^{0,-}(\kappa)) + 2Qkav_m^-(\kappa)\right) \kappa^{-1} J_{|m|}(\kappa\rho)d\kappa = M_m^l \rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty ikav_m^-(\kappa)\kappa^{-1} J_{|m|}(\kappa\rho)d\kappa = P_m^r \rho^{-|m|} & (\rho>1) \end{cases} \quad (67)$$

$$\begin{cases} \int\limits_0^\infty \left(ika(u_m^+(\kappa)+u_m^{0,+}(\kappa))+2Qi\gamma(\kappa)u_m^+(\kappa)\right)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = \mathrm{sgn}(m)M_m^l\rho^{|m|} & (\rho<1) \\ \int\limits_0^\infty \gamma(\kappa)u_m^+(\kappa)\kappa^{-1}J_{|m|}(\kappa\rho)d\kappa = \mathrm{sgn}(m)P_m^r\rho^{-|m|} & (\rho>1) \end{cases} \quad (68)$$

Note that Equations (65), (66) do not differ from the Equations (57), (58), which can be reduced to the following integral equations, as shown above:

$$\gamma^{1/2-\varepsilon}(\lambda)u_m^-(\lambda) = i\gamma^{1/2-\varepsilon}(\lambda)\int\limits_0^\infty \kappa^{-1}\left(w(\kappa)u_m^-(\kappa) + 2Rkau_m^-(\kappa)\right.$$
$$\left.+\gamma(\kappa)u_m^{0,-}(\kappa)\right) S_{|m|-1/2}(\kappa,\lambda)d\kappa - iA_m^l\sqrt{2}\frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)}\frac{\gamma^{1/2-\varepsilon}(\lambda)}{\lambda^{1/2}}J_{|m|+1/2}(\lambda) \quad (69)$$

$$\gamma^{1/2-\varepsilon/2}(\lambda)\lambda^{1/2-\varepsilon/2}v_m^+(\lambda) =$$
$$-\frac{ka}{2R}\frac{\lambda^{1-\varepsilon/2}}{\gamma^{1/2+\varepsilon/2}(\lambda)}\int\limits_0^\infty \kappa^{-1/2}\left(v_m^+(\kappa)+v_m^{0,+}(\kappa)\right)S_{|m|+1}(\kappa,\lambda)d\kappa - 2mD_m^r\frac{\lambda^{1/2-\varepsilon/2}}{\gamma^{1/2+\varepsilon/2}(\lambda)}J_{|m|}(\lambda) \quad (70)$$

with the additional "coupling" equations to find (to exclude) the coupling constants A_m^l and D_m^r:

$$A_m^l\frac{1}{\sqrt{\pi}}\frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)} - (ka)^{-1}D_m^r\frac{2}{\sqrt{\pi}}\frac{\Gamma(|m|+1/2)}{\Gamma(|m|)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \kappa^{-3/2} (w(\kappa) u_m^-(\kappa) + 2Rka u_m^-(\kappa) + \gamma(\kappa) u_m^{0,-}(\kappa)) J_{|m|+1/2}(\kappa) d\kappa \quad (71)$$

$$\frac{i}{2R} \operatorname{sgn}(m) A_m^l + \operatorname{sgn}(m) D_m^r = \frac{ka}{2R} \int_0^\infty \kappa^{-1} \left(v_m^+(\kappa) + v_m^{0,+}(\kappa) \right) J_{|m|}(\kappa) d\kappa \quad (72)$$

Also note that Equations (67) and (68) have the same type as Equations (65), (66). And hence they can be similarly reduced to following:

$$\begin{aligned} &\gamma^{1/2-\varepsilon}(\lambda) v_m^-(\lambda) \\ &= i\gamma^{1/2-\varepsilon}(\lambda) \int_0^\infty \kappa^{-1} \left(w(\kappa) v_m^-(\kappa) + 2Qka v_m^-(\kappa) + \gamma(\kappa) v_m^{0,-}(\kappa) \right) S_{|m|-1/2}(\kappa, \lambda) d\kappa \\ &\quad - i M_m^l \sqrt{2} \frac{\Gamma(|m|+1)}{\Gamma(|m|+1/2)} \frac{\gamma^{1/2-\varepsilon}(\lambda)}{\lambda^{1/2}} J_{|m|+1/2}(\lambda) \end{aligned} \quad (73)$$

$$\begin{aligned} &\gamma^{1/2-\varepsilon/2}(\lambda) \lambda^{1/2-\varepsilon/2} u_m^+(\lambda) \\ &= -\frac{ka}{2Q} \frac{\lambda^{1-\varepsilon/2}}{\gamma^{1/2+\varepsilon/2}(\lambda)} \int_0^\infty \kappa^{-1/2} (u_m^+(\kappa) \\ &\quad + u_m^{0,+}(\kappa)) S_{|m|+1}(\kappa, \lambda) d\kappa - 2m \cdot P_m^r \frac{\lambda^{1/2-\varepsilon/2}}{\gamma^{1/2+\varepsilon/2}(\lambda)} J_{|m|}(\lambda) \end{aligned} \quad (74)$$

with additional "coupling" equations to find (to exclude) the coupling constants M_m^l and P_m^r:

$$\begin{aligned} &M_m^l \frac{1}{\sqrt{\pi}} \frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)} - (ka)^{-1} P_m^r \frac{2}{\sqrt{\pi}} \frac{\Gamma(|m|+1/2)}{\Gamma(|m|)} \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \kappa^{-3/2} \big(w(\kappa) v_m^-(\kappa) + 2Qka v_m^-(\kappa) + \gamma(\kappa) v_m^{0,-}(\kappa) \big) J_{|m|+1/2}(\kappa) d\kappa \end{aligned} \quad (75)$$

$$\frac{i}{2Q} \operatorname{sgn}(m) M_m^l + \operatorname{sgn}(m) P_m^r = \frac{ka}{2Q} \int_0^\infty \kappa^{-1} \left(u_m^+(\kappa) + u_m^{0,+}(\kappa) \right) J_{|m|}(\kappa) d\kappa \quad (76)$$

Equations (69), (70), together with the "coupling" equations (71), (72), and also Equations (73), (74), together with "coupling" equations (75), (76), are two sets of the coupled Fredholm second kind integral equations in $L_2(\mathbb{R}_+)$ for the unknown functions $\gamma^{1/2-\varepsilon}(\lambda) u_m^-(\lambda)$, $\gamma^{1/2-\varepsilon}(\lambda) v_m^-(\lambda)$ and $\gamma^{1/2-\varepsilon/2}(\lambda) \lambda^{1/2-\varepsilon/2} u_m^+(\lambda)$, $\gamma^{1/2-\varepsilon/2}(\lambda) \lambda^{1/2-\varepsilon/2} v_m^+(\lambda)$. Therefore, from the Fredholm alternative, there exists the unique solution of these equations.

8. DISCUSSIONS AND CONCLUSIONS

Discuss the way to build the numerical solutions of the sets of coupled Fredholm second kind integral equations shortly. One can show that each of the obtained sets of coupled equations can be reduced to the following short form:

$$x(\lambda) + (K_{1,1}x)(\lambda) = -(K_{1,1}x_0)(\lambda) + c_1 \cdot G_1(\lambda) \tag{77}$$

$$\begin{array}{l} M_{1,1}c_1 + M_{1,2}c_2 = F_1(x) + F_1(x_0) \\ M_{2,1}c_1 + M_{2,2}c_2 = F_2(y) + F_2(y_0) \end{array} \tag{78}$$

$$y(\lambda) + (K_{2,2}y)(\lambda) = -(K_{2,2}y_0)(\lambda) + c_2 \cdot G_2(\lambda) \tag{79}$$

Here, $x(\lambda) = u_m^-(\lambda)/\lambda$ or $x(\lambda) = v_m^-(\lambda)/\lambda$ and $y(\lambda) = \gamma(\lambda)u_m^+(\lambda)/\lambda$ or $y(\lambda) = \gamma(\lambda)v_m^+(\lambda)/\lambda$ are unknown functions; c_1 and c_2 are unknown constants, x_0 and y_0 are given functions (generated by the incident field), $K_{1,1}$ and $K_{2,2}$ are given integral operators, $G_1(\lambda)$ and $G_2(\lambda)$ are given functions; $M_{i,j}$ are given "coupling" equations coefficients and, F_1 and F_2 are given functionals. One can reduce this set of integral equations to the Fredholm second kind infinite matrix analog, $(\underline{I} + \underline{A})\bar{X} = \bar{B}$ (a set of linear algebraic equations), by using any reasonable discretization scheme. This can be

1) Galerkin method. For example, with basis functions $\left\{\lambda^{-3/2}J_{2n+|m|+1/2}(\lambda)\right\}_{n=0}^{\infty}$ for $x(\lambda)$ and $\left\{\lambda^{-1/2}J_{2n+|m|-1/2}(\lambda)\right\}_{n=0}^{\infty}$ (in the PEC disk case) or $\left\{\lambda^{-1}J_{2n+|m|}(\lambda)\right\}_{n=0}^{\infty}$ (in another cases) for $y(\lambda)$ function [9, 13, 20].
2) Nyström method, with an infinite grid on the interval $(0, \infty)$.

Moreover, such kind of integral Equations (77)–(79) guarantee that $\underline{A}$ is a block-type three-diagonal matrix:

$$\underline{A} = \begin{pmatrix} \boxed{\underline{K}_{1,1}} & -\overline{G}_1 & \overline{0} & \underline{0} \\ -\overline{F}_1 & M_{1,1}-1 & M_{1,2} & \underline{0} \\ \overline{0} & M_{2,1} & M_{2,2}-1 & -\overline{F}_2 \\ \underline{0} & \overline{0} & -\overline{G}_2 & \boxed{\underline{K}_{2,2}} \end{pmatrix}$$

with the first diagonal block $\underline{K}_{1,1}$ corresponding to the Equation (77), the third diagonal block $\underline{K}_{2,2}$ corresponding to the Equation (79), and the second block (central 2×2 block) corresponding to the "coupling" equations (78). The features of the Fredholm second kind matrix equation guarante the uniqueness and existence of its solution. In our case it can be found numerically by iterative inversion of $\underline{I} + \underline{K}_{1,1}^N$ and $\underline{I} + \underline{K}_{2,2}^N$ $N \times N$ blocks of the full matrix (here $\underline{K}_{1,1}^N$ and $\underline{K}_{2,2}^N$ are two truncated $\underline{K}_{1,1}$ and $\underline{K}_{2,2}$ blocks) and one 2×2 matrix, instead of the $2N \times 2N$ matrix inversion in the case if we project the CDIEs (12) and (13) on some basis functions [13, 14].

Thus, the problem of arbitrary electromagnetic wave scattering is reduced to the coupled Fredholm second kind integral equations. The unknowns are jumps and average values of the images of the normal to the disk scattered field components. The features of such kind equations guarantee the existence of the solutions which can be found numerically after discretization and the following matrix truncation. Introduction of the coupling constants null mostly all of matrix "coupling" blocks elements. This minimizes the computational resources for the matrix inversion.

ACKNOWLEDGMENT

This work has been supported by National Academy of Sciences of Ukraine (NASU) and Ministère de l'Éducation nationale de l'enseignement supérieur et de la Recherche (MENESR)/France via "Research and Training Network: Advanced numerical modeling and design of dielectric lens antennas" and NASU and Royal Society/UK via join project: "Advanced modelling of single and periodic active dielectric resonators for microlasers".

REFERENCES

1. Chew, W. C. and J. A. Kong, "Resonance of axial symmetric modes in microstrip disk resonators," *J. Math. Phys.*, Vol. 21, No. 3, 582, 1980.
2. Chew, W. C. and J. A. Kong, "Analysis of a circular microstrip disk antenna with a thick dielectric substrate," *IEEE Trans. Antennas Propag.*, Vol. 29, 68–76, 2005.
3. Bliznyuk, N. Y. and A. I. Nosich, "Numerical analysis of a lossy circular microstrip antenna," *Radio Physics and Electronics*, Vol. 4, No. 3, 125–128, Kharkov IRE Press, 1999. Translated to English in *Telecommunications and Radio Engineering*, Vol. 55, No. 8, 15–23, Begell House Publ., 2001.
4. Ulaby, F. T., K. Sarabandi, K. McDonald, M. Whitt, and M. Craig, "Michigan microwave canopy scattering model," *International Journal of Remote Sensing*, Vol. 11, 1223–1253, 1990.
5. Frateschi, N. C. and A. F. J. Levi, "Resonant modes and laser

spectrum of microdisk lasers," *Appl. Phys. Lett.*, Vol. 66, No. 22, 2932–2934, 1995.

6. Liu, X., W. Fang, Y. Huang, X. H. Wu, S. T. Ho, H. Cao, and R. P. H. Chang, "Optically pumped ultraviolet microdisk laser on a silicon substrate," *Appl. Phys. Lett.*, Vol. 84, No. 14, 2488–2490, 2004.
7. Nosich, A. I., E. I. Smotrova, S. V. Boriskina, T. M. Benson, and P. Sewell, "Trends in microdisk laser research and linear optical modeling," *Optical and Quantum Electronics*, Vol. 39, No. 15, 1253–1272, 2007.
8. Boriskin, A. V., A. Rolland, R. Sauleau, and A. I. Nosich, "Assessment of FDTD accuracy in the compact hemielliptic dielectric lens antenna analysis," *IEEE Trans. Antennas Propag.*, Vol. 56, 758–764, 2008.
9. Tranter, C. J., "A further note on dual integral equations and an application to the diffraction of electromagnetic waves," *Quart J. Mech. and Appl. Math.*, Vol. 7, No. 3, 317–325, 1954.
10. Lugovoy, A. V. and V. G. Sologub, "Scattering of electromagnetic waves by a disk placed over lossy dielectric halfspace," *URSI-B Symp. Electromagnetic Theory*, 198–200, 1974.
11. Nosich, A. I., "The MAR in wave-scattering and eigenvalue problems: Foundations and review of solutions," *IEEE Antennas Propagat. Magazine*, Vol. 41, No. 3, 25–49, 1999.
12. Nosich, A. I., "Method of analytical regularization based on the static part inversion in wave scattering by imperfect thin screens," *J. Telecommunications and Information Technology*, No. 3, 72–79, NIT Press, Warsaw, 2001.
13. Panariello, G., F. Schettino, L. Verolino, R. Araneo, and S. Celozzi, "Analysis of microstrip antennas by means of regularization via Neumann series," *Review of Radio Science*, W. Ross Stone (ed.), 2002.
14. Hongo, K. and Q. A. Naqvi, "Diffraction of electromagnetic wave by disk and circular hole in a perfectly conducting plane," *Progress In Electromagnetics Research*, PIER 68, 113–150, 2007.
15. Bleszynski, E., M. Bleszynski, and T. Jaroszewicz, "Surface-integral equations for electrmagnetic scattering from impenetrable and penetrable sheets," *IEEE Antennas Propagat. Magazine*, Vol. 35, 14–24, 1993.
16. Braver, I., P. Fridberg, K. Garb, and I. Yakover, "The behavior of the electromagnetic field near the edge of a resistive half-plane," *IEEE Trans. Antennas Propag.*, Vol. 36, 1760–1768, 1988.

17. Chew, W. C. and T. B. Habashy, "The use of vector transforms in solving some electromagnetic scattering problems," *IEEE Trans. Antennas Propag.*, Vol. 56, No. 1, July 1986.
18. Ali, S. M., W. C. Chew, and J. A. Kong, "Vector Hankel transform analysis of annular-ring microstrip antenna," *IEEE Trans. Antennas Propag.*, Vol. 30, No. 4, 637–644, July 1982.
19. Khizhnyak, A. N., "Diffraction of a plane wave by a thin disk," *Sov. Physics Acoustics*, Vol. 35, No. 5, 539–541, 1989.
20. Bliznyuk, N. Y., A. I. Nosich, and A. N. Khizhnyak, "Accurate computation of a circular-disk printed antenna axysimmetrically excited by an electric dipole," *Microwave and Optical Technology Lett.*, Vol. 25, No. 3, 211–216, 2000.
21. Mandal, B. N. and N. Mandal, "Advances in dual integral equations," *Research Notes in Mathematics*, Chapman & Hall-CRC, New York, 1999.
22. Cooke, J. C., "A solution of Transfer's dual integral equations problem," *Quart J. Mech. and Appl. Math.*, Vol. 9, No. 1, 103–110, 1956.