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DUAL OF THE FUNCTION ALGEBRA $A^{-\infty}(D)$ AND REPRESENTATION OF FUNCTIONS IN DIRICHLET SERIES

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Abstract. In this paper we present the following results: a description, via the Laplace transformation of analytic functionals, of the dual to the (DFS)-space $A^{-\infty}(D)$ (D being either a bounded C^2-smooth convex domain in $\mathbb{C}^N$, with $N > 1$, or a bounded convex domain in $\mathbb{C}$) as an (FS)-space $A_D^{-\infty}$ of entire functions satisfying a certain growth condition; an explicit construction of a countable sufficient set for $A_D^{-\infty}$; and a possibility of representating functions from $A^{-\infty}(D)$ in the form of Dirichlet series.

1. Introduction

1.1. **Basic notation and definitions.** $\mathcal{O}(D)$ (D being a domain in $\mathbb{C}^N$) denotes the space of functions that are holomorphic in D, with the topology of uniform convergence on compact subsets of D.

$\mathcal{O}(K)$ (respectively $C^\infty(K)$), with K being a compact set in $\mathbb{C}^N$, denotes the space of germs of functions holomorphic on K, endowed with the topology of inductive limit, $\mathcal{O}(K) = \lim \operatorname{ind} \mathcal{O}(\omega)$, where ω are open neighborhoods of K (respectively the space of functions that are infinitely differentiable on K).

If $z, \zeta \in \mathbb{C}^N$, then $|z| = (z_1\bar{z}_1 + \cdots + z_N\bar{z}_N)^{1/2}$ and $\langle z, \zeta\rangle = z_1\zeta_1 + \cdots + z_N\zeta_N$.

The supporting function of a convex set M in $\mathbb{C}^N$ is

$$H_M(\xi) := \sup_{z\in M} \operatorname{Re}\langle z, \xi\rangle, \quad \xi \in \mathbb{C}^N$$

(see, e.g., [8]). This is a positively homogeneous, semi-additive function in $\mathbb{C}^N$.

For a set $E \subset \mathbb{C}^N$ (such that $0 \in E$) we denote by $\widetilde{E}$ the conjugate set of E; i.e.,

$$\widetilde{E} := \{w \in \mathbb{C}^N : \langle z, w\rangle \neq 1 \text{ for any } z \in E\}.$$

In the case where E is open, its conjugate $\widetilde{E}$ is a compact set and plays the role of "the exterior" in the duality of Martineau and Aĭzenberg ([3], [15]).

1.2. **The function spaces $A^{-\infty}(D)$ and $A_D^{-\infty}$.** Let D be a bounded domain in $\mathbb{C}^N$. Put

$$d(\lambda) := \inf_{\zeta\in\partial D} |\lambda - \zeta|, \quad \lambda \in D,$$

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the minimum Euclidean distance between λ and the boundary ∂D of D. The space $A^{-\infty}(D)$ is defined as follows:

$$A^{-\infty}(D) := \left\{ f \in \mathcal{O}(D) : \exists\, n, C > 0 \text{ such that } \sup_{\lambda \in D} |f(\lambda)|\ [d(\lambda)]^n \leq C \right\}.$$

Notice that the condition in the definition of $A^{-\infty}(D)$ is the familiar polynomial growth condition $\sup\limits_{\lambda \in D}(1-|\lambda|)^n |f(\lambda)| \leq C$ if the domain D is the open unit ball.

The space $A^{-\infty}(D)$ can be thought of as the union of the Banach spaces

$$A^{-n}(D) := \left\{ f \in \mathcal{O}(D) : \|f\|_n = \sup_{\lambda \in D} |f(\lambda)|\ [d(\lambda)]^n < +\infty \right\};$$

that is,

$$A^{-\infty}(D) = \bigcup_{n=1}^{\infty} A^{-n}(D).$$

We can endow the space $A^{-\infty}(D)$ with the natural topological structure of inductive limit of spaces $A^{-n}(D)$.

Now let D be convex. Without loss of generality, we can assume that $0 \in D$. Define a space

$$A_D^{-\infty} := \left\{ f \in \mathcal{O}(\mathbb{C}^N) : |f|_n = \sup_{z \in \mathbb{C}^N} \frac{|f(z)|(1+|z|)^n}{\exp H_D(z)} < \infty \text{ for all } n \in \mathbb{N} \right\},$$

where H_D is the supporting function of D, endowed with the topology given by the system of norms $(|\cdot|_n)_{n=1}^{\infty}$.

It is easy to see that $A_D^{-\infty}$ is a Fréchet-Schwartz space (briefly, an (FS)-space) while $A^{-\infty}(D)$ is a dual Fréchet-Schwartz space (briefly, a (DFS)-space). We refer the reader to [24] and [16] for detailed information on these notions.

The aim of this paper is to establish that the dual of the space $A^{-\infty}(D)$ is isomorphic, via the Laplace transformation, to $A_D^{-\infty}$. As an application of this result, we give an explicit construction of a sufficient set for the Fréchet space $A_D^{-\infty}$; due to the duality description, we are able to show that any function from the algebra $A^{-\infty}(D)$ can always be represented in the form of Dirichlet series.

It should be noted that the duality problem for the space $A^{-\infty}(D)$ has been studied by several authors, using different methods. In particular, Bell [5], Straube [20], Barret [4], Kiselman [11], and others established the duality between $A^{-\infty}(D)$ and the space $A^{\infty}(\overline{D})$ of holomorphic functions in D that are in $C^{\infty}(\overline{D})$, and therefore their results are quite different from ours. Also, the representation problem is never treated in above-mentioned papers.

One final remark is that similar problems for the space $A_D^{-\infty}$ (establishing its dual via the Laplace transformation and the possibility of representating functions from this space in Dirichlet series) have been studied in our recent paper [2]. Also note that some of our results were announced in [1].

2. The space $A_D^{-\infty}$ is a dual of $A^{-\infty}(D)$

The Laplace transformation of an analytic functional φ on the space $A^{-\infty}(D)$ is defined as follows:

$$\mathcal{F}(\varphi)(z) := \varphi_\lambda\big(e^{\langle z,\lambda\rangle}\big), \quad \varphi \in (A^{-\infty}(D))', z \in \mathbb{C}^N.$$

Theorem 2.1. *Let D be either a bounded convex domain with C^2 boundary in $\mathbb{C}^N$, for $N>1$, or an arbitrary bounded convex domain in $\mathbb{C}$. The Laplace transformation establishes a topological isomorphism between the strong dual $\left(A^{-\infty}(D)\right)_b'$ of $A^{-\infty}(D)$ and the space $A_D^{-\infty}$.*

For $N=1$ this theorem was obtained by Melikhov [18]. To prove it for $N>1$, we need some notation and auxiliary results.

For the sake of simplicity we write $H(z)$ instead of $H_D(z)$. We also let $r := \min_{z\in S} H(z)$ and $R := \max_{z\in S} H(z)$, where $S := \{z \in \mathbb{C}^N : |z| = 1\}$. It is clear that $0 < r \le R < \infty$ and $0 < d(\lambda) < R$ for every $\lambda \in D$.

Lemma 2.2. *Let*

$$a_n := \min\left\{\left(\frac{r}{e}\right)^n, \left(\frac{nr}{eR}\right)^n\right\}, \quad A_n := e^R\left(\frac{n}{e}\right)^n.$$

For every $n \in \mathbb{N}$,

$$a_n \frac{e^{H(z)}}{(1+|z|)^n} \le \left\|e^{\langle z,\cdot\rangle}\right\|_n \le A_n \frac{e^{H(z)}}{(1+|z|)^n},\ \forall z \in \mathbb{C}^N. \tag{2.1}$$

Proof. Obviously, $S_d(\lambda) := \left\{\xi \in \mathbb{C}^N : |\xi-\lambda| = d(\lambda)\right\} \subset \overline{D}$ for any $\lambda \in D$. Then for each $z \in \mathbb{C}^N$ and $\xi \in S_d(\lambda)$,

$$\mathrm{Re}\langle z,\lambda\rangle = \mathrm{Re}\langle z,\xi\rangle + \mathrm{Re}\langle z,\lambda-\xi\rangle \le H(z) + \mathrm{Re}\langle z,\lambda-\xi\rangle.$$

Hence

$$\mathrm{Re}\langle z,\lambda\rangle \le H(z) + \inf_{\xi\in S_d(\lambda)} \mathrm{Re}\langle z,\lambda-\xi\rangle = H(z) - |z|d(\lambda).$$

From this it follows that

$$\begin{aligned}\|e^{\langle z,\cdot\rangle}\|_n &= \sup_{\lambda\in D} e^{\mathrm{Re}\langle z,\lambda\rangle}[d(\lambda)]^n \le e^{H(z)} \sup_{\lambda\in D} e^{-|z|d(\lambda)}[d(\lambda)]^n \\ &\le e^{H(z)+R} \sup_{t\ge 0} t^n e^{-(|z|+1)t} = A_n \frac{e^{H(z)}}{(1+|z|)^n}.\end{aligned}$$

For the lower bound in (2.1), write $\overline{B}_r = \{z \in \mathbb{C}^N : |z| \le r\}$ and consider $\overline{D}_t := \{\lambda \in D : d(\lambda) \ge t\}$ with $0<t<r$. Since

$$D = \Big(1-\frac{t}{r}\Big)D + \frac{t}{r}\overline{D} \supset \Big(1-\frac{t}{r}\Big)D + \frac{t}{r}\overline{B}_r,$$

we have $\overline{D}_t \supset \Big(1-\frac{t}{r}\Big)\overline{D}$. From this it follows that for each $z \in \mathbb{C}^N$ there exists $\xi \in \partial\overline{D}_t$ such that

$$\mathrm{Re}\langle z,\xi\rangle \ge \Big(1-\frac{t}{r}\Big)H(z) \ge H(z) - \frac{t}{r}R|z|.$$

Clearly, $d(\xi) = t$. Therefore, for $|z| > \frac{n}{R}$ we have

$$\|e^{\langle z,\cdot\rangle}\|_n \ge \sup_{0<t<r} \sup_{d(\lambda)=t} e^{\mathrm{Re}\langle z,\lambda\rangle} t^n \ge e^{H(z)} \sup_{0<t<r} t^n e^{-t\frac{R}{r}|z|} = \Big(\frac{nr}{eR}\Big)^n \frac{e^{H(z)}}{|z|^n}.$$

On the other hand, for $|z| \le \frac{n}{R}$ we have

$$\|e^{\langle z,\cdot\rangle}\|_n = \sup_{\lambda\in D} e^{\mathrm{Re}\langle z,\lambda\rangle}[d(\lambda)]^n \ge [d(0)]^n = r^n \ge \Big(\frac{r}{e}\Big)^n e^{H(z)} \ge \Big(\frac{r}{e}\Big)^n \frac{e^{H(z)}}{(1+|z|)^n}.$$

Thus

$$\|e^{\langle z,\cdot\rangle}\|_n \geq a_n \frac{e^{H(z)}}{(1+|z|)^n},\ \forall z \in \mathbb{C}^N. \qquad \square$$

Proposition 2.3. *The system $\mathcal{E}^N := \left\{e^{\langle z,\cdot\rangle} : z \in \mathbb{C}^N\right\}$ is complete in $A^{-\infty}(D)$.*

Proof. Let $\varphi \in (A^{-\infty}(D))'$. Then for each $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$|\varphi(f)| \leq C_n \|f\|_n,\ \forall f \in A^{-n}(D).$$

Define a Banach space of continuous functions

$$C_0^{-n}(D) := \{f \in C(D) : \|f\|_n < \infty,\ f(\lambda)[d(\lambda)]^n \to 0 \text{ as } \lambda \to \partial D\}.$$

By the Hahn–Banach theorem, φ can be extended as a continuous linear functional on $C_0^{-n}(D)$ for every $n \in \mathbb{N}$. Obviously, $A^{-n+1}(D) \hookrightarrow C_0^{-n}(D)$ (here and in the sequel, the symbol $\hookrightarrow$ denotes the continuous embedding). From this it follows that for each $n \in \mathbb{N}$ there exists a Borel complex measure $\mu_{\varphi,n}$ on D with $\int_D [d(\lambda)]^{-n}\, d|\mu_{\varphi,n}|(\lambda) < \infty$ such that

$$\varphi(f) = \int_D f(\lambda)\, d\mu_{\varphi,n}(\lambda),\ \forall f \in A^{-n+1}(D).$$

Note that $d(\lambda)$ is a concave function on D (see, e.g., [9, Theorem 2.1.24]). Therefore, $d(\gamma\lambda) \geq \gamma d(\lambda)$ for all $0 \leq \gamma \leq 1$ and $\lambda \in D$. Then $f(\gamma\cdot) \in L(D, \mu_{\varphi,n})$ for any $f \in A^{-n+1}(D)$ and $\gamma \in [0,1]$, where $L(D, \mu_{\varphi,n})$ is the space of all $\mu_{\varphi,n}$-integrable functions on D. By the Lebesgue dominated convergence theorem, we have

$$\lim_{\gamma\uparrow 1} \varphi\big(f(\gamma\cdot)\big) = \lim_{\gamma\uparrow 1} \int_D f(\gamma\lambda)\, d\mu_{\varphi,n}(\lambda) = \int_D f(\lambda)\, d\mu_{\varphi,n}(\lambda) = \varphi(f).$$

So

$$\varphi(f) = \lim_{\gamma\uparrow 1} \varphi\big(f(\gamma\cdot)\big) \quad \text{for all } f \in A^{-\infty}(D). \tag{2.2}$$

Furthermore, it is clear that $f(\gamma\cdot) \in \mathcal{O}(\overline{D})$ for any $\gamma \in (0,1)$ and that $\mathcal{O}(\overline{D}) \hookrightarrow A^{-\infty}(D)$. From this and (2.2) we conclude that $\mathcal{O}(\overline{D})$ is dense in $A^{-\infty}(D)$.

To finish the proof it remains to note that the system $\mathcal{E}^N$ is complete in $\mathcal{O}(\overline{D})$. $\square$

Proposition 2.4. *Let D be a bounded convex domain with C^2 boundary in $\mathbb{C}^N$, for $N > 1$. For each $f \in A_D^{-\infty}$ there exists $\varphi \in (A^{-\infty}(D))'$ such that $\mathcal{F}(\varphi) = f$.*

Proof. Define

$$\rho(\lambda) := \begin{cases} -d(\lambda), & \lambda \in D, \\ d(\lambda), & \lambda \notin D. \end{cases}$$

Since D has C^2 boundary, $\rho(\lambda) \in C^2$ in some neighborhood of ∂D. Take $\delta > 0$ sufficiently small so that $\rho \in C^2(\overline{D} \setminus D_\delta)$, where $D_\delta = \{\lambda \in D : d(\lambda) > \delta\}$.

We put

$$\nabla_\lambda \rho \ := \ \left(\frac{\partial\rho}{\partial\lambda_1},\dots,\frac{\partial\rho}{\partial\lambda_N}\right),$$

$$R_j(\lambda) \ := \ \det\begin{pmatrix} \partial\rho/\partial\lambda_1 & \cdots & \partial\rho/\partial\lambda_N \\ \partial^2\rho/\partial\bar\lambda_1\partial\lambda_1 & \cdots & \partial^2\rho/\partial\bar\lambda_1\partial\lambda_N \\ \cdots & [j] & \cdots \\ \partial^2\rho/\partial\bar\lambda_N\partial\lambda_1 & \cdots & \partial^2\rho/\partial\bar\lambda_N\partial\lambda_N \end{pmatrix},$$

$$\bar\omega(\lambda,\nabla_\lambda\rho) := \langle\lambda,\nabla_\lambda\rho\rangle^{-N}\sum_{j=1}^N R_j(\lambda)d\bar\lambda_1\wedge\dots[j]\dots\wedge d\bar\lambda_N\wedge d\lambda_1\wedge\dots\wedge d\lambda_N,$$

where $\wedge$ is the symbol for exterior multiplication,

$$u(\lambda) \ := \ \langle\lambda,\nabla_\lambda\rho\rangle^{-1}\nabla_\lambda\rho = (u_1(\lambda),\dots,u_N(\lambda)),$$

and

$$\mathcal{R}_j(\lambda) := \langle\lambda,\nabla_\lambda\rho\rangle^{-N}\sum_{k=1}^N \frac{\partial\bar u_j}{\partial\bar\lambda_k}(\lambda)(-1)^{k-1}R_k(\lambda), \quad j=1,\dots,N.$$

Now let f be a fixed function in $A_{\widetilde D}^{-\infty}$. For each $w\in\partial\widetilde D$ choose $\xi\in\mathbb{C}$ with $|\xi|=1$, $\operatorname{Re}\xi>0$ and $H(\xi w)=\operatorname{Re}\xi$. Then, for $u=\gamma w$ with $0\le\gamma\le 1$, put

$$F(u) := \frac{\xi^{N-1}}{(N-1)!}\int_0^\infty f(t\xi u)t^{N-1}e^{-t\xi}dt.$$

By [15] and [22], F is a holomorphic function in $\operatorname{int}\widetilde D$ and the value of $F(u)$, for $u\in\operatorname{int}\widetilde D$, does not depend on the choice of ξ above. In [2, Lemma 2.7], it was shown that F is infinitely differentiable on $\widetilde D$ as a function of $2N$ real variables. By Whitney's extension theorem for C^∞ functions, [23, Theorem I], there exists an infinitely differentiable function $\widetilde F$ on $\mathbb{R}^{2N}$ such that $\widetilde F|_{\widetilde D}=F$ and $\operatorname{supp}\widetilde F\subset\widetilde D_\delta$.

Consider

$$\langle g,f\rangle := \frac{(N-1)!}{(2\pi i)^N}\int_{D\setminus D_\delta} g(\lambda)\sum_{j=1}^N\frac{\partial\widetilde F}{\partial\bar u_j}(u(\lambda))\mathcal{R}_j(\lambda)\,d\bar\lambda\wedge d\lambda, \quad g\in A^{-\infty}(D).$$

We claim that $\langle g,f\rangle$ is a linear continuous functional on $A^{-\infty}(D)$. Indeed, it is well known that $u(\lambda)$ maps ∂D onto $\partial\widetilde D$. In this case, for each $\lambda\in D\setminus D_\delta$ we have

$$\operatorname{dist}\big(u(\lambda),\partial\widetilde D\big) = \inf_{w\in\partial\widetilde D}|u(\lambda)-w| = \inf_{\xi\in\partial D}|u(\lambda)-u(\xi)| \le C_\rho\inf_{\xi\in\partial D}|\lambda-\xi| = C_\rho d(\lambda),$$

where C_ρ is some constant depending only on ρ.

Next, since $\bar\partial\widetilde F=\bar\partial F=0$ on $\widetilde D$, it follows from Taylor's formula that

$$\left|\frac{\partial\widetilde F}{\partial\bar u_j}(u(\lambda))\right| \le C_n\big(\operatorname{dist}(u(\lambda),\partial\widetilde D)\big)^n \le C_n(C_\rho)^n[d(\lambda)]^n,\ \forall\lambda\in D\setminus D_\delta,\ \forall n\in\mathbb{N}.$$

Also, since $\mathcal{R}_j$ is continuous on $D\setminus D_\delta$, there exists $B>0$ such that $|\mathcal{R}_j(\lambda)|\le B$ for all $\lambda\in D\setminus D_\delta$.

So

$$|\langle g,f\rangle| \le \frac{BC_nC_\rho^nN!}{(2\pi)^N}\operatorname{mes}(D)\|g\|_n,\ \forall g\in A^{-n}(D),\ \forall n\in\mathbb{N},$$

where $\operatorname{mes}(D)$ is the Lebesgue measure of D. Therefore, $\varphi:=\langle\cdot,f\rangle\in(A^{-\infty}(D))'$.

Now let $g \in \mathcal{O}(\overline{D})$. By direct calculations, we can verify that

$$g(\lambda)\sum_{j=1}^{N}\frac{\partial \widetilde{F}}{\partial \bar{u}_j}(u(\lambda))\mathcal{R}_j(\lambda)d\bar{\lambda}\wedge d\lambda = d\left(g(\lambda)\widetilde{F}(u(\lambda))\bar{\omega}(\lambda,\nabla_\lambda\rho)\right).$$

In this case, by the Green–Stokes formula,

$$\langle g, f\rangle = \frac{(N-1)!}{(2\pi i)^N}\int_{\partial D} g(\lambda)F(u(\lambda))\bar{\omega}(\lambda,\nabla_\lambda\rho).$$

Hence, by Martineau's projective formula ([3], [15]) we have

$$\langle e^{\langle z,\cdot\rangle}, f\rangle = \frac{(N-1)!}{(2\pi i)^N}\int_{\partial D} e^{\langle\lambda,z\rangle}F(u(\lambda))\bar{\omega}(\lambda,\nabla_\lambda\rho) = f(z),\ \forall z\in\mathbb{C}^N;$$

that is, $\mathcal{F}(\varphi) = f$. □

Proof of Theorem 2.1. From Lemma 2.2 it obviously follows that $\mathcal{F}\colon (A^{-\infty}(D))'_b \to A_D^{-\infty}$ is a continuous mapping. By Proposition 2.3, this mapping is injective, while by Proposition 2.4, it is surjective. Hence, by the open mapping theorem, $\mathcal{F}\colon (A^{-\infty}(D))'_b \to A_D^{-\infty}$ is a topological isomorphism. □

3. Discrete sufficient sets in the (FS)-space $A_D^{-\infty}$

Let D be a bounded convex domain in $\mathbb{C}^N$ ($N \ge 1$) such that $0 \in D$, with the supporting function $H(z)$.

Let S be a subset of $\mathbb{C}^N$. For each $f \in A_D^{-\infty}$ define

$$|f|_{n,S} := \sup_{z\in S}\frac{|f(z)|(1+|z|)^n}{\exp H(z)}, \quad n = 1,2,\dots.$$

Notice that $|\cdot|_{n,S}$ is in general a semi-norm and that the space $A_D^{-\infty}$ can also be endowed with the topology given by the system of semi-norms $(|\cdot|_{n,S})_{n=1}^\infty$. Obviously, this topology is weaker than the global "original" topology given by the system of norms $(|\cdot|_n)_{n=1}^\infty$. If these two topologies coincide, S is called a sufficient set for $A_D^{-\infty}$.

Sufficient sets for some spaces of entire and infinitely differentiable functions were considered by Ehrenpreis [7], Taylor [21], Schneider [19] and others. It should be noted that almost all results concern the case where the spaces are inductive limits of countable sets of weighted Banach spaces.

For our space $A_D^{-\infty}$ the definition of a sufficient set is as follows.

Definition 3.1. Let D be a bounded convex domain in $\mathbb{C}^N$. A subset $S \subset \mathbb{C}^N$ is said to be *sufficient* for the space $A_D^{-\infty}$ if

$$\forall p\in\mathbb{N}\ \exists m = m(p)\in\mathbb{N},\ \exists C = C(p) > 0:$$
$$\sup_{z\in\mathbb{C}^N}\frac{|f(z)|(1+|z|)^p}{e^{H(z)}} \le C\sup_{z\in S}\frac{|f(z)|(1+|z|)^m}{e^{H(z)}},\ \forall f\in A_D^{-\infty}.$$

In particular, a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}^N$ is said to be *sufficient* for the space $A_D^{-\infty}$ if

$$\forall p\in\mathbb{N}\ \exists m = m(p)\in\mathbb{N},\ \exists C = C(p) > 0:$$
$$\sup_{z\in\mathbb{C}^N}\frac{|f(z)|(1+|z|)^p}{e^{H(z)}} \le C\sup_{k\ge 1}\frac{|f(\lambda_k)|(1+|\lambda_k|)^m}{e^{H(\lambda_k)}},\ \forall f\in A_D^{-\infty}.$$

Below we present an explicit construction, in the form of an algorithm, of a discrete sufficient sequence in the space $A_D^{-\infty}$. This construction exploits the method given in [10] for the case of inductive limit spaces (for $N=1$ see also [13]).

For $t>0$ let $\mathcal{S}_t := \{z \in \mathbb{C}^N : H(z)=t\}$ and $M_t(f) := \sup_{z\in\mathcal{S}_t} |f(z)|$.

Algorithm 3.2.

Step 1. Take a sequence $0<(t_k)\uparrow\infty$ *such that*

$$\lim_{k\to\infty}\frac{t_{k+1}}{t_k}=1. \tag{3.1}$$

Step 2. Take a sequence $0<(s_k)\uparrow\infty$ *such that*

$$\lim_{k\to\infty}\frac{1}{s_k(t_{k+1}-t_k)}=0. \tag{3.2}$$

Step 3. Take and fix some natural number $N_0\geq 1$*; on each* $\mathcal{S}_{t_k}$ *with* $k=N_0,N_0+1,\ldots$*, mark* ℓ_k *points* $z_{k,j}$ $(j=1,2,\ldots,\ell_k)$*, which form a* $1/s_k$*-net on* $\mathcal{S}_{t_k}$*.*

Step 4. We re-enumerate the obtained system of points $\{z_{k,j} : 1\leq j\leq \ell_k, k\geq N_0\}$ *in one sequence, denoted by* $\Lambda=\big(\lambda_n\big)_{n=1}^{\infty}$*, by writing first all the points with* $k=N_0$ *and then those with* $k=N_0+1$*, etc.*

We will show that the countable set thus obtained is as desired.

In order to simplify the exposition, in the sequel we take $t_k=k$ $(k=1,2,\ldots)$, although the result is valid for the general case.

Observe that $r|z|\leq H(z)=t\leq R|z|$, $\forall z\in\mathcal{S}_t$. Then, writing $\alpha:=\log\min(1,\frac{1}{R})$ and $\beta:=\log\max(1,\frac{1}{r})$, we have

$$\alpha+\log(1+t)\leq\log(1+|z|)\leq\beta+\log(1+t),\ \forall z\in\mathcal{S}_t,\forall t>0. \tag{3.3}$$

Therefore,

$$|f|_p=\sup_{z\in\mathbb{C}^N}\frac{|f(z)|}{e^{H(z)-p\log(1+|z|)}}\leq e^{\beta p}\sup_{t>0}\frac{M_t(f)}{e^{t-p\log(1+t)}}=e^{\beta p}\sup_{t>0}\frac{M_t(f)}{e^{g_p(t)}}, \tag{3.4}$$

where $g_p(t)=t-p\log(1+t)$ $(t>0)$.

Obviously, $(g_p)_{p\in\mathbb{N}}$ is a decreasing sequence and $g_p(t)$ is decreasing on $(0,p-1]$ and increasing on $[p-1,\infty)$. Moreover,

$$g_p(t+1)-g_p(t)=1-p\log\frac{t+2}{t+1}<1,\ \forall t>0,\ \forall p\in\mathbb{N}. \tag{3.5}$$

We estimate the quantity

$$\sup_{t>0}\frac{M_t(f)}{e^{g_p(t)}},\quad f\in A_D^{-\infty}.$$

For $t\in(0,p-1]$,

$$\frac{M_t(f)}{e^{g_p(t)}}\leq\frac{M_{p-1}(f)}{e^{g_p(p-1)}}\leq\frac{M_{p-1}(f)}{e^{g_{p+1}(p-1)}}. \tag{3.6}$$

Furthermore, due to (3.5), for $t\in(\ell,\ell+1]$ with $\ell\geq p-1$ we have

$$\frac{M_t(f)}{e^{g_p(t)}}\leq\frac{M_{\ell+1}(f)}{e^{g_p(\ell)}}\leq e\frac{M_{\ell+1}(f)}{e^{g_{p+1}(\ell+1)}}. \tag{3.7}$$

Now let $J \geq p$; we have

$$\begin{aligned} \mathcal{X}_J := \sup_{0<t\leq J} \frac{M_t(f)}{e^{g_p(t)}} &\leq \max\left\{ \sup_{0<t\leq p-1} \frac{M_t(f)}{e^{g_p(t)}}, \sup_{p-1<t\leq J} \frac{M_t(f)}{e^{g_p(t)}} \right\} \\ &\leq e \max\left\{ \frac{M_{p-1}(f)}{e^{g_{p+1}(p-1)}}, \sup_{p-1\leq \ell\leq J-1} \frac{M_{\ell+1}(f)}{e^{g_{p+1}(\ell+1)}} \right\} \\ &= e \sup_{p-1\leq \ell\leq J} \frac{M_\ell(f)}{e^{g_{p+1}(\ell)}} := e\mathcal{Y}_J. \end{aligned}$$

For each ℓ there exists $w_\ell \in \mathcal{S}_\ell$ such that $|f(w_\ell)| = M_\ell(f)$. We can find, by Step 3 of Algorithm 3.2, some point $z_{\ell,j_0} \in \mathcal{S}_\ell$ that satisfies $|w_\ell - z_{\ell,j_0}| \leq 1/s_\ell$. Now use the following result from [10].

Lemma 3.3. *For any numbers $0 < s < t$, if $z, w \in \mathcal{S}_s$ and $f \in \mathcal{O}(\mathbb{C}^N)$, then*

$$|f(z) - f(w)| \leq \frac{RN\sqrt{N}}{t-s} M_t(f)|z-w|. \tag{3.8}$$

By this lemma, we have

$$\begin{aligned} M_\ell(f) - |f(z_{\ell,j_0})| &\leq |f(w_\ell) - f(z_{\ell,j_0})| \\ &\leq RN\sqrt{N} M_{\ell+1}(f)|w_\ell - z_{\ell,j_0}| \leq \frac{RN\sqrt{N}}{s_\ell} M_{\ell+1}(f), \end{aligned}$$

which, due to (3.5), implies that

$$\begin{aligned} \frac{M_\ell(f)}{e^{g_{p+1}(\ell)}} &\leq \frac{|f(z_{\ell,j_0})|}{e^{g_{p+1}(\ell)}} + \frac{RN\sqrt{N}}{s_\ell} \cdot \frac{M_{\ell+1}(f)}{e^{g_{p+1}(\ell)}} \\ &\leq \frac{|f(z_{\ell,j_0})|}{e^{g_{p+1}(\ell)}} + \frac{eRN\sqrt{N}}{s_\ell} \cdot \frac{M_{\ell+1}(f)}{e^{g_{p+1}(\ell+1)}}. \end{aligned}$$

Then we get

$$\begin{aligned} \mathcal{Y}_J \leq \sup_{p-1\leq \ell\leq J} \frac{|f(z_{\ell,j_0})|}{e^{g_{p+1}(\ell)}} &+ \sup_{p-1\leq \ell\leq J-1} \left\{ \frac{eRN\sqrt{N}}{s_\ell} \cdot \frac{M_{\ell+1}(f)}{e^{g_{p+1}(\ell+1)}} \right\} \\ &+ \frac{eRN\sqrt{N}}{s_J} \cdot \frac{M_{J+1}(f)}{e^{g_{p+1}(J+1)}}. \end{aligned}$$

Define

$$\mathcal{T}_J := \sup_{p-1\leq \ell\leq J} \sup_{\lambda_k \in \mathcal{S}_\ell} \frac{|f(\lambda_k)|}{e^{g_{p+1}(\ell)}}$$

and notice that $z_{\ell,j_0} \in \mathcal{S}_\ell$. Then, assuming from the beginning that $s_1 > 4eRN\sqrt{N}$, we obtain

$$\begin{aligned} \mathcal{Y}_J &\leq \mathcal{T}_J + \frac{1}{4} \sup_{p-1\leq \ell\leq J-1} \left\{ \frac{M_{\ell+1}(f)}{e^{g_{p+1}(\ell+1)}} \right\} + \frac{eRN\sqrt{N}}{s_J} \cdot \frac{M_{J+1}(f)}{e^{g_{p+1}(J+1)}} \\ &\leq \mathcal{T}_J + \frac{1}{4}\mathcal{Y}_J + \frac{eRN\sqrt{N}}{s_J} \cdot \frac{M_{J+1}(f)}{e^{g_{p+1}(J+1)}}, \end{aligned}$$

or equivalently

$$\frac{3}{4}\mathcal{Y}_J \leq \mathcal{T}_J + \frac{eRN\sqrt{N}}{s_J} \cdot \frac{M_{J+1}(f)}{e^{g_{p+1}(J+1)}}. \tag{3.9}$$

At this point we pause in our estimates for a moment and present the following fact:

Lemma 3.4. *For each nontrivial function* $f \in A_D^{-\infty}$,

$$\liminf_{k\to\infty} \frac{M_{k+1}(f)}{M_k(f)} \le e. \tag{3.10}$$

Proof. Assuming (3.10) is not true, we can then find $r_0 > 1$ and $k_0 \in \mathbb{N}$ so that

$$M_{k+1}(f) > e^{r_0} M_k(f),\ \forall k \ge k_0.$$

Therefore for every $k \ge 1$,

$$M_{k_0+k}(f) > \quad e^{kr_0} M_{k_0}(f).$$

Hence,

$$\liminf_{k\to\infty} \frac{\log M_k(f)}{k} \ge r_0 > 1. \tag{3.11}$$

On the other hand, using (3.3), we have

$$|f|_p = \sup_{z\in\mathbb{C}^N} \frac{|f(z)|}{e^{H(z)-p\log(1+|z|)}} \ge e^{\alpha p} \sup_{t>0} \frac{M_t(f)}{e^{t-p\log(1+t)}} = e^{\alpha p} \sup_{t>0} \frac{M_t(f)}{e^{g_p(t)}}. \tag{3.12}$$

Thus,

$$\log M_k(f) \le \log |f|_p - \alpha p + g_p(k),\ \forall k \ge 1.$$

This implies

$$\limsup_{k\to\infty} \frac{\log M_k(f)}{k} \le \lim_{k\to\infty} \frac{g_p(k)}{k} = 1,$$

which contradicts (3.11). □

Now we return to estimate (3.9). Take and fix a number $\sigma > 1$. By Lemma 3.4, there is a sequence $(k_j) \uparrow \infty$ with $s_{k_1} > 4e^{\sigma+1}RN\sqrt{N}$ and $k_1 \ge p$ such that

$$M_{k_j+1}(f) \le e^{\sigma} M_{k_j}(f),\ \forall j \ge 1.$$

In (3.9) putting $J = k_j, k_{j+1}, \dots$ we obtain

$$\begin{aligned} \frac{3}{4}\mathcal{Y}_{k_j} &\le \mathcal{T}_{k_j} + \frac{eRN\sqrt{N}}{s_{k_j}} \cdot \frac{M_{k_j+1}(f)}{e^{g_{p+1}(k_j+1)}} \\ &\le \mathcal{T}_{k_j} + \frac{e^{\sigma+1}RN\sqrt{N}}{s_{k_j}} \cdot \frac{M_{k_j}(f)}{e^{g_{p+1}(k_j)}} \le \mathcal{T}_{k_j} + \frac{1}{4}\mathcal{Y}_{k_j}. \end{aligned}$$

This means that $\mathcal{Y}_{k_j} \le 2\mathcal{T}_{k_j}$ for all j.

By (3.3), for all j,

$$\begin{aligned} \mathcal{T}_{k_j} &= \sup_{p-1\le \ell \le k_j} \sup_{\lambda_k\in\mathcal{S}_\ell} \frac{|f(\lambda_k)|}{e^{g_{p+1}(\ell)}} \\ &\le e^{-(p+1)\alpha} \sup_{p-1\le \ell \le k_j} \sup_{\lambda_k\in\mathcal{S}_\ell} \frac{|f(\lambda_k)|}{e^{H(\lambda_k)-(p+1)\log(1+|\lambda_k|)}} \\ &\le e^{-(p+1)\alpha} \sup_{k\ge 1} \frac{|f(\lambda_k)|}{e^{H(\lambda_k)-(p+1)\log(1+|\lambda_k|)}} = e^{-(p+1)\alpha}|f|_{p+1,\Lambda}. \end{aligned}$$

Thus, for all j,

$$\sup_{0<t\le k_j} \frac{M_t(f)}{e^{g_p(t)}} = \mathcal{X}_{k_j} \le e\mathcal{Y}_{k_j} \le 2e\mathcal{T}_{k_j} \le 2e^{1-(p+1)\alpha}|f|_{p+1,\Lambda}.$$

Letting $k_j \to \infty$ in the last inequality, we obtain

$$\sup_{t>0} \frac{M_t(f)}{e^{g_p(t)}} \leq 2e^{1-(p+1)\alpha} |f|_{p+1,\Lambda}.$$

Combining this and (3.4) yields

$$|f|_p \leq e^{\beta p} \sup_{t>0} \frac{M_t(f)}{e^{g_p(t)}} \leq 2e^{\beta p} e^{1-(p+1)\alpha} |f|_{p+1,\Lambda}.$$

This means that the desired estimate in Definition 3.1 of a sufficient set for $A_D^{-\infty}$ holds with $m = p+1$ and $C = 2e^{\beta p + 1 - \alpha(p+1)}$.

4. Representation of functions from $A^{-\infty}(D)$ by Dirichlet series

In this section, as an application of the results obtained above, we show that any function from the space $A^{-\infty}(D)$ can always be represented in the form of a Dirichlet series,

$$f(z) = \sum_{k=1}^{\infty} c_k e^{\langle \lambda_k, z \rangle}, \ \forall z \in D,$$

that converges absolutely in the space $A^{-\infty}(D)$.

Recall that, in a general setting (see, e.g., Korobeinik [12]), a sequence $\big(x_k\big)$ of nonzero elements of a locally convex space H is said to be an *absolutely representing system* in H if any element x from H can be represented in the form of a series

$$x \ = \ \sum_{1}^{\infty} c_k x_k$$

which converges absolutely in the topology of H.

It should be noted that this concept is more general than the concept of basis, for which uniqueness of representation is essentially required.

In studying representing systems, Korobeinik obtained criteria for a countable system to be absolutely representing in Fréchet and (DFS)-spaces. Later those results were proved for more practical spaces. In particular, the following result, which follows directly from [14, Chapter 2, Theorem 7] and [17, Theorem 5], is of use to our case considered in this paper.

Proposition 4.1. *Let $E = \lim \operatorname{ind}(E_n, \|\cdot\|_n)$ be a (DFS)-space, where E_n are Banach spaces with $E_1 \subset E_2 \subset \dots$, and let $X = (x_k)_{k=1}^{\infty}$ be a system of elements from E_1 such that*

$$\lim_{k\to\infty} \frac{\|x_k\|_{n+1}}{\|x_k\|_n} = 0, \quad n = 1, 2, \dots. \tag{4.1}$$

The system X is absolutely representing in E if and only if

$$\forall p \in \mathbb{N} \ \exists m = m(p) \in \mathbb{N}, \ \exists C = C(p) > 0:$$
$$\sup_{x \in E_p, \|x\|_p \leq 1} |\varphi(x)| \leq C \sup_{k\geq 1} \frac{|\varphi(x_k)|}{\|x_k\|_m}, \ \forall \varphi \in E'.$$

Now return to the situation considered in the present paper. For each $\lambda_k \in \mathbb{C}^N$ $(k = 1, 2, \dots)$ write $x_k(z) = e^{\langle \lambda_k, z \rangle}$, $z \in D$. We first study the question of whether the conditions in Proposition 4.1 are satisfied by the system $\big(x_k(z)\big)$.

By Lemma 2.2, for each $k = 1, 2, \dots$ we have

$$\|x_k(\cdot)\|_1 = \|e^{\langle \lambda_k, \cdot \rangle}\|_1 = \sup_{z\in D} |e^{\langle \lambda_k, z\rangle}| d(z) \le A_1 \frac{e^{H(\lambda_k)}}{1+|\lambda_k|} < +\infty,$$

which shows that $x_k(z) \in A^{-1}(D)$.

For condition (4.1) we notice, by Lemma 2.2 again, that

$$\frac{a_{n+1}}{A_n} \cdot \frac{1}{1+|\lambda_k|} \le \frac{\|x_k(\cdot)\|_{n+1}}{\|x_k(\cdot)\|_n} \le \frac{A_{n+1}}{a_n} \cdot \frac{1}{1+|\lambda_k|},$$

which implies that condition (4.1) holds if and only if $|\lambda_k| \to \infty$ as $k \to \infty$.

Applying Proposition 4.1 to the spaces $A_D^{-\infty}$ and $A^{-\infty}(D)$, by Theorem 2.1 we obtain the following criterion.

Proposition 4.2. *Let D be either a bounded convex domain with C^2 boundary in $\mathbb{C}^N$, for $N > 1$, or an arbitrary bounded convex domain in $\mathbb{C}$. Further, let $\lambda_k \in \mathbb{C}^N$ $(k = 1, 2, \dots)$ with $|\lambda_k| \to \infty$ as $k \to \infty$. The system $\left(e^{\langle \lambda_k, z\rangle}\right)_{k=1}^{\infty}$ is absolutely representing in the space $A^{-\infty}(D)$ if and only if*

$$\forall p \in \mathbb{N}\ \exists m = m(p) \in \mathbb{N},\ \exists C = C(p) > 0 :$$
$$\sup_{z\in\mathbb{C}^N} \frac{|f(z)|(1+|z|)^p}{e^{H_D(z)}} \le C \sup_{k\ge 1} \frac{|f(\lambda_k)|(1+|\lambda_k|)^m}{e^{H_D(\lambda_k)}},\ \forall f \in A_D^{-\infty}$$

or, in other words, if and only if the sequence of frequencies $\left(\lambda_k\right)_{k=1}^{\infty}$ is sufficient for the space $A_D^{-\infty}$.

Combining Algorithm 3.2 and Proposition 4.2 yields the following representation result for the function algebra $A^{-\infty}(D)$.

Theorem 4.3. *Let D be either a bounded convex domain with C^2 boundary in $\mathbb{C}^N$, for $N > 1$, or an arbitrary bounded convex domain in $\mathbb{C}$. There is an explicit construction of $\Lambda = \left(\lambda_k\right)_{k=1}^{\infty} \subset \mathbb{C}^N$ such that the system $\left(e^{\langle \lambda_k, z\rangle}\right)_{k=1}^{\infty}$ is absolutely representing in the space $A^{-\infty}(D)$; that is, any function $f \in A^{-\infty}(D)$ can be represented in the form of a Dirichlet series*

$$f(z) = \sum_{k=1}^{\infty} c_k e^{\langle \lambda_k, z\rangle},\ \forall z \in D, \tag{4.2}$$

which converges absolutely in the space $A^{-\infty}(D)$.

5. Concluding remarks

Summarizing the results on this subject that have been obtained in our papers [2] and the present one (together with [6]), we see that for the well-known function algebra $A^{-\infty}(D)$ and the newly introduced space $A_D^{-\infty}$ of entire functions in $\mathbb{C}^N$ satisfying a certain growth condition, the following statements hold:

(1) There is a mutual duality between $A^{-\infty}(D)$ and $A_D^{-\infty}$; specifically, the Laplace transformation of analytic functionals establishes a topological isomorphism between the following spaces:
   (a) the strong dual $\left(A^{-\infty}(D)\right)_b'$ of $A^{-\infty}(D)$ and the space $A_D^{-\infty}$;
   (b) the strong dual $\left(A_D^{-\infty}\right)_b'$ of $A_D^{-\infty}$ and the space $A^{-\infty}(D)$.

(2) In both spaces $A^{-\infty}(D)$ and $A_D^{-\infty}$ there exists an absolutely representing system of exponentials $\mathcal{E}_\Lambda = \left(e^{\langle \lambda_k, z\rangle}\right)_{k=1}^\infty$; that is, any function $f(z)$ from either space can be represented in the form of a Dirichlet series

$$f(z) = \sum_{k=1}^{\infty} c_k e^{\langle \lambda_k, z\rangle}$$

that converges absolutely in the topology of the corresponding space.

Equivalently, there exists a sequence $\left(\lambda_k\right)_{k=1}^\infty$, where

(a) $(\lambda_k) \subset \mathbb{C}^N$ for the space $A^{-\infty}(D)$,

(b) $(\lambda_k) \subset D$ for the space $A_D^{-\infty}$,

which is (weakly) sufficient for the corresponding space. Moreover, the frequencies (λ_k) can be constructed explicitly.

These results not only allow us to study deeper properties of functions from the spaces $A^{-\infty}(D)$ and $A_D^{-\infty}$ but may also have important applications to functional equations and approximations of functions in those spaces. Problems in this direction are being investigated and will be presented in our forthcoming work.

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