DUAL REPRESENTATIONS OF BANACH ALGEBRAS

BY

F. F. BONSALL and J. DUNCAN

The University, Edinburgh, Scotland, and the University, Aberdeen, Scotland (1)

The representation theory for Banach algebras has three main branches that are only rather loosely connected with each other. The Gelfand representation of a commutative algebra represents the given algebra by continuous complex valued functions on a space built from the multiplicative linear functionals on the algebra. A Banach star algebra is represented by operators on a Hilbert space, the Hilbert space being built by means of positive Hermitian functionals on the algebra. Finally, for general non-commutative Banach algebras, an extension of the Jacobson theory of representations of rings is available. In this general theory, the representations are built in terms of irreducible operator representations on Banach spaces, and, on the face of it, no part is played by the linear functionals on the algebra. There is some evidence that the concepts involved in the general theory are not sufficiently strong to exploit to the full the Banach algebra situation.

The purpose of the present paper is to develop a new unified general representation theory that is more closely related than the Jacobson theory to the special theories for commutative and star algebras. The central concept is that of a dual representation on a pair of Banach spaces in normed duality. It is found that each continuous linear functional on a Banach algebra gives rise to a dual representation of the algebra, and thus the dual space of the algebra enters representation theory in a natural way. One may ask of a dual representation that it be irreducible on each of the pair of spaces in duality, and thus obtain a concept of irreducibility stronger than the classical one. Correspondingly one obtains a stronger concept of density. For certain pairs of spaces in duality, topological irreducibility on one of the spaces implies topological irreducibility on the other. However, we show that this is very far from being the case in general. We also consider a further concept of irreducibility, namely uniform strict transitivity, which is stronger than strict irreducibility. For certain pairs of spaces in duality, uniform strict transitivity on one of the spaces implies

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uniform strict transitivity (and hence strict irreducibility) on the other; but it is not true for such spaces that strict irreducibility on one of the spaces implies strict irreducibility on the other. We are also concerned to relate the concept of a dually strictly irreducible dual representation to the ideal structure of the algebra, and we introduce the dual radical of a Banach algebra as the intersection of the kernels of all such representations.

We lean heavily on the thorough account of representation theory given by C. E. Rickart in his book [5]; and, moreover, the present theory had its origin in a course of lectures given by Rickart in London in 1961, in which he drew attention to the need for a stronger general representation theory. A special case of the concept of dual representation is already familiar in the established theory of representations of primitive algebras with minimal ideals [5, pp. 62–70]. Our construction of dual representations in terms of continuous linear functionals also appears in a recent paper by J. M. G. Fell [4]. His interest however is mainly in the case in which the representing spaces have finite dimension. In this case the dual irreducibility questions with which we are concerned are trivial.

1. Representations of Banach algebras

In this section, we collect together a few propositions concerning representations of Banach algebras that we shall need. Most of this material is well known, and can be found, though not precisely in the present form, in Rickart [5, Chapter II]. Standard definitions and all proofs are accordingly omitted.

Let $\mathbf{F}$ denote either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$, and let $\mathfrak{A}$ denote a Banach algebra over $\mathbf{F}$. It is not assumed that $\mathfrak{A}$ is commutative, nor that $\mathfrak{A}$ has an identity. Let X be a Banach space over $\mathbf{F}$ such that $X \neq (0)$, and let $\mathfrak{B}(X)$ denote the Banach algebra of all bounded linear operators on X, with the usual operator norm.

PROPOSITION 1. *Let $a \to T_a$ be a representation of $\mathfrak{A}$ on X, let $u \in X$, and let $L = \{a : T_a u = 0\}$.*

(i) *If $T_e u = u$ for some e in $\mathfrak{A}$, then e is a right identity* (mod L).
(ii) *If u is a strictly cyclic vector, then L is a modular left ideal.*
(iii) *If u is a strictly cyclic vector and L is a maximal left ideal, then $a \to T_a$ is strictly irreducible.*
(iv) *If $a \to T_a$ is strictly irreducible, and $u \neq 0$, then L is a maximal modular left ideal.*

PROPOSITION 2. *Let $a \to T_a$ be a topologically irreducible representation of $\mathfrak{A}$ on X, and let there exist a strictly cyclic vector u. Then $L = \{a : T_a u = 0\}$ is a maximal modular left ideal, and the representation $a \to T_a$ is strictly irreducible.*

PROPOSITION 3. *Let L be a closed left ideal of $\mathfrak{A}$ such that $\mathfrak{A}^2 \nsubseteq L$, and let $a \to T_a$ be the left regular representation on $\mathfrak{A} - L$.*

(i) *If L is modular with e a right identity* (mod L), *then the L-coset e' is a strictly cyclic vector for the representation, and $L = \{a : T_a e' = 0\}$.*
(ii) *The representation is strictly irreducible if and only if L is maximal.*
(iii) *The representation is topologically irreducible if and only if L is maximal closed.*

PROPOSITION 4. *Let the scalar field $\mathbf{F}$ be the complex field $\mathbf{C}$, and let $a \to T_a$ be a strictly irreducible representation of $\mathfrak{A}$ on X. Then the representation is strictly dense on X; i.e. given $T \in \mathfrak{B}(X)$ and given a finite dimensional subspace U of X, there exists $a \in \mathfrak{A}$ such that*

$$T_a|_U = T|_U, \quad i.e. \quad (T_a - T)\, U = (0).$$

2. Notation and elementary properties of dual representations

Throughout this paper $(X, Y, \langle\, ,\rangle)$ will denote a pair of non-zero Banach spaces X, Y in normed duality with respect to a bilinear form $\langle\, ,\rangle$ (see Rickart [5, Definition 2.4.8]). Given such a pair $(X, Y, \langle\, ,\rangle)$, there are two associated natural mappings $x \to \hat{x}$, $y \to \hat{y}$ defined by

$$\hat{x}(y) = \langle x, y\rangle \quad (y \in Y),$$

$$\hat{y}(x) = \langle x, y\rangle \quad (x \in X).$$

It is clear that $\hat{x} \in Y'$ $(x \in X)$ and $\hat{y} \in X'$ $(y \in Y)$.

The following two routine propositions describe the nature of two Banach spaces in normed duality.

PROPOSITION 5. *Let $(X, Y, \langle\, ,\rangle)$ be Banach spaces in normed duality.*

(i) *The mapping $y \to \hat{y}$ is a continuous monomorphism from Y into X' whose image $\hat{Y}$ is weak* dense in X'.*
(ii) *If X is reflexive, then $\hat{Y}$ is norm dense in X'.*
(iii) *$\hat{Y}$ is norm closed in X' if and only if $y \to \hat{y}$ is bicontinuous.*

Similar statements hold for the mapping $x \to \hat{x}$.

PROPOSITION 6. *Let X be a Banach space and Y a weak* dense subspace of X'. If Y is a Banach space under a norm dominating the usual norm on X', then X, Y are in normed duality with respect to the natural bilinear form $(\, ,)$.*

It is significant in parts of dual representation theory to know when $\hat{X}$ and $\hat{Y}$ are norm closed. We leave the reader to produce examples in which (i) neither $\hat{X}$ nor $\hat{Y}$ is norm closed, (ii) $\hat{X}$ is norm closed, $\hat{Y}$ is not norm closed.

Given Banach spaces X, Y in normed duality with respect to $\langle\,,\rangle$, operators $T\in\mathfrak{B}(X)$, $S\in\mathfrak{B}(Y)$ are said to be *adjoint* with respect to $\langle\,,\rangle$ if

$$\langle Tx, y\rangle = \langle x, Sy\rangle \quad (x\in X, y\in Y).$$

The non-degeneracy of the form $\langle\,,\rangle$ implies that there is at most one $S\in\mathfrak{B}(Y)$ adjoint to a given $T\in\mathfrak{B}(X)$ (and at most one $T\in\mathfrak{B}(X)$ adjoint to a given $S\in\mathfrak{B}(Y)$). The unique S, if it exists, is denoted by T^* and is called *the adjoint* of T with respect to $\langle\,,\rangle$. The adjoint S^* of a given $S\in\mathfrak{B}(Y)$ is similarly defined. It is easy to verify that if elements T, U of $\mathfrak{B}(X)$ have adjoints on Y with respect to $\langle\,,\rangle$, then so do λT $(\lambda\in\mathbf{F})$, $T+U$, TU, and

$$(\lambda T)^* = \lambda T^*, \quad (T+U)^* = T^* + U^*, \quad (TU)^* = U^*T^*.$$

It is also clear that T^* has an adjoint $(T^*)^*$ on X, and that $(T^*)^* = T$. We denote by $\mathfrak{B}(X, Y, \langle\,,\rangle)$ the algebra of all operators $T\in\mathfrak{B}(X)$ that have adjoints $T^*\in\mathfrak{B}(Y)$ with respect to $\langle\,,\rangle$. The mapping $T\to T^*$ need not be continuous, but it has a closed graph in the sense that

$$\lim_{n\to\infty} |T_n - T| = 0, \quad \lim_{n\to\infty} |T_n^* - S| = 0 \quad \Rightarrow \quad S = T^*.$$

This follows since

$$\langle Tx, y\rangle = \lim_{n\to\infty} \langle T_n x, y\rangle = \lim_{n\to\infty} \langle x, T_n^* y\rangle = \langle x, Sy\rangle \quad (x\in X, y\in Y).$$

We observe in passing that $\mathfrak{B}(X, X', (\,,)) = \mathfrak{B}(X)$.

Given any Banach space X, we denote by $F(X)$ the algebra of all bounded operators on X of finite rank. Given any pair $(X, Y, \langle\,,\rangle)$ we write

$$F(X, Y, \langle\,,\rangle) = F(X) \cap \mathfrak{B}(X, Y, \langle\,,\rangle).$$

Given non-zero $x\in X$ and non-zero $y\in Y$, we denote by $x\otimes y$ the bounded operator of rank one defined by

$$(x\otimes y)(u) = \langle u, y\rangle x \quad (u\in X).$$

It is clear that $x\otimes y\in F(X, Y, \langle\,,\rangle)$ and that

$$(x\otimes y)^*(v) = \langle x, v\rangle y \quad (v\in Y).$$

PROPOSITION 7. *$\mathfrak{B}(X, Y, \langle , \rangle)$ is a Banach algebra under the norm*

$$\|T\| = \max (|T|, |T^*|).$$

Proof. This follows easily from the fact that $T \to T^*$ has a closed graph.

PROPOSITION 8. *The following statements are equivalent.*

(i) *$\mathfrak{B}(X, Y, \langle , \rangle)$ is closed in $\mathfrak{B}(X)$.*
(ii) *There is a real constant k such that*

$$|T^*| \leqslant k|T| \quad (T \in \mathfrak{B}(X, Y, \langle , \rangle)).$$

(iii) *$\hat{Y}$ is norm closed in X'.*

Similar remarks apply with X and Y interchanged.

Proof. This is a straightforward exercise.

Algebras $A \subseteq \mathfrak{B}(X)$, $B \subseteq \mathfrak{B}(Y)$ are said to be a *dual pair of operator algebras* on $(X, Y, \langle , \rangle)$ if $A \subseteq \mathfrak{B}(X, Y, \langle , \rangle)$ and $A^* = \{T^* : T \in A\} = B$. It follows in this case that each $S \in B$ has an adjoint in $\mathfrak{B}(X)$ with respect to $\langle , \rangle$, and that $B^* = A$. Thus the concept of a dual pair of operator algebras is symmetrical with respect to X and Y.

A *dual representation* of a Banach algebra $\mathfrak{A}$ on $(X, Y, \langle , \rangle)$ is a mapping $a \to T_a$ of $\mathfrak{A}$ into $\mathfrak{B}(X, Y, \langle , \rangle)$ such that $a \to T_a$ is a representation of $\mathfrak{A}$ on X. The following proposition shows that the concept of dual representation is symmetrical with respect to X and Y.

PROPOSITION 9. *Let $a \to T_a$ be a dual representation of $\mathfrak{A}$ on $(X, Y, \langle , \rangle)$. Let $A = \{T_a : a \in \mathfrak{A}\}$, $B = \{T_a^* : a \in \mathfrak{A}\}$.*

(i) *(A, B) is a dual pair of operator algebras on $(X, Y, \langle , \rangle)$.*
(ii) *$a \to T_a^*$ is an anti-representation of $\mathfrak{A}$ on Y.*
(iii) *The kernel of the representation $a \to T_a$ is also the kernel of the anti-representation $a \to T_a^*$.*

Proof. (i) and (iii) are trivial, (ii) follows readily from the closed graph theorem.

COROLLARY. *Let $\|\cdot\|$ be the norm on $\mathfrak{B}(X, Y, \langle , \rangle)$ given in Proposition 7. Then $a \to T_a$ is continuous with respect to the norm $\|\cdot\|$ on $\mathfrak{B}(X, Y, \langle , \rangle)$.*

Every Banach algebra admits dual representations. In fact, since $\mathfrak{B}(X, X', (,)) = \mathfrak{B}(X)$, every representation of $\mathfrak{A}$ on X is also a dual representation of $\mathfrak{A}$ on $(X, X', (,))$. For this reason, dual representations are of interest only when both the representation $a \to T_a$ and the anti-representation $a \to T_a^*$ have spatial properties on X and Y respectively, for example, when both are strictly or topologically irreducible.

We say that a dual representation $a \to T_a$ is *dually strictly (topologically) irreducible* if $a \to T_a$ and $a \to T_a^*$ are both strictly (topologically) irreducible.

The following result will be useful in subsequent sections.

PROPOSITION 10. *The identity mapping is a dually strictly irreducible dual representation of $\mathfrak{B}(X, Y, \langle\,,\rangle)$ on $(X, Y, \langle\,,\rangle)$.*

Proof. The required irreducibility follows from the abundance of operators of rank one.

Remark. Evidently $F(X, Y, \langle\,,\rangle)$ is strictly irreducible on X and $(F(X, Y, \langle\,,\rangle))^*$ is strictly irreducible on Y.

3. Characterisations of irreducible adjoint algebras

The following theorem gives a property analagous to the strict density of $a \to T_a$ that corresponds to the strict irreducibility of $a \to T_a^*$.

THEOREM 1. *Let the scalar field* $\mathbf{F}$ *be* $\mathbf{C}$. *Let $a \to T_a$ be a dual representation of $\mathfrak{A}$ on $(X, Y, \langle\,,\rangle)$. Then the following conditions are equivalent.*

(i) *$a \to T_a^*$ is strictly irreducible on Y.*

(ii) *$a \to T_a^*$ is strictly dense on Y.*

(iii) *Given a $\sigma(X, Y, \langle\,,\rangle)$ closed linear subspace U of X of finite codimension, and given $T \in \mathfrak{B}(X, Y, \langle\,,\rangle)$, there exists $a \in \mathfrak{A}$ such that $T_a = T \pmod{U}$, i.e. $(T_a - T)X \subseteq U$.*

(iv) *Given $\sigma(X, Y, \langle\,,\rangle)$ closed linear subspaces U, V of X of finite codimensions m, n with $n \leqslant m$, there exists $a \in \mathfrak{A}$ such that $T_a^{-1} U = V$.*

(v) *Condition* (iv) *holds whenever $n \leqslant m = 1$.*

Remarks. (1) If Y is the dual space X' of X and $\langle\,,\rangle$ is the natural bilinear form, then all norm closed linear subspaces of X are closed in $\sigma(X, Y, \langle\,,\rangle)$.

(2) We denote the dimension and codimension of a subspace E by $\dim(E)$ and $\operatorname{codim}(E)$ respectively, and we have $\operatorname{codim}(U) = \dim(X - U)$.

(3) Given subsets E, F of X, Y respectively, let

$$E^0 = \{y : \langle x, y\rangle = 0 \ (x \in E)\}, \qquad {}^0F = \{x : \langle x, y\rangle = 0 \ (y \in F)\}.$$

It is well known that for a $\sigma(X, Y, \langle\,,\rangle)$ closed linear subspace U of X, we have $U = {}^0(U^0)$ and that $\operatorname{codim}(U) = \dim(U^0)$.

Proof of Theorem 1. We prove one of the implications and leave the rest to the reader.

(iii) $\Rightarrow$ (iv). Let U, V be linear subspaces of X with the properties stated in (iv). We have $\dim(U^0) = m$, $\dim(V^0) = n$, $n \leqslant m$. By Propositions 10 and 4, $(\mathfrak{B}(X, Y, \langle\,,\rangle))^*$ is strictly dense on Y. Thus there exists $T \in \mathfrak{B}(X, Y, \langle\,,\rangle)$ such that $T^* U^0 = V^0$. By (iii), there exists $a \in \mathfrak{A}$ such that $(T_a - T)X \subseteq U$. Since

$$\langle x, (T_a^* - T^*)y\rangle = 0 \quad (x \in X,\ y \in U^0),$$

it follows that $(T_a^* - T^*)U^0 = (0)$ and so $T_a^* U^0 = V^0$. We now have

$$\langle T_a x, y\rangle = \langle x, T_a^* y\rangle = 0 \quad (x \in V,\ y \in U^0).$$

Therefore $T_a V \subseteq {}^0(U^0) = U$, and so $V \subseteq T_a^{-1} U$. Also,

$$\langle x, T_a^* y\rangle = \langle T_a x, y\rangle = 0 \quad (x \in T_a^{-1} U,\ y \in U^0),$$

and, since $T_a^* U^0 = V^0$, this gives,

$$\langle x, y\rangle = 0 \quad (x \in T_a^{-1} U,\ y \in V^0),$$

from which $T_a^{-1} U \subseteq {}^0(V^0) = V$.

4. A correspondence between linear functionals and dual representations

The following notation will remain fixed throughout. $\mathfrak{A}$ denotes, as before, a Banach algebra, $\mathfrak{A}'$ denotes its dual space of all continuous linear functionals on $\mathfrak{A}$ (as a Banach space). For each f in $\mathfrak{A}'$, we write

$$N_f = \{x : f(x) = 0\}, \quad L_f = \{x : \mathfrak{A}x \subseteq N_f\},$$

$$K_f = \{x : x\mathfrak{A} \subseteq N_f\}, \quad P_f = \{x : \mathfrak{A}x\mathfrak{A} \subseteq N_f\}.$$

Clearly, L_f is a closed left ideal, K_f is a closed right ideal, and P_f is a closed two-sided ideal.

Given a subset E of $\mathfrak{A}$, the *right quotient* of E is the set $\{x : x\mathfrak{A} \subseteq E\}$, denoted by $E : \mathfrak{A}$. Similarly, the *left quotient* of E is the set $\{x : \mathfrak{A}x \subseteq E\}$, and we denote this by $E :' \mathfrak{A}$ to distinguish it from the right quotient of E. With this notation,

$$L_f = N_f :' \mathfrak{A}, \quad K_f = N_f : \mathfrak{A}, \quad P_f = L_f : \mathfrak{A} = K_f :' \mathfrak{A}.$$

We denote the Banach spaces $\mathfrak{A} - L_f$, $\mathfrak{A} - K_f$ by X_f and Y_f respectively, and define a form $\langle\,,\rangle_f$ on $X_f \times Y_f$ by taking

$$\langle x', y'\rangle_f = f(yx) \quad (x \in x' \in X_f,\ y \in y' \in Y_f).$$

This form is well-defined, for if $x_1, x_2 \in x'$ and $y_1, y_2 \in y'$, then $x_1 - x_2 \in L_f$ and $y_1 - y_2 \in K_f$, and so $(y_1 - y_2)x_1 \in N_f$ and $y_2(x_1 - x_2) \in N_f$, from which

$$f(y_1 x_1) = f(y_2 x_1) = f(y_2 x_2).$$

We denote the left regular representation on X_f by $a \to T^f_a$, and the right regular representation on Y_f by $a \to S^f_a$.

We recall that an ideal is *left primitive* if it is the right quotient of a maximal modular left ideal, and it is *right primitive* if it is the left quotient of a maximal modular right ideal. Given a closed two-sided ideal P that is both left and right primitive, we say that a linear functional f belonging to $\mathfrak{A}'$ is *appropriate* for P if L_f is a maximal modular left ideal, K_f is a maximal modular right ideal, and $P_f = P$. We shall also say loosely that f is an appropriate functional if there is some left and right primitive ideal P of $\mathfrak{A}$ for which f is appropriate.

THEOREM 2. *Given $f \in \mathfrak{A}'$ with $f(\mathfrak{A}^3) \neq (0)$, the mapping $a \to T^f_a$ is a dual representation of $\mathfrak{A}$ on $(X_f, Y_f, \langle\,,\rangle_f)$, S^f_a is the adjoint of T^f_a on Y_f with respect to $\langle\,,\rangle_f$, and the following statements hold.*

(i) *If $L_f(K_f)$ is modular, then there is a strictly cyclic vector in $X_f(Y_f)$.*

(ii) *The representation is dually strictly irreducible if and only if L_f and K_f are maximal.*

(iii) *The representation is dually topologically irreducible if and only if L_f and K_f are maximal closed.*

Proof. This is routine verification together with an application of Proposition 3.

Given $i = 1, 2$, let $a \to T^i_a$ be dual representations of $\mathfrak{A}$ on $(X_i, Y_i, \langle\,,\rangle_i)$. We say that these dual representations are *equivalent* if there exist bicontinuous linear isomorphisms U, V of X_1 on to X_2 and of Y_1 on to Y_2 respectively such that

(i) $UT^1_a = T^2_a U \quad (a \in \mathfrak{A})$,

(ii) $\langle x_1, y_1 \rangle_1 = \langle Ux_1, Vy_1 \rangle_2 \quad (x_1 \in X_1, y_1 \in Y_1)$.

In the first corollary to the following theorem, we give conditions under which a dual representation is equivalent to a dual representation $a \to T^f_a$ associated as in Theorem 2 with a linear functional f. The theorem is a halfway house.

THEOREM 3. *Let $a \to T_a$ be a dual representation of $\mathfrak{A}$ on $(X, Y, \langle\,,\rangle)$ such that there exist topologically cyclic vectors $x_0 \in X$, $y_0 \in Y$. Then there exist $f \in \mathfrak{A}'$ and continuous linear monomorphisms U, V of X_f, Y_f on to dense linear subspaces of X, Y respectively, such that $f(\mathfrak{A}^3) \neq (0)$ and*

(i) $L_f = \{a : T_a x_0 = 0\}$, $K_f = \{a : T_a{}^* y_0 = 0\}$,
(ii) $UT^f_a = T_a U$, $VS^f_a = T_a{}^* V$ $(a \in \mathfrak{A})$,
(iii) $\langle x', y' \rangle_f = \langle Ux', Vy' \rangle$ $(x' \in X_f,\ y' \in Y_f)$.

Proof. Let $f(a) = \langle T_a x_0, y_0 \rangle$ $(a \in \mathfrak{A})$. It is clear that $f \in \mathfrak{A}'$. Also,

$$f(ba) = \langle T_{ba} x_0, y_0 \rangle = \langle T_b T_a x_0, y_0 \rangle = \langle T_a x_0, T_b{}^* y_0 \rangle.$$

Since y_0 is topologically cyclic, this shows that $a \in L_f$ if and only if $T_a x_0 = 0$. Thus $L_f = \{a : T_a x_0 = 0\}$ and, similarly, $K_f = \{a : T_a{}^* y_0 = 0\}$. We define U and V by

$$Ux' = T_x x_0 \quad (x \in x' \in X_f), \qquad Vy' = T_y{}^* y_0 \quad (y \in y' \in Y_f).$$

The rest of the proof is routine verification.

COROLLARY 1. *If x_0 and y_0 are strictly cyclic, then the dual representation $a \to T^f_a$ is equivalent to the dual representation $a \to T_a$, and L_f and K_f are modular.*

Proof. Let x_0 and y_0 be strictly cyclic. Then U maps X_f on to X, and therefore, by Banach's isomorphism theorem, is bicontinuous. Similarly, V is a bicontinuous mapping of Y_f on to Y. Thus the dual representations are equivalent. Since $L_f = \{a : T_a x_0 = 0\}$, Proposition 1 (ii) shows that L_f is modular, and similarly for K_f.

COROLLARY 2. *If $a \to T_a$ is dually strictly irreducible, then L_f is a maximal modular left ideal, and K_f is a maximal modular right ideal.*

Proof. Proposition 1 (iv).

THEOREM 4. *Let $g \in \mathfrak{A}'$ be such that $g(\mathfrak{A}^3) \neq (0)$ and L_g and K_g are maximal left and right ideals respectively. Then there exists $f \in \mathfrak{A}'$ such that $f(\mathfrak{A}^3) \neq (0)$ and*

(i) *L_f and K_f are maximal modular left and right ideals respectively,*
(ii) $P_f = P_g$,
(iii) *the dual representations $a \to T^f_a$, $a \to T^g_a$ are equivalent.*

Proof. By Theorem 2 (ii), $a \to T^g_a$ is a dually strictly irreducible dual representation of $\mathfrak{A}$. The theorem is now an immediate consequence of Theorem 3 Corollary 2.

THEOREM 5. *Let P be the kernel of a dually strictly irreducible dual representation of $\mathfrak{A}$. Then P is left and right primitive, and there exists an appropriate functional for P.*

Proof. Let $a \to T_a$ be the given dual representation with kernel P. By Theorem 3 Corollary 2, there exists $f \in \mathfrak{A}'$ such that $a \to T^f_a$ is an equivalent dual representation, and

L_f, K_f are maximal modular ideals. We have $T_a^f=0$ if and only if $T_a=0$, and so $P=P_f$. This shows that P is left and right primitive and that f is appropriate for P.

In the next theorem we shall characterise the existence of appropriate functionals in terms of the ideal structure of the Banach algebra. We remark that if $f\in\mathfrak{A}'$ is such that L_f is a proper modular left ideal, then $P_f\subseteq L_f$ and so it is automatic that $f(\mathfrak{A}^3)\neq(0)$.

LEMMA 1. *Let L be a maximal modular left ideal of $\mathfrak{A}$.*

(i) *If $a\notin L$, there exists $f\in\mathfrak{A}'$ with $f(L)=(0)$ and $f(a)=1$.*

(ii) *For each non-zero $f\in\mathfrak{A}'$ with $f(L)=(0)$, we have $L_f=L$.*

Proof. (i). This follows directly from the Hahn–Banach theorem since maximal modular left ideals are closed in a Banach algebra. (ii) Let $f\in\mathfrak{A}'$ be such that $f\neq0$ and $f(L)=(0)$. Since L is modular, there exists e in $\mathfrak{A}$ such that $a-ae\in L$ $(a\in\mathfrak{A})$. Therefore $f(a)=f(ae)$ $(a\in\mathfrak{A})$ and so $e\notin L_f$. Since L is a left ideal, we have $\mathfrak{A}L\subseteq L\subseteq N_f$ and so $L\subseteq L_f$. By the maximality of L we conclude that $L=L_f$.

A similar result clearly holds for maximal modular right ideals.

THEOREM 6. *Let L be a maximal modular left ideal in $\mathfrak{A}$. Then the following statements are equivalent.*

(i) *There exists $f\in\mathfrak{A}'$ such that $L_f=L$ and K_f is a maximal modular right ideal.*

(ii) *There exists a maximal right ideal K such that $\overline{L+K}\neq\mathfrak{A}$.*

Proof. (i) $\Rightarrow$ (ii). Let f satisfy the conditions of (i). Let e_1 be a right identity $(\operatorname{mod} L_f)$ and e_2 a left identity $(\operatorname{mod} K_f)$. Let $g(a)=f(e_2ae_1)$ $(a\in\mathfrak{A})$. Then $g\in\mathfrak{A}'$. Since $L_fe_1\subseteq L_f$ we have $f(e_2L_fe_1)=(0)$ and thus $L_f\subseteq N_g$. Similarly we have $K_f\subseteq N_g$. It follows that $\overline{L_f+K_f}\subseteq N_g$. Since $L_f=L$ and K_f is maximal, it is now sufficient to show that $g\neq0$. But if $g=0$, then $\langle(ae_1)', e_2'\rangle_f=0$ $(a\in\mathfrak{A})$ and so $\langle X_f, e_2'\rangle_f=(0)$, which is a contradiction.

(ii) $\Rightarrow$ (i). Let K satisfy the conditions of (ii). By the Hahn–Banach theorem there exists $g\in\mathfrak{A}'$ such that $g\neq0$ and $g(L+K)=(0)$. We have $g(L)=(0)$ and therefore $L_g=L$ by Lemma 1. Also, $g(K)=(0)$ so that $g(K\mathfrak{A})=(0)$ and thus $K\subseteq K_g$. We have

$$K_f=\mathfrak{A}\Leftrightarrow g(\mathfrak{A}^2)=(0)\Leftrightarrow L_g=\mathfrak{A}.$$

Since K is maximal we conclude that $K_g=K$. Since L_g is a maximal modular left ideal, $g(\mathfrak{A}^3)\neq(0)$, and so by Theorem 2, $a\to T_a^g$ is dually strictly irreducible. Let e be a right identity $(\operatorname{mod} L)$ and let $y\notin K$. Let

$$f(a)=\langle T_a^g e', y'\rangle_g \quad (a\in\mathfrak{A}).$$

Clearly $f\in\mathfrak{A}'$, and by the argument of Theorem 3 we have

$$L_f = \{a : T_a^g e' = 0\} = \{a : ae \in L\} \supseteq L.$$

Since $a \to T_a^g$ is strictly irreducible, L_f is proper and hence $L_f = L$. Finally, the fact that K_f is a maximal modular right ideal follows from Theorem 3 Corollary 2.

THEOREM 7. *Let P be a left and right primitive ideal of $\mathfrak{A}$. Then* ([0]) *is a left and right primitive ideal of $\mathfrak{A}/P$, and the following statements are equivalent.*

(i) *There exists an appropriate f in $\mathfrak{A}'$ for P.*
(ii) *There exists an appropriate g in $(\mathfrak{A}/P)'$ for* ([0]).

Proof. Apply Theorem 6.

As far as the existence of appropriate functionals is concerned, Theorem 7 has essentially reduced the problem to the case of a Banach algebra which is both left and right primitive. It is still an open question as to whether every left primitive Banach algebra is also right primitive. G. M. Bergman, [1], has given an example of a ring primitive on the right but not on the left, but his construction seems to have no analogue for Banach algebras. Accordingly, our basic starting point for the next section is left primitive Banach algebras. It is well known that such algebras are continuously isomorphic with strictly irreducible algebras of bounded linear operators on some Banach space. In fact we are also interested in the weaker situation of topologically irreducible algebras of operators.

5. Analysis of dual pairs of operator algebras

The main purpose of this section is to examine the following question.

"Given that (A, B) is a dual pair of operator algebras on $(X, Y, \langle\,,\rangle)$ with A topologically irreducible on X, what irreducibility properties has B on Y?"

PROPOSITION 11. *Let (A, B) be a dual pair of operator algebras on $(X, Y, \langle\,,\rangle)$ with A topologically irreducible on X. Let V be a non-zero invariant subspace of Y for B, and let*

$$Z = \{g : g \in Y',\ (V, g) = (0)\}.$$

Then Z is a weak closed subspace of Y', $Z \cap \hat{X} = (0)$, and $\bar{V} = {}^0Z = \{y : y \in Y,\ (y, Z) = (0)\}$. Further, $(A, B|_{\bar{V}})$ is a dual pair on $(X, \bar{V}, \langle\,,\rangle)$.*

Proof. It is well known that Z is weak* closed and that $\bar{V} = {}^0Z$. Let $z \in Z \cap \hat{X}$ so that $z = \hat{x}$ for some $x \in X$. Since $(V, z) = (0)$, we have $\langle x, V\rangle = (0)$. Since B is invariant on V, we have $\langle x, BV\rangle = (0)$ and thus $\langle Ax, V\rangle = (0)$. If $x \neq 0$, then $\overline{Ax} = X$ and so $V = (0)$. This shows that $Z \cap \hat{X} = (0)$ as required.

Since ${}^{0}V=(0)$, the Banach spaces X and $\overline{V}$ are in normed duality with respect to $\langle\, ,\rangle$. Since B is invariant on $\overline{V}$, it follows that $(A, B|_{\overline{V}})$ is a dual pair on $(X, \overline{V}, \langle\, ,\rangle)$.

We now consider a very special condition on the pair $(X, Y, \langle\, ,\rangle)$. We say that Y *represents* X' *through* $\langle\, ,\rangle$ if for each $f\in X'$ there exists $y_f\in Y$ such that

$$f(x)=\langle x, y_f\rangle \quad (x\in X).$$

This condition is equivalent to $X'\subseteq \hat{Y}$. It follows from Proposition 5 and Banach's isomorphism theorem that $y\to\hat{y}$ is thus a bicontinuous isomorphism of Y with X'. In other words, the pair is essentially $(X, X', (\, ,))$. We may similarly speak of X representing Y' through $\langle\, ,\rangle$, and then the pair is essentially $(Y', Y, (\, ,))$.

PROPOSITION 12. *Let* $(X, Y, \langle\, ,\rangle)$ *be such that* X *represents* Y' *through* $\langle\, ,\rangle$. *Let* (A, B) *be a dual pair on* $(X, Y, \langle\, ,\rangle)$ *with* A *topologically irreducible on* X. *Then* B *is topologically irreducible on* Y.

Proof. This follows easily from Proposition 11.

PROPOSITION 13. *Let* $(X, Y, \langle\, ,\rangle)$ *be such that* B *is topologically irreducible on* Y *whenever* (A, B) *is a dual pair on* $(X, Y, \langle\, ,\rangle)$ *with* $A\subseteq F(X)$ *and* A *strictly irreducible on* X. *Then* X *represents* Y' *through* $\langle\, ,\rangle$.

Proof. Suppose that Y' is not represented by X. Then there is some $f\in Y'$ that is not represented by any element of X. Let $V=N_f$ so that V is a closed subspace of Y with $(0)\neq V\neq Y$. We have

$$\langle x, V\rangle=(0)\Rightarrow(V,\hat{x})=(0)\Rightarrow N_f\subseteq N_{\hat{x}}.$$

Since f is not represented by any element of X we must have $N_{\hat{x}}=Y$. This gives $\hat{x}=0$ and so $x=0$. It follows that the Banach spaces $(X, V, \langle\, ,\rangle)$ are in normed duality. By the remark after Proposition 10, $F(X, V, \langle\, ,\rangle)$ is dually strictly irreducible on $(X, V, \langle\, ,\rangle)$. Let $A=F(X, V, \langle\, ,\rangle)$. Then $A\subseteq\mathfrak{B}(X, Y, \langle\, ,\rangle)$ and (A, A^*) is a dual pair on $(X, Y, \langle\, ,\rangle)$, but A^* is not topologically irreducible on Y. This contradiction completes the proof.

THEOREM 8. *A Banach space* X *is reflexive if and only if whenever* $A\subseteq F(X)$ *is topologically irreducible on* X, A^* *is topologically irreducible on* X'.

Proof. Recall that (A, A^*) is a dual pair on $(X, X', (\, ,))$ for every $A\subseteq F(X)$. The Banach space X is reflexive if and only if X represents X'' through $(\, ,)$. The result follows immediately from Propositions 12 and 13.

Proposition 12 has an analogue for the case of a pair $(X, Y, \langle\, ,\rangle)$ such that the linear space $Y' - \hat{X}$ is finite dimensional. (In fact, using the results of Dixmier, [3], one can show that such a pair is essentially of the form $(Q', j(Q) \oplus Z, (\, ,))$, where Q is a non-reflexive Banach space and Z is a finite dimensional subspace of Q''.) Recall, [2], that a Banach space X is *quasi-reflexive* of *order* n if $X'' - j(X)$ is of (finite) dimension n. By simple extensions of the techniques employed above we obtain the following result.

THEOREM 9. *A Banach space X is quasi-reflexive of order n if and only if whenever $A \subseteq F(X)$ is topologically irreducible on X, A^* is topologically irreducible on a closed subspace of X' of finite deficiency k, where the maximum of such k is n.*

We shall now consider two examples of dual pairs of operator algebras in which the irreducibility properties are completely unsymmetrical.

In what follows we denote the set of positive integers by $\mathbf{P}$ and the n-th prime number by p_n. For each $n \in \mathbf{P}$ we denote the usual factorisation of n by $n = \prod p_i^{\alpha_i}$. If $m, n \in \mathbf{P}$, we write $m \mid n$ to denote that m divides n. Given $m, n \in \mathbf{P}$, we denote the highest common factor of m and n by (m, n).

Given $k \in \mathbf{P}$, let z_k be the element of l^∞ defined by

$$z_k(n) = \begin{cases} 1 & \text{if} \quad k \mid n, \\ 0 & \text{if} \quad k \nmid n. \end{cases}$$

Let Z_n be the subspace of l^∞ generated by $z_1, \ldots, z_n$, and let $V_n = {}^0Z_n = \{y : y \in l, (y, Z_n) = (0)\}$.

LEMMA 2. $\bigcap \{V_n : n \in \mathbf{P}\} = (0)$.

Proof. Let Ω denote the Stone–Čech compactification of $\mathbf{P}$, i.e. the Gelfand carrier space of the Banach algebra l^∞, and for each $z \in l^\infty$ let $\tilde{z}$ denote its unique continuous extension to Ω. For all i, j in $\mathbf{P}$, we have $z_i z_j = z_k$, where k is the least common multiple of i and j. Thus the closed real linear hull A of $\{z_i : i \in \mathbf{P}\}$ is a real subalgebra of l^∞. Let $B = \tilde{A}$, and given $\varphi, \psi \in \Omega$ let $\varphi \sim_B \psi$ denote that

$$f(\varphi) = f(\psi) \quad (f \in B).$$

Then by the Stone–Weierstrass theorem, we have

$$B = \{f : f \in C_{\mathbf{R}}(\Omega) \text{ such that } f(\varphi) = f(\psi) \text{ whenever } \varphi \sim_B \psi\}.$$

Given i, j in $\mathbf{P}$ with $j \nmid i$, we have

$$z_j(i) = 0, \quad z_j(j) = 1.$$

Therefore the elements of $\mathbf{P}$ belong to distinct equivalence classes under $\sim_B$; and given $n \in \mathbf{P}$ and real numbers $\lambda_1, \ldots, \lambda_n$, there exists $g \in B$ such that $g(k) = \lambda_k$ $(1 \leqslant k \leqslant n)$. Let

$$\mu = \sup \{|\lambda_k| : 1 \leqslant k \leqslant n\}.$$

Let $h = (g \wedge \mu) \vee (-\mu)$, and $f = h|_{\mathbf{P}}$. Then $f \in A$, $\|f\| = \mu$, and $f(k) = \lambda_k$ $(1 \leqslant k \leqslant n)$.

Let $y \in \bigcap \{V_n : n \in \mathbf{P}\}$, so that $(y, A) = (0)$. Let $r, n \in \mathbf{P}$ with $r \leqslant n$. Let $\lambda_r = -1$, $\lambda_i = 0$ $(1 \leqslant i \leqslant n, i \neq r)$. Let $f \in A$ be as above. Then we have $(y, f) = 0$ and therefore

$$y(r) = \sum_{i=n+1}^{\infty} f(i)\, y(i).$$

Hence,
$$|y(r)| \leqslant \|f\| \sum_{i=n+1}^{\infty} |y(i)|.$$

Since $y \in l$, it follows that $y(r) = 0$ for each r in $\mathbf{P}$ and so $y = 0$.

Remark. The above lemma may also be proved by a combinatorial argument.

Let $A_{\mathbf{P}} = \{T : T \in \mathfrak{B}(c_0),\ T^* V_n \subseteq V_n\ (n \in \mathbf{P})\}$. It is easily verified that $A_{\mathbf{P}}$ is a closed subalgebra of $\mathfrak{B}(c_0)$. Also, given $T \in \mathfrak{B}(c_0)$, we have $T \in A_{\mathbf{P}}$ if and only if $T^{**} Z_n \subseteq Z_n$ $(n \in \mathbf{P})$, where T^{**} denotes the usual second adjoint of T and so belongs to $\mathfrak{B}(l^\infty)$ (with the usual abuse of notation). Given any one-to-one mapping φ of $\mathbf{P}$ into itself, we define T_φ on c_0 by

$$T_\varphi x = x \circ \varphi, \text{ i.e. } (T_\varphi x)(n) = x(\varphi(n)) \quad (n \in \mathbf{P}).$$

Clearly $T_\varphi \in \mathfrak{B}(c_0)$. We shall call φ *admissible* if $T_\varphi \in A_{\mathbf{P}}$. It is easily seen that, given $t \in \mathbf{P}$,

$$(T_\varphi^{**} z_t)(r) = \begin{cases} 1 & \text{if } t \mid \varphi(r), \\ 0 & \text{if } t \nmid \varphi(r). \end{cases}$$

In particular, if there is $s \in \mathbf{P}$ such that $t \mid \varphi(r) \Leftrightarrow s \mid r$, then $T_\varphi^{**} z_t = z_s$. For any φ we thus have $T_\varphi^{**} z_1 = z_1$.

LEMMA 3. *Given $k \in \mathbf{P}$, let $\psi : \mathbf{P} \to \mathbf{P}$ be defined by*

$$\psi(n) = \prod p_{kp_i}^{\alpha_i}, \quad \textit{where } n = \prod p_i^{\alpha_i}.$$

(i) $m \mid n \Leftrightarrow \psi(m) \mid \psi(n)$.

(ii) *Given $n \in \mathbf{P}$ with $n > 1$, either $\{r : n \mid \psi(r)\} = \emptyset$, or there exists $m \in \mathbf{P}$ such that $\{r : n \mid \psi(r)\} = \{r : m \mid r\}$ and $km \leqslant n$.*

Proof. (i). Let $m = \prod p_i^{\beta_i}$, $n = \prod p_i^{\alpha_i}$. Then

$$m \mid n \Leftrightarrow \beta_i \leqslant \alpha_i \quad \text{for all } i \Leftrightarrow \psi(m) \mid \psi(n).$$

(ii). We have $\{r : n \mid \psi(r)\} = \emptyset$ if and only if there exists $t \in \mathbf{P}$ such that $p_t \mid n$ and $t \notin \{kp_i : i \in \mathbf{P}\}$. The remaining integers n are of the form $n = \prod p_{kp_i}^{\alpha_i}$, so that $n = \psi(m)$, where $m = \prod p_i^{\alpha_i}$. Then

$$\{r : n \mid \psi(r)\} = \{r : \psi(m) \mid \psi(r)\} = \{r : m \mid r\}.$$

To see that $km \leqslant n$, it is sufficient to observe that some $\alpha_i \geqslant 1$, and for each i in $\mathbf{P}$ we have $kp_i \leqslant p_{kp_i}$.

LEMMA 4. *For each k in $\mathbf{P}$ there exists an admissible φ in each of the following classes.*

(i) $\varphi(1) = k$.

(ii) *φ is monotonic,* $\varphi(1) = 1$, $\varphi(2) = p_{k+1}$.

(iii) $\varphi(k) = 1$.

Proof. (i). Let $\varphi(s) = ks$ $(s \in \mathbf{P})$. Let $t \in \mathbf{P}$ and let $h = (t, k)$. Then $t = ha$, $k = hb$, with $(a, b) = 1$. Thus

$$\{s : t \mid \varphi(s)\} = \{s : ha \mid hbs\} = \{s : a \mid bs\} = \{s : a \mid s\},$$

since a and b are coprime. We thus have $T_\varphi{}^{**} z_t = z_a$ with $a \leqslant t$. It follows immediately that $T_\varphi{}^{**} Z_n \subseteq Z_n$ $(n \in \mathbf{P})$, so that φ is admissible. We denote the corresponding T_φ by T_k^1, and we note that $(T_k^1 x)(n) = x(nk)$ $(n \in \mathbf{P})$.

(ii). Let

$$\varphi(s) = \prod p_{i+k}^{\alpha_i}, \quad \text{where} \quad s = \prod p_i^{\alpha_i} \in \mathbf{P}.$$

Then φ satisfies the conditions of (ii), and it is clear from the argument of Lemma 3 (i), that $a \mid s \Leftrightarrow \varphi(a) \mid \varphi(s)$. Let $t \in \mathbf{P}$ with $t > 1$. If there does not exist $a \in \mathbf{P}$ with $\varphi(a) = t$, then $\{s : t \mid \varphi(s)\} = \emptyset$, and so $T_\varphi{}^{**} z_t = 0$. If there is $a \in \mathbf{P}$ with $\varphi(a) = t$, then

$$\{s : t \mid \varphi(s)\} = \{s : \varphi(a) \mid \varphi(s)\} = \{s : a \mid s\},$$

and so $T_\varphi{}^{**} z_t = z_a$ with $a \leqslant t$. It is now clear that φ is admissible. We denote the corresponding T_φ by T_k^2, and we note that $(T_k^2 x)(1) = x(1)$, and $(T_k^2 x)(n) = x(\varphi(n))$ where $\varphi(n) \geqslant p_{k+1}$ $(n \geqslant 2)$.

(iii). Let ψ be as in Lemma 3. Let

$$\varphi(s) = \begin{cases} \psi(s/k) & \text{if } k \mid s, \\ k\psi(s) & \text{if } k \nmid s. \end{cases}$$

Since for every i in $\mathbf{P}$, p_{kp_i} is greater than every prime factor of k, φ is a one-to-one mapping of $\mathbf{P}$ into itself. Also $\varphi(k) = 1$. Let $t \in \mathbf{P}$ with $t > 1$. We have $\{s : t \mid \varphi(s)\} = E \cup F$, where

$$E = \{s : k \mid s \,\&\, t \mid \psi(s/k)\},$$

$$F = \{s : k \nmid s \,\&\, t \mid k\psi(s)\} = \{s : t \mid k\psi(s)\} \diagdown \{s : t \mid k\psi(s) \,\&\, k \mid s\}.$$

By Lemma 3 (ii), either $E = \emptyset$, or there exists $b \in \mathbf{P}$ such that $kb \leqslant t$ and

$$E = \{s : k \mid s \ \&\ b \mid s/k\} = \{s : kb \mid s\}.$$

Let $h = (t, k)$, so that $t = hc$, $k = hd$, with $(c, d) = 1$. Then

$$F = \{s : hc \mid hd\psi(s)\} \diagdown \{s : hc \mid hd\psi(s) \ \&\ k \mid s\} = \{s : c \mid \psi(s)\} \diagdown \{s : c \mid \psi(s) \ \&\ k \mid s\}.$$

By Lemma 3 (ii) again, either $F = \emptyset$, or there exists $m \in \mathbf{P}$ such that $km \leqslant c$ and

$$F = \{s : m \mid s\} \diagdown \{s : m \mid s \ \&\ k \mid s\}.$$

Let $q = (m, k)$, so that $m = qu$, $k = qv$, with $(u, v) = 1$. Then

$$\begin{aligned} F &= \{s : m \mid s\} \diagdown \{s : qu \mid s \,\&\, qv \mid s\} \\ &= \{s : m \mid s\} \diagdown \{s : quv \mid s\} \\ &= \{s : m \mid s\} \diagdown \{s : ku \mid s\}. \end{aligned}$$

We have $m \leqslant km \leqslant c \leqslant t$ and $ku \leqslant km \leqslant c \leqslant t$. It is now clear that φ is admissible. We denote the corresponding T_φ by T_k^3, and we note that $T_k^3 e_1 = e_k$, where $\{e_n : n \in \mathbf{P}\}$ denotes the usual basis for c_0.

PROPOSITION 14. *$A_{\mathbf{P}}$ is topologically irreducible on c_0, while $A_{\mathbf{P}}^*$ is not topologically irreducible on any non-zero closed subspace of l.*

Proof. Let x be any non-zero element of c_0. Then there exists $r \in \mathbf{P}$ such that $x(i) = 0$ $(1 \leqslant i < r)$, and $x(r) \neq 0$. Let $T_k = (x(r))^{-1} T_k^2 T_r^1$, so that $T_k \in A_{\mathbf{P}}$ $(k \in \mathbf{P})$. Let φ be as in Lemma 4 (ii). Then

$$\|T_k x - e_1\| = \sup \{(|x(r)|)^{-1} |x(r\varphi(n))| : n \geqslant 2\} \leqslant (|x(r)|)^{-1} \sup \{|x(n)| : n \geqslant p_{k+1}\}.$$

It follows that $T_k x \to e_1$ as $k \to \infty$. Given $y \in c_0$, let $S_k = \sum_{i=1}^k y(i) T_i^3$, so that $S_k \in A_{\mathbf{P}}$ $(k \in \mathbf{P})$. Then

$$\|S_k e_1 - y\| = \|\sum_{i=1}^{k} y(i)\, e_i - y\|,$$

so that $S_k e_1 \to y$ as $k \to \infty$. It follows easily that $A_{\mathbf{P}}$ is topologically irreducible on c_0.

Suppose that $A_{\mathbf{P}}^*$ is topologically irreducible on some non-zero closed subspace V of l. By Proposition 11, $(c_0, V, (\,,))$ are Banach spaces in normed duality and so V is infinite dimensional. We thus have $V \cap V_n \neq (0)$ $(n \in \mathbf{P})$, for otherwise V would be finite dimensional. Let v_n be any non-zero element of $V \cap V_n$. Then $V = \overline{A_{\mathbf{P}}^* v_n} \subseteq V_n$. We thus have $V \subseteq V_n$ $(n \in \mathbf{P})$ and so $V = (0)$ by Lemma 2. This contradiction completes the proof.

It is not known whether or not $A_{\mathbf{P}}$ is strictly irreducible on c_0. To obtain an example in which strict irreducibility obtains we proceed as follows. Let $X = Y = l$, and

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x(n)\, y(n) \quad (x, y \in l).$$

It is easily verified that $(X, Y, \langle\,,\rangle)$ is a pair of Banach spaces in normed duality. As above, let φ be any one-to-one mapping of $\mathbf{P}$ into itself, and define T_φ on l by $T_\varphi x = x \circ \varphi$ $(x \in l)$. Define φ^* on $\mathbf{P}$ by

$$\varphi^*(n) = \begin{cases} \varphi^{-1}(n) & \text{if } n \in \varphi(\mathbf{P}), \\ 0 & \text{if } n \notin \varphi(\mathbf{P}). \end{cases}$$

We then have $T_\varphi \in \mathfrak{B}(l, l, \langle\,,\rangle)$ with $T_\varphi^* = T_{\varphi^*}$. We now regard the operators T_k^1, T_k^2, T_k^3 $(k \in \mathbf{P})$ of Lemma 4 as elements of $\mathfrak{B}(l, l, \langle\,,\rangle)$. These operators generate a countable family of finite products, $\{T_{\varphi_n} : n \in \mathbf{P}\}$, say. We thus have $T_{\varphi_n} \in \mathfrak{B}(l, l, \langle\,,\rangle)$ $(n \in \mathbf{P})$, and, since $T_\varphi T_\psi = T_{\psi \circ \varphi}$, it is not difficult to see that

$$|T_{\varphi_n}| = |T_{\varphi_n}^*| = 1 \quad (n \in \mathbf{P}).$$

Given $x \in l$, let

$$T_x = \sum_{n=1}^{\infty} x(n)\, T_{\varphi_n},$$

and let $B_{\mathbf{P}}$ be the image of l under the mapping $x \to T_x$. It follows simply from Proposition 7 that $B_{\mathbf{P}} \subseteq \mathfrak{B}(l, l, \langle\,,\rangle)$. Since $T_{\varphi_n}^* V_k \subseteq V_k$ $(n, k \in \mathbf{P})$, we also have $T_x^* V_k \subseteq V_k$ $(k \in \mathbf{P},\, x \in l)$. We define a second norm on $B_{\mathbf{P}}$ by

$$\|T_x\| = \inf \{\|y\| : y \in l,\ T_y = T_x\}.$$

PROPOSITION 15. (i) *$B_{\mathbf{P}}$ is a Banach algebra with unit under $\|\cdot\|$.* (ii) *$B_{\mathbf{P}}$ is strictly irreducible on l, while $B_{\mathbf{P}}^*$ is not strictly irreducible on any non-zero subspace of l.*

Proof. (i). It is clear that $x \to T_x$ is a linear homomorphism of l on to $B_{\mathbf{P}}$ such that $|T_x| \leqslant \|x\|$ $(x \in l)$. The kernel $N = \{x : T_x = 0\}$ is thus a closed subspace of l, so that $l - N$ is a Banach space under the infimum norm. The norm $\|\cdot\|$ on $B_{\mathbf{P}}$ is precisely this infimum norm transferred to $B_{\mathbf{P}}$. Thus $B_{\mathbf{P}}$ is a Banach space under $\|\cdot\|$, and $|T_x| \leqslant \|T_x\|$ $(x \in l)$.

Next, $$T_x T_y = \sum_{n=1}^{\infty} x(n)\, T_{\varphi_n} \sum_{n=1}^{\infty} y(n)\, T_{\varphi_n} = \sum_{m,n=1}^{\infty} x(n)\, y(m)\, T_{\varphi_m \circ \varphi_n}.$$

Since, $$\sum_{m,n=1}^{\infty} |x(n)\, y(m)| \leqslant \sum_{n=1}^{\infty} |x(n)| \sum_{m=1}^{\infty} |y(m)|,$$

it follows that $T_x T_y \in B_{\mathrm{P}}$. Also, $\|T_x T_y\| \leqslant \|x\| \|y\|$, and so $\|T_x T_y\| \leqslant \|T_x\| \|T_y\|$. Finally, $I = T_1^1 \in B_{\mathrm{P}}$ and we easily see that $\|I\| = 1$. Thus B_{P} is a Banach algebra with unit under $\|\cdot\|$.

(ii). If we now argue as in Proposition 14 with c_0 replaced by l, we see that B_{P} is topologically irreducible on l. But in this case, e_1 is a strictly cyclic vector, for if $y \in l$, then

$$T = \sum_{n=1}^{\infty} y(n)\, T_n^3 \in B_{\mathrm{P}} \quad \text{and} \quad Te_1 = \lim_{n \to \infty} \sum_{i=1}^{n} y(i)\, T_i\, e_1 = \lim_{n \to \infty} \sum_{i=1}^{n} y(i)\, e_i = y.$$

By Proposition 2, B_{P} is thus strictly irreducible on l. By a slight modification of the argument of Proposition 14 we see that B_{P}^* is not strictly irreducible on any non-zero subspace of l.

Remarks. (1) We note that B_{P} is a left primitive Banach algebra. It is still an open question as to whether or not B_{P} is right primitive. (2) We observe that B_{P} admits a dual representation on $(l, c_0, (\,,))$ with associated dual pair (A, B) such that A is strictly irreducible on l and B is topologically, but not strictly, irreducible on c_0. This follows immediately from Proposition 12 when we note that each $T_x \in B_{\mathrm{P}}$ has an adjoint on c_0 with respect to the natural bilinear form $(\,,)$, and that the proper subspace l of c_0 is invariant for the adjoint algebra. This observation also shows how far removed topological irreducibility may be from strict irreducibility. In fact the adjoint algebra is topologically irreducible on c_0 and yet has a chain of invariant subspaces with zero intersection.

We close this section with a question. For which Banach spaces X does the following statement hold?

"If $A \subseteq \mathfrak{B}(X)$ is topologically irreducible on X, then A^* is topologically irreducible on some non-zero closed subspace of X'."

6. Uniformly transitive representations of Banach algebras

It is well known that any topologically irreducible *-representation of a B*-algebra is automatically strictly irreducible. We have seen that a dually topologically irreducible representation of a Banach algebra need not be dually strictly irreducible. In fact, by Remark (2) after Proposition 15, the Banach algebra B_{P} admits a dual representation $a \to T_a$ on a pair $(X, X', (\,,))$ such that $a \to T_a$ is topologically, but not strictly, irreducible on X, and $a \to T_a^*$ is strictly irreducible on X'. In this section we strengthen even further the concept of irreducibility, and for this concept we obtain dual irreducibility for dual representations on pairs of the form $(X, X', (\ ,\ ))$.

Let $a \to T_a$ be a representation of a Banach algebra $\mathfrak{A}$ on a Banach space X. Let $\mathfrak{A}_1$, X_1 denote the closed unit balls in $\mathfrak{A}$, X. Let $u \in X$, $u \neq 0$, and let $\alpha > 0$.

(i) We say that u is *pointwise boundedly topologically cyclic* (*p.b.t.c.*) if $\overline{T_{\mathfrak{A}_1} u}$ is absorbent, i.e. for each $x \in X$, there exists a bounded sequence $\{a_n\}$ in $\mathfrak{A}$ such that $T_{a_n} u \to x$.

(ii) We say that u is *uniformly topologically cyclic* with *bound* α (α-*u.t.c.*) if $X_1 \subseteq \alpha \overline{T_{\mathfrak{A}_1} u}$, i.e. for each $x \in X_1$ and each $\varepsilon > 0$, there exists $a \in \alpha \mathfrak{A}_1$ such that $\|T_a u - x\| < \varepsilon$.

(iii) We say that u is *uniformly strictly cyclic* with *bound* α (α-*u.s.c.*) if $X_1 \subseteq \alpha T_{\mathfrak{A}_1} u$, i.e. for each $x \in X_1$, there exists $a \in \alpha \mathfrak{A}_1$ with $T_a u = x$.

(iv) We say that $a \to T_a$ is *uniformly topologically transitive* with *bound* α (α-*u.t.t.*) if each $x \in X$ with $\|x\| = 1$ is α-*u.t.c.*

(v) We say that $a \to T_a$ is *uniformly strictly transitive* with *bound* α (α-*u.s.t.*) if each $x \in X$ with $\|x\| = 1$ is α-*u.s.c.*

If $a \to T_a$ is α-*u.s.t.* (α-*u.t.t.*), then evidently $a \to T_a$ is strictly (topologically) irreducible on X. It is also clear that if u is α-*u.s.c.*, then u is α-*u.t.c.* Further, if u is α-*u.t.c.*, then u is *p.b.t.c.* In fact, these three conditions on u are almost equivalent as the next two propositions indicate.

PROPOSITION 16. *Let u be p.b.t.c. for $a \to T_a$. Then u is α-u.t.c. for some $\alpha > 0$.*

Proof. This is a straightforward application of the Baire category theorem.

PROPOSITION 17. *Let u be α-u.t.c. for $a \to T_a$. Then u is $(\alpha + \varepsilon)$-u.s.c. for every $\varepsilon > 0$.*

Proof. This follows readily by the method employed in [5] Theorem 4.9.10.

COROLLARY 1. *If u is strictly cyclic for $a \to T_a$, then u is α-u.s.c. for some $\alpha > 0$.*

Proof. If u is strictly cyclic, then u is clearly *p.b.t.c.* and so the result follows from Propositions 16 and 17.

COROLLARY 2. *If $a \to T_a$ is α-u.t.t., then $a \to T_a$ is $(\alpha + \varepsilon)$-u.s.t. for every $\varepsilon > 0$.*

PROPOSITION 18. *Let $\mathfrak{A}$ be a closed subalgebra of $\mathfrak{B}(X)$ such that $\alpha \mathfrak{A}_1$ is dense in the closed unit ball of $\mathfrak{B}(X)$ with respect to the weak operator topology. Then $\mathfrak{A}$ is $(\alpha + \varepsilon)$-u.s.t. for every $\varepsilon > 0$.*

Proof. Routine.

PROPOSITION 19. *Let H be a complex Hilbert space, and let $\mathfrak{A}$ be a strictly irreducible self-adjoint closed subalgebra of $\mathfrak{B}(H)$. Then $\mathfrak{A}$ is $(1+\varepsilon)$-u.s.t. for every $\varepsilon > 0$.*

Proof. By a theorem of Kaplansky (see [5], Theorem 4.9.10), the unit ball of $\mathfrak{A}$ is dense in the unit ball of $\mathfrak{B}(H)$ in the strong operator topology. The result follows easily.

We have already seen (Proposition 12) that if the anti-representation $a \to T_a^*$ of $\mathfrak{A}$ on X' is topologically irreducible, then the representation $a \to T_a$ of $\mathfrak{A}$ on X is topologically irreducible. If $a \to T_a^*$ is strictly irreducible on X', then $a \to T_a$ need not be strictly irreducible on X as the Banach algebra B_P shows. The situation is more satisfactory for uniformly strictly transitive representations.

THEOREM 10. *Let $a \to T_a$ be a representation of $\mathfrak{A}$ on X such that $a \to T_a^*$ is α-u.s.t. on X'. Then $a \to T_a$ is $(\alpha^2+\varepsilon)$-u.s.t. on X for every $\varepsilon > 0$.*

Proof. Given $u \in X$, $u \neq 0$, $\|u\| \leqslant 1$, and $\beta > 0$, let

$$E_\beta(u) = \beta \overline{T_{\mathfrak{A}_1} u}, \quad p(u) = \sup\{\|T_a u\| : a \in \mathfrak{A}_1\}.$$

We have
$$p(u) \geqslant \alpha^{-1}\|u\|. \tag{*}$$

For there exists $f \in X'$ such that $\|f\| = 1$ and $f(u) = \|u\|$. Since $a \to T_a^*$ is α-u.s.t. on X', there is $a \in \alpha\mathfrak{A}_1$ such that $T_a^* f = f$. Then $\alpha^{-1}a \in \mathfrak{A}_1$, and so

$$p(u) \geqslant \|\alpha^{-1}T_a u\| \geqslant \alpha^{-1}(T_a u, f) = \alpha^{-1}(u, T_a^* f) = \alpha^{-1}(u, f) = \alpha^{-1}\|u\|.$$

We prove next that if there exists $y \in X_1 \setminus E_{\beta\alpha}(u)$, then

$$p(u) \leqslant \beta^{-1}. \tag{**}$$

In fact, given such y, since $E_{\beta\alpha}(u)$ is a closed convex set, there exists $f \in X'$ with $\|f\| = 1$, such that

$$\operatorname{Re} f(x) \leqslant \operatorname{Re} f(y) \quad (x \in E_{\beta\alpha}(u)).$$

Given $x \in E_1(u)$ and $\varphi \in X'$ with $\|\varphi\| = 1$, there exists $b \in \alpha\mathfrak{A}_1$ such that $T_b^* f = \varphi$. Then

$$\beta(x, \varphi) = \beta(x, T_b^* f) = (\beta T_b x, f).$$

Since $\beta b\mathfrak{A}_1 \subseteq \beta\alpha\mathfrak{A}_1$, we have $\beta T_b x \in E_{\beta\alpha}(u)$, and thus

$$\operatorname{Re} \beta\varphi(x) \leqslant \operatorname{Re} f(y) \leqslant 1.$$

Since this holds for every $\varphi \in X'$ with $\|\varphi\| = 1$, we have

$$\beta\|x\| \leqslant 1 \quad (x \in E_1(u)).$$

Finally, since $T_a u \in E_1(u)$ $(a \in \mathfrak{A}_1)$, this proves (**).

Combining (*) and (**), we see that if $X_1 \nsubseteq E_{\beta\alpha}(u)$, then $\|u\| \leqslant \beta^{-1}\alpha$. Thus, whenever $\|u\| = 1$ and $\beta > \alpha$, we have $X_1 \subseteq E_{\beta\alpha}(u)$, i.e. u is $\beta\alpha$-*u.t.c.* By Proposition 17, u is then $(\alpha^2 + \varepsilon)$-*u.s.c.* for every $\varepsilon > 0$, and the result follows.

COROLLARY. *Let $a \to T_a$ be an α-u.s.t. representation of $\mathfrak{A}$ on a reflexive Banach space X. Then $a \to T_a$ is dually strictly irreducible on $(X, X', (\,,\,))$.*

Remark. Let $a \to T_a$ be the dual representation of B_P on $(c_0, c_0', (\,,\,))$ given in Remark (2) after Proposition 15. Then $a \to T_a^*$ is strictly irreducible on c_0', but it follows from the above theorem that $a \to T_a^*$ is not α-*u.s.t.* on c_0' for any $\alpha > 0$.

7. The dual radical

The *dual radical* of a Banach algebra $\mathfrak{A}$ is defined to be the intersection of the kernels of all dually strictly irreducible dual representations of $\mathfrak{A}$. We denote the dual radical by R_d and we say that $\mathfrak{A}$ is dually semi-simple if $R_d = (0)$.

Let R denote the Jacobson radical of $\mathfrak{A}$. We denote by R_p the intersection of all the ideals of $\mathfrak{A}$ which are both left and right primitive. It is easily seen that R_p is also a "radical" in that $\mathfrak{A}/R_p$ is "semi-simple" in the corresponding sense.

THEOREM 11. *Let $\mathfrak{A}$ be a Banach algebra, and let Ω be the set of all appropriate functionals in $\mathfrak{A}'$ with norm one.*

(i) $R_d = \bigcap\{P_f : f \in \Omega\} = \bigcap\{L_f : f \in \Omega\} = \bigcap\{K_f : f \in \Omega\}$.

(ii) $R \subseteq R_p \subseteq R_d$.

(iii) *$\mathfrak{A}/R_d$ is dually semi-simple.*

Proof. Routine.

It is still an open question as to whether there are Banach algebras in which the radicals R, R_p, R_d are distinct. It would also be interesting to have more intrinsic algebraic and topological characterisations of the dual radical.

We say that $\mathfrak{A}$ is *dually primitive* if it admits a faithful dually strictly irreducible dual representation. It is clear from Theorem 7 and the argument of [5] Theorem 2.6.1 that every dually semi-simple Banach algebra is continuously isomorphic with a normed sub-direct sum of dually primitive Banach algebras. It is also of interest to give an operator representation as in the next theorem.

THEOREM 12. *Let $\mathfrak{A}$ be a dually semi-simple Banach algebra. Then there exists a faithful dual representation $a \to T_a$ on a pair $(X, Y, \langle , \rangle)$ such that $|T_a| \leq \|a\|$, $|T_a^*| \leq \|a\|$ $(a \in \mathfrak{A})$.*

Proof. Since $\mathfrak{A}$ is dually semi-simple there exists $\Omega \subseteq \mathfrak{A}'$ such that $\|f\| = 1$ $(f \in \Omega)$ and $\bigcap \{P_f : f \in \Omega\} = (0)$. Let X be the normed sub-direct sum of $\sum \{X_f : f \in \Omega\}$ consisting of all functions x on Ω such that $x(f) \in X_f$ $(f \in \Omega)$ and $\|x\| = \sum \{\|x(f)\| : f \in \Omega\} < \infty$. It is easily seen that X is a Banach space. Let Y be the normed sub-direct sum of $\sum \{Y_f : f \in \Omega\}$ consisting of all functions y on Ω such that $y(f) \in Y_f$ $(f \in \Omega)$ and $\|y\| = \sup \{\|y(f)\| : f \in \Omega\} < \infty$. It is also easily seen that Y is a Banach space. Let

$$\langle x, y \rangle = \sum \{\langle x(f), y(f) \rangle_f : f \in \Omega\} \quad (x \in X, y \in Y).$$

We then have $|\langle x, y \rangle| \leq \|x\| \|y\|$ and it follows simply that X and Y are in normed duality with respect to $\langle , \rangle$.

For each $a \in \mathfrak{A}$ we define T_a, S_a as follows.

$$(T_a x)(f) = T_a^f x(f) \quad (f \in \Omega,\ x \in X).$$

$$(S_a y)(f) = S_a^f y(f) \quad (f \in \Omega,\ y \in Y).$$

The rest of the proof is straightforward.

8. Examples

The first part of the following theorem states that any left primitive complex Banach algebra with minimal one-sided ideals is dually primitive. The result is well known, only the terminology is new.

THEOREM 13. *Let $\mathfrak{A}$ be a left primitive complex Banach algebra with minimal one-sided ideals.*

(i) *$\mathfrak{A}$ admits a faithful dually strictly irreducible dual representation $a \to T_a$ on some pair $(X, Y, \langle , \rangle)$ such that the image of $\mathfrak{A}$ under $a \to T_a$ contains all operators of the form $x \otimes y$ $(x \in X, y \in Y)$.*

(ii) *Let L be any maximal modular left ideal with $L : \mathfrak{A} = (0)$, and K any maximal modular right ideal with $K :' \mathfrak{A} = (0)$. Then there exist $x \in X$, $y \in Y$ such that*

$$L = \{a : T_a x = 0\}, \quad K = \{a : T_a^* y = 0\},$$

and $\mathfrak{A}-L$, $\mathfrak{A}-K$ are bicontinuously isomorphic with X, Y respectively. Further

$$L+K=\overline{L+K}=\{a:\langle T_a x, y\rangle=0\},$$

and so is a maximal proper linear subspace of $\mathfrak{A}$.

Proof. (i). This is [5] Theorem 2.4.12.

(ii). The first part follows by standard arguments.

It is clear that

$$\overline{L+K}\subseteq\{a:\langle T_a x, y\rangle=0\}.$$

Suppose that $\langle T_a x, y\rangle=0$. If $T_a x=0$, then $a\in L\subseteq L+K$. If $T_a x\neq 0$, we may choose $y_1\in Y$ such that $\langle T_a x, y_1\rangle=1$. By (i), there exists $c\in\mathfrak{A}$ such that $T_c=T_a x\otimes y_1$. We have

$$T_{ca-a}x=T_cT_ax-T_ax=(T_ax\otimes y_1)(T_ax)-T_ax=0.$$

Therefore $ca-a\in L$. Since also $T_c^*y=\langle T_a x, y\rangle y_1=0$, we have $c\in K$. Thus $ca\in K$ and so $a\in L+K$. It is now immediate that

$$L+K=\overline{L+K}=\{a:\langle T_a x, y\rangle=0\},$$

and so is a maximal proper linear subspace, being the null space of a continuous linear functional.

THEOREM 14. *Let L be a maximal modular left ideal of a Banach algebra $\mathfrak{A}$ such that $r(L)=\{a:La=(0)\}\nsubseteq R$. Then there exists $f\in\mathfrak{A}'$ such that $L_f=L$ and K_f is a maximal modular right ideal.*

Proof. Since $R=R:\mathfrak{A}$, $\mathfrak{A}r(L)\nsubseteq R$, and so there exists a maximal modular right ideal K such that $\mathfrak{A}r(L)\nsubseteq K$. Hence there exist $a\in\mathfrak{A}$ and $u\in r(L)$ such that $au\notin K$. By the Hahn-Banach theorem there exists $g\in\mathfrak{A}'$ with $g(K)=(0)$ and $g(au)=1$. Let $f(x)=g(xu)$ $(x\in\mathfrak{A})$, so that $f\in\mathfrak{A}'$ and $f(a)=1$. Since $Lu=(0)$, we have $f(L)=(0)$ and so $L_f=L$ by Lemma 1. Also $x\in K$ implies $xu\in K$ so that $f(x)=g(xu)=0$ $(x\in K)$. By the analogue of Lemma 1 for right ideals, we have $K_f=K$ and the proof is complete.

COROLLARY. *Let $\mathfrak{A}$ be a Banach algebra with a family $\{L_\lambda:\lambda\in\Lambda\}$ of maximal modular left ideals such that $r(L_\lambda)\neq(0)$ $(\lambda\in\Lambda)$ and $\bigcap\{L_\lambda:\lambda\in\Lambda\}=(0)$. Then $\mathfrak{A}$ is dually semi-simple.*

We turn finally to complex Banach *-algebras. Recall that a *-representation is made on a normed self-dual space X (see [5], Definition 4.3.1). Recall also that $F\in\mathfrak{A}'$ is Hermitian if

$$F(a^*) = \overline{F(a)} \quad (a \in \mathfrak{A}).$$

Associated with F, there is the dual representation $a \to T_a^F$ on $(X_F, Y_F, \langle\, , \rangle_F)$. There is also the *-representation $a \to T_a^F$ on $(X_F, X_F, (\, ,)_F)$, where

$$(x', y')_F = F(y^*x) \quad (x \in x' \in X_F,\ y \in y' \in X_F).$$

There is a natural conjugate linear isomorphism U from X_F on to Y_F. If the involution is continuous, then U is bicontinuous, but if the involution is not continuous, then U need not be continuous and so the representations might be quite different topologically.

We point out that there are dually semi-simple Banach *-algebras for which no appropriate functional is Hermitian. The next result shows, however, that the condition for the existence of appropriate functionals is simplified when the functional is Hermitian.

THEOREM 15. *Let L be a maximal modular left ideal in a Banach *-algebra $\mathfrak{A}$. Let $F \in \mathfrak{A}'$ be such that $F \neq 0$, $F(L) = (0)$, and F is Hermitian. Then F is appropriate for $L:\mathfrak{A}$.*

Proof. We have $L_F = L$ by Lemma 1, and $K_F = L_F^*$.

We remark that the above proof requires only the weaker Hermitian condition that $F(xy^*) = F(y^*x)$ $(x, y \in \mathfrak{A})$. If $F \in \mathfrak{A}'$ is positive, i.e. $F(x^*x) \geqslant 0$ $(x \in \mathfrak{A})$, then this condition is automatically satisfied. It follows immediately from [5] Theorem 4.7.14 that if $\mathfrak{A}$ is a symmetric Banach *-algebra with locally continuous involution, then $R_d = R$. In particular, any B*-algebra is dually semi-simple.

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