H. J. YOO
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DUAL SPACES OF RESTRICTIONS IN THE REPRODUCING KERNEL HILBERT SPACES IN DISCRETE SETS

Hyun Jae Yoo*

Abstract

We characterize the dual spaces of restrictions of a dual pair of reproducing kernel Hilbert spaces in a discrete set. Consequently, we give a canonical dense subset to the restriction spaces. As applications, we reprove a variational principle in a dual pair of reproducing kernel Hilbert spaces. Also we give a geometric representation for the existence and ergodicity condition of equilibrium Glauber and Kawasaki dynamics for some determinantal point processes.

1. Introduction

In this paper we discuss the linear function spaces on discrete sets. Given a countable set E, we define a dual pair of reproducing kernel Hilbert spaces with a priori given kernel functions. We are interested in the restrictions of the functions to any subsets of E. The restriction theory for the reproducing kernel Hilbert spaces (in short RKHS's) is well explained by Aronszajn in [1]. Nevertheless, we will further investigate, in particular, the dual spaces of the restrictions. Though RKHS's are Hilbert spaces themselves, in many aspects their behavior is not so apparent as much as that of the usual Hilbert space $l^2(E)$. For instance, any restriction of a vector in $l^2(E)$ to a subset of E may be regarded as an element of $l^2(E)$, but it is not all the case for RKHS's (see an example in Section 5). Therefore, some problems, although obvious in the $l^2(E)$-theory, are not easy to see the result. In Section 3 we discuss one such a problem.

One more motivation for this study came from a construction of the equilibrium dynamics which leave invariant a priori given a probability measure. To say little more concretely, the kernel operator used in the RKHS's in this paper will define a certain determinantal point process, which is a probability measure on the configuration space with state space E. We want to construct the so called Glauber and Kawasaki dynamics with the determinantal point

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process being a symmetrizing measure [2]. For the construction of Fellerian Markov process, and further, to discuss the ergodicity of the dynamics, the theory of RKHS's and the restriction theories play important roles. In Section 3, we discuss some part of them.

We organize this paper as follows. In Section 2, we introduce a basic construction method for the dual pair of RKHS's and then state the main results. In Section 3, we discuss above mentioned applications. Section 4 is devoted to the proofs. In the final Section 5, we discuss an open problem.

2. Preliminaries and results

In this Section we briefly introduce the reproducing kernel Hilbert spaces and give the main results. We first recall the definition of RKHS's from ref. [1].

A (complex) Hilbert space $\mathscr{H}$ consisting of functions on a set E and equipped with an inner product $(\cdot,\cdot)$ (assumed linear for the second argument) is called a reproducing kernel Hilbert space with reproducing kernel (shortly RK), say $K(x,y)$, $x,y \in E$, if

(i) For every $x \in E$, the function $K(\cdot,x)$ belongs to $\mathscr{H}$;
(ii) The reproducing property: for every $x \in E$ and $f \in \mathscr{H}_+$, $f(x) = (K(\cdot,x), f)$.

In this paper we deal only with discrete spaces. Thus from now on we let E be any fixed countable set and let $\mathscr{H}_0 := l^2(E)$ be the Hilbert space of square summable functions (sequences) on E equipped with the usual inner product:

$$(f,g)_0 := \sum_{x \in E} \overline{f(x)} g(x), \quad f,g \in \mathscr{H}_0. \tag{2.1}$$

Let A be any positive definite, bounded linear operator on $\mathscr{H}_0$. Notice that A is a Hermitian operator. We assume that $\operatorname{Ker} A = \{0\}$, thus $\operatorname{Ran} A$ is dense in $\mathscr{H}_0$. Let $B := \{e_x : x \in E\}$ be the usual basis of $\mathscr{H}_0$, i.e., $e_x \in \mathscr{H}_0$ is the unit vector whose component is 1 at x and 0 at all other sites.

We define two additional norms on $\mathscr{H}_0$ and on the range of A, respectively. First on $\mathscr{H}_0$, we define a new inner product $(\cdot,\cdot)_-$ as follows:

$$(f,g)_- := (f, Ag)_0, \quad f,g \in \mathscr{H}_0. \tag{2.2}$$

On $\operatorname{Ran} A$, we define another inner product $(\cdot,\cdot)_+$ by

$$(f,g)_+ := (f, A^{-1}g)_0, \quad f,g \in \operatorname{Ran} A. \tag{2.3}$$

Let us denote by $\|\cdot\|_-$ and $\|\cdot\|_+$ the corresponding induced norms. Finally, let $\mathscr{H}_-$ be the completion of $\mathscr{H}_0$ w.r.t. $\|\cdot\|_-$ and $\mathscr{H}_+$ the completion of $\operatorname{Ran} A$ w.r.t. $\|\cdot\|_+$. Then we obtain the following rigging of Hilbert spaces.

$$\mathscr{H}_- \supset \mathscr{H}_0 \supset \mathscr{H}_+. \tag{2.4}$$

Let $A(x,y)$, $x,y \in E$, be the matrix elements of A w.r.t. the basis $\{e_x\}_{x \in E}$:

$$A(x,y) := (e_x, Ae_y)_0, \quad x,y \in E. \tag{2.5}$$

It is easily seen that $\mathscr{H}_+$ is a reproducing kernel Hilbert space with reproducing kernel $A(x,y)$. On the other hand, it should be noted that some of the elements of $\mathscr{H}_-$ may not be represented as functions on E in general. This is so called a functional completion problem [1] and we will assume the following:

Hypothesis (H): We suppose that $\mathscr{H}_-$ is functionally completed, i.e., any vector of $\mathscr{H}_-$ can be represented as a function on E.

In [4], we gave some sufficient conditions on the operator A so that the above hypothesis is satisfied. Now the space $\mathscr{H}_-$ being functionally completed, $\mathscr{H}_-$ itself is a reproducing kernel Hilbert space. For this fact we refer to [1, p 343 and p 347]. Denote the RK of $\mathscr{H}_-$ by $B(x,y)$, $x, y \in E$. Formally $B = A^{-1}$, which is not a bounded operator in general.

The main merit of the rigging in (2.4) is that the spaces $\mathscr{H}_-$ and $\mathscr{H}_+$ are the dual spaces to each other [4, Proposition 2.2]. The purpose of this paper is to characterize the dual spaces of the restrictions of the RKHS's $\mathscr{H}_-$ and $\mathscr{H}_+$. For this purpose, we briefly recall the restriction theory for RKHS's from the reference [1, Section 5, Part I].

Let $\mathscr{H}$ be any RKHS (on E) with RK $K(x,y)$. Let $R \subset E$ be any (finite or infinite) subset of E, and let $K_R(x,y)$, $x, y \in R$, denote the restriction of K to the set $R \times R$. As $K(x,y)$ is a positive definite function, and the same is true for the restriction $K_R(x,y)$, the kernel $K_R(x,y)$ itself is a unique RK for a RKHS on the set R, which we denote by $\mathscr{H}_{R,K_R}$ [1]. It turns out that $\mathscr{H}_{R,K_R}$ is in fact the restriction space of $\mathscr{H}$ to the set R. That is, $\mathscr{H}_{R,K_R}$ consists of all functions $f : R \to \mathbf{C}$ such that there is a vector $\tilde{f} \in \mathscr{H}$ with

$$\pi_R \tilde{f} = f, \tag{2.6}$$

where π_R is the restriction operator on the function space on E to the function space on R defined by

$$\pi_R f(x) = f(x), \quad x \in R,$$

for any function f on E. The norm of $\mathscr{H}_{R,K_R}$ is defined by

$$\|f\|_{R,K_R} := \inf\{\|\tilde{f}\|_K : \pi_R \tilde{f} = f\}, \tag{2.7}$$

where $\|\cdot\|_K$ is the norm for $\mathscr{H}$. We notice that for any $f \in \mathscr{H}_{R,K_R}$, there is a (unique) $f' \in \mathscr{H}$ s.t. $\pi_R f' = f$ and

$$\|f\|_{R,K_R} = \|f'\|_K. \tag{2.8}$$

We refer to [1, Part I, Section 5] for the details. By (2.7) and (2.8), we see that the operator $\pi_R : (\mathscr{H}, \|\cdot\|_K) \to (\mathscr{H}_{R,K_R}, \|\cdot\|_{R,K_R})$ is bounded and the operator norm is 1.

Recall that $\mathscr{H}_+$ and $\mathscr{H}_-$ are RKHS's with RK's A and B, respectively. Given any subset $R \subset E$, we denote the restriction spaces of $\mathscr{H}_+$ and $\mathscr{H}_-$ to the set R by $\mathscr{H}_{R,A_R}$ and $\mathscr{H}_{R,B_R}$, respectively. We would like to characterize the dual spaces of them. As usual $l^2(R)$ denotes the space of square summable functions on R. Since we have $\mathscr{H}_0 \equiv l^2(E) \supset \mathscr{H}_+$, the restriction space $\mathscr{H}_{R,A_R}$ is a subspace of $l^2(R)$:

$$l^2(R) \supset \mathscr{H}_{R,A_R}. \tag{2.9}$$

As an inverse operation to π_R, we let $\imath_R$ be the embedding operation mapping a function f on R to the function on E as follows:

$$\imath_R f(x) = \begin{cases} f(x), & x \in R, \\ 0, & x \in E \backslash R. \end{cases} \tag{2.10}$$

Since $\imath_R l^2(R) \subset \mathscr{H}_0 \subset \mathscr{H}_-$, we see that

$$\mathscr{H}_{R,B_R} \supset l^2(R). \tag{2.11}$$

We want to first characterize the dual space $\mathscr{H}'_{R,B_R}$ of $\mathscr{H}_{R,B_R}$. It is shown in [4, p 337] that there is a positive definite bounded linear operator B_R^{-1} on $l^2(R)$ s.t.

$$(f,f)_{R,B_R} = (f, B_R^{-1} f)_0, \quad f \in l^2(R), \tag{2.12}$$

where $(\cdot\,,\cdot)_{R,B_R}$ is the inner product for $\mathscr{H}_{R,B_R}$ and, by abuse of notation, $(\cdot\,,\cdot)_0$ is the usual inner product in $l^2(R)$. For each $f \in l^2(R)$, since $\imath_R f \in \mathscr{H}_0 \subset \mathscr{H}_-$ and $\pi_R(\imath_R f) = f$ we see that

$$\begin{aligned} (f, B_R^{-1} f)_0 &= (f,f)_{R,B_R} \\ &\leq (\imath_R f, \imath_R f)_- \\ &= (\imath_R f, A\imath_R f)_0 \\ &= (f, A_R f)_0. \end{aligned}$$

Therefore we get

$$B_R^{-1} \leq A_R. \tag{2.13}$$

By (2.12) and the polarization identity we see that the matrix components of $B_R^{-1}(x,y)$ is given by

$$B_R^{-1}(x,y) = (e_x, B_R^{-1} e_y)_0 = (e_x, e_y)_{R,B_R}, \quad x, y \in R. \tag{2.14}$$

Let us denote by $\mathscr{H}_{R,B_R^{-1}}$ the RKHS on R with RK $B_R^{-1}(x,y)$. The first characterization result is the following, which extends [4, Lemma 3.6]:

Theorem 2.1. *Let the hypothesis* (H) *be satisfied. Then for any* $R \subset E$, *we have*

$$\mathscr{H}'_{R,B_R} = \mathscr{H}_{R,B_R^{-1}} = \pi_R(\imath_R(l^2(R)) \cap \mathscr{H}_+).$$

We notice that the space $\imath_R(l^2(R)) \cap \mathscr{H}_+$ consists of elements $\mathscr{H}_+$ that is supported on R.

In order to characterize $\mathscr{H}'_{R,A_R}$, we introduce a notation. For any subset $S \subset E$, we define

$$\mathsf{F}_0(S) := \{f \in \mathscr{H}_- : f(x) = 0 \text{ for all } x \in S\}. \tag{2.15}$$

As in the case for $\mathscr{H}_{R,B_R}$, let $A_R^{-1}(x,y)$ be the kernel function on R defined by

$$A_R^{-1}(x,y) := (e_x, e_y)_{R,A_R}, \quad x, y \in R, \tag{2.16}$$

where $(\cdot\,,\cdot)_{R,A_R}$ denotes the inner product in $\mathscr{H}_{R,A_R}$. The inner product in (2.16) is well defined because $e_x \in \mathscr{H}_+$ for all $x \in E$ (see [4, Proposition 2.2]) and $\|e_x\|_{R,A_R} \le \|e_x\|_+$ for each $x \in E$.

Obviously $A_R^{-1}(x,y)$, $x, y \in R$, is a positive definite function and we let $\mathscr{H}_{R,A_R^{-1}}$ be the RKHS with RK $A_R^{-1}(x,y)$. The second characterization is as follows:

Theorem 2.2. *Under the hypothesis* (H), *for any* $R \subset E$ *we have*

$$\mathscr{H}'_{R,A_R} = \mathscr{H}_{R,A_R^{-1}} = \pi_R(\mathsf{F}_0(R^c)).$$

Moreover, the above spaces are equal to (*the restriction of*) *span* $\{e_x : x \in R\}$ *with respect to* $\|\cdot\|_-$-*norm.*

The final result is for the hierachies of the function spaces.

Theorem 2.3. *Suppose that the hypothesis* (H) *is satisfied. Then for any* $R \subset E$, *as for the functions on the set* R *we have the inclusions*:

$$\mathscr{H}_{R,B_R} \supset \mathscr{H}'_{R,A_R} \supset l^2(R) \supset \mathscr{H}_{R,A_R} \supset \mathscr{H}'_{R,B_R}.$$

The embedding $\mathscr{H}_{R,B_R} \supset l^2(R)$ *is dense.*

The proofs of the theorems are given in the section 4.

Remark 2.4. When the operator A has a bounded inverse $B := A^{-1}$, all the results in Theorems 2.1–2.3 can be proven without difficulty. The theory of RKHS's helps us extend the results when A is not boundedly invertible.

3. Applications

3.1. A variational principle in the pair of $\mathscr{H}_-$ and $\mathscr{H}_+$

In [4], we have shown a variational principle in the dual pair of $\mathscr{H}_-$ and $\mathscr{H}_+$. Its proof was rather long. By using the characterization theorem, Theorem 2.1, we can reprove it. Let us briefly introduce it. For each finite subset $\Lambda \subset E$ (denoted by $\Lambda \subset\subset E$ hereafter) let

$$\mathsf{F}_{\mathrm{loc},\Lambda} := \text{the linear space spanned by } \{e_x : x \in \Lambda\}. \tag{3.1}$$

Let $x_0 \in E$ be a fixed point and let $E = \{x_0\} \cup R_1 \cup R_2$ be any partition of E (one of R_1 and R_2 may be the empty set). For each $\Lambda \subset\subset E$, define

$$\alpha_\Lambda := \inf_{f \in \mathsf{F}_{\mathrm{loc},\Lambda\cap R_1}} \|e_{x_0} - f\|_-^2 \quad\text{and}\quad \beta_\Lambda := \inf_{g \in \mathsf{F}_{\mathrm{loc},\Lambda\cap R_2}} \|e_{x_0} - g\|_+^2. \tag{3.2}$$

Obviously, $\{\alpha_\Lambda\}_{\Lambda\subset\subset E}$ and $\{\beta_\Lambda\}_{\Lambda\subset\subset E}$ are decreasing nets of nonnegative numbers. Consequently we define

$$\alpha := \lim_{\Lambda\uparrow E} \alpha_\Lambda \quad \text{and} \quad \beta := \lim_{\Lambda\uparrow E} \beta_\Lambda \tag{3.3}$$

The variational principle in [4] reads as follows: no matter how we take a partition $E = \{x_0\} \cup R_1 \cup R_2$, the product of α and β is equal to 1 (see [4, Theorem 2.4] and also [3]):

$$\alpha\beta = 1. \tag{3.4}$$

In order to prove (3.4) we recall the bilinear functional on $\mathscr{H}_- \times \mathscr{H}_+$ introduced in [4]. First, for $f \in \mathscr{H}_0$ and $g \in \operatorname{Ran} A$, define

$$_-\langle f, g\rangle_+ := (f, g)_0 = \sum_{x\in E} \overline{f(x)}g(x) \tag{3.5}$$

It is not hard to see that

$$|_-\langle f, g\rangle_+| \leq \|f\|_-\|g\|_+, \tag{3.6}$$

thus it continuously extends to $\mathscr{H}_- \times \mathscr{H}_+$. By abuse of notation, we denote the extension by the same notation $_-\langle\cdot,\cdot\rangle_+$. For convenience we denote the complex conjugate of it by $_+\langle\cdot,\cdot\rangle_-$, i.e.,

$$_+\langle g, f\rangle_- := \overline{_-\langle f, g\rangle_+}, \quad f \in \mathscr{H}_-,\ g \in \mathscr{H}_+. \tag{3.7}$$

Now we are in a position to prove (3.4). In [4], we have noticed that there are vectors $a_2 \in \mathscr{H}_+$ and $b_1 \in \mathscr{H}_-$ such that $\operatorname{supp} a_2 \subset R_2$ (meaning that $a_2(x) = 0$ for $x \in R_2^c$) and $\operatorname{supp} b_1 \in R_1$, and moreover the following equality holds (see [4, eq. (3.41)]):

$$1 = \alpha\beta + {}_+\langle a_2, b_1\rangle_-. \tag{3.8}$$

By using (3.8), the relation (3.4) follows from the following proposition.

Proposition 3.1. *Suppose that R_1 and R_2 are disjoint subsets of E. If $a_2 \in \mathscr{H}_+$ is supported on R_2 and $a_1 \in \mathscr{H}_-$ is supported on R_1, then $_+\langle a_2, a_1\rangle_- = 0$.*

Proof. Like the bilinear form $_-\langle\cdot,\cdot\rangle_+$ on $\mathscr{H}_- \times \mathscr{H}_+$, we denote the dual pairing on $\mathscr{H}_{R_2,B_{R_2}} \times \mathscr{H}'_{R_2,B_{R_2}}$ by $_{R_2,B_{R_2}}\langle\cdot,\cdot\rangle'_{R_2,B_{R_2}}$. Notice that for $f \in l^2(R_2)$ and $g \in \mathscr{H}'_{R_2,B_{R_2}} \subset l^2(R_2)$, we have

$$_{R_2,B_{R_2}}\langle f, g\rangle'_{R_2,B_{R_2}} = \sum_{x\in R_2} \overline{f(x)}g(x). \tag{3.9}$$

Let $\{f_n\} \subset \mathscr{H}_0 = l^2(E)$ be any sequence that converges to a_1 in $\mathscr{H}_-$, i.e., converging in $\|\cdot\|_-$-norm. Since $a_2 \in \mathscr{H}_+$ is supported on R_2, i.e., $a_2 \in \iota_{R_2}(l^2(R_2)) \cap \mathscr{H}_+$, by Theorem 2.1 we see that $\pi_{R_2}a_2 \in \mathscr{H}'_{R_2,B_{R_2}}$. By (3.9), and by using the continuity of the restriction operator $\pi_R : (\mathscr{H}_-, \|\cdot\|_-) \to (\mathscr{H}_{R,B_R}, \|\cdot\|_{R,B_R})$ for any $R \subset E$ (see (2.7)), we get

$$\begin{aligned}
{}_-\langle a_1, a_2\rangle_+ &= \lim_{n\to\infty} {}_-\langle f_n, a_2\rangle_+ \\
&= \lim_{n\to\infty} \sum_{x\in R_2} \overline{f_n(x)} a_2(x) \\
&= \lim_{n\to\infty} {}_{R_2,B_{R_2}}\langle \pi_{R_2} f_n, \pi_{R_2} a_2\rangle'_{R_2,B_{R_2}} \\
&= {}_{R_2,B_{R_2}}\langle \pi_{R_2} a_1, \pi_{R_2} a_2\rangle'_{R_2,B_{R_2}} \\
&= 0,
\end{aligned}$$

since $\pi_{R_2} a_1 = 0$. □

3.2. Interdependencies of flip rates of Glauber and Kawasaki dynamics for determinantal point processes: a Hilbertian, geometric representation

In [2], we have constructed Glauber and Kawasaki dynamics for determinantal point processes in discrete sets. To construct the equilibrium dynamics that leaves certain point process invariant, the Papangelou intensities of the point process, which are conditional probability densities, play a central role (see [2] for the details). In order to get a Fellerian Markov process, and also to get an ergodicity of the process, it is needed to control the inter-dependencies of the flip rates. We focus only on the application of the result of this paper, so we introduce just the key expressions, referring the details to [2]. The Papangelou intensities are turned out to be the numbers α in (3.3). More concretly, let $x \in E$ be any element and let $\xi \subset E\backslash\{x\}$ be any subset (configuration). Replacing x_0 and R_1 in (3.2) by x and ξ, respectively, let us denote the resulting number α in (3.3) by $\alpha(x;\xi)$. The flip rates for Glauber and Kawasaki dynamics which leave the law of the determinantal point process invariant are determined by the numbers $\alpha(x;\xi)$. From the definition, this number $\alpha(x;\xi)$ has already a geometric interpretation. Namely, $\alpha(x;\xi)$ is the square of the distance (in $\mathscr{H}_-$) from the vector e_x to the subspace spanned by $\{e_y : y \in \xi\}$. What we have called the inter-dependency has the following expression for Glauber dynamics (and similarly for Kawasaki dynamics):

$$\sup_{x\in E} \sum_{u\neq x} \sup_{\xi \not\ni x,u} |\alpha(x;\xi) - \alpha(x;u\xi)|, \tag{3.10}$$

where we used a short-handed expression, $u\xi := \{u\} \cup \xi$. Therefore we need to understand the quantity $|\alpha(x;\xi) - \alpha(x;u\xi)|$ more concretely as much as possible.

For each $\xi \subset E$, we let P_ξ the orthogonal projection in $\mathscr{H}_-$ onto the subspace span$\{e_y : y \in \xi\}$. The following proposition gives several ways of interpretation to the difference $\alpha(x;\xi) - \alpha(x;u\xi)$.

Proposition 3.2. *For any $x \neq u \in E$ and $x, u \notin \xi \subset E$, we have the following representations*:

$$\begin{aligned}\alpha(x;\xi) - \alpha(x;u\xi) &= \|P_{u\xi}e_x - P_\xi e_x\|_-^2 \\ &= |(e_x,(I-P_\xi)e_u)_-|^2 \cdot \|(I-P_\xi)e_u\|_-^{-2} \\ &= |(e_x,e_u)_{\xi^c,B_{\xi^c}}|^2 \cdot \|e_u\|_{\xi^c,B_{\xi^c}}^{-2}.\end{aligned}$$

In particular, in a formal level, we also have the representation:

$$\alpha(x;\xi) - \alpha(x;u\xi) = |A(x,u) - A(x,\xi)A(\xi,\xi)^{-1}A(\xi,u)|^2 \cdot \alpha(u;\xi)^{-1}. \tag{3.11}$$

Proof. Recall that $\alpha(x;\xi)$ is the square of the distance between the vector e_x and the space $P_\xi\mathscr{H}_-$. That is,

$$\alpha(x;\xi) = \|(I-P_\xi)e_x\|_-^2. \tag{3.12}$$

By using this fact and the theorem of three perpendiculars we get the first identity. To proceed, we next show the equality of the second and the last expressions. By Theorem 2.2, $P_\xi\mathscr{H}_-$ is equal to the space $\mathsf{F}_0(\xi^c)$, the subspace of $\mathscr{H}_-$ consisting of the functions that vanish on ξ^c. On the other hand, the orthogonal complement to $\mathsf{F}_0(\xi^c)$ is isometrically equivalent to the space $\mathscr{H}_{\xi^c,B_{\xi^c}}$ (see [1, Section 5, Part I]). The correspondence is via the relation (2.8), i.e., for any vector $f \in \mathscr{H}_{\xi^c,B_{\xi^c}}$, there is a unique $f' \in \mathsf{F}_0(\xi^c)^\perp$ s.t. $\pi_{\xi^c}f' = f$ and $\|f'\|_- = \|f\|_{\xi^c,B_{\xi^c}}$. Since e_x and e_u are supported on ξ^c we may simply write $\pi_{\xi^c}e_x = e_x$ and $\pi_{\xi^c}e_u = e_u$, and then we have

$$\alpha(x;\xi) = \|(I-P_\xi)e_x\|_-^2 = \|e_x\|_{\xi^c,B_{\xi^c}}^2. \tag{3.13}$$

Now we recall from [4] that

$$\alpha(x;\xi) = \lim_{\Delta\uparrow E}\frac{\det A(x\xi_\Delta, x\xi_\Delta)}{\det A(\xi_\Delta,\xi_\Delta)}, \tag{3.14}$$

where $\xi_\Delta = \xi \cap \Delta$ and $A(\xi_\Delta,\xi_\Delta)$ is the matrix $(A(x,y))_{x,y\in\xi_\Delta}$. Similarly we have

$$\begin{aligned}\alpha(x;u\xi) &= \lim_{\Delta\uparrow E}\frac{\det A(xu\xi_\Delta, xu\xi_\Delta)}{\det A(u\xi_\Delta,u\xi_\Delta)} \\ &= \lim_{\Delta\uparrow E}\frac{\det A(xu\xi_\Delta, xu\xi_\Delta)/\det A(\xi_\Delta,\xi_\Delta)}{\det A(u\xi_\Delta,u\xi_\Delta)/\det A(\xi_\Delta,\xi_\Delta)}.\end{aligned} \tag{3.15}$$

The denominator in (3.15) converges to $\alpha(u;\xi) = \|e_u\|_{\xi^c,B_{\xi^c}}^2$. In a very similar way we see that the numerator converges to

$$\det\begin{pmatrix}\|e_x\|_{\xi^c,B_{\xi^c}}^2 & (e_x,e_u)_{\xi^c,B_{\xi^c}} \\ (e_u,e_x)_{\xi^c,B_{\xi^c}} & \|e_u\|_{\xi^c,B_{\xi^c}}^2\end{pmatrix}. \tag{3.16}$$

From (3.15) and (3.16) we see that

$$\alpha(x;u\xi) = \|e_x\|_{\xi^c,B_{\xi^c}}^2 - |(e_x,e_u)_{\xi^c,B_{\xi^c}}|^2 \cdot \|e_u\|_{\xi^c,B_{\xi^c}}^{-2}. \tag{3.17}$$

From (3.13) and (3.17) we get

$$\alpha(x;\xi) - \alpha(x;u\xi) = |(e_x, e_u)_{\xi^c, B_{\xi^c}}|^2 \cdot \|e_u\|^{-2}_{\xi^c, B_{\xi^c}}, \tag{3.18}$$

which is the last expression in the proposition. For the second equality we notice that

$$\begin{aligned} |(e_x, (I - P_\xi)e_u)_-|^2 &= |((I - P_\xi)e_x, (I - P_\xi)e_u)_-|^2 \\ &= |(e_x, e_u)_{\xi^c, B_{\xi^c}}|^2, \end{aligned} \tag{3.19}$$

where we have used the relation (3.13) and the polarization identity. From (3.13) and (3.18)–(3.19), we see that the second and the last expressions are the same. Finally we check the relation (3.11) in a formal level. Since (informally) $B = A^{-1}$,

$$\begin{aligned} (e_x, e_u)_{\xi^c, B_{\xi^c}} &= (e_x, (B_{\xi^c})^{-1} e_u)_0 \\ &= (e_x, [(A^{-1})_{\xi^c}]^{-1} e_u)_0 \\ &= (e_x, [A(\xi^c, \xi^c) - A(\xi^c, \xi)A(\xi, \xi)^{-1} A(\xi, \xi^c)]e_u)_0 \\ &= A(x, u) - A(x, \xi)A(\xi, \xi)^{-1} A(\xi, u). \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) we get (3.11). This completes the proof. □

4. Proofs

In this section we provide with the proofs for the theorems in Section 2. We start by showing the last assertion in Theorem 2.3, which says that $l^2(R)$ is densely embedded in $\mathscr{H}_{R, B_R}$. Since it is worthy to notice we state it as a proposition.

Proposition 4.1. *Under the hypothesis (H), $l^2(R)$ is densely embedded in $\mathscr{H}_{R, B_R}$ for any $R \subset E$.*

Proof. Let $f \in \mathscr{H}_{R, B_R}$ be any element and let $\tilde{f} \in \mathscr{H}_-$ be an element so that $f = \pi_R \tilde{f}$. Let $\{f_n\} \subset \mathscr{H}_0 = l^2(E)$ be any sequence that converges to $\tilde{f}$ in $\mathscr{H}_-$. As noticed before, since $\pi_R : (\mathscr{H}_-, \|\cdot\|_-) \to (\mathscr{H}_{R, B_R}, \|\cdot\|_{R, B_R})$ is continuous, we see that $\pi_R f_n \to \pi_R \tilde{f} = f$ in $\mathscr{H}_{R, B_R}$. Notice that $\pi_R f_n \in l^2(R)$ since $f_n \in l^2(E)$. This proves that $l^2(R)$ is dense in $\mathscr{H}_{R, B_R}$. □

Proof of Theorem 2.1. We first show the equality $\mathscr{H}'_{R, B_R} = \pi_R(\iota_R(l^2(R)) \cap \mathscr{H}_+)$. The half inclusion $\mathscr{H}'_{R, B_R} \subset \pi_R(\iota_R(l^2(R)) \cap \mathscr{H}_+)$ was shown in [4, Lemma 3.6], and there it was also shown that for $g \in \mathscr{H}'_{R, B_R}$ the equality

$$\|g\|'_{R, B_R} = \|\iota_R(g)\|_+ \tag{4.1}$$

holds. So, let $g \in \imath_R(l^2(R)) \cap \mathscr{H}_+$. For any element $f \in l^2(R)$, since $\imath_R(f) \in l^2(E) \subset \mathscr{H}_-$, we let $f' \in \mathscr{H}_-$ be the orthogonal projection of $\imath_R(f)$ onto $\mathsf{F}_0(R)^\perp$. Recall that $\pi_R f' = \pi_R(\imath_R(f)) = f$ and $\|f'\|_- = \|f\|_{R,B_R}$ (see (2.8)). We define a (conjugate) linear functional on $l^2(R)$ by

$$l^2(R) \ni f \mapsto {}_-\langle f', g\rangle_+. \tag{4.2}$$

By Schwarz inequality we see that this functional is bounded by

$$|{}_-\langle f', g\rangle_+| \le \|f'\|_- \cdot \|g\|_+ = \|f\|_{R,B_R} \cdot \|g\|_+. \tag{4.3}$$

This shows that the functional in (4.2) is a bounded linear functional on $l^2(R)$ equipped with the $\|\cdot\|_{R,B_R}$-norm. Since $l^2(R)$ is dense in $\mathscr{H}_{R,B_R}$ by Propsition 4.1, we conclude that g belongs to $\mathscr{H}'_{R,B_R}$. By the identity of the norms in (4.1) we have proven the equality $\mathscr{H}'_{R,B_R} = \pi_R(\imath_R(l^2(R)) \cap \mathscr{H}_+)$. In order to see the equality $\mathscr{H}_{R,B_R^{-1}} = \pi_R(\imath_R(l^2(R)) \cap \mathscr{H}_+)$, it is enough to see that

$$(e_x, e_y)_{R,B_R^{-1}} = (e_x, e_y)_+, \quad x, y \in R. \tag{4.4}$$

But both are equal to the value $B(x, y)$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. It is not hard to see that the operator A_R, the restriction of A onto $l^2(R)$ satisfies the hypothesis (H) when the set E is replaced by R. In fact this follows by noticing that if a sequence $\{f_n\} \subset l^2(R)$ is a Cauchy sequence in the sense that $(f_n - f_m, A_R(f_n - f_m)) \to 0$ as $m, n \to \infty$, then $\{\imath_R(f_n)\}$ is a Cauchy sequence in $\mathscr{H}_-$, which is functionally completed. Therefore as in the dual relation between $\mathscr{H}_-$ and $\mathscr{H}_+$, we see that $\mathscr{H}_{R,A_R^{-1}}$ and $\mathscr{H}_{R,A_R}$ are dual spaces to each other, or $\mathscr{H}_{R,A_R^{-1}} = \mathscr{H}'_{R,A_R}$. It remains to show that $\mathscr{H}_{R,A_R^{-1}} = \pi_R(\mathsf{F}_0(R^c))$.

For this, let $f \in l^2(R)$ be any element. Then

$$\|f\|^2_{R,A_R^{-1}} = (f, A_R f) = \|\imath_R(f)\|^2_-. \tag{4.5}$$

This shows that $\imath_R(f) \in \mathsf{F}_0(R^c)$ and $f = \pi_R(\imath_R(f))$. Since $l^2(R)$ is dense in $\mathscr{H}_{R,A_R^{-1}}$ we see that $\mathscr{H}_{R,A_R^{-1}} \subset \pi_R(\mathsf{F}_0(R^c))$. Now suppose that $f \in \mathsf{F}_0(R^c)$. We want to show

$$\pi_R f \in \mathscr{H}_{R,A_R^{-1}} = \mathscr{H}'_{R,A_R}. \tag{4.6}$$

We follow the same method used in the proof of Theorem 2.1. For each $g \in \mathscr{H}_{R,A_R}$, let $g' \in \mathscr{H}_+$ be the unique element such that $\pi_R g' = g$ and $\|g'\|_+ = \|g\|_{R,A_R}$. Since f is supported on R, regarding it as $\pi_R f$, we define a linear functional on $\mathscr{H}_{R,A_R}$ by

$$g \mapsto {}_-\langle f, g'\rangle_+. \tag{4.7}$$

Then $|{}_-\langle f, g'\rangle_+| \le \|f\|_- \|g'\|_+ = \|f\|_- \|g\|_{R,A_R}$. This shows that $\pi_R f \in \mathscr{H}'_{R,A_R} = \mathscr{H}_{R,A_R^{-1}}$ and

$$\|\pi_R f\|'_{R, A_R} \le \|f\|_{-}. \tag{4.8}$$

By (4.5), we already know that for $\pi_R f \in \mathscr{H}'_{R, A_R}$, $\|\pi_R f\|'_{R, A_R} = \|f\|_{-}$. Thus we conclude that $\mathscr{H}'_{R, A_R} = \pi_R(\mathsf{F}_0(R^c))$. The last assertion follows by noticing that $l^2(R)$ is dense in $\mathscr{H}_{R, A_R^{-1}}$ and $\|\pi_R e_x\|_{R, A_R^{-1}} = \|e_x\|_{-}$ for each $x \in R$. $\square$

Proof of Theorem 2.3. First we notice that $l^2(R)$ is dense in $\mathscr{H}_{R, B_R}$ and in $\mathscr{H}_{R, A_R^{-1}}$ respectively in the corresponding norms. By (2.13), we see that for all $f \in l^2(R)$,

$$(f, B_R^{-1} f)_0 \le (f, A_R f)_0, \tag{4.9}$$

or

$$\|f\|_{R, B_R} \le \|f\|_{R, A_R^{-1}} \tag{4.10}$$

This shows that

$$\mathscr{H}_{R, B_R} \supset \mathscr{H}_{R, A_R^{-1}} = \mathscr{H}'_{R, A_R} \tag{4.11}$$

By the duality characterization theorems, Thoerem 2.1 and 2.2, we get the inclusions stated in the theorem. The last statement has been already shown in Proposition 4.1. $\square$

5. Open problem: the Shauder basis

Since $\{e_x\}_{x \in E}$ is a basis for $\mathscr{H}_0 = l^2(E)$ and $\mathscr{H}_0$ is dense in $\mathscr{H}_{-}$, by Gram-Schmidt orthogonalization procedure, we can construct an orthonormal basis for $\mathscr{H}_{-}$ from the set $\{e_x\}_{x \in E}$. Now let $\{\Lambda_n\}_{n=1}^{\infty}$ be any increasing sequence of finite subsets of E such that $\bigcup_{n=1}^{\infty} \Lambda_n = E$. Let $f \equiv (f(x))_{x \in E} \in \mathscr{H}_{-}$ be any element and for each $N \in \mathbf{N}$ let $f_N := \sum_{x \in \Lambda_N} f(x) e_x \in \mathscr{H}_{-}$. The following is an open problem:

Open Problem: In the above, is the following true or not?

$$\lim_{N \to \infty} f_N = f \quad (\text{in } \mathscr{H}_{-}). \tag{5.1}$$

Similarly, for $g \equiv (g(x))_{x \in E} \in \mathscr{H}_{+}$, we define $g_N := \sum_{x \in \Lambda_N} g(x) e_x \in \mathscr{H}_{+}$ and also ask whether the limit

$$\lim_{N \to \infty} g_N = g \quad (\text{in } \mathscr{H}_{+}) \tag{5.2}$$

holds or not.

For these problems we make some remarks. First, if it is true, then it says that $\{e_x\}_{x \in E}$ is a Shauder basis for $\mathscr{H}_{-}$ (and also for $\mathscr{H}_{+}$). Second, it is well known that (see [1, Theorem I, p 362])

$$\lim_{N \to \infty} \|f_N\|_{\Lambda_N, B_{\Lambda_N}} = \|f\|_{-}. \tag{5.3}$$

Though the norms $\|f_N\|_{\Lambda_N, B_{\Lambda_N}}$ increases as N increases, we don't know any monotonicity or convergence for the sequence $\{\|f_N\|_-\}$. Finally, as mentioned in the introduction, a restriction of a vector in a RKHS may not belong to the original space. For instance, let A be the operator in [4, Example 2.5] defined by $A := B^*B$ on $l^2(E)$ with $E := \mathbf{N}$, and B is defined by

$$Be_n = \begin{cases} e_1, & n = 1, \\ \dfrac{1}{n}(e_1 + e_n), & n \geq 2, \end{cases}$$

and by a linear extension. Let $\mathscr{H}_+$ be the RKHS with kernel $A(x, y)$. We can show that $e_1 \notin \mathscr{H}_+$. Thus for any $n \geq 1$ and $\Lambda \subset\subset E$ with $1 \in \Lambda$ we have $\imath_\Lambda((Ae_n)_\Lambda) \notin \mathscr{H}_+$, though $Ae_n \in \mathscr{H}_+$. In other words, the questions (5.1) or (5.2) might be meaningless for some cases. But under our hypothesis (H), two questions are well posed and surely they are interesting.

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Hyun Jae Yoo
Department of Applied Mathematics
Hankyong National University
67 Seokjeong-dong, Anseong-si, Gyeonggi-do 456-749
Korea
E-mail: yoohj@hknu.ac.kr