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A DUAL THEORY FOR FREE FERMIONS

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ABSTRACT

A wave equation for free fermions is proposed based on the structure of the dual theory for bosons. Its formal properties preserve the role played by the Virasoro algebra. Additional Ward like identities, compatible with the equation, are shown to exist. Its solution lie on linear trajectories. In particular the parent is shown to be doubly degenerate, but these solutions lie on different sheets of the cut j-plane.

INTRODUCTION

In spite of its obvious theoretical appeal, the dual model[1] has been denied full acceptance (credibility) because of its failure to include fermions. In this paper we present an extension of the model to encompass half integer spin states by making use of a structure evident in the dual theory of free bosons[2]. Namely, we found that the following view of duality led to no contradiction with existing results: Each "free" boson appearing in the theory is a state of a complex system. Its structure can be parametrized in terms of an internal motion which is periodic in an internal time coordinate so that each observable of the system is the average over a cycle of the internal motion of suitably generalized operators. In this way operators appearing in the description of point particles in conventional theories must be thought of as averages over some internal motion when applied to a hadronic system. The system then becomes a point particle in the limit of the internal cycle going to zero. These precepts are illustrated by their application to the bosonic case in Section I. We use these guidelines to introduce a generalization of the Dirac matrices and postulate a Dirac wave equation for the free fermionic system. Its formal properties are studied in Section II. The final section will be devoted to a detailed study of its solutions.

I

In order to set the notation and illustrate the ideas behind our interpretation, it is desirable to first consider the (already known) free boson theory. The free hadronic system is described in terms of an internal motion generated by the Nambu[3] Hamiltonian

$$H_B = \frac{1}{2} \sum_{n=o}^{\infty} [p^{(n)} . p^{(n)} + \omega_n^2 q^{(n)} . q^{(n)}] \tag{I-1}$$

with

$$\omega_{n+1} - \omega_n = \omega \qquad n = 0, 1, 2 \ldots \tag{I-2}$$

and the normal mode coordinates are four-vector operators satisfying the usual commutation relations

$$[q_\alpha^{(n)}, q_\beta^{(m)}] = [p_\alpha^{(n)}, p_\beta^{(m)}] = 0$$

$$[q_\alpha^{(m)}, p_\beta^{(n)}] = - i g_{\alpha\beta} \delta^{m,n} \qquad m,n = 0, 1 \ldots \tag{I-3}$$

where we use $g_{\alpha\beta} = (1, -1, -1, -1)$ for the Lorentz metric. The internal system carries a total momentum

$$P_\mu = \sum_{n=o}^{\infty} p_\mu^{(n)} \tag{I-4}$$

corresponding to a coordinate

$$Q_\mu = \sum_{n=o}^{\infty} q_\mu^{(n)} \tag{I-5}$$

The variable τ which describes the evolution of the internal motions is introduced by means of the Heisenberg equations

$$[H_B, f] = i \frac{df}{d\tau} \tag{I-6}$$

where f is any operator. It is important to note that in the limit $\omega_o \to 0$, the lowest mode becomes translational while the internal motions generated by the higher modes become periodic. We consider this to be the physical limit. It is our observation that in this limit all observables of the system can be written as averages over the period of internal motions. We take the fundamental cycle of the internal motion to be the interval

$$-\frac{\pi}{\omega} \leq \tau \leq \frac{\pi}{\omega} \tag{I-7}$$

so that the average of an operator $A(\tau)$ is defined as

$$<A(\tau)> \equiv \frac{\omega}{2\pi}\int_{-\pi/\omega}^{+\pi/\omega} d\tau\; A(\tau) \tag{I-8}$$

In particular this means that each operator appearing in usual theories must be expressible as the average of a more general operator over the internal motion of the system it describes. We now proceed to give several examples The momentum of the boson is

$$p_\mu = <P_\mu(\tau)> \tag{I-9a}$$

while its position is

$$x_\mu = <Q_\mu(\tau)> \tag{I-9b}$$

We thus call P_μ and Q_ρ the generalized momenta and position respectively. The generalization of the Klein-Gordon operator is obtained by this correspondence principle

$$(p^2 - m^2) = <P>\cdot<P> - m^2 \to \;<P\cdot P> - m^2 \tag{I-10}$$

The solutions of the generalized Klein-Gordon equation are the states of the free bosonic system. This equation is of course the usual one. It should be noted that we require normal ordering of the periodic modes to eliminate the zero point energy.

Similarly the operators

$$M^{B}_{\alpha\beta} = < Q_\alpha P_\beta - Q_\beta P_\alpha > \tag{I-11}$$

explicitely satisfy the commutation relations of the Lorentz group. An amusing application of this correspondence principle is to consider the usual ghost killing conditions and its generalization

$$0 = p \cdot a^{(n)} = < P_\mu > \; < e^{i\omega_n \tau} P_\mu > \;\rightarrow\; < P^2 \, e^{i\omega_n \tau} > = 0 \tag{I-12}$$

which can be explicitely seen to be exactly the condition found to hold by Virasoro[4]. We stress that in the absence of interactions, this condition is independent of the mass.

It should be noted that the operators

$$L^{B}_{\pm n} = < e^{\pm i\omega_n \tau} P^2(\tau) > \tag{I-13}$$

generate an infinite-dimensional Lie algebra and play a central role in the construction of suitable interaction terms[5]. Before proceeding to the fermion case we mention the formula

$$[\, P_\alpha(\tau),\; P_\beta(\tau') \,] = -\frac{i}{2}\, g_{\alpha\beta} \frac{d}{d\tau}\, \delta[\, \frac{\omega}{2\pi} (\tau - \tau') \,] \,,\ \mathrm{mod}\, (\frac{2\pi}{\omega}) \tag{I-14}$$

which we shall use in the following section.

II

In close analogy to the usual procedure where Dirac matrices are introduced to describe half integer spin point particles, we keep all the features encountered in the bosonic case and define over the space of internal motions a generalization of the Dirac matrices, $\Gamma_\mu(\tau)$. We require that its average over a cycle of the internal system be equal to the usual Dirac matrix[6], namely

$$< \Gamma_\mu(\tau) > = \gamma_\mu \tag{II-1}$$

The equation

$$\{ \Gamma_\mu(\tau), \Gamma_\nu(\tau') \} = 2 g_{\mu\nu} \delta [\frac{\omega}{2\pi} (\tau - \tau')] , \bmod (\frac{2\pi}{\omega}) \tag{II-2}$$

for the anticommutator seems to be the simplest one consistent with that obeyed by the γ's. Similarly we require

$$\Gamma_\mu(\tau)^\dagger = \gamma_o \Gamma_\mu(\tau) \gamma_o \tag{II-3}$$

on the grounds of simplicity. These last three requirements are sufficient to determine the explicit form of $\Gamma_\mu(\tau)$ in an almost unique way. This is done by remarking that $\Gamma_\mu(\tau)$ can be written, by assumption, as a Fourier series over the fundamental cycle. The Fourier coefficients are then determined by taking various projections of Eq. (II-2) along the components $e^{\pm i \omega n \tau}$ and integrating over τ. Then requiring (II-3) to hold suffices to set all the higher Fourier coefficients. We find

$$\Gamma_\mu(\tau) = \gamma_\mu + i\omega_o \tau \delta_\mu + i\sqrt{2} \gamma_5 \sum_{n=1}^{\infty} [b_\mu^{(n)\dagger} e^{i\omega_n \tau} + b_\mu^{(n)} e^{-i\omega_n \tau}] \tag{II-4}$$

where the b's are operators obeying the anticommutation relations

$$\{ b_\mu^{(n)}, b_\nu^{(m)} \} = \{ b_\mu^{(n)\dagger}, b_\nu^{(m)\dagger} \} = 0$$

$$\{ b_\mu^{(n)}, b_\nu^{(m)\dagger} \} = -g_{\mu\nu} \delta^{n,m} \quad n, m = 1, \ldots \tag{II-5}$$

In what follows we shall assume that δ_μ is not singular in the limit $\omega_o \to 0$, and we thus neglect it. We emphasize that these simple requirements lead to a unique form. Note the appearance of γ_5 which is essential for (II-2) to hold since it is the only 4 x 4 matrix to anticommute with all γ_μ's. We propose the following generalization of the Dirac equation

$$[< \Gamma_\mu(\tau) \, P_\mu(\tau) > - m] \, | \Psi > = 0 \tag{II-6}$$

as suggested by our correspondence principle. In terms of creation and annihilation operator it is given by

$$\gamma \cdot p - m - \gamma_5 \sum_{n=1}^{\infty} \sqrt{\omega_n} [a^{(n)\dagger} \cdot b^{(n)} - b^{(n)\dagger} \cdot a^{(n)}] \tag{II-7}$$

It is easy to see that one recovers a familiar spectrum. Use of the anticommutation relations (II-2), the periodicity of Γ_μ, and of Eq. (I-14), yields

$$[< \Gamma \cdot p > < \Gamma \cdot p > - m^2] \, | \Psi > = [< P^2 > - \frac{i}{4} < \Gamma \cdot \dot{\Gamma} > - m^2] \, | \Psi > = 0 \tag{II-8}$$

where the dot denotes differentiation with respect to τ. This expression explicitely reduces to

$$(p^2 - m^2 + \omega \sum_{n=1}^{\infty} n[\ a^{(n)\dagger} \cdot a^{(n)} + b^{(n)\dagger} \cdot b^{(n)}]\ |\Psi> \ = 0 \qquad \text{(II-9)}$$

which leads to linear trajectories.

The consistency of our interpretation requires the relativistic Hamiltonian to be the generator of the internal motion. Indeed we see that

$$[\ -\frac{i}{4} < \Gamma \cdot \dot{\Gamma} > \ ,\ \Gamma_\rho(\tau)\ ] = i\ \frac{d}{d\tau} \Gamma_\rho\ (\tau) \qquad \text{(II-10)}$$

by direct use of Eq. (II-2). In analogy to the usual Dirac equation, we check that the operators

$$M^F_{\alpha\beta} = \frac{i}{2} < \Gamma_\alpha \Gamma_\beta > \qquad \text{(II-11)}$$

satisfy the Lorentz group commutation relations and

$$\left[M^F_{\alpha\beta},\ \Gamma_\mu(\tau)\right] = i\left(g_{\beta\mu}\ \Gamma_\alpha(\tau) - g_{\alpha\mu}\Gamma_\beta(\tau)\right) \qquad \text{(II-12)}$$

where [a's and b's are taken to commute] the total Lorentz generators

$$M_{\alpha\beta} = M^F_{\alpha\beta} + M^B_{\alpha\beta} \qquad \text{(II-13)}$$

leave the generalized Dirac operator invariant, thereby showing that they are the relevant representations of the Lorentz group acting on the solutions of (II-6).

We next introduce the operators

$$L_{\pm n}^{F} = -\frac{i}{4} < e^{\pm i\omega_n \tau} \Gamma_\mu(\tau)\dot{\Gamma}_\mu(\tau) > \tag{II-14}$$

which obey the infinite dimensional Lie algebra

$$[L_n^F , L_m^F] = \omega(m-n) L_{m+n}^F - \frac{n^2\omega^2}{8} \delta_{n,-m} \tag{II-15}$$

so that we define the generalization of the Virasoro operators by

$$L_{\pm n} = L_{\pm n}^F + L_{\pm n}^B \tag{II-16}$$

It is clear that the condition

$$L_n | \Psi > = 0 \tag{II-17}$$

is compatible with the equation for the spectrum i. e., the square of the generalized Dirac equation, and that it holds for any mass. Another Ward-like identity is obtained by considering the operator

$$F_{\pm n} = < e^{\pm i\omega_n \tau} \Gamma_\mu(\tau)P_\mu(\tau) > \tag{II-18}$$

which obeys

$$[L_n , F_m] = \frac{1}{2} \omega (2m-n) F_{n+m} \tag{II-19a}$$

$$\{ F_n , F_m \} = 2 L_{n+m} \tag{II-19b}$$

These are now used to obtain compatible new relations. Now

$$0 = F_n(F_o-m) | \Psi> = [2L_n - (F_o-m)F_n] | \Psi > \tag{II-20a}$$

so that we have an additional identity, namely

$$F_n | \Psi> = 0 \tag{II-20b}$$

This one can be seen to be the generalization of $p \cdot b^{(n)} \Psi = 0$ according to our correspondence principle (or $\gamma \cdot a^{(n)} = 0$). The interaction terms must have specific transformation properties under the algebras (II-15) and (II-19) for these equations to hold for the system in interaction.

III

We now turn to a discussion of the solutions of the equation. It is convenient to introduce the notation

$$N = \sum_{n=1}^{\infty} \sqrt{n\omega}\,[\, a^{(n)\dagger} \cdot b^{(n)} - b^{(n)\dagger} \cdot a^{(n)} \,] \qquad \text{(III-1)}$$

Then the generalized Dirac equation is

$$[\, \gamma \cdot p - m - \gamma_5 N \,] \mid \Psi > = 0 \qquad \text{(III-2)}$$

The spectrum equation (II-9) shows that in the occupation number space spanned by the a's and the b's the masses of the excited states obey

$$m_\ell^2 = m^2 + \ell\omega \qquad \ell = 0, 1, \ldots \qquad \text{(III-3)}$$

It is easy to show that we can write the positive energy solutions corresponding to the ℓ th mass level as

$$\mid \Psi_\ell^{(i)}(k) > = \frac{1}{\sqrt{2E(E+m)}}\,[\, m + <\Gamma \cdot P> \,] \mid U_\ell^{(i)}(k) > \quad (i = 1, 2). \qquad \text{(III-4)}$$

with

$$N^2 \mid U_\ell^{(i)}(k) > \; = -\ell\omega \mid U_\ell^{(i)}(k) > \quad i = 1, 2, 3, 4. \tag{III-5}$$

and $\mid U_\ell^{(i)}(k) >$ is a 4-spinor operator with non zero element in the ith column only. The negative energy solutions are

$$\mid \Psi_\ell^{(i)}(k) > \; = \frac{1}{\sqrt{2E(E-m)}} [\, m + < \Gamma \cdot P > \,] \mid U_\ell^{(i)}(k) > \quad (i = 3, 4) \tag{III-6}$$

The norm of these states is positive only when the space components (or an even number of time components) is involved. We proceed to give several examples.

<u>$\ell = 1$</u>

we have two candidates

$$\mid U_{11}^{(i)}(k) > \; = \; a_\rho^{(1)\dagger} e^{-ik\cdot q} \mid 0 > U^{(i)}$$

$$\mid U_{12}^{(i)}(k) > \; = b_\rho^{(1)\dagger} e^{-ik\cdot q} \mid 0 > U^{(i)} \tag{III-7}$$

The corresponding states $\mid \Psi_{11}^{(i)} >$ and $\mid \Psi_{12}^{(i)} >$ have spin 3/2 and 1/2 components as can be checked by applying on them the Lorentz generators. Hence the parent trajectory is doubly degenerate. As discussed earlier, the spin 1/2 components can be removed by requiring Eqs. (II-17 and II-20) to hold.

<u>$\ell = 2$</u>

There are four candidates

$$| U^{(i)}_{21}(k) > = \frac{1}{\sqrt{2}} a^{(1)\dagger}_{\rho} a^{(1)\dagger}_{\sigma} e^{-ik\cdot q} | 0 > U^{(i)}$$

$$| U^{(i)}_{22}(k) > = \frac{1}{\sqrt{2}}\left(a^{(1)\dagger}_{\rho} b^{(1)\dagger}_{\sigma} + a^{(1)\dagger}_{\sigma} b^{(1)\dagger}_{\rho}\right) e^{-ik\cdot q} | 0 > U^{(i)}$$

$$| U^{(i)}_{23}(k) > = a^{(2)\dagger}_{\rho} e^{-ik\cdot q} | 0 > U^{(i)}$$

$$| U^{(i)}_{24} (k) > = b^{(2)\dagger}_{\rho} e^{-ik\cdot q} | 0 > U^{(i)} \qquad \text{(III-8)}$$

so that we have two spin 5/2, 4 spin 3/2 and six spin 1/2 states. Some are eliminated through the Ward-like identities.

In the general case, we have

$$| U^{(i)}_{\ell}(k) > \prod_{j=1} \frac{[a^{(j)\dagger}]^{n_j}}{(\sqrt{n_j\,!})^{n_j}} [b^{(j)\dagger}]^{\epsilon_j} e^{-ik\cdot q} | 0 > U^{(i)} \qquad \text{(III-9)}$$

where

$$\ell = \sum_{j=1}^{\infty} (jn_j + j\epsilon_j) \qquad \text{(III-10)}$$

and $\epsilon_j = 0$ or 1.

It follows that there are two spin $J = (\ell + 1/2)$ positive energy solutions at this level (those with $n_1 = \ell$ and those with $n_1 = \ell - 1, \epsilon_1 = 1$) so that the parent trajectory is doubly degenerate.

Before interpreting this degeneracy, consider the propagator

$$S_F(p) = \frac{1}{<\Gamma\cdot P> - m + i\epsilon}$$

$$= \frac{<\Gamma\cdot P> + m}{L_o - m^2 + 2i\epsilon m} \qquad \text{(III-11)}$$

Notice that, unlike usual infinite component equations, the imaginary part is positive definite, thereby rendering the sign of the mass matrix unimportant for considerations of extra "parity ghosts"[7]. Also, for on mass shell states, the numerator is positive definite.

A possible explanation of the degeneracy is presented by considering the possible expectation values of the matrix $\gamma_5 N$ between the states on the parent trajectory. We find that for positive energy solution, it has two eigenvalues

$$\lambda_{\pm} = \pm i \sqrt{(J-\frac{1}{2})\,\omega\,\frac{k^2}{E^2}} \tag{III-12}$$

It is seen that the sign can be accounted for by taking one solution to lie on the second sheet of the cut J-plane. Finally we wish to say that the degeneracy structure of the solution is comparable to that encountered in the boson case.

IV. CONCLUSION

Although we have not presented a treatment of the system in interaction, we hope that the wave equation will prove to lead to such a formulation in the near future. At present we can only understand how to introduce the electromagnetic interaction of our base systems by means a minimal coupling scheme. However the more important self interactions are still a mystery to us.

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