Dual unitary matrices and unit dual quaternions

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Abstract. In this study, the dual complex numbers defined as the dual quaternions have been considered as a generalization of complex numbers. In addition, the dual unitary matrices that are more general form than unitary matrices were obtained. Finally, the group of the dual symplectic matrices was attained by using symplectic structure upon dual quaternions. In particular, the group of symplectic matrices that are isomorphic to dual unitary matrices was studied.

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1 Introduction

The algebra $\mathbf{H} = \{q = a_0 1 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbf{R}\}$ of quaternions is defined as the 4-dimensional vector space over $\mathbf{R}$ having a basis $\{1, i, j, k\}$ with the following properties

$$i^2 = j^2 = k^2 = -1\ ,\ ij = -ji = k.$$

It is clear that $\mathbf{H}$ is an associative and not commutative algebra and 1 is the identity element of $\mathbf{H}$. A quaternion $q = a_0 1 + a_1 i + a_2 j + a_3 k$ is pieced into two parts with scalar piece $S_q = a_0$ and vectorial piece $\overrightarrow{V_q} = a_1 i + a_2 j + a_3 k$. We also write $q = S_q + \overrightarrow{V_q}$. The conjugate of $q = S_q + \overrightarrow{V_q}$ is then defined as $\overline{q} = S_q - \overrightarrow{V_q}$. We call a quaternion pure if its scalar part vanishes. The pure quaternions form the three-dimensional linear subspace $Im\mathbf{H} = \{a_1 i + a_2 j + a_3 k : a_1, a_2, a_3 \in \mathbf{R}\} = \{q \in H : \overline{q} = -q\}$ of H spanned by $\{i, j, k\}$. Summation of two quaternions $q = S_q + \overrightarrow{V_q}$ and $p = S_p + \overrightarrow{V_p}$ is defined as $q + p = (S_q + S_p) + \left(\overrightarrow{V_q} + \overrightarrow{V_p}\right)$ .Multiplication of a quaternion $q = S_q + \overrightarrow{V_q}$ with a scalar $\lambda \in \mathbf{R}$ is defined as $\lambda q = \lambda S_q + \lambda \overrightarrow{V_q}$. In addition,quaternionic multiplication of two quaternions $q = a_0 1 + a_1 i + a_2 j + a_3 k$ and $p = b_0 + b_1 i + b_2 j + b_3 k$ is defined

$$qp = S_q S_p - \left\langle \overrightarrow{V_q}, \overrightarrow{V_p} \right\rangle + S_q \overrightarrow{V_p} + S_p \overrightarrow{V_q} + \overrightarrow{V_q} \Lambda \overrightarrow{V_p},$$

where $\left\langle \overrightarrow{V_q}, \overrightarrow{V_p} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$, $\overrightarrow{V_q} \Lambda \overrightarrow{V_p} = (a_2 b_3 - a_3 b_2)\, i - (a_1 b_3 - a_3 b_1)\, j + (a_1 b_2 - a_2 b_1)\, k$. Thus, with this multiplication operation, $\mathbf{H}$ is called real quaternion

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algebra[5]. Quaternionic multiplication satisfies the following properties as in [6]: For any two quaternions q and p we have $\overline{qp} = \overline{p}\overline{q}$ and the formula for the inner product $\langle q, p\rangle = \frac{\overline{q}p+\overline{p}q}{2}$. In particular, if $q = p$, we obtain $|q|^2 = \langle q, q\rangle = \overline{q}q$, the usual Length-Identity. With this, the quaternionic inverse of a nonzero quaternion $q \in \mathbf{H}$ can be written as $q^{-1} = \frac{\overline{q}}{|q|^2}$. The 3-sphere $\mathbf{S}^3 \subset \mathbf{H}$ in quaternionic calculus is like the unit circle $\mathbf{S}^1 \subset \mathbf{C}$ in complex calculus. In fact,

$$\mathbf{S}^3 = \{q \in \mathbf{H} : |q| = 1\}$$

constitutes a group under quaternionic multiplication; this is an immediate consequence of the fact that quaternionic multiplication is normed as $|qp| = |q|\,|p|$.

Definition 1. Each element of the set

$$\begin{aligned}\mathbf{D} &= \left\{A = a + \varepsilon a^* : a, a^* \in \mathbf{R} \text{ and } \varepsilon \neq 0, \varepsilon^2 = 0\right\} \\ &= \{A = (a, a^*) : a, a^* \in \mathbf{R}\}\end{aligned}$$

is called a dual number. A dual number $A = a + \varepsilon a^*$ can be expressed in the form $A = ReA + \varepsilon DuA$, where $ReA = a$ and $DuA = a^*$.The conjugate of $A = a + \varepsilon a^*$ is defined as $\overline{A} = a - \varepsilon a^*$. Summation and multiplication of two dual numbers are defined as similar to the complex numbers but it is must be forgotten that $\varepsilon^2 = 0$.Thus, $\mathbf{D}$ is a commutative ring with a unit element [3].

Definition 2. The ring

$$\mathbf{H}_D = \{Q = A_0 + A_1 i + A_2 j + A_3 k : A_0, A_1, A_2, A_3 \in \mathbf{D}\}$$

of is defined as the 4-dimensional vector space over $\mathbf{D}$ having a basis $\{1, i, j, k\}$ with the same multiplication property of basis elements in real quaternions. Each element of $\mathbf{H}_D$ is called a dual quaternion,dual numbers A_0, A_1, A_2, A_3 are called components of dual quaternion Q. A quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k$ is pieced into two parts with scalar piece $S_Q = A_0$ and vectorial piece $\overrightarrow{V_Q} = A_1 i + A_2 j + A_3 k$. We also write $Q = S_Q + \overrightarrow{V_Q}$. Moreover, any dual quaternion Q can be written in the form $Q = q + \varepsilon q^*$,where $q, q^* \in \mathbf{H}.$ The conjugate of $Q = S_Q + \overrightarrow{V_Q}$ is then defined as $Q = S_Q - \overrightarrow{V_Q}$. We call a dual quaternion pure if its scalar part vanishes. The pure dual quaternions form the three-dimensional linear subspace

$$Im\mathbf{H}_D = \{A_1 i + A_2 j + A_3 k : A_1, A_2, A_3 \in \mathbf{D}\} = \left\{Q \in H_D : \overline{Q} = -Q\right\}$$

of $\mathbf{H}_D$ spanned by $\{i, j, k\}$ [4].

Summation of two dual quaternions $Q = S_Q + \overrightarrow{V_Q}$ and $P = S_P + \overrightarrow{V_P}$ is defined as $Q + P = (S_Q + S_P) + \left(\overrightarrow{V_Q} + \overrightarrow{V_P}\right)$. Multiplication with a scalar $\lambda \in \mathbf{R}$ of a dual quaternion $Q = S_Q + \overrightarrow{V_Q}$ is defined as $\lambda Q = (\lambda A_0) + (\lambda A_1)i + (\lambda A_2)j + (\lambda A_3)k$.In addition,dual quaternionic multiplication of two dual quaternions $Q = S_Q + \overrightarrow{V_Q}$ and $P = S_P + \overrightarrow{V_P}$ is defined

$$QP = S_Q S_P - \left\langle \overrightarrow{V_Q}, \overrightarrow{V_P} \right\rangle + S_Q \overrightarrow{V_P} + S_P \overrightarrow{V_Q} + \overrightarrow{V_Q} \Lambda \overrightarrow{V_P}.$$

Thus, with this multiplication operator, $\mathbf{H}_D$ is called dual quaternion algebra[4]. Dual quaternionic multiplication satisfies the following properties: For any two quaternions Q and P we have $\overline{QP} = \overline{PQ}$ and the formula for the inner product $\langle Q, P\rangle = \frac{\overline{Q}P+\overline{P}Q}{2}$. In particular, if $Q = P$, we obtain $|Q|^2 = \langle Q, Q\rangle = \overline{Q}Q$, the usual Lengith-Identity. With this, the dual quaternionic inverse of a dual quaternion $Q \in \mathbf{H}_D$ that its scalar part is nonzero can be written as $Q^{-1} = \frac{\overline{Q}}{|Q|^2}$. The 3-dual sphere

$$\mathbf{S}_D^3 = \{Q \in \mathbf{H}_D \colon |Q| = 1\} \subset \mathbf{H}_D$$

constitutes a group under dual quaternionic multiplication; this is an immediate consequence of the fact that dual quaternionic multiplication is normed as $|QP| = |Q|\,|P|$.

The dual quaternion operator. Multiplication of two unit dual quaternions $\overrightarrow{A_0}$ and $\overrightarrow{B_0}$ is as follows;

$$\overrightarrow{A_0} \times \overrightarrow{B_0} = -\left\langle \overrightarrow{A_0}, \overrightarrow{B_0}\right\rangle + \overrightarrow{A_0}\Lambda\overrightarrow{B_0} = -\cos\Theta + \overrightarrow{S}\sin\Theta$$

where $\Theta = \theta + \theta^*$ is the dual angle between the unit dual quaternions $\overrightarrow{A_0}$ and $\overrightarrow{B_0}$, and $\overrightarrow{S} = s_0 + \varepsilon s_0^* = \frac{\overrightarrow{A_0}\Lambda\overrightarrow{B_0}}{\left\|\overrightarrow{A_0}\Lambda\overrightarrow{B_0}\right\|}$ is a unit vector which is ortogonal to both $\overrightarrow{A_0}$ and $\overrightarrow{B_0}$. Also each unit dual quaternion corresponds to a directed line. In adition, the conjugate $K(\overrightarrow{A_0} \times \overrightarrow{B_0})$ of $\overrightarrow{A_0}\Lambda\overrightarrow{B_0}$ is

$$K(\overrightarrow{A_0} \times \overrightarrow{B_0}) = \overrightarrow{B_0} \times \overrightarrow{A_0} = -(\cos\Theta + \overrightarrow{S}\sin\Theta) = -\overrightarrow{Q_0}$$

and the inverses $\left(\overrightarrow{A_0}\right)^{-1}$ and $\left(\overrightarrow{B_0}\right)^{-1}$ respectively of $\overrightarrow{A_0}$ and $\overrightarrow{B_0}$ are

$$\left(\overrightarrow{A_0}\right)^{-1} = \frac{K_{\overrightarrow{A_0}}}{N_{\overrightarrow{A_0}}} = -\overrightarrow{A_0} \quad , \quad \left(\overrightarrow{B_0}\right)^{-1} = \frac{K_{\overrightarrow{B_0}}}{N_{\overrightarrow{B_0}}} = -\overrightarrow{B_0}.$$

Thus, we can write

$$\overrightarrow{Q_0} = -(\overrightarrow{B_0} \times \overrightarrow{A_0}) = \left(\overrightarrow{B_0}\right)^{-1} \times \overrightarrow{A_0} = \overrightarrow{B_0} \times \left(\overrightarrow{A_0}\right)^{-1},$$

where $\overrightarrow{Q_0} = \cos\Theta + \overrightarrow{S}\sin\Theta$ is unit dual quaternion.Thus, the following operator $\Theta \to \overrightarrow{Q_0} = \cos\Theta + \overrightarrow{S}\sin\Theta$ is called dual quaternion operator. Hence, we can say that the expression $\overrightarrow{Q_0} = \overrightarrow{B_0} \times (\overrightarrow{A_0})^{-1}$ than $\overrightarrow{B_0} = Q_0 \times \overrightarrow{A_0}$ which is found by left side multiplication of $\overrightarrow{A_0}$ by $\overrightarrow{Q_0}$ rotates $\overrightarrow{A_0}$ around the axes $\overrightarrow{S}$ with a dual angle Θ. Since $\Theta = \theta + \varepsilon\theta^*$, a rotation of angle θ, a slide of θ^* occurs and $\frac{\theta^*}{\theta}$ is the step. This statement, we can show in the following figure.

2 Dual unitary matrices and unit dual quaternions

In this section, we will firstly define the dual complex numbers similar to the complex numbers and express the dual quaternions as the dual complex numbers.

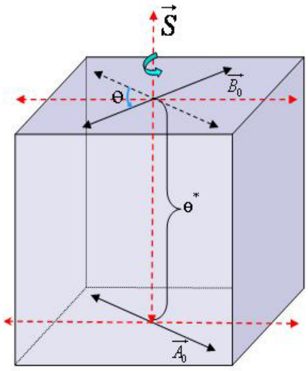

Figure 1: Screw motion

Definition 3. We know that the complex numbers set

$$\mathbf{C} = \left\{ z = a + bi : \; a, b \in \mathbf{R} \text{ and } i^2 = -1 \right\} = \{ z = (a, b) : \; a, b \in \mathbf{R} \}$$

has field structure. Now define the set $\mathbf{C}_D$ as

$$\mathbf{C}_D = \left\{ Z = A + Bi : \; A, B \in \mathbf{D} \text{ and } i^2 = -1 \right\} = \{ Z = (A, B) : \; A, B \in \mathbf{D} \}.$$

Each element of $\mathbf{C}_D$ is called a dual complex number. A dual complex number $Z = A + Bi$ can be expressed in the form $Z = DuZ + \varepsilon ImZ$, where $DuZ = A$ and $ImZ = B$. The conjugate of $Z = A + Bi$ is defined as $\overline{Z} = A - Bi$. Summation and multiplication of any two dual complex numbers $Z = A + Bi$ and $W = C + Di$ are defined in the following ways,

$$Z + W = (A + C) + (B + D)i = (A + C, B + D)$$

and

$$Z.W = (A + Bi)(C + Di) = (AC - BD) + (AD + BC)i = (AC - BD, AD + BC).$$

We can give matrix representation of multiplying the dual complex numbers Z and W as

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}.$$

Thus, $\mathbf{C}_D$ is a commutative ring with a unit element. Since for any dual complex number $Z = A + Bi$, where $A, B \in \mathbf{D}$ we can write $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$, where $a, a^*, b, b^* \in \mathbf{R}$. In particular it $a^* = b^* = 0$, then $Z = A + Bi$ becomes $Z = a + bi$ since $A = a$ and $B = b$, where $a, b \in \mathbf{R}$ and $i^2 = -1$. Namely, Z is a complex number, i.e. $Z \in \mathbf{C}$. Hence we obtain $\mathbf{C} \subset \mathbf{C}_D$.

We can give the polar form of dual complex numbers as complex numbers. Let $Z = A + Bi$ be a unit dual complex number. Then, we get $A^2 + B^2 = 1$, since $|Z| = 1$. Writing $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$, we obtain $a^2 + b^2 = 1, ab^* + ba^* = 0$. Thus, the unit dual complex number $Z = A + Bi$ can also be written as $Z = \cos\Theta + i \sin\Theta =$

$e^{i\Theta}$.Let $Z = A_0 + A_1 i$ and $W = B_0 + B_1 i$ be unit dual complex numbers. Thus, we can write $W = e^{i\theta} Z$ such that hence $e^{i\theta}$ is called the dual complex number operator, Now, we show this. The unit dual complex numbers $Z = A_0 + A_1 i$ and $W = B_0 + B_1 i$ can also be writen as $Z = \cos\Theta_Z + i\sin\Theta_Z = e^{i\Theta_Z}$ and $W = \cos\Theta_W + i\sin\Theta_W = e^{i\Theta W}$,where $\cos\Theta_Z = A_0$, $\sin\Theta_Z = A_1$ and $\cos\Theta_W = B_0$, $\sin\Theta_W = B_1$. Then

$$\begin{aligned} \frac{W}{Z} &= \frac{\cos\Theta_W + i\sin\Theta_W}{\cos\Theta_Z + i\sin\Theta_Z} \\ &= \cos(\Theta_W - \Theta_Z) + i\sin(\Theta_W - \Theta_Z) = e^{i(\Theta_W - \Theta_Z)}. \end{aligned}$$

For $\Theta = \Theta_W - \Theta_Z$, one can obtain

$$\frac{W}{Z} = e^{i\Theta} \Rightarrow W = e^{i\Theta} Z.$$

Thus, the operator $\Theta \to e^{i\Theta} = \cos\Theta + i\sin\Theta$ is called *dual complex number operator*.

Multiplication a dual complex number by $e^{i\Theta}$ means a rotation by the dual angle Θ around the origin of this dual complex number in dual complex plane. If the dual pieces of unit dual complex numbers $Z = A_0 + A_1 i$ and $W = B_0 + B_1 i$ are taken zero, we obtain $z = a_0 + a_1 i$ and $w = b_0 + b_1 i$, where $a_0, a_1, b_0, b_1 \in \mathbf{R}$. Hence, it can be founded that

$$Z = \cos\theta_Z + i\sin\theta_Z = e^{i\theta_Z} \text{ and } W = \cos\theta_W + i\sin\theta_W = e^{i\theta_W}.$$

For $\theta = \theta_W - \theta_Z$ it can be written

$$\frac{W}{Z} = e^{i(\theta_W - \theta_Z)} = e^{i\theta} \Rightarrow W = e^{i\theta} Z.$$

This latter statement is usualy known to be the rotation by an real angle θ around the origin on complex plane.

Matrix representation of the dual quaternions

We define the set

$$\mathbf{H}_{D_1} = \left\{ Q = A_0 + A_1 i : \ A_0, A_1 \in \mathbf{D} \text{ and } i^2 = -1 \right\}$$

as a subset of dual quaternions $\mathbf{H}_D$. $\mathbf{H}_{D_1}$ is a subalgebra with the same operations of $\mathbf{H}_D$. Furthermore, there is a one-to-one correspondence between every dual quaternion $(A_0 + A_1 i) \in \mathbf{D}$ and complex number $\left(A_0 + A_1\sqrt{-1}\right) \in \mathbf{C}_D$. Hence, $\mathbf{H}_{D_1}$ and $\mathbf{C}_D$ are isomorphic, i.e. $\mathbf{H}_{D_1} \cong \mathbf{C}_D$. Thus, the algebra $\mathbf{H}_D$ contains a field which is isomorphic to $\mathbf{C}_D$. Therefore, with the operation

$$\underset{(Q,}{\mathbf{H}_D} \times \underset{B+C\sqrt{-1})}{\mathbf{C}_D} \to \underset{Q(B+C\sqrt{-1})}{\mathbf{H}_D}$$

$\mathbf{H}_D$ is a module over $\mathbf{C}_D$. However, we will consider this structure as a vector space. Since any dual quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k$ over dual complex numbers can be written in the form $Q = (A_0 + A_1 i) + j(A_2 - A_3 i)$. Hence, the vector space $\mathbf{H}_D$ onto $\mathbf{C}_D$ is 2-dimensional. From here it is clear that $\mathbf{H}_D = Sp\{1, j\}$. If q^* is taken to be 0, i.e $q^* = 0$,since $Q = q$ in any dual quaternion $Q = q + \varepsilon q^*$;$q, q^* \in \mathbf{H}$ then $\mathbf{H} \subset \mathbf{H}_D$. Thus, ıf $\mathbf{H}_D$ and $\mathbf{H}$ correspond to $\mathbf{C}_D$ and $\mathbf{C}$, respectively, we obtain $\mathbf{H} \cong \mathbf{C}$ since the real quaternion $q = a_0 1 + a_1 i + a_2 j + a_3 k$ can be written in the form $q = (a_0 + a_1 i) + j(a_2 - a_3 i)$.

The following operator

$$T : \mathbf{H}_D \to Hom(\mathbf{H}_D, \mathbf{H}_D)$$

is used to obtain matrix representation of dual quaternions.For every $Q \in \mathbf{H}_D$, we describe the transformation as,

$$T_Q : \underset{P}{\mathbf{H}_D} \to \underset{T_Q(P)=QP}{\mathbf{H}_D}$$

If dual quaternion product is considered, it can be easily seen that the operators linear. Thus, the set

$$Hom(\mathbf{H}_D, \mathbf{H}_D) = \{T_Q : Q \in \mathbf{H}_D\}$$

becomes a module over $\mathbf{C}_D$ that will be considered as a vector space. If $Q = A_0 + A_1 i + A_2 j + A_3 k$ for any $Q \in \mathbf{H}_D$, take $A = A_0 + A_1 i$ and $B = A_2 - A_3 i$, since for any $Q \in \mathbf{H}_D$ we can write $Q = 1A + jB \in \mathbf{H}_D$, then

$$\begin{aligned} T_Q(1) &= Q = 1A + jB \\ T_Q(j) &= Qj = (1A + jB)j = 1(-B) + jA \end{aligned}$$

where T_Q is the corresponding matrix over $\mathbf{C}_D$ obtained by the transformation. Hence,

$$T_Q = \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}$$

Thus, the transformation $Q \leftrightarrow T_Q$ is the 2×2 matrix representation of the algebra $\mathbf{H}_D$ over the dual complex numbers module $\mathbf{C}_D$ If the dual pieces of the dual numbers A_0, A_1, A_2, A_3 in dual quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k$ are taken zero, a real quaternion in the form $Q = q = a_0 1 + a_1 i + a_2 j + a_3 k$ is obtained. It can be written as $Q = q = 1(a_0 + a_1 i) + j(a_2 - a_3 i)$ or $Q = q = 1a + jb$ since $\mathbf{H} \cong \mathbf{C}$,where $a, b \in \mathbf{C}$ and $a = a_0 + a_1 i, b = a_2 - a_3 i$.The matrix corresponding to the real quaternion $Q = q$ can be found as

$$T_Q = T_q = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$$

Thus, the 2×2 matrix representation of algebra $\mathbf{H}$ over complex numbers field $\mathbf{C}$ is obtained. Consider the set of matrices,

$$M_2(\mathbf{C}_D) = \left\{ \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix} : A, B \in \mathbf{C}_D \right\}$$

define an transformation as,

$$f : \underset{Q}{\mathbf{H}_D} \to \underset{f(Q)=T_Q}{M_2(\mathbf{C}_D)}.$$

Thus, let us show that $f(QP) = f(Q)f(P)$ or $T_{QP} = T_Q T_P$. For $Q = A_0 + A_1 i + A_2 j + A_3 k$ and $P = B_0 + B_1 i + B_2 j + B_3 k$

$$\begin{aligned} QP &= S_Q S_P - \left\langle \overrightarrow{V_Q}, \overrightarrow{V_P} \right\rangle + S_Q \overrightarrow{V_P} + S_P \overrightarrow{V_Q} + \overrightarrow{V_Q} \Lambda \overrightarrow{V_P} \\ &= A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3) + (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2)\, i \\ &\quad + (A_0 B_2 + A_2 B_0 + A_3 B_1 - A_1 B_3)\, j + (A_0 B_3 + A_1 B_2 - A_2 B_1)\, k \end{aligned}$$

is obtained. Then, we can write

$$\begin{aligned} Q &= (A_0 + A_1 i) + j\,(A_2 - A_3 i)\,, P = (B_0 + B_1 i) + j\,(B_2 - B_3 i) \\ QP &= [(A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3)) + (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2)\, i] \\ &\quad + j\,[(A_0 B_2 + A_2 B_0 + A_3 B_1 - A_1 B_3) - (A_0 B_3 + A_3 B_0 + A_1 B_2 - A_2 B_1)\, i]\,. \end{aligned}$$

Hence,

$$T_Q = \begin{bmatrix} A_0 + A_1 i & -A_2 - A_3 i \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix}, T_P = \begin{bmatrix} B_0 + B_1 i & -B_2 - B_3 i \\ B_2 - B_3 i & B_0 - B_1 i \end{bmatrix}$$

$$T_{QP} = [\begin{matrix} A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3) + (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2)\, i \\ (A_0 B_2 + A A_2 B_0 + A_3 B_1 - A_1 B_3) - (A_0 B_3 + A_3 B_0 + A_1 B_2 - A_2 B_1)\, i \end{matrix}$$

$$\begin{matrix} -(A_0 B_2 + A_2 B_0 + A_3 B_1 - A_1 B_3) - (A_0 B_3 + A_3 B_0 + A_1 B_2 - A_2 B_1)\, i \\ A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3) - (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2)\, i \end{matrix}]$$

is found. Since

$$\begin{aligned} T_Q T_P &= \begin{bmatrix} A_0 + A_1 i & -A_2 - A_3 i \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} \begin{bmatrix} B_0 + B_1 i & -B_2 - B_3 i \\ B_2 - B_3 i & B_0 - B_1 i \end{bmatrix} \\ &= [\begin{matrix} (A_0 + A_1 i)(B_0 + B_1 i) + (-A_2 - A_3 i)(B_2 - B_3 i) \\ (A_2 - A_3 i)(B_0 + B_1 i) + (A_0 - A_1 i)(B_2 - B_3 i) \end{matrix} \\ &\quad \begin{matrix} (A_0 + A_1 i)(-B_2 - B_3 i) + (-A_2 - A_3 i)(B_0 - B_1 i) \\ (A_2 - A_3 i)(-B_2 - B_3 i) + (A_0 - A_1 i)(B_0 - B_1 i) \end{matrix}] \\ &= [\begin{matrix} A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3) + (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2)\, i \\ (A_0 B_2 + A_2 B_0 + A_3 B_1 - A_1 B_3) - (A_0 B_3 + A_3 B_0 + A_1 B_2 - A_2 B_1)\, i \end{matrix} \\ &\quad \begin{matrix} -(A_0 B_2 + A_2 B_0 + A_3 B_1 - A_1 B_3) - (A_0 B_3 + A_3 B_0 + A_1 B_2 - A_2 B_1)\, i \\ A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3) - (A_0 B_1 + A_1 B_0 + A_2 B_3 - A_3 B_2)\, i \end{matrix}] \end{aligned}$$

, $T_{QP} = T_Q T_P$ or $f(QP) = f(Q)f(P)$. Thus, the isomorphism $\mathbf{H}_D \cong M_2(\mathbf{C}_D)$ is found. If for M_D the set of matrices $M_2(\mathbf{C}) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbf{C} \right\}$ and for $\mathbf{H}_D$ the algebra of real quaternions $\mathbf{H}$ are substituted, since the dual pieces of Q and P are equal to zero $Q = q = a_0 + a_1 i + a_2 j + a_3 k, P = p = b_0 + b_1 i + b_2 j + b_3 k$ will be obtained. Hence, the special condition $T_{qp} = T_q T_p$ is obtained that gives the isomorphism $\mathbf{H} \cong M_2(\mathbf{C})$.

Matrix representation of the unit dual sphere

The set

$$\begin{aligned}\mathbf{S}_D^3 &= \{Q = (A_0 + A_1 i) + j(A_2 - A_3 i) : Q \in \mathbf{H}_D\} \\ &= \left\{Q = A_0 + A_1 i + A_2 j + A_3 k : Q \in \mathbf{H}_D, A_0^2 + A_1^2 + A_2^2 + A_3^2 = 1\right\}\end{aligned}$$

is named as unit dual sphere in $\mathbf{H}_D$. The set $\mathbf{S}_D^3$ is a group with quaternion product. Also taken the special dual uniter group

$$\mathbf{SU}_2(\mathbf{D}) = \left\{K = \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix} : K \in M_D, \overline{(K^T)}K = I_2, \det K = 1\right\}$$

instead of M_D.

Theorem 4. *The groups $\mathbf{S}_D^3$ and $\mathbf{SU}_D(2)$ are isomorphic.*

Proof. We define the transformation

$$g : \underset{Q}{\mathbf{S}_D^3} \to \underset{g(Q)=T_Q}{\mathbf{SU}_D(2)}$$

Firstly; this transformation is well defined. Because, for $Q = A_0 + A_1 i + A_2 j + A_3 k \in S_D^3$, $A_0^2 + A_1^2 + A_2^2 + A_3^2 = 1$. Hence,

$$T_Q = \begin{bmatrix} A_0 + A_1 i & -A_2 - A_3 i \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} \Rightarrow \overline{\left(T_Q^T\right)} = \begin{bmatrix} A_0 - A_1 i & A_2 + A_3 i \\ -A_2 + A_3 i & A_0 + A_1 i \end{bmatrix}.$$

Thus, it is found that $T_Q \in \mathbf{SU}_2(\mathbf{D})$ since

$$\begin{aligned}\overline{\left(T_Q^T\right)} T_Q &= \begin{bmatrix} A_0 - A_1 i & A_2 + A_3 i \\ -A_2 + A_3 i & A_0 + A_1 i \end{bmatrix}\begin{bmatrix} A_0 + A_1 i & -A_2 - A_3 i \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} \\ &= [\begin{matrix} (A_0 - A_1 i)(A_0 + A_1 i) + (A_2 + A_3 i)(A_2 - A_3 i) \\ (-A_2 + A_3 i)(A_0 + A_1 i) + (A_0 + A_1 i)(A_2 - A_3 i) \end{matrix} \\ &\qquad \begin{matrix} (A_0 - A_1 i)(-A_2 - A_3 i) + (A_2 + A_3 i)(A_0 - A_1 i) \\ (-A_2 + A_3 i)(-A_2 - A_3 i) + (A_0 + A_1 i)(A_0 - A_1 i) \end{matrix}] \\ &= \begin{bmatrix} A_0^2 + A_1^2 + A_2^2 + A_3^2 & 0 \\ 0 & A_0^2 + A_1^2 + A_2^2 + A_3^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2\end{aligned}$$

and

$$\begin{aligned}\det(T_Q) &= \det\begin{bmatrix} A_0 + A_1 i & -A_2 - A_3 i \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} \\ &= (A_0 + A_1 i)(A_0 - A_1 i) - (-A_2 - A_3 i)(A_2 - A_3 i) \\ &= A_0^2 + A_1^2 + A_2^2 + A_3^2 = 1.\end{aligned}$$

Now, we show that $g(QP) = g(Q)g(P)$ or $T_{QP} = T_Q T_P$ for the function

$$g : \underset{Q}{\mathbf{S}_D^3} \to \underset{g(Q)=T_Q}{\mathbf{SU}_D(2)},$$

where any $Q, P \in \mathbf{S}_D^3$. We know that for any $Q, P \in \mathbf{S}_D^3; Q, P \in \mathbf{H}_D$ then $f(QP) = f(Q) f(P)$ or $T_{QP} = T_Q T_P$.From here $g(QP) = g(Q) g(P)$ is obtained. Thus, the isomorphism $\mathbf{S}_D^3 \cong \mathbf{SU}_D(2)$ is found. □

If the dual pieces of dual quaternion $Q \in \mathbf{S}_D^3$ are taken zero, $Q = q = a_0 1 + a_1 i + a_2 j + a_3 k$ becomes a real quaternion. Hence, $Q = q \in \mathbf{S}^3$. So, for the matrix

$$T_Q = T_q = \begin{bmatrix} a_0 + a_1 i & -a_2 - a_3 i \\ a_2 - a_3 i & a_0 - a_1 i \end{bmatrix},$$

$\overline{(Tq^T)} T_q = I_2$ and $\det(T_q) = 1$. This give us the isomorphism $\mathbf{S}^3 \cong \mathbf{SU}(2)$.

If the imaginary pieces of real quaternion

$$\begin{aligned} q &= a_0 1 + a_1 i + a_2 j + a_3 k \\ &= (a_0 + a_1 i) + j(a_2 - a_3 i) \end{aligned}$$

are taken zero, the complex number $q = a_0 + j a_2 \in \mathbf{C}$ is obtained, where $a_0, a_2 \in \mathbf{R}$ and $a_0^2 + a_2^2 = 1$. Then,

$$T_q = \begin{bmatrix} a_0 & -a_2 \\ a_2 & a_0 \end{bmatrix}$$

,which gives $\overline{(Tq^T)} = T_q^T, \left(Tq^T\right)(T_q) = \ I_2$ and $\det(T_q) = 1$. Thus, the isomorphism $\mathbf{S}^1 \cong \mathbf{SO}(2)$ is found.

Definition 5. Let $\mathbf{H}_D$ be the algebra of dual quaternions; n being an integer greater than zero, we denote by $\mathbf{H}_D^n$ the product of n sets identical to $\mathbf{H}_D$. Elements of the set

$$\begin{aligned} \mathbf{H}_D^n &= \mathbf{H}_D \times \mathbf{H}_D \times ... \times \mathbf{H}_D \\ &= \left\{ \overrightarrow{U} = (u^1, u^2, ..., u^n) : u^i \in \mathbf{H}_D, 1 \leq i \leq n \right\} \end{aligned}$$

will be called a (dual quaternionic) vector; $u^1, u^2, ..., u^n$ will be called the coordinates of this vector[2].

The addition of vectors and the scalar product in $\mathbf{H}_D$ is defined by

$$\underset{(\overrightarrow{U},\ \ \overrightarrow{V})}{+ : \mathbf{H}_D^n \times \mathbf{H}_D^n} \to \underset{\overrightarrow{U}+\overrightarrow{V}=(u^1,u^2,...,u^n)+(v^1,v^2,...,v^n)}{\mathbf{H}_D^n}$$

and

$$\underset{(\overrightarrow{U},\ \ Q)}{\cdot : \mathbf{H}_D^n \times \mathbf{H}_D} \to \underset{\overrightarrow{U}Q=(u^1 q, u^2 q, ..., u^n q)}{\mathbf{H}_D}.$$

Thus, $\mathbf{H}_D^n$ becomes a modul over $\mathbf{H}_D$. But with this structure we will call a vector space for $\mathbf{H}_D^n$. The transformation

$$\underset{(\overrightarrow{U},\ \ \overrightarrow{V})}{\langle , \rangle : \mathbf{H}_D^n \times \mathbf{H}_D^n} \to \underset{\left\langle \overrightarrow{U}, \overrightarrow{V} \right\rangle = \sum_{i=1}^{n} K_{u^i} v^i}{\mathbf{H}_D}$$

is symplectic product onto $\mathbf{H}_D^n$, where $U = (u^1, u^2, ..., u^n)$ and $V = (v^1, v^2, ..., v^n)$ are dual quaternionic vectors. Symplectic product over $\mathbf{H}_D^n$ has similar properties with Hermitian product. That is to say; The symplectic product have the following properties

$$\begin{aligned}\left\langle \overrightarrow{U}+\overrightarrow{V},\overrightarrow{W}\right\rangle &= \left\langle \overrightarrow{U},\overrightarrow{W}\right\rangle + \left\langle \overrightarrow{V},\overrightarrow{W}\right\rangle \\ \left\langle \overrightarrow{U},\overrightarrow{V}+\overrightarrow{W}\right\rangle &= \left\langle \overrightarrow{U},\overrightarrow{V}\right\rangle + \left\langle \overrightarrow{U},\overrightarrow{W}\right\rangle \\ \left\langle \overrightarrow{U},\overrightarrow{V}Q\right\rangle &= \left\langle \overrightarrow{U},\overrightarrow{V}\right\rangle Q \ , \ \left\langle \overrightarrow{U}Q,\overrightarrow{V}\right\rangle = K_Q \left\langle \overrightarrow{U},\overrightarrow{V}\right\rangle ,\end{aligned}$$

where any vectors $\overrightarrow{U},\overrightarrow{V},\overrightarrow{W} \in \mathbf{H}_D^n$ and any quaternion $Q \in \mathbf{H}_D$. Hence

$$\left\langle \overrightarrow{U},\overrightarrow{U}\right\rangle = \sum_{i=1}^{n} K_{u^i} u^i = \left\| \overrightarrow{U} \right\| ,$$

where $\left\| \overrightarrow{U} \right\|$ is a dual number and this number is called the norm of $\overrightarrow{U}$.

Definition 6. The vector space $\mathbf{H}_D$ on which a symplectic product is defined is called a dual symplectic vector space.

A linear endomorphism

$$\sigma : \mathbf{H}_D \to \mathbf{H}_D$$

of the vector space $\mathbf{H}_D$ is a mapping that have the following properties

$$\sigma\left(\overrightarrow{U}+\overrightarrow{V}\right) = \sigma\left(\overrightarrow{U}\right) + \sigma\left(\overrightarrow{V}\right), \sigma\left(\overrightarrow{U}Q\right) = \sigma\left(\overrightarrow{U}\right) Q,$$

where any vectors $\overrightarrow{U},\overrightarrow{V} \in \mathbf{H}_D^n$ and any quaternion $Q \in \mathbf{H}_D$. Hence, the linear endomorphism of $\mathbf{H}_D^n$ are in a one-to-one correspondence with the matrices (Q_{ij}) which components are in $\mathbf{H}_D$. We shall denote by the same symbol σ the endomorphism itself and the corresponding matrix. If $\sigma = (Q_{ij})$ and $\tau = (P_{ij})$ are two of these matrices, we denote (as usual) by $\sigma\tau$ the matrix (s_{ij}) with

$$(s_{ij}) = \left(\sum_{k=1}^{n} Q_{ij} P_{kj}\right)$$

and then we have $\sigma\tau = \sigma \circ \tau$, i.e. $(\sigma\tau)\left(\overrightarrow{U}\right) = \sigma\left(\tau\overrightarrow{U}\right)$ for every vector $\overrightarrow{U} \in \mathbf{H}_D$. the set of all matrices of degree n with coefficients in $\mathbf{H}_D$ will be denoted by $M_n\left(\mathbf{H}_D\right)$[2].

Definition 7. Let $\mathbf{H}_D^n$ be a dual symplectic vector space. If the linear endomorphism $\sigma : \mathbf{H}_D^n \to \mathbf{H}_D^n$ has the following property

$$\left\langle \sigma\left(\overrightarrow{U}\right), \sigma\left(\overrightarrow{V}\right)\right\rangle = \left\langle \overrightarrow{U},\overrightarrow{V}\right\rangle$$

for every vectors $\overrightarrow{U},\overrightarrow{V} \in \mathbf{H}_D^n$, σ is called a symplectic mapping over $\mathbf{H}_D$[1].

Proposition 8. *If σ is a symplectic mapping, σ has the invers σ^{-1} and σ^{-1} is also a symplectic mapping[1].*

Theorem 9. Let *σ be a symplectic mapping over $\mathbf{H}_D^n$. If $Q = (q_{ij})$ is the corresponding matrix of σ, Q satisfies the following equality*

$$\left(K_Q^t\right) Q = I_n,$$

where I_n is the unit matrix with n type.

Proof. Let the vector space $\mathbf{H}_D^n$ be spanned by the n linearly independent vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_n}$; where $\overrightarrow{e_i}$ is the vector whose j th coordinate is δ_{ij}. We may write $\langle \sigma\overrightarrow{e_1}, \sigma\overrightarrow{e_j} \rangle = \langle \overrightarrow{e_i}, \overrightarrow{e_j} \rangle$ to be the linear endomorphism $\sigma : \mathbf{H}_D^n \to \mathbf{H}_D^n$ for $\sigma\overrightarrow{e_i} = \sum_{j=1}^{n} \overrightarrow{e_j} q_{ji}$ and $\sigma\overrightarrow{e_k} = \sum_{s=1}^{n} \overrightarrow{e_s} q_{sk}$. Hence ,we obtain

$$\begin{aligned}
\left\langle \sum_{j=1}^{n} \overrightarrow{e_j} q_{ji}, \sum_{s=1}^{n} \overrightarrow{e_s} q_{sk} \right\rangle &= (\delta_{ik}) \\
\sum_{j,s=1}^{n} K_{q_{ji}} \langle \overrightarrow{e_j}, \overrightarrow{e_s} \rangle q_{sk} &= \sum_{j=1}^{n} K_{q_{ji}} q_{jk} = (\delta_{ik}) \\
\sum_{j=1}^{n} \left(K_{q_{ji}}\right)^t q_{jk} &= (K_Q)^t Q = I.
\end{aligned}$$

□

Definition 10. The group of symplectic mappings onto $\mathbf{H}_D^n$ which are isomorphic to the group of matrices are called group of symplectic matrices and denoted in the following form as in[1]

$$\mathbf{S}_p(n, \mathbf{H}_D) = \left\{ \sigma \in M_n(\mathbf{H}_D) : (K_\sigma)^t \sigma = I_n \right\}.$$

If we choose $n = 1$ as a special case, we obtain

$$\begin{aligned}
\mathbf{S}_p(1, \mathbf{H}_D) &= \left\{ \sigma \in M_1(\mathbf{H}_D) : (K_\sigma)^t \sigma = 1 \right\} \\
&= \{ \sigma \in M_1(\mathbf{H}_D) : N_\sigma = 1 \} \\
&= S_D^3.
\end{aligned}$$

Then, $\mathbf{S}_p(1, \mathbf{H}_D) \cong \mathbf{S}_D^3$ is found. If $\sigma \in \mathbf{S}_p(1, \mathbf{H}_D)$, it is the form

$$\sigma = \begin{bmatrix} A_0 + A_1 i & -A_2 - A_3 i \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} \text{ and } (K_\sigma)^t \sigma = 1.$$

If the dual pieces of σ are taken zero, $\sigma \in \mathbf{S}_p(1, \mathbf{H})$ since

$$\sigma = \begin{bmatrix} A_0 & -A_2 \\ A_2 & A_0 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 i & a_2 - a_3 i \\ a_2 - a_3 i & a_0 - a_1 i \end{bmatrix}.$$

Furthermore, it is found $\sigma \in \mathbf{S}^3$ since $(K_\sigma)^t \sigma = 1$ or $N_\sigma = 1$. Hence, the isomorphism $\mathbf{S}_p(1, \mathbf{H}) \cong \mathbf{S}^3$ is obtained. Now, let the imaginary pieces of σ be zero. For $\sigma = \begin{bmatrix} a_0 & -a_2 \\ a_2 & a_0 \end{bmatrix}$, we find $K_\sigma = \sigma$. Since $(K_\sigma)^t \sigma = 1$ than $\sigma^t \sigma = 1$, we find $\sigma \in \mathbf{S}^1$. Thus, the isomorphism $\mathbf{S}_p(1,\mathbf{C}) \cong \mathbf{S}^1$ is obtained. Finally, we may write the followings ın general:

$$\mathbf{SU}_D\left(2\right) \cong \mathbf{S}_D^3 \cong \mathbf{S}_p\left(1,\mathbf{H}_D\right).$$

As a special case of this statement we find

$$\mathbf{SU}\left(2\right) \cong \mathbf{S}^3 \cong \mathbf{S}_p\left(1,\mathbf{H}\right).$$

It also obtained as a special case of this last statement that

$$\mathbf{SO}\left(2\right) \cong \mathbf{S}^1 \cong \mathbf{S}_p\left(1,\mathbf{C}\right).$$

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