DUAL VARIABLE METRIC ALGORITHMS FOR CONSTRAINED OPTIMIZATION†

Shih-Ping Han

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Department of Computer Science
Cornell University
Ithaca, NY 14853

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Abstract:

We present a class of algorithms for solving constrained optimization problems. In the algorithm non-negatively constrained quadratic programming subproblems are iteratively solved to obtain estimates of Lagrange multipliers and with these estimates a sequence of points which converges to the solution is generated. To achieve a superlinear rate of convergence the matrix appearing in the subproblem is required to be an approximate inverse of the Hessian of the Lagrangian. Some well-known variable metric updates such as the BFGS update are employed to generate the matrix and the resulting algorithm converges locally with a superlinear rate. When the penalty Lagrangian developed by Hestenes, Powell and Rockafellar is incorporated in the algorithm. it turns out to be closely related to the recently developed the method of multipliers. Unlike the method of multipliers, our algorithm possesses a superlinear rate of convergence even without requiring a penalty parameter goint to infinity and therefore avoids the numerical instability so caused.

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1. Introduction

The techniques for solving quadratic programming problems have been developed so extensively that it becomes feasible to deal with the general nonlinear programming problem by reducing it to a sequence of quadratic programming subproblems. Adopted by many authors [17, 18, 22, 23] and shown very effective, this approach allows us to approximate the nonlinear programming problem quadratically and affords an extension of Newton's and Newton-like methods to constrained optimization. Following this approach, we present in this work a class of algorithms in which we iteratively solve non-negatively constrained quadratic programming subproblems to obtain estimates of Lagrange multipliers and with these estimates generate a sequence of points which converges to the solution. To achieve a superlinear rate of convergence the matrix appearing in the subproblem is required to be an approximate inverse of the Hessian of the Lagragian. We suggest variable metric updates to generate these matrices and justify our suggestion by showing that, when some well-known updates such as the BFGS update are employes in this context, the algorithm converges locally with a superlinear rate. The penalty Lagrangian developed by Hestenes [24], Powell [33] and Rockafellar [37] may also be incorporated into the algorithm to replace the ordinary Lagragian; the resulting algorithm turns

2. Algorithm

In this paper we consider the following nonlinear programming problem

$$(P) \qquad \min_{x} \; f(x) \quad \text{s.t.} \quad g(x) \leqq 0, \quad h(x) = 0$$

where f, g, and h are functions from R^n into R, R^m, and R^q respectively. The Lagrangian of problem (P) is the real-valued function $L(x,u,v) = f(x) + u^T g(x) + v^T h(x)$ defined on R^{n+m+q}, and a Kuhn-Tucker triple is a vector $z^* = (x^*,u^*,v^*)$ in R^{n+m+q} which satisfies the first-order Kuhn-Tucker conditions [28]. We define a quadratic programming problem DQ(x,A)

$$\text{(2.1)} \quad \min_{(u,v)} \; \frac{1}{2}(\nabla f(x)+\nabla g(x)u+\nabla h(x)v)^T A(\nabla f(x)+\nabla g(x)u+\nabla h(x)v)-u^T g(x)-v^T h(x) \quad \text{s.t.} \quad u \geqq 0$$

associated with any x in R^n and any A in $L(R^n)$.

Definition 2.1 A vector $z = (\hat{x},\hat{u},\hat{v})$ in R^{n+m+q} is a z-solution of DQ(x,A) if $(\hat{u},\hat{v})$ is a Kuhn-Tucker point of DQ(x,A) and

$$\text{(2.2)} \qquad \hat{x} = x - A(\nabla f(x) + \nabla g(x)\hat{u} + \nabla h(x)\hat{v}). \quad \square$$

It is noted that DQ(x,A) has no constraint at all if (P) has out to be closely related to the recently developed multiplier method [8, 24, 33, 37, 38] — a very promising method which has lately attracted a great deal of attention. Unlike the multiplier method, our algorithm has a superlinear rate of convergence even without requiring a penalty parameter going to infinity and therefore avoids the numerical instability so caused.

In Section 2 we state the algorithm and compare it with the related results in the literature of nonlinear programming. Sufficient conditions for convergence of the algorithm and for superlinear rates of convergence are presented in Sections 3 and 4 respectively. In Section 5 we embed the BFGS and some other updates into the algorithm and with results obtained in Sections 3 and 4 show that the algorithm converges locally with a superlinear rate. In Section 6 the algorithm is modified by replacing the Lagrangian by a penalty Lagrangian in order to relax some assumptions in the convergence theorems. Some comments and computational results are contained in Section 7.

We note here that all vectors are column vectors and a row vector will be indicated by superscript T. For convenience a column vector in R^{n+m+q} is sometimes written as (x,u,v). We use x^i to denote different vectors; i.e., x^1 and x^2. To avoid some cumbersome constants we restrict ourselves to the ℓ_2 vector norm and operator norm and denote it by $||\cdot||$. An ε-neighborhood $N(x,\varepsilon)$ of a point x in R^n is the set $N(x,\varepsilon) = \{y \in R^n : || y-x || < \varepsilon\}$. We use $L(R^n)$ to indicate the set of n×n real matrices and write $f \in LC^2[x]$ if the function f has Lipschitz continuous second-order derivatives in a neighborhood of x.

no inequality constraint. Now we can state the algorithm as follows.

Algorithm step 1. Start with an estimate of a Kuhn-Tucker triple $z^0 = (x^0, u^0, v^0)$ of Problem (P) and an estimate of A_0 of the inverse of the Hessian of the Lagrangian.

step 2. Set $k = 0$.

step 3. Find a z-solution of $DQ(x^k, A_k)$ and call this z-solution $z^{k+1} = (x^{k+1}, u^{k+1}, v^{k+1})$. If there is more than one such z-solution, choose one which is closest to z^k.

step 4. If $z^{k+1} = (x^{k+1}, u^{k+1}, v^{k+1})$ satisfies a prescribed convergence criterion, stop; otherwise, update A_{k+1} by some scheme, set $k = k+1$ and go to step 3. □

In the algorithm, with an estimate (u^{k+1}, v^{k+1}) of the Lagrange multipliers obtained from solving $QD(x^k, A_k)$, we find a new point x^{k+1} by taking one step of a gradient method to minimize the Lagrangian $L(x, u^{k+1}, v^{k+1})$; when $A_k = \nabla_{xx} L(x^k, u^{k+1}, v^{k+1})^{-1}$, a Newton step is carried out. In this paper we are more interested in the variable metric way to generate the matrix A_k; for example, the very successful BFGS update in unconstrained optimization can be so employed here. It is perhaps worth mentioning that the updated matrix A_k is used to find not only x^{k+1} but also the multipliers (u^{k+1}, v^{k+1}).

By Dorn's duality theorem [28] and under the assumption that A is symmetric and positive definite, the quadratic program DQ(x,A) is dual to the quadratic program

$$
(2.3)\qquad \begin{array}{ll} \min\limits_{s} & \nabla f(x)^T s + \frac{1}{2} s^T A^{-1} s \\ \text{s.t.} & g(x) + \nabla g(x)^T s \leqq 0 \\ & h(x) + \nabla h(x)^T s = 0, \end{array}
$$

which can be viewed as a quadratic approximation to problem (P) if A^{-1} is the Hessian of the Lagrangian. Some efficient algorithms [17, 18, 21, 22] based on (2.3) have been developed; however, our algorithm seems more promising since the subproblem (2.1) has only non-negative constraints and no constraints at all if problem (P) has only equality constraints. Moreover, some unconstrained optimization updating schemes are more naturally incorporated in (2.1) than in (2.3).

The algorithm is related to the dual, feasible direction algorithm developed by Mangasarian [29]; but unlike it we do not require the generated points to be feasible for the original problem (P) and therefore never need an anti-zigzag procedure to avoid the jamming situation.

For solving $DQ(x^k, A_k)$ there are a number of effective methods in the extensive literature of quadratic programming. These include Beale's method [2], Wolfe's method [40], Ritter's method [36] and the principal pivoting method [10, 11].

and for $i = 1, \cdots m$, we have

$$w_i \hat{u}_i = 0. \tag{3.4}$$

It also follows from Definition 2.1 that

$$A(\nabla f(\bar{x}) + \nabla g(\bar{x})\hat{u} + \nabla h(\bar{x})\hat{v}) + (\hat{x} - \bar{x}) = 0. \tag{3.5}$$

Thus,

$$w = -(g(\bar{x}) + \nabla g(\bar{x})^T(\hat{x} - \bar{x}))$$

which in conjunction with (3.4) implies that for $i = 1, \cdots m$

$$\hat{u}_i(g_i(\bar{x}) + \nabla g_i(\bar{x})^T(\hat{x} - \bar{x})) = 0. \tag{3.6}$$

From (3.3) and (3.5) we have

$$h(\bar{x}) + \nabla h(\bar{x})^T(\hat{x} - \bar{x}) = 0 \tag{3.7}$$

which combines with (3.5) and (3.6) to lead to the desired result. □

Corollary 3.2 Let A in $L(R^n)$ be symmetric and non-singular and $\hat{z} = (\hat{x}, \hat{u}, \hat{v})$ be a z-solution of $DQ(\bar{x}, A)$. If $\hat{x} = \bar{x}$ then $\hat{z} = (\hat{x}, \hat{u}, \hat{v})$ is a Kuhn-Tucker triple of problem (P), □

Let $z^* = (x^*, u^*, v^*)$ be a Kuhn-Tucker triple of problem (P); the non-singularity of $\nabla_z G(x^*, A, z^*)$ with $A = \nabla_{xx} L(x^*, u^*, v^*)^{-1}$ is essential for establishing our convergence theorems. To ensure such non-singularity we need the following condition, which was first studied by Fiacco and McCormick [14] and has been called "the Jacobian uniqueness condition" [27].

3. Convergence Theorems

In this section we shall show that under suitable conditions the algorithm will generate a sequence of vectors in R^{n+m+q} which converge to a Kuhn-Tucker triple of problem (P). First, we define the following function $G(\tilde{x},A,\cdot): R^{n+m+q} \to R^{n+m+q}$,

$$G(\tilde{x},A,z) = \begin{bmatrix} A(\nabla f(\tilde{x})+\nabla g(\tilde{x})u+\nabla h(\tilde{x})v) + (x-\tilde{x}) \\ u_1(g_1(\tilde{x})+\nabla g_1(\tilde{x})^T(x-\tilde{x})) \\ \vdots \\ u_m(g_m(\tilde{x})+\nabla g_m(\tilde{x})^T(x-\tilde{x})) \\ h_1(\tilde{x})+\nabla h_1(\tilde{x})^T(x-\tilde{x}) \\ \vdots \\ h_q(\tilde{x})+\nabla h_q(\tilde{x})^T(x-\tilde{x}) \end{bmatrix} \tag{3.1}$$

which is associated with any $\tilde{x}$ in R^n and any A in $L(R^n)$. The function $G(\tilde{x},A,\cdot)$ is related to the equalities of the Kuhn-Tucker conditions of problem $DQ(\tilde{x},A)$ by the following lemma.

Lemma 3.1 If A in $L(R^n)$ is symmetric and $\hat{z} = (\hat{x},\hat{u},\hat{v})$ in R^{n+m+q} is a z-solution of $DQ(\bar{x},A)$ then $G(\bar{x},A,\hat{z}) = 0$. □

Proof: If $\hat{z} = (\hat{x},\hat{u},\hat{v})$ is a z-solution of $DQ(\bar{x},A)$, then by Definition 2.1 $(\hat{u},\hat{v})$ is a Kuhn-Tucker point of $DQ(\bar{x},A)$; hence, there exists a vector w in R^m such that $w \geqq 0$ and

$$\nabla g(\bar{x})^T A(\nabla f(\bar{x})+\nabla g(\bar{x})\hat{u}+\nabla h(\bar{x})\hat{v}) - g(\bar{x}) - w = 0, \tag{3.2}$$

$$\nabla h(\bar{x})^T A(\nabla f(\bar{x})+\nabla g(\bar{x})\hat{u}+\nabla h(\bar{x})\hat{v}) - h(\bar{x}) = 0, \tag{3.3}$$

<u>Definition 3.3</u> A Kuhn-Tucker triple $z^* = (x^*,u^*,v^*)$ of problem (P) satisfies the Jacobian uniqueness condition if the following three conditions are simultaneously satisfied

(a) $u_i^* > 0$ if $i \in I(x^*) = \{j : g_j(x^*) = 0\}$

(b) the gradients $\{\nabla g_i(x^*)\}$ (all $i \in I(x^*)$), $\{\nabla h_j(x^*)\}$, $j = 1,\ldots q$ are linearly independent

(c) for every non-zero vector y satisfying $y^T\nabla g_i(x^*) = 0$ for all $i \in I(x^*)$ and $y^T\nabla h_j(x^*) = 0$, $j = 1,\ldots q$, it follows that $y^T \nabla_{xx}L(z^*)y > 0$. □

We note here that conditions (a) and (c) have also been called the strict complementarity condition and the second order sufficiency condition respectively.

Our convergence theorems also need the following two lemmas; the proof of the first one follows from the mean value theorem and appears in [21].

<u>Lemma 3.4</u> If $z^* = (x^*,u^*,v^*) \in R^{n+m+q}$ and f, g and $h \in LC^2[x^*]$, then there exists a neighborhood $N(x^*,\varepsilon)$ and two positive numbers $\bar{K}$ and $\hat{K}$ such that for any $\bar{x}$ and $\hat{x}$ in $N(x^*,\varepsilon)$ and any $(\hat{u},\hat{v})$ in R^{m+q} we have

$$(3.8)\qquad ||\nabla_x L(\tilde{x},\hat{u},\hat{v})-\nabla_x L(\bar{x},\hat{u},\hat{v})-\nabla_{xx}L(x^*,u^*,v^*)(\tilde{x}-\bar{x})||$$

$$\leq (\bar{K}\max\{||\tilde{x}-x^*||, ||\bar{x}-x^*||\} + \tilde{K}||(\hat{u},\hat{v})-(u^*,v^*)||)||\tilde{x}-\bar{x}||. \quad \square$$

<u>Corollary 3.5</u> If all the assumptions of Lemma 3.4 hold and $\nabla_{xx}L(z^*)$ is non-singular then there exist positive numbers ε, η and ξ such that whenever $\bar{x}$, $\tilde{x} \in N(x^*,\varepsilon)$ and $(\hat{u},\hat{v}) \in N((u^*,v^*),\varepsilon)$ then

$$\eta||\bar{x}-\tilde{x}|| \leqq ||\nabla_x L(\tilde{x},\hat{u},\hat{v})-\nabla_x L(\bar{x},\hat{u},\hat{v})|| \leqq \xi||\bar{x}-\tilde{x}||. \qquad \square$$

Lemma 3.6 If f, g and h $\in LC^2[x^*]$ and $\nabla_{xx}L(z^*)$ is non-singular, then for any $\alpha > 0$ there exist two positive numbers ε and δ such that for any x in R^n and any A in $L(R^n)$ satisfying $||x-x^*|| \leqq \varepsilon$ and $||A-\nabla_{xx}L(z^*)^{-1}|| \leqq \delta$ it follows that

$$||A\nabla_x L(x,u^*,v^*)+(x^*-x)|| \leqq \alpha||x^*-x||.$$

Proof. Let $\alpha > 0$ be given and let

$$\lambda = \max\{||\nabla_{xx}L(z^*)^{-1}||,\ ||\nabla_{xx}L(z^*)||\}. \tag{3.9}$$

Choose ε and δ such that

$$\delta < 1/2, \tag{3.10}$$

$$(\delta+\lambda)(\bar{K}\varepsilon+\frac{\lambda^2\delta}{1-\lambda\delta}) \leqq \alpha \tag{3.11}$$

where $\bar{K}$ is the constant defined in Lemma 3.4. Since $||A-\nabla_{xx}L(z^*)^{-1}|| \leqq \delta$, it follows from (3.9), (3.10) and the perturbation Lemma [25] that A is non-singular and

$$||A^{-1}- {}_{xx}L(z^*)|| \leqq \frac{\lambda^2\delta}{1-\lambda\delta}. \tag{3.12}$$

Then

$$\begin{aligned}||A\nabla_{xx}L(x,u^*,v^*)+(x^*-x)|| &\leqq ||A||\ ||\nabla_x L(x,u^*,v^*)+A^{-1}(x^*-x)||\\ &\leqq ||A||\ ||\nabla_x L(x,u^*,v^*)-\nabla_{xx}L(z^*)(x^*-x)||\\ &+||A||\ ||\nabla_{xx}L(z^*)-A^{-1}||\ ||x^*-x||\end{aligned}$$

where G is defined in (3.1) and I_n is the $n \times n$ identity matrix. Let C^* denote C_A when $A = \nabla_{xx}L(z^*)^{-1}$; under the Jacobian uniqueness condition it can be shown that C^* is non-singular.

Let $r \in (0,1)$ be given, define

(3.13) $$\lambda = \max \{||C^{*-1}||,\ ||\nabla g(x^*)||+||\nabla h(x^*)||,\ ||\nabla_{xx}L(z^*)||,\ ||\nabla_{xx}L(z^*)^{-1}||\}$$

and

(3.14) $$\tau = \frac{\lambda}{1-r}$$

We first choose $\bar{\varepsilon} > 0$ such that the following conditions are satisfied

(3.15) (a) for any z and $\tilde{z}$ in $N(z^*,\bar{\varepsilon})$ and any A in $L(R^n)$ with $||A|| \leqq \frac{r}{\lambda^2} + \lambda$ we have $||\nabla_z G(\tilde{x},A,z)-C_A|| \leqq \frac{1}{2\tau}$, and for all $i = 1,\ldots m$

(b) $g_i(x^*) < 0$ implies $g_i(\tilde{x}) + g_i(\tilde{x})^T(x-\tilde{x}) < 0$,

(c) $u_i^* > 0$ implies $u_i > 0$.

Then choose $\varepsilon(r)$ and $\delta(r)$ to satisfy the following conditions, where for simplicity we write henceforth ε and δ for $\varepsilon(r)$ and $\delta(r)$ respectively.

$$\leqq (\delta+\lambda)(\bar{K}\varepsilon+\frac{\lambda^2\delta}{1-\lambda\delta})||x^*-x|| \quad \text{(by Lemma 3.4 and 3.12)}$$

$$\leqq \alpha||x^*-x||. \quad \text{(by 3.11)} \; \square$$

The following theorem which guarantees that under suitable conditions the algorithm generates a better estimate of a Kuhn-Tucker point of problem (P) in each iteration.

Theorem 3.7 Let f, g and h $\in LC^2[x^*]$. If a Kuhn-Tucker triple $z^* = (x^*,u^*,v^*)$ of problem (P) satisfies the Jacobian uniqueness condition and $\nabla_{xx}L(z^*)$ is non-singular, then for any $r \in (0,1)$ there exist two positive numbers $\varepsilon(r)$ and $\delta(r)$ such that if $||\tilde{z}-z^*|| \leqq \varepsilon(r)$ and A is a symmetric $n\times n$ matrix with $||A-\nabla_{xx}L(z^*)^{-1}|| \leqq \delta(r)$ then a closest z-solution $\bar{z}$ of $DQ(\tilde{x},A)$ to $\tilde{z}$ exists and $||\bar{z}-z^*|| \leqq r||\tilde{z}-z^*||$.

Proof. For any A in $L(R^n)$ let C_A in $L(R^{n+m+q})$ be defined as follows

$$C_A = \nabla_z \, G(x^*,A,z)\Big|_{z^*}$$

$$= \begin{bmatrix} I_n & A\nabla g(x^*) & A\nabla h(x^*) \\ u_1^*\nabla g_1(x^*)^T & & \\ \vdots & \operatorname{diag}(g_i(x^*)) \;\; i=1,\dots m & 0 \\ u_m^*\nabla g_m(x^*)^T & & \\ \nabla h(x^*)^T & 0 & 0 \end{bmatrix}$$

(3.16) (a) $\text{Max } \{\lambda\delta, \lambda^2\delta\} \leqq r$

(b) $\varepsilon \leqq \frac{\bar{\varepsilon}}{3}$

(c) for any $\tilde{z}$ in R^{n+m+q} and any A in $L(R^n)$ with $||\tilde{z}-z^*|| \leqq \varepsilon$ and $||A-\nabla_{xx}L(z^*)^{-1}|| \leqq \delta$ we have that $||G(\tilde{x},A,z^*)|| \leqq \frac{r}{2\tau}||x-x^*||$.

The existence of such ε and δ follows from Lemma 3.6 and by observing that $u^*_i\, g_i(x^*) = 0$, $(i = 1,\ldots m)$ and $h(x^*) = 0$.

Assume that a vector $\tilde{z} = (\tilde{x},\tilde{u},\tilde{v})$ in R^{n+m+q} and a symmetric A in $L(R^n)$ satisfy $||\tilde{z}-z^*|| \leqq \varepsilon$ and $||A-\nabla_{xx}L(z^*)^{-1}|| \leqq \delta$, then we have

$$||C_A-C^*|| \leqq ||A-\nabla_{xx}L(z^*)^{-1}||\,(||\nabla g(x^*)||+||\nabla h(x^*)||) \leqq \delta\lambda \leqq r < 1.$$

(by 3.13 and 3.15.a)

Hence by the non-singularity of C* and the perturbation Lemma [25] C_A is also non-singular and

(3.17) $$||C_A^{-1}|| \leqq \frac{\lambda}{1-\lambda^2\delta} \leqq \frac{\lambda}{1-r} \leqq \tau. \qquad \text{(by 3.16 and 3.14)}$$

Define the function $S_{\tilde{x},A} : R^{n+m+q} \to R^{n+m+q}$, associated with $\tilde{x}$ and A as follows

$$S_{\tilde{x},A}(z) = z - C_A^{-1}G(\tilde{x},A,z).$$

For any z in $N(z^*,\bar{\varepsilon})$ we have that

$$||\nabla_z S_{\tilde{x},A}(z)|| = ||I - C_A^{-1}\nabla_z G(\tilde{x},A,z)|| \leq ||C_A^{-1}||\ ||C_A^{-1} - \nabla_z G(\tilde{x},A,z)||$$

$$\leq \tau \frac{1}{2\tau} \leq \frac{1}{2}, \quad \text{(by 3.15.a and 3.17)}$$

which implies that $S_{\tilde{x},A}$ is a contraction mapping in $N(z^*,\bar{\varepsilon})$. Since from (3.16.b), (3.16.c) and (3.17) we also have

$$||z^* - S_{\tilde{x},A}(z^*)|| \leq \tau||G(\tilde{x},A,z^*)|| \leq \frac{\bar{\varepsilon}}{2} .$$

Thus, the contraction mapping theorem [26] implies that $S_{\tilde{x},A}$ has a unique fixed point, say $\bar{z}$, in $N(z^*,\bar{\varepsilon})$ which satisfies

$$(3.18) \qquad ||\bar{z}-z^*|| \leq 2\tau||G(\tilde{x},A,z^*)|| \leq r||\tilde{x}-x^*|| \leq r||\tilde{z}-z^*||.$$

We now show that $\bar{z}$ is a z-solution of $DQ(\tilde{x},A)$. Since $\bar{z}$ is the unique fixed point of $S_{\tilde{x},A}$ in $N(x^*,\bar{\varepsilon})$, $\bar{z}$ is the unique zero of $G(\tilde{x},A,\cdot)$ in $N(z^*,\bar{\varepsilon})$. Thus

$$(3.19) \qquad A(\nabla f(\tilde{x})+\nabla g(\tilde{x})\bar{u}+\nabla h(\tilde{x})\bar{v}) + (\bar{x}-\tilde{x}) = 0,$$

and for $i = 1,\ldots m$,

$$(3.20) \qquad \bar{u}_i(g_i(\tilde{x})+\nabla g_i(\tilde{x})^T(\bar{x}-\tilde{x})) = 0,$$

and for $j = 1,\ldots q$,

$$(3.21) \qquad h_j(\tilde{x})+\nabla h_j(\tilde{x})^T(\bar{x}-\tilde{x}) = 0.$$

By (3.15.b) and (3.15.c) in the choice of $\bar{\varepsilon}$ and also by (3.20) we have to $\tilde{z}$. Suppose that $\hat{z}$ is another z-solution of $DQ(\tilde{x},A)$. Then by Lemma 3.1 we have that $\hat{z}$ is a zero of $G(\tilde{x},A,\cdot)$. It follows from the uniqueness of the zero of $G(\tilde{x},A,\cdot)$ in $N(z^*,\bar{\varepsilon})$ that $||\hat{z}-z^*|| \geq \bar{\varepsilon}$; hence

$$||\hat{z}-\tilde{z}|| \geq ||\hat{z}-z^*||-||z^*-\tilde{z}|| > \bar{\varepsilon} - \bar{\varepsilon}/3 = 2\bar{\varepsilon}/3. \quad \text{(by 3.16.b)}$$

However,

$$||z-\tilde{z}|| \leq ||z-z||+||\tilde{z}-z|| \leq (r+1)\ ||\tilde{z}-z|| \leq 2\bar{\varepsilon}/3.$$

Therefore $\bar{z}$ is the closest z-solution of $DQ(\tilde{x},A)$ to $\tilde{z}$ and the proof is complete. □

Since the quadratic programs $DQ(\tilde{x},A)$ and $DQ(\tilde{x},\frac{1}{2}(A+A^T))$ have identical Kuhn-Tucker points, Theorem 3.7 and Theorem 3.10 below are also true for a non-symmetric A if $\bar{x}$ is generated by $\bar{x} = \tilde{x} - \frac{1}{2}(A+A^T)(\nabla f(\tilde{x})+\nabla g(\tilde{x})\bar{u}+\nabla h(\tilde{x})\bar{v})$ rather than by (2.2).

We note here that a sequence $\{z^k\}$ converges Q-linearly to a point z^* if $||z^{k+1}-z^*|| \leq r\ ||z^k-z^*||$ for some r in (0,1); and it converges Q-superlinearly if $||z^{k+1}-z^*|| \leq \theta_k ||z^k-z^*||$ with $\lim_{k\to\infty} \theta_k = 0$. Therefore, it follows from Theorem 3.7 that if the starting point is close to a solution and the sequence $\{A_k\}$ of matrices remains close to the inverse of the Hessian of the Lagrangian, then our algorithm will generate a sequence of points which converges to the solution with at least a Q-linear rate. The following result is an immediate consequence of Theorem 3.7 and of the above remarks.

(3.22) $\bar{u} \geqq 0$

and

(3.23) $g(\tilde{x})+\nabla g(\tilde{x})^T(\bar{x}-\tilde{x}) \leqq 0.$

If w in R^m be defined by

(3.24) $w = -(g(\tilde{x})+\nabla g(\tilde{x})^T(\bar{x}-\tilde{x}))$

then clearly,

(3.25) $w \geqq 0.$

Premultiplying (3.19) by $\nabla g(\tilde{x})^T$ and taking (3.24) into account, we then have

(3.26) $\nabla g(\tilde{x})^T A(\nabla f(\tilde{x})+\nabla g(\tilde{x})\bar{u}+\nabla h(\tilde{x})\bar{v}) - g(\tilde{x}) - w = 0.$

Similarly, from (3.19) and (3.21) we can get

(3.27) $\nabla h(\tilde{x})^T A(\nabla f(\tilde{x})+\nabla g(\tilde{x})\bar{u}+\nabla h(\tilde{x})\bar{v}) - h(\tilde{x}) = 0.$

From (3.20) and (3.24) it is also clear that for i = 1,...m

$$\bar{u}_i w_i = 0$$

which in conjunction with (3.26), (3.27), (3.22) and (3.25) imply that $(\bar{u},\bar{v})$ is a Kuhn-Tucker point of $DQ(\tilde{x},A)$ with Lagrange multiplier vector w; therefore, it follows from (3.19) that $\bar{z}$ is a z-solution of $DQ(\tilde{x},A)$.

We next show that $\bar{z}$ is the closest z-solution of $DQ(\tilde{x},A)$

<u>Corollary 3.8</u> Let the assumptions of Theorem 3.7 hold and let $\{j_k\}$ be a subsequence of positive integers with $j_k \leq k$. If z^0 is sufficiently close to z^* and $\|A_k - {}_{xx}L(z^{j_k})^{-1}\| \leq \alpha_k$ where $\{\alpha_k\}$ is a sequence of non-negative numbers bounded by a sufficiently small number, then the sequence $\{z^k\}$ generated by the algorithm exists and converges to z^* with at least a Q-linear rate. Furthermore, if $j_k \to \infty$ and $\alpha_k \to 0$, then $\{z^k\}$ converges to z^* with at least a Q-superlinear rate. □

The simplest way to generate the matrices $\{A_k\}$ in the algorithm is by setting $A_k = \nabla_{xx}L(z^{j_k})^{-1} + \alpha_k = 0$ and $j_k = k$ we abtain a Newton-type method which can be shown to possess a quadratic rate of convergence; and for the equality constraint problem this method turns out to be similar to a method studied by Polyak [32].

Inequality (3.18) in the proof of Theorem 3.7 shows that we actually can get the sharper result $\|\bar{z}-z^*\| \leq r\|\tilde{x}-x^*\|$, and thus the following corollary.

<u>Corollary 3.9</u> Let all the assumptions of Theorem 3.7 hold. Then for any $r \in (0,1)$ there exists two positive numbers $\varepsilon(r)$ and $\delta(r)$ such that if $\|\tilde{x}-x^*\| \leq \varepsilon(r)$ and $\|A-\nabla_{xx}L(z^*)\| \leq \delta(r)$ then an unique Kuhn-Tucker point $(\bar{u},\bar{v})$ of $DQ(\tilde{x},A)$ in $N((u^*,v^*), \varepsilon(r))$ exists and $\|(\bar{u},\bar{v})-(u^*,v^*)\| \leq r\,\|\tilde{x}-x^*\|$. Moreover, $\bar{u}_i = 0$ if $u_i^* = 0$. □

The result of Corollary 3.9 has nothing to do with the way we generate $\bar{x}$, and hence can be applied to establish the convergence theorem for some other methods in which x^{k+1} is generated by another way. Indeed, this corollary has been so employed in [23].

To achieve convergence, according to Theorem 3.7 the matrices $\{A_k\}$ in the algorithm are required to remain close to $\nabla_{xx}L(z^*)^{-1}$. In the theorem below we give a sufficient condition which ensures such closeness and at the same time can be satisfied by some variable metric updates. This condition was first studied by Broyden, Dennis and Moré [7] for non-linear system of equations and unconstrained optimization and some techniques of their proof will be employed here. Throughout this work $||\cdot||'$ denotes any fixed matrix norm which may be different from $||\cdot||$.

<u>Theorem 3.10</u> Let $z^* = (x^*,u^*,v^*)$ be a Kuhn-Tucker triple of problem (P) and let f, g and h $\in LC^2[x^*]$ and $\nabla_{xx}L(z^*)$ be non-singular. Let the Jacobian uniqueness condition hold at z^* and let there exist two non-negative numbers α_1 and α_2 such that for an update which generates symmetric matrices the following condition holds

$$(3.28)\quad ||A_{k+1}-\nabla_{xx}L(z^*)^{-1}||' \leq (1+\alpha_1||z^k-z^*||)||A_k-\nabla_{xx}L(z^*)^{-1}||' + \alpha_2||z^k-z^*||.$$

Then for any $r \in (0,1)$ there exist two positive numbers $\varepsilon(r)$ and $\delta(r)$ such that if $||z^0-z^*|| \leq \varepsilon(r)$ and $||A_0-\nabla_{xx}L(z^*)^{-1}||' \leq \delta(r)$ then the sequence $\{z^k\}$ generated by the algorithm is well defined and converges Q-linearly to z^*. Furthermore, $\varepsilon(r)$ and $\delta(r)$ can be chosen small enough to ensure the non-singularity of all the updated matrices $\{A_k\}$ and the uniform boundedness of $\{A_k^{-1}\}$.

Proof. By the equivalence of matrix norms, there exist two positive numbers of d and d' such that for any A in $L(R^n)$ we have

$$(3.29) \qquad d\|A\|' \geqq \|A\| \quad \text{and} \quad d'\|A\| \geqq \|A\|'.$$

Let $r \in (0,1)$ be given. By Theorem 3.7 there exist two positive numbers $\bar{\varepsilon}$ and $\bar{\delta}$ such that if $\|\bar{z}-z^*\| \leqq \bar{\varepsilon}$ and $\|A-\nabla_{xx}L(z^*)^{-1}\| \leqq \bar{\delta}$ then the closest z-solution $\hat{z}$ of $DQ(\bar{x},A)$ exists and $\|\hat{z}-z^*\| \leqq r\,\|\bar{z}-z^*\|$. We choose two positive numbers ε and δ such that the following conditions hold

(3.30) (a) $\varepsilon \leqq \bar{\varepsilon}$

(b) $2dd'\delta \leqq \bar{\delta}$

(c) $(2\alpha_1 \delta d' + \alpha_2)\,\frac{\varepsilon}{1-r} \leqq d'\delta.$

If we can show that for each k

$$(3.31) \qquad \|z^k-z^*\| \leqq r^k \varepsilon$$

and

$$(3.32) \qquad \|A_k-\nabla_{xx}L(z^*)^{-1}\|' \leqq 2d'\delta,$$

then $\|z^k-z^*\| \leqq \bar{\varepsilon}$ and $\|A_k-\nabla_{xx}L(z^*)^{-1}\| \leqq \bar{\delta}$, and the theorem follows immediately from Theorem 3.7.

We prove (3.31) and (3.32) by induction. They are obviously true for $k = 0$. Assume that they are true for j, $0 \leqq j \leqq k$; then it follows from (3.28) that

$$\|A_{j+1}-\nabla_{xx}L(z^*)^{-1}\|' - \|A_j-\nabla_{xx}L(z^*)^{-1}\|' \leqq 2\alpha_1 d'\varepsilon\delta r^j + \alpha_2 \varepsilon r^j,$$

and by taking the sum from $j = 0$ to $j = k$,

$$||A_{k+1}-\nabla_{xx}L(z^*)^{-1}|| \leq ||A_0-\nabla_{xx}L(z^*)^{-1}|| + (2\alpha_1 d'\delta+\alpha_2)\frac{\varepsilon}{1-r}$$

$$\leq d'\delta+d'\delta \leq 2d'\delta.$$

Therefore, (3.32) is true for $j = k+1$. Moreover, we have $||A_{k+1}-\nabla_{xx}L(z^*)^{-1}|| \leq \bar{\delta}$, and by the induction hypothesis and (3.30.c) we have $||z^k-z^*|| \leq r^k\varepsilon \leq \bar{\varepsilon}$. Thus, it follows from Theorem 3.7 that z^{k+1} exists and $||z^{k+1}-z^*|| \leq r||z^k-z^*|| \leq r^{k+1}\varepsilon$.

The second part of the theorem follows directly from (3.32) and the perturbation Lemma. □

4. Superlinear Rate of Convergence

In this section sufficient conditions are given which guarantee a superlinear rate of convergence for the sequence of points generated by the algorithm. We first introduce a Lemma which is due to Mangasarian [30] and is closely related to a result of Dennis and Moré [13]; its proof can be found in [22].

Lemma 4.1 Let z^* be a Kuhn-Tucker triple of problem (P) satisfying the Jacobian uniqueness condition and f, g and h $\in LC^2[x^*]$. A sequence $\{z^k\}$ converges to z^* with a Q-superlinear rate of $\{z^k\}$ converges to z^* and

$$\lim_{k\to\infty} \frac{||E(z^{k+1})||}{||z^{k+1}-z^k||} = 0 \tag{4.1}$$

where $E : R^{n+m+q} \to R^{n+m+q}$,

$$E(z) = \begin{bmatrix} \nabla_x L(z) \\ u_1 g_1(x) \\ \vdots \\ u_m g_m(x) \\ h_1(x) \\ \vdots \\ h_q(x) \end{bmatrix}. \qquad \square$$

In the Lemma above there is no specification on how the sequence $\{z^k\}$ is generated; nevertheless, if the sequence is generated by our algorithm then we can derive from Lemma 4.1 some other criteria, which are contained in the following two theorems.

Theorem 4.2 Let all the assumptions of Lemma 4.1 hold. If a sequence $\{z^k\}$ constructed by the algorithm converges to z^* and

(4.2) $$\lim_{k\to\infty} \frac{||\nabla_x L(z^{k+1})||}{||z^{k+1}-z^k||} = 0,$$

then $\{z^k\}$ converges Q-superlinearly to z^*.

Proof By Lemma 4.1 we need only to establish (4.1). By virture of (3.6) and (3.7) in the proof of Lemma 3.1 we have

$$||E(z^{k+1})|| \leqq ||\nabla_x L(x^{k+1},u^{k+1},v^{k+1})|| + ||\nabla h(x^{k+1})|| + \sum_{i=1}^{m} |u_i^{k+1} g_i(x^{k+1})$$

$$\leqq ||\nabla_x L(x^{k+1},u^{k+1},v^{k+1})|| + ||h(x^{k+1})-h(x^k)-\nabla h(x^k)^T(x^{k+1}-x^k$$

$$+ \sum_{i=1}^{m} |u_i^{k+1}(g_i(x^{k+1})-g_i(x^k)-\nabla g_i(x^k)^T(x^{k+1}-x^k))|$$

$$\leqq ||\nabla_x L(x^{k+1},u^{k+1},v^{k+1})|| + \circ(||x^{k+1}-x^k||).$$

Hence (4.1) follows. □

Theorem 4.3 Let all the assumptions of Lemma 4.1 hold; furthermore, let $\nabla_{xx}L(z^*)$ be non-singular and let $\{z^k\}$ be a sequence of points generated by the algorithm with respect to a sequence of non-singular symmetric matrices $\{A_k\}$ with $\{A_k^{-1}\}$ uniformly bounded. If $\{z_k\}$ converges to z^*, and

$$(4.3) \qquad \lim_{k\to\infty} \frac{||(A_k-\nabla_{xx}L(z^*)^{-1})y^k||}{||y^k||} = 0$$

where $y^k=\nabla_x L(x^{k+1},u^{k+1},v^{k+1})-\nabla_x L(x^k,u^{k+1},v^{k+1})$, then $\{z^k\}$ converges Q-superlinearly to z^*.

Proof By the assumptions z^{k+1} is a z-solution and

$$A_k \nabla_x L(x^k,u^{k+1},v^{k+1}) + (x^{k+1}-x^k) = 0,$$

which yields

$$(A_k-\nabla_{xx}L(z^*)^{-1})y^k = A_k\nabla_x L(z^{k+1})+(x^{k+1}-x^k)-\nabla_{xx}L(z^k)^{-1}y^k$$

and in turn implies

$$(5.2)\quad \bar{A} = A + \frac{(s-Ay)s^T + s(s-Ay)^T}{s^Ty} - \frac{y^T(s-Ay)ss^T}{(s^Ty)^2}$$

$$(5.3)\quad \bar{A} = A + \frac{(s-Ay)y^T + y(s-Ay)^T}{y^Ty} - \frac{y^T(s-Ay)yy^T}{(y^Ty)^2}$$

These updates are well known in unconstrained optimization where y is defined as $y=\nabla f(x^{k+1})-\nabla f(x^k)$; in this context update (5.2) has been studied by Broyden [6], Fletcher [16], Goldfarb [19] and Shanno [39], and is often referred to as the BFGS update. Update (5.3) is one of Greenstadt's methods [20].

In the sequel we establish superlinear convergence theorems for Algorithms D1 and D2 by utilizing the techniques developed by Broyden, Dennis and Moré [7] and Dennis and Moré [13]. It is noted here that for any non-singular $n\times n$ matrix M the matrix norm $||\cdot||_M$ is defined in such a way that for any $n\times n$ matrix A

$$||A||_M = \text{trace}[(MAM)^T(MAM)].$$

We state two Lemmas below; their proofs are in [13] and [7] respectively.

<u>Lemma 5.1</u> Let $\{a_k\}$ and $\{b_k\}$ be sequences of non-negative numbers and $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$ such that

$$a_{k+1} \leqq (1+\alpha_1 b_k)a_k + \alpha_2 b_k$$

and

$$\nabla_x L(z^{k+1}) = A_k^{-1}(A_k-\nabla_{xx}L(z^*)^{-1})y^k + A_k^{-1}(\nabla_{xx}L(z^*)^{-1}y^k-(x^{k+1}-x^k)).$$

By the uniform boundedness of $\{A_k^{-1}\}$ there exists $\lambda > 0$ such that

$$(4.4)\quad ||\nabla_x L(z^{k+1})|| \leqq \lambda||(A_k-\nabla_{xx}L(z^*)^{-1})y^k||+\lambda||\nabla_{xx}L(z^*)^{-1}|| \; ||y^k-\nabla_{xx}L(z^*)(x^{k+1}-x^k)||.$$

On the other hand, since f, g and h $\in LC^2[x^*]$ and $z^k \to z^*$, there exists some $\alpha > 0$ such that

$$(4.5)\quad ||y^k|| \leqq \alpha||x^{k+1}-x^k|| \leqq \alpha||z^{k+1}-z^k||.$$

By (4.3), (4.4) and Lemma 3.4, taking (4.5) into account, we get (4.2) and complete the proof. □

5) Updates

The updates which we consider for generating matrices in the algorithm are of the following form

$$(5.1)\quad \bar{A} = A + \frac{(s-Ay)d^T+d(s-Ay)^T}{d^Ty} - \frac{y^T(s-Ay)dd^T}{(d^Ty)^2}$$

where $\bar{A} = A_{k+1}$, $A = A_k$, $s = x^{k+1}-x^k$ and $y = \nabla_x L(x^{k+1},u^{k+1},v^{k+1})-\nabla_x L(x^k,u^{k+1},v^{k+1})$ and d is any vector in R^n with $y^Td \neq 0$. A particular algorithm is determined once d is specified. The algorithm will be called Algorithm D1 when d = s and Algorithm D2 when d = y. Thus, we have the following updates:

$$\sum_{k=1}^{\infty} b_k < \infty$$

then $\{a_k\}$ converges. □

Lemma 5.2 Let A be any n×n symmetric matrix and s, d and y be vectors in R^n with $d^T y \neq 0$ and define $\bar{A}$ by (5.1). If M is a non-singular symmetric n×n matrix with

$$(5.4) \qquad ||Md-M^{-1}y|| \leq \beta ||M^{-1}y||$$

for some $\beta \in [0,1/3]$, then for any symmetric n×n matrix B with $B \neq A$ we have

$$(5.5) \quad ||\bar{A}-B||_M \leq \left((1-\lambda\theta^2)^{1/2} + \lambda_1 \frac{||Md-M^{-1}y||}{||M^{-1}y||}\right) ||A-B||_M + \lambda_2 \frac{||s-By||}{||y||}$$

where $\lambda \in (0,1)$, and λ_1 and λ_2 are constants which only depend on M and n, and

$$(5.6) \qquad \theta = \frac{||M(A-B)y||}{||A-B||_M ||M^{-1}y||}$$

if $A \neq B$ and $\theta = 0$ otherwise. □

The following theorem gives a sufficient condition for the superlinear convergence of the algorithm with an update of form (5.1); this condition turns out to be satisfied by updates (5.2) and (5.3).

Theorem 5.3 Let $z^* = (x^*,u^*,v^*)$ be a Kuhn-Tucker triple of problem (P) satisfying the Jacobian uniqueness condition and f, g and h $\in LC^2[x^*]$. Suppose that $\nabla_{xx}L(z^*)$ is non-singular and in the algorithm the matrices $\{A_k\}$ are updated by formula (5.1) with any d^k such that for $y^k \neq 0$,

$$(5.7) \quad \frac{||Md^k - M^{-1}y^k||}{||M^{-1}y^k||} \leqq \mu \max \{||z^k - z^*||, ||z^{k+1} - z^*||\}$$

for a constant μ and an arbitrary but fixed non-singular symmetric matrix M. If z^0 and A_0 are sufficiently close to z^* and $\nabla_{xx}L(z^*)^{-1}$ respectively then the sequence $\{z^k\}$ generated by the algorithm is well defined and converges Q-superlinearly to z^*.

Proof For any $r \in (0,1)$ let $\varepsilon(r)$ and $\delta(r)$ be defined as in Theorem 3.10 with matrix norm $||\cdot||'$ as $||\cdot||_M$. Now set $\alpha_1 = \lambda_1\mu_1$, $\alpha_2 = \frac{\lambda_2}{\eta}(K+\tilde{K})||\nabla_{xx}L(z^*)^{-1}||$ where $\bar{K}$ and $\tilde{K}$ are the constants defined Lemma 3.4 and λ_1 and λ_2 are as in Lemma 5.2 and η is as in Corollary 3.5. We further require $\varepsilon(r)$ to satisfy

$$(5.8) \qquad \varepsilon(r) \leqq \tfrac{1}{3}\mu.$$

We first show by induction that if $||z^0 - z^*|| \leqq \varepsilon(r)$ and $||A_0 - \nabla_{xx}L(z^*)^{-1}|| \leqq \delta(r)$ then the generated sequence $\{z^k\}$ exists and converges Q-linearly to z^*; that is, Lemma 3.4 yields

$$||s^k - By^k|| \leq ||B||(\bar{K}+\tilde{K})||z^k - z^*|| \;\; ||s^k||$$

and Corollary 3.5 implies that for some $\eta > 0$

$$\eta||s^k|| \leq ||y^k||.$$

Therefore,

$$\frac{||s^k - By^k||}{||y^k||} \leq \frac{1}{\eta}||B||(\bar{K}+\tilde{K})||z^k - z^*|| \;.$$

which in conjunction with (5.10) and the fact that $(1-\lambda\theta^2)^{1/2} \leq 1-\frac{\lambda}{2}\theta^2$ yields

$$(5.11) \quad ||A_{k+1}-B||_M \leq (1-\frac{\lambda}{2}\theta_k^2+\alpha_1||z^k-z^*||)||A_k-B||_M+\alpha_2||z^k-z^*||.$$

Hence the existence of z^{k+2} and $||z^{k+2}-z^*|| \leq r||z^{k+1}-z^*||$ follow from Theorem 3.10 and (5.11) immediately.

So far we have shown that the sequence $\{z^k\}$ exists and converges to z^* with at lease a Q-linear rate; we are going to prove that the rate of convergence is actually Q-superlinear. By Lemma (5.1) and (5.11) the sequence $\{||A_k-B||_M\}$ has a limit, say p. If $p = 0$ then the desired result follows directly from Theorem 4.3. Assume $p \neq 0$. It follows from (5.11) that

$$\frac{\lambda}{2}\theta_k^2 p_k \leq p_k - p_{k+1} + (\alpha_1 p_k + \alpha_2)||z^k - z^*||$$

where $p_k = ||A_k-B||_M$. By taking the sum of both sides over $k = 0,1,2,\ldots$ and taking the Q-linear convergence of $\{||z^k-z^*||\}$

$$(5.9)\qquad ||z^{j+1}-z^*|| \leqq r\,||z^j-z^*||.$$

When $j = 0$ the existence of z^{j+1} and (5.9) follows directly from Theorem 3.7 and the choice of $\varepsilon(r)$ and $\delta(r)$. Assume z^{j+1} exists for all $j \leq k$ and that (5.9) holds. We show that z^{k+2} exists and that (5.9) is also true for $j = k+1$. Assume $y^k \neq 0$, for if $y^k = 0$ then Corollary 3.5 implies that $s^k = 0$ which by Corollary 3.2 in turn implies that $z^{k+1} = (x^{k+1},u^{k+1},v^{k+1})$ is a Kuhn-Tucker triple of (P). On the other hand the Jacobian uniqueness condition yields that z^* is the unique Kuhn-Tucker triple in $N(z^*,\varepsilon(r))$ and hence we have $z^{k+1} = z^*$. Therefore, in case $y^k = 0$ the sequence $\{z^k\}$ converges to z^* in a finite number of steps. When $y^k \neq 0$ it follows from (5.7) and (5.8) that

$$\frac{||Md^k-M^{-1}y^k||}{||M^{-1}y^k||} \leqq \mu \max\{||z^k-z^*||,||z^{k+1}-z^*||\} < \mu\,\varepsilon(r) \leqq 1/3.$$

Let $B = \nabla_{xx}L(z^*)^{-1}$; then by Lemma 5.2 we have

$$(5.10)\qquad ||A_{k+1}-B||_M \leqq ((1-\lambda\ \theta_k^2)^{1/2}+\lambda_1\mu||z^k-z^*||)||A_k-B||_M + \lambda_2\frac{||s^k-By^k||}{||y^k||}$$

where

$$\theta_k = \frac{||M(A_k-B)y^k||}{||A_k-B||_M||M^{-1}y^k||}$$

and the boundedness of $\{p_k\}$ into consideration we have

$$\sum_{i=0}^{\infty} \lambda_k^2 p_k < \infty \quad .$$

Since $\lambda \in (0,1)$ and $p_k \to p$ with $p \neq 0$, we must have $\lim_{k\to\infty} \theta_k = 0$

which implies

$$\lim_{k\to\infty} \frac{||(A_k - \nabla_{xx} L(z^*)^{-1})y^k||}{||y^k||} = 0.$$

Hence the result also follows from Theorem 4.3.

Our main results are contained in the following theorem which shows that Algorithms D1 and D2 possess local superlinear convergence properties. □

<u>Theorem 5.4</u> Let $z^* = (x^*,u^*,v^*)$ be a Kuhn-Tucker triple of (P) satisfying the Jacobian uniqueness condition and f, g and h $\in LC^2[x^*]$. If $\nabla_{xx}L(z^*)$ is non-singular and the starting point z^0 and the starting matrix A_0 are sufficiently close to z^* and $\nabla_{xx}L(z^*)^{-1}$ respectively then the sequence $\{z^k\}$ generated by Algorithm D2 exists and converges Q-superlinearly to z^*. If further assume $\nabla_{xx}L(z^*)$ to be positive definite then the conclusion is also true for Algorithm D2.

<u>Proof</u> By Theorem 5.3 it is sufficient to establish (5.7) for some suitable matrix M and constant μ. Since it is obviously valid for Algorithm D2, we only need to verify it for Algorithm D1.

With $\nabla_{xx}L(z^*)$ positive definite we can set $M = (\nabla_{xx}L(z^*))^{1/2}$. By Lemma 3.4 we have

$$||Ms^k - M^{-1}y^k|| \leq ||M^{-1}||\ ||y^k - \nabla_{xx}L(z^*)s^k||$$

$$\leq ||M^{-1}||(\bar{K}+\tilde{K})\ \max\{||z^k - z^*||, ||z^{k+1} - z^*||\}\ ||s^k||$$

and by Corollary 3.5 we have that for some $\xi > 0$, $||s^k|| \leq \xi||y^k||$. Therefore,

$$||Ms^k - M^{-1}y^k|| \leq ||M^{-1}||(\bar{K}+\tilde{K})\ \max\{||z^k - z^*||, ||z^{k+1} - z^*||\}\ \xi||M||\ ||M^{-1}y^k||.$$

Thus (5.7) is true with $\mu = \xi||M||\ ||M^{-1}||\ (\bar{K}+\tilde{K})$. □

We note here that local superlinear convergence can also be achieved for the algorithm if the following updates are used [21],

(5.12) $$\bar{A} = A - \frac{Ayy^TA}{y^TAy} + \frac{ss^T}{s^Ty},$$

(5.13) $$\bar{A} = A + \frac{(s-Ay)s^T + s(s-Ay)^T}{2\ s^Ty},$$

(5.14) $$\bar{A} = A + \frac{(s-Ay)y^T + y(s-Ay)^T}{2\ y^Ty}$$

The non-symmetric updates such as

(5.15) $$A = A + \frac{(s-Ay)s^T}{s^Ty},$$

Theorems 5.3 and 5.4 can be relaxed if the function F replaces the Lagrangian L. Moreover, with a large α the function F has the property of penalizing the infeasible points, so the domain of convergence is likely to be enlarged.

For reasons which will become clear later on we consider seperately the equality constraint problem

$$(6.2) \qquad \min_x \; f(x) \quad \text{s.t.} \quad h(x) = 0$$

and the inequality constraint problem

$$(6.3) \qquad \min_x \; f(x) \quad \text{s.t.} \quad g(x) \leqq 0;$$

the function F and the Jacobian uniqueness condition are also defined accordingly. We note that the modified algorithms are applicable to the general problem with constraints of mixed type.

For problem (6.1), Algorithms D1 and D2 can be modified as follows.

<u>Algorithm M</u> Step 1. Start with a penalty parameter α, an estimate $z^0 = (x^0, v^0)$ of a Kuhn-Tucker pair $z^* = (x^*, v^*)$ of (6.1) and an estimate A_0 of $\nabla_{xx}F(x^*, v^*, \alpha)^{-1}$.

Step 2. Set $k = 0$.

$$\bar{A} = A + \frac{(s-Ay)y^T}{y^Ty} \tag{5.16}$$

can also be shown to possess Q-linear rates of convergence; however, we have not succeeded in establishing Q-superlinear rates for them, though such results are predicted. In unconstrained optimization (5.12) is the famous Daviden-Fletcher-Powell update and (5.15) and (5.16) have been studied by Broyden [5] and McCormick [31] respectively.

6. Modification via a penalty Lagrangian

Considerable attention has been given recently to a penalty Lagrangian developed by Hestenes [24], Powell [33] and Rockafellar [37]. $F : R^{n+m+q+1} \to R$, is defined by

$$F(x,u,v,\alpha) = f(x)+\frac{1}{2}\alpha \sum_{i=1}^{m} ((\alpha g_i(x)+u_i)_+^2-u_i^2)+v^Th(x)+\frac{\alpha}{2}h(x)^Th(x) \tag{6.1}$$

where $(\alpha g_i(x)+u_i)_+ = \max\{0,\alpha g_i(x)+u_i\}$. A very attractive feature of this function is that a local convexification procedure can be carried out by choosing a sufficient large penalty parameter α. We state this result in the following lemma which is due to Arrow, Gould and Howe [1].

Lemma 6.1 Let f, g and $h \in LC^2[x^*]$ and $z^* = (x^*,u^*,v^*)$ be a Kuhn-Tucker triple which satisfies the Jacobian uniqueness condition. Then there exists an $\bar{\alpha} > 0$ such that if $\alpha \geq \bar{\alpha}$ then $\nabla_{xx}F(x^*,u^*,v^*,\alpha)$ is positive definite. □

With this result the assumptions on the Lagrangian L in

Step 3. Solve the system of linear equations

(6.4) $$Bv = b$$

where $B = \nabla h(x^k)^T A_k \nabla h(x^k)$,

$b = h(x^k) - \nabla h(x^k)^T A_k (\nabla f(x^k) + \alpha \nabla h(x^k) h(x^k))$,

and let the solution be v^{k+1}. Set

(6.5) $$x^{k+1} = x^k - A_k \nabla_x F(x^k, v^{k+1}, \alpha).$$

Step 4. Check convergence; if not, generate A_{k+1} from A_k, $s^k = x^{k+1} - x^k$ and $y^k = \nabla_x F(x^{k+1}, v^{k+1}, \alpha) - \nabla_x F(x^k, v^{k+1}, \alpha)$ either by (5.2) or by (5.3). Set $k = k+1$ and go to Step 3. □

To show the superlinear convergence of Algorithm M, we consider the following auxiliary problem

(6.6) $$\min_x \; f(x) + \frac{\alpha}{2} h(x)^T h(x)$$

$$\text{s.t. } h(x) = 0.$$

It is evident that problems (6.2) and (6.6) have the same Kuhn-Tucker pairs and furthermore, the function $F(x,v,\alpha)$ is the Lagrangian of problem (6.6). When Algorithm D1 or D2 is adopted to solve (6.6), the resulting algorithm is just Algorithm M. Therefore, taking Lemma 6.1 into consideration, the following results follows from Theorem 5.4.

<u>Theorem 6.2</u> Let $z^* = (x^*, v^*)$ be a Kuhn-Tucker pair of (6.2) satisfying the Jacobian uniqueness condition and let f, g and $h \in LC^2[x^*]$. If the penalty parameter α is sufficiently large

and if the starting point $z^0 = (x^0, v^0)$ and the starting matrix A_0 are sufficiently close to z^* and $\nabla_{xx}F(x^*, v^*, \alpha)^{-1}$ respectively then the sequence $\{z^k\}$ generated by Algorithm M exists and converges Q-superlinearly to z^*. □

For the inequality constraint problem (6.3) the modification is the following.

Algorithm M' Step 1. Start with a positive number α and an estimate $z^0 = (x^0, u^0)$ of a Kuhn-Tucker pair $z^* = (x^*, u^*)$ of (6.3) and an estimate A_0 of $\nabla_{xx}F(x^*, u^*, \alpha)^{-1}$.

Step 2. Set $k = 0$.

Step 3. Solve the following quadratic programming subproblem

(6.7)
$$\min_u \quad \tfrac{1}{2}\phi_k(u)^T A_k \, \phi_k(u) - u^T g(x^k)$$
$$\text{s.t. } u \geqq 0$$

where $\phi_k = \nabla f(x^k) + \alpha \sum_{i \in I_k} g_i(x^k) \nabla g_i(x^k) + \nabla g(x^k) u$ and $I_k = \{i : g_i(x^k) \geqq - u_k/\alpha\}$; let its solution be u^{k+1} and set

(6.8)
$$x^{k+1} = x^k - A_k \nabla_x F(x^k, u^{k+1}, \alpha).$$

Step 4. Check convergence; if not, generate matrix A_{k+1} from A_k, $s^k = x^{k+1} - x^k$ and $y^k = \nabla_x F(x^{k+1}, u^{k+1}, \alpha) - \nabla_x F(x^k, u^{k+1}, \alpha)$ either by (5.2) or by (5.3). Set $k = k+1$ and go to Step 3. □

Theorem 6.3 because of the local convexification property of the function F.

We would like to compare our modified algorithms with the recently developed multiplier method in which we generate $z^{k+1} = (x^{k+1}, u^{k+1}, v^{k+1})$ from $z^k = (x^k, u^k, v^k)$ by

$$(6.10) \qquad u_i^{k+1} = \max \{0, u_i^k + \alpha g_i(x^k)\} \qquad \text{for } i=1,\dots m,$$

$$(6.11) \qquad v_j^{k+1} = v_j^k + \alpha h_j(x^k) \qquad \text{for } j=1,\dots q,$$

and

$$(6.12) \qquad F(x^{k+1}, u^{k+1}, v^{k+1}, \alpha) = \min_x F(x, u^{k+1}, v^{k+1}, \alpha).$$

This method has been shown superlinearly convergent if α is replaced by a sequence $\{\alpha_k\}$ required to go to infinity [3]. However, this usually causes numerical instability. It has been shown [4] that (6.10) and (6.11) are a steepest ascent step for finding a maximum point of the function $\psi_\alpha(u,v) = \min_x F(x,u,v,\alpha)$. With α bounded the multiplier method has only a linear rate of convergence. To avoid the numerical instability caused by large α and at the same time to achieve a superlinear rate we need a more accurate scheme for updating (u^k, v^k) than (6.10) and (6.11). An appropriate candidate is (6.4) and (6.7) this approach results in our modified algorithms. Moreover, to find x^{k+1} we need only take one step of a variable metric method to minimize $F(\cdot, u^{k+1}, v^{k+1}, \alpha)$ with an updated matrix A_k which has already obtained in the stage of finding multiplier vector (u^{k+1}, v^{k+1}); however, in the

Theorem 6.3 Let $z^* = (x^*,u^*)$ be a Kuhn-Tucker pair of (6.3) satisfying the Jacobian uniqueness condition and f, g and $h \in LC^2[x^*]$. If the penalty parameter α is sufficiently large and the starting point $z^0 = (x^0,u^0)$ and the starting matrix A_0 are sufficiently close to z^* and $\nabla_{xx}F(x^*,u^*,\alpha)^{-1}$ respectively, then the sequence $\{z^k\}$ generated by Algorithm M' exists and converges Q-superlinearly to z^*.

Proof Consider the following auxiliary problem

$$(6.9) \qquad \min_x \quad f(x) + \frac{\alpha}{2} \sum_{i \in I^*} g_i(x)^2$$

$$\text{s.t.} \quad g(x) \leqq 0$$

where $I^* = \{i : g_i(x^*) = 0\}$. If $z = (x,u)$ and $z^k = (x^k,u^k)$ are sufficiently close to z^* and α is sufficiently large, then it is easy to check that $\phi_k(u) = \nabla_x F(x^k,u,\alpha)$. If we further assume that $u_i = 0$ for all $i \notin I^*$, then $F(x,u,\alpha)$ turns out to be the Lagrangian of (6.9). Therefore, with the second part of Corollary 3.9 taken into account, Algorithm M' is equivalent to Algorithm D1 or D2 applied to problem (6.9). Since z^* is also a Kuhn-Tucker pair of (6.9), the theorem follows immediately from Theorem 3.4 and Lemma 6.1. □

We note that problem (6.9) is only used for the proof of convergnece for Algorithm M'; it is never used in practice. Since the set I^* is not known a priorie, it is usually not abtainable. We also point out that the assumptions of non-singularity and positive definiteness on the Hessian of the Lagrangian L are not needed in multiplier method we need to do a whole unconstrained minimization process. It is noted that a similar approach can be found in [23].

7. Comments and Computational Experiences

(1). The algorithm based on the subproblem

$$\min_x \quad \nabla f(x^k)^T(x-x^k) + \frac{1}{2}(x-x^k)^T H_k (x-x^k)$$

$$\text{s.t.} \quad g(x^k) + \nabla g(x^k)^T(x-x^k) \leqq 0$$

$$h(x^k) + \nabla h(x^k)^T(x-x^k) = 0$$

can be viewed as primal to the algorithm discussed in this paper because its quadratic programming subproblem is primal to subproblem (2.1). To achieve local superlinear convergence for the primal algorithm the matrices $\{H_k\}$ need to be good estimates to the Hessian of the Lagrangian rather than to the inverse of the Hessian. In [22] local superlinear convergence has been established for the primal algorithm when $\{H_k\}$ are updated by the following schemes

$$(7.1) \qquad \overline{H} = H + \frac{(y-Hs)y^T+y(y-Hs)^T}{y^Ts} - \frac{s^T(y-Hs)yy^T}{(y^Ts)^2}$$

(unconstrained case: Davidon-Fletcher-Powell [12,15])

$$(7.2) \qquad \overline{H} = H + \frac{(y-Hs)s^T+S(y-Hs)^T}{s^Ts} - \frac{s^T(y-Hs)ss^T}{(s^Ts)^2}$$

(unconstrained case: Powell [34,35])

where s and y are defined as in (5.1). It is noted that updates (7.1) and (7.2) are dual to updates (5.2) and (5.3) respectively in the sense of Fletcher [16]. The duality of updating schemes and the duality of mathematical programming have been defined and used in two different contexts; it is very interesting that in our approach they are coincidentally connected to each other. We also note that though some theorems in this paper are analogous to those in [22], there is no direct implication among them.

(2). As the primal algorithm our algorithm is in a sense a natural extension of variable metric algorithms to general non-linear programming; this extension provides a fruitful field of future research. A lot of results in the extensive literature of variable metric algorithms need to be investigated and developed for non-linear programming and the whole theory can be treated in a unified way in both constrained and unconstrained optimization.

(3). All the results in this paper are local. One approach studied by this author for achieving global convergence is to determine a stepsize in each iteration which maintains a monotone decrease of an exact penalty function or the penalty Lagrangian defined in (6.1). Some global convergence results have already been established [21].

Computational tests of the algorithms in this paper have been performed and are still going on. A report on the tests results is expected to be published in the near future. However, it would be unfair to finish without at least giving some idea of the power of these algorithms in practice. We state in the table

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below the test results of Algorithm D1 and D2 for Colville's test problems 1 and 2. The computations were done on the UNIVAC 1110 system at the University of Wisconsin, Madison. The principal pivoting method [10,11] was used in solving the quadratic programming subproblems.

Table 1

Prob.	Algorithm	Obj. Fct. Value	Standard Time Ratio
1	D1	-32.3487	.00448 [1)]
	D2	-32.3486	.00906
2 [2)]	D1	-32.3488	.2133
	D2	-32.3488	.6311

1) This result is better than any one reported in Colville's report [9].

2. Infeasible starting point.

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