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Dualities in Baryon-Antibaryon Scattering

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Discussions are given of possible solutions for the Finite Energy Sum Rules (FESR), applied to baryon-antibaryon scattering, with or without assuming resonances in exotic channels. Exotic resonances, assuming they exist, are classified, as a result of factorization, according to 36-plet, 18-plet, $\overline{18}$-plet and nonet. It is then shown that these mesons are required to obey an equal spacing mass formula and nonet-like couplings, which are direct extensions of the ideal nonet scheme for 1^- and 2^+ mesons.

When there are no resonances in exotic channels, the idea of duality between ordinary meson resonances is not applicable to baryon-antibaryon scattering. However, by extending the concept of the duality, it is possible to find a solution of the FESR. We construct an explicit formula with such extended duality, where Regge cuts as well as Regge poles appear in the amplitude in question.

§ 1. Introduction

Many interesting results on spectra and couplings of hadrons have recently been obtained from the analyses of the Finite Energy Sum Rules (FESR), applied to meson-meson and meson-baryon scattering.[1)~3)] The main assumptions underlying these analyses may be summarized as follows:

1) The two-component hypothesis introduced by Freund[2)] and Harari.[3)] It states that the scattering amplitudes at high energies are decomposed into two components, non-diffractive and diffractive parts and that the FESR is assumed to hold individually for these two components. The non-diffractive amplitudes are Regge-behaved at high energies, and the FESR is satisfied by taking into account only the resonating contributions at low energies. The diffractive part, on the other hand, is assumed to be constant at high energies and represents the Pomeranchuk contribution. It is related through the FESR to the s-channel non-resonating background.

2) The absence of exotic resonances. Exotic states for mesons and baryons are usually defined by referring to the simple quark model. For mesons, for instance, there are two kinds of exotic states.[4)] Exotic states of the first kind are those belonging to the $SU(3)$ representations other than octets and singlets. Exotic states of the second kind, on the other hand, are those with odd CP and with normal parity $P=(-1)^J$, and a state with $J^{PC}=0^{--}$. The simple $q\bar{q}$ pair model with l-excitations for mesons forbids the states with exotic quantum numbers of both the first and second kinds. In this paper, we are primarily con-

cerned with excluding exotic resonances of the first kind. Many similar results, however, may also be obtained from the analysis of scattering processes, requiring the absence of exotic mesons of the second kind.[5)]

3) Saturation of the FESR by narrow resonances. Absorptive parts of the non-diffractive amplitudes are assumed to be saturated by the narrow resonances.

A number of applications of these ideas to meson-baryon scattering have been made during the last few years.[6)] They have shown that in many cases the last assumption 3) is indeed consistent with experiment, at least in the first approximation. Among the interesting consequences obtained from such applications, we note the generalized exchange degeneracies between meson trajectories and also between baryon trajectories. Moreover we obtain the ideal nonet scheme for couplings of the meson trajectories.[1)] For meson systems, these predictions are extremely well satisfied in nature.

The dual resonance model proposed by Veneziano[7)] is an explicit example that satisfies all the foregoing sets of assumptions. The implications of the Veneziano formula are not yet fully understood, but there have been many investigations in recent times. It is a remarkable fact that the Veneziano formula is well visualized by the duality diagrams[8)] based on the quark model. Many interesting features of the amplitudes are, in fact, easily read off from such diagrams.

However, there are troubles also. It has been noted earlier that the foregoing assumptions, when applied to baryon-antibaryon scattering, are mutually inconsistent.[1)] Such difficulty is also expected when we draw the duality diagram for baryon-antibaryon scattering, as illustrated in Fig. 1. In Fig. 1, the ordinary meson trajectories represented by $q\bar{q}$ pair are exchanged in t-channel, while at the same time when it is viewed from s-channel there are objects represented by $qq\bar{q}\bar{q}$. These objects should be exotic mesons if they appear as resonances, but they can equally be identified as two (ordinary) meson states. Whichever states they may be, it is impossible for the FESR for baryon-antibaryon scattering amplitudes to have any consistent solution with the assumptions 1), 2) and 3).

In order to remove the inconsistency mentioned above, it will be necessary to abandon either the assumption 2) or 3). If the assumption 2) is dropped and instead of it exotic mesons of the first kind are introduced, it is then easy to find a solution and this will certainly be the simplest approach to our problem. A question then arises. What kind of exotic resonances, if any, should be realized in nature? We try, in § 2, to find a minimal set of such exotic meson trajectories necessary to dissolve the inconsistency. It will be shown there that the FESR in question has a unique solution if we introduce 36-plet,[9)] 18-plet, $\overline{18}$-plet and nonet meson trajectories with both signatures and the extended nonet coupling ansatz for these exotic mesons. Mass spectra and possible selection rules for decays of these mesons are then examined in § 3 by applying the extended nonet ansatz. The spacing laws operating among these exotic meson trajectories are fixed, although their intercepts are not uniquely given. The results we obtain are direct generalizations of the nonet spacing law already found in the analysis of ordinary meson-meson scattering.[10] It is also argued there that similar results are obtainable by examining the Veneziano-type formula for the scattering processes involving exotic mesons as external particles.

One may still ask the question whether there are any other simple ways of avoiding the difficulty without introducing exotic resonances. In particular one would like to know whether it is possible to extend the idea of duality between s- and t-channel (narrow) resonances to a "duality" among resonances in one channel and continua (cuts) in another channel,[11] as suggested from the duality diagram. Guided again by the duality diagram, we propose in § 4, an explicit formula which indeed has such "pole-cut" duality. Using the formula developed in § 4, it will be possible to construct a solution for the FESR without introducing exotic resonances.

Finally in § 5, discussions about possible models for hadrons will be given in the light of our results.

The appendix is devoted to a detailed discussion of the derivation of the amplitude proposed in § 4.

§ 2. Exotic resonances and baryon-antibaryon scattering

With assumptions 1) and 3), the flatness of total cross sections for meson-meson and meson-baryon scattering processes in exotic channels is directly connected with the absence of exotic resonances at low energies in the same channels. In Regge theory, on the other hand, the constant total cross sections are explained by the exchange degenerate nonet trajectories upon which $V(1^-)$ and $T(2^+)$ nonet mesons lie. The coupling relations among the residues of these trajectories are found to be consistent with those derived from the "nonet coupling ansatz" of Okubo in good agreement with experiment.[12]

For baryon-antibaryon scattering, the situation is different. Assuming the exchange degenerate nonet trajectories with the nonet coupling residues, we calculate the amplitudes of baryon-antibaryon scattering at high energies in exotic channels and find that there is in fact energy dependences for some of these amplitudes (10, $\overline{10}$ and 27).[13] The FESR then implies that there must be some sort of resonating contributions at low energies with exotic quantum numbers. In this section we assume that these contributions come from exotic meson resonances coupled to baryon-antibaryon channels and examine what kind of exotic trajectories, if any, will appear as a solution of the FESR.

Consider first the scattering of decuplet baryons ($D_{\alpha_1\alpha_2\alpha_3}$) from decuplet antibaryons, as illustrated, in terms of the duality diagram, in Fig. 1.[*] The

[*] In Fig. 1, we consider only the diagram in which ordinary nonet mesons are exchanged in t-channel. s-t crossing symmetry for the whole amplitude can be easily incorporated by adding a similar term in which exotic mesons are exchanged in t-channel.

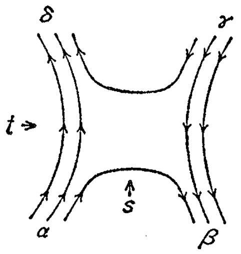

Fig. 1. Duality diagram for the process
$D_{\alpha_1\alpha_2\alpha_3}+\bar{D}^{\beta_1\beta_2\beta_3}\rightarrow\bar{D}_{\gamma_1\gamma_2\gamma_3}+D^{\delta_1\delta_2\delta_3}$.

diagram explicitly displays the duality between $q\bar{q}$ states, identified with the nonet vector and tensor trajectories exchanged in t-channel, and $qq\bar{q}\bar{q}$ states, which we are going to interpret as trajectories with exotic quantum numbers.

In order to investigate further the coupling schemes for mesons which lie on these exotic trajectories, we introduce a model*) with factorizable internal symmetry[14] and look for a solution of the FESR in which ordinary nonets are exchanged in t-channel.

We write the scattering amplitude for the process $D_{\alpha_1\alpha_2\alpha_3}+\bar{D}^{\beta_1\beta_2\beta_3}\rightarrow\bar{D}_{\gamma_1\gamma_2\gamma_3}+D^{\delta_1\delta_2\delta_3}$ as follows:

$$A_{\alpha\gamma}^{\beta\delta}(s,t)=R_{\alpha\gamma}^{\beta\delta}F(s,t), \tag{2·1}$$

where the three quark indices are associated to each decuplet. We have adopted the convention that the quark index is a subscript while the anti-quark index is a superscript for incoming particles and the quark index is a superscript while the antiquark index is a subscript for outgoing particles. R is a matrix in unitary space. As is seen from Fig. 1, we have the following $SU(3)$ invariant form for the matrix R:

$$R_{\alpha\gamma}^{\beta\delta}=\sum_{P(123)}\delta_{\alpha_1}^{\delta_2}\delta_{\alpha_2}^{\delta_3}\delta_{\alpha_3}^{\beta_1}\delta_{\gamma_1}^{\beta_2}\delta_{\gamma_2}^{\beta_3}\delta_{\gamma_3}^{\delta_1}, \tag{2·2}$$

where the right-hand side is to be symmetrized with respect to 1, 2 and 3 for α, β, γ and δ.

Let us now investigate the factorization of the matrix R. In t-channel it is factorized as

$$R_{\alpha\gamma}^{\beta\delta}=\sum_{x,y}V_{x\alpha}^{\delta y}V_{y\gamma}^{\beta x}, \tag{2·3}$$

where

$$V_{x\alpha}^{\delta y}=\sum_{P(123)}\delta_x{}^{\delta_1}\delta_{\alpha_1}^{\delta_2}\delta_{\alpha_2}^{\delta_3}\delta_{\alpha_3}^{y},$$

$$V_{y\gamma}^{\beta x}=\sum_{P(123)}\delta_y{}^{\beta_1}\delta_{\gamma_1}^{\beta_2}\delta_{\gamma_2}^{\beta}\delta_{\gamma_3}^{x}.$$

*) More specifically, we keep in mind the duality amplitude such as that proposed by Veneziano, but its explicit form is not necessary for the following discussions,

Equation (2·3) is to be understood as a summation over the exchanged nonets with indices x and y. The couplings of Regge residues for these nonet trajectories are described by the following $SU(3)$-invariant interaction:

$$\bar{D}^{ijk}D_{ijk'}M_k^{k'}, \tag{2·4}$$

where $M_k^{k'}$ stands for $V(1^-)$ and $T(2^+)$ nonet trajectories. This coupling is of course consistent with the nonet coupling ansatz.*)

The factorization in s-channel, on the other hand, may be written as

$$R_{\alpha\gamma}^{\beta\delta}=\sum_{x,y} V_{\alpha x}^{y\beta}V_{\gamma y}^{x\delta}\,, \tag{2·5}$$

where

$$V_{\alpha x}^{y\beta}=\sum_{P(123)}\delta_{\alpha_1}^{y_1}\delta_{\alpha_2}^{y_2}\delta_{\alpha_3}^{\beta_1}\delta_{x_1}^{\beta_2}\delta_{x_2}^{\beta_3}\,,$$

$$V_{\gamma y}^{x\delta}=\sum_{P(123)}\delta_{\gamma_1}^{x_1}\delta_{\gamma_2}^{x_2}\delta_{\gamma_3}^{\delta_1}\delta_{y_1}^{\delta_2}\delta_{y_2}^{\delta_3}\,.$$

Note that the V's are symmetric with respect to x_1, x_2 and y_1, y_2. Consequently the intermediate states in s-channel should be described by a tensor E_{kl}^{ij} symmetric in upper and lower indices, respectively, and the couplings of the Regge residues are written as

$$\bar{D}^{ijk}D_{ij'k'}E_{j\ k}^{j'k'}. \tag{2·6}$$

It can easily be shown that the tensor E_{kl}^{ij} belongs to the 36-dimensional representation, reducible to $1\oplus8\oplus27$ in $SU(3)$ and that the coupling scheme described in (2·6) is a generalization of the nonet coupling scheme, extended here to the 36-plet meson trajectories.[9]

It is possible to modify Eq. (2·1) slightly so that one can introduce the $SU(3)$ breaking in the intercepts of Regge trajectories. Thus not all of the 36-plet trajectories are necessarily degenerate. There are restrictions, however. For instance, if the degeneracy between ρ-A_2 and ω-f trajectories of the nonet is assumed, which is one of the results of the FESR, as applied to meson-meson scattering, it is then concluded that α_T-α_V, ρ-A_2 and ω-f members**) of the 36-plet trajectories must also be degenerate. The same conclusion is derivable from the FESR as directly applied to Δ-$\bar{\Delta}$ scattering in $SU(2)$.

*) Let us briefly recapitulate the nonet coupling ansatz of Okubo. One assigns $V(1^-)$ and $T(2^+)$ mesons to $SU(3)$ nonets with the $SU(6)$ mixing angle (ideal nonets). The ansatz then requires that, for meson vertices, coupling terms involving the trace of these nonets should vanish. This ansatz can be extended to the couplings involving baryons, if we further assume that the quark indices associated with the nonet mesons are allowed to couple *directly* to the quark indices of the baryons. Consequently, even for the octet baryons, one should use the three-quark wave functions $B_{ij,k}$ instead of $B_j{}^i$. One can then decouple ϕ and f' from nonstrange octet baryons, in a manner similar to decoupling them from nonstrange mesons. It is interesting to note that, in terms of the quark model, these coupling schemes are realized when one allows only connected quark line vertices, as emphasized by Iizuka in reference 8).

**) For notations, see Table I.

We note, however, that the relative location of the intercepts among 36-plet trajectories in broken $SU(3)$ is not yet specified, and we shall come back to this point in the next section.

So far we have considered only decuplet baryon-decuplet antibaryon scattering. The scattering involving octet baryons can be dealt with in a manner similar to that which we have discussed above. For the scattering of octet baryons from octet antibaryons we should introduce, besides the 36-plet, the 18-plet, $\overline{18}$-plet and another nonet, reducible to $8\oplus 10$, $8\oplus\overline{10}$ and $1\oplus 8$, respectively. For the scattering of octet baryons from decuplet antibaryons we need the 36-plet and the 18-plet. Altogether we have 81-plet (=36-plet+18-plet+$\overline{18}$-plet+9-plet) trajectories in s-channel and their coupling schemes are described by the "extended nonet scheme" using tensors corresponding to 2 quark and 2 antiquark indices. The $SU(2)$ contents of each multiplet are summarized in Tables I, II and III.

§ 3. Further properties of exotic resonances

In this section we shall discuss some of the properties of the representations with exotic quantum numbers.

a) *Structures of Exotic Regge Trajectories*

To begin with we consider the mass spectra of the 36-plet mesons in broken $SU(3)$, and investigate the $SU(3)$ structure of Regge trajectories on which these 36-plet mesons should lie.

The content of the 36-plet representation, as expressed in terms of $SU(3)$ tensor E^{ij}_{kl} symmetric in upper and lower indices, may be divided into the following six groups,

$$E^{ab}_{cd},\ E^{3a}_{bc},\ E^{3a}_{3b},\ E^{33}_{3a},\ E^{33}_{33} \text{ and } E^{33}_{ab}. \qquad (a, b, c, d=1, 2) \tag{3·1}$$

The $SU(2)$ contents of individual groups are summarized in Table I.

It was noted in the previous section that the Regge trajectories for the members belonging to each group are to have exact degeneracies and consequently each of them is described by a single trajectory. The trajectories for different groups, on the other hand, are not necessarily degenerate but it is difficult to predict, from the simple arguments such as those given in the previous section, relative locations of the trajectories among different groups.

On the other hand, Okubo's "ansatz", which we have used to determine the coupling schemes of nonets and 36-plets, can also be applied to predicting mass splittings among these multiplets. For 36-plet mesons, the mass formula consistent with the ansatz is

$$m^2 = aE^{ij}_{kl}E^{kl}_{ij} + b(E^{i3}_{jk}E^{jk}_{i3} + E^{ij}_{k3}E^{k3}_{ij}), \tag{3·2}$$

Table I. 36-plet.

Tensor	Hypercharge (Y)	Isospin Contents	Trajectory
E^{ab}_{cd}	0	$\alpha_T(I=2)$; $A_2(I=1)$; $f(I=0)$	α_0
E^{3a}_{bc}	1	$K_T{}^{3/2}(I=\frac{3}{2})$; $K_T(I=\frac{1}{2})$	α_1
E^{3a}_{3b}	0	$A'_2(I=1)$; $f'(I=0)$	α_2
E^{33}_{3a}	1	$K_T'(I=\frac{1}{2})$	α_3
E^{33}_{33}	0	$f''(I=0)$	α_4
E^{33}_{ab}	2	$\beta_T(I=1)$	α_β

The particle states are tentatively labeled by referring to the corresponding members of 2^+ nonets. Antiparticles are not listed in this table. For detailed expression of particles in terms of tensors, see the last reference in 9).

from which the following mass formulae[9] are derived:*)

$$
\begin{aligned}
&m^2(\alpha_T)=m^2(A_2)=m^2(f),\\
&m^2(A^{2\prime})=m^2(f')=m^2(\beta),\\
&m^2(K_T{}^{3/2})=m^2(K_T), \qquad (3\cdot3)\\
&m^2(f'')-m^2(K_T')=m^2(K_T')-m^2(f')=m^2(f')-m^2(K_T)=m^2(K_T)-m^2(f).
\end{aligned}
$$

These mass relations then require that the members belonging to the same group, given in (3·1), have degeneracy in masses, thus implying the degenerate Regge trajectories, consistent with the previous arguments. The last relation of (3·3) suggests in fixing the relations among the relevant trajectories*) belonging to different groups

$$
\begin{aligned}
&\alpha_4(s)-\alpha_3(s)=\alpha_3(s)-\alpha_2(s)=\alpha_2(s)-\alpha_1(s)=\alpha_1(s)-\alpha_0(s),\\
&\alpha_2(s)=\alpha_\beta(s). \qquad (3\cdot4)
\end{aligned}
$$

It is interesting to note that the structures we have obtained above are also suggested by the arguments[10] based on the Veneziano-type amplitudes applied to the scattering of 0^- octet mesons and the 27-plet mesons with appropriate normality. We write the beta-function formulae without satellite terms for these reactions and require for them Adler's self-consistency condition. One then obtains the constraints for the intercepts among the Regge trajectories that appear in those reactions, which imply the equal spacing rule given in (3·4).

Finally, we notice that the equal spacing rule similar to that for 36-plet meson trajectories holds also for the trajectories belonging to 18-plet, $\overline{18}$-plet and nonet introduced in the previous section. The $SU(2)$ contents of these multiplets are summarized in Tables II and III.

*) For notations see Table I.

Table II. $\overline{18}$-plet.

Tensor	Hypercharge	Isospin Contents
E^{ab}_{3c}	-1	$I=\frac{3}{2}$, $I=\frac{1}{2}$
E^{ab}_{cd}	0	$I=1$
E^{3a}_{bc}	1	$I=\frac{1}{2}$
E^{3a}_{3b}	0	$I=1$, $I=0$
E^{33}_{3a}	1	$I=\frac{1}{2}$
E^{33}_{ab}	2	$I=0$

Note that the tensor in this table is symmetric in upper indices and antisymmetric in lower indices. 18-plet is just conjugate of this tensor.

Table III. Nonet.

Tensor	Hypercharge	Isospin Contents
E^{ab}_{cd}	0	$I=0$
E^{3a}_{bc}	1	$I=\frac{1}{2}$
E^{3a}_{3b}	0	$I=1$, $I=0$

b) *Couplings of exotic mesons*

Let us again consider the problem of factorization in D-$\bar{D}$ scattering. We have given in the previous section an explicit solution for the factorization in t- and s-channels. However, besides the form given there, there is still another way of achieving the factorization in t-channel, namely

$$R^{\beta\delta}_{\alpha\gamma}=\sum_{x,y} V^{\delta y}_{x\alpha} V^{\beta x}_{y\gamma}\,, \tag{3·5}$$

where

$$V^{\delta y}_{x\alpha}=\sum_{P(123)} \delta^{\delta_1}_{x_1}\delta^{\delta_2}_{x_2}\delta^{\delta_3}_{\alpha_1}\delta^{y_1}_{\alpha_2}\delta^{y_2}_{\alpha_3}\,,$$

$$V^{\beta x}_{y\gamma}=\sum_{P(123)} \delta^{\beta_1}_{y_1}\delta^{\beta_2}_{\gamma_1}\delta^{\beta_3}_{\gamma_2}\delta^{x_1}_{\gamma_3}\delta^{x_2}_{y_2}\,.$$

This factorization is illustrated in Fig. 2. The coupling of the first vertex (the left vertex in Fig. 2) is identical to that of Eq. (2·6), while the coupling of

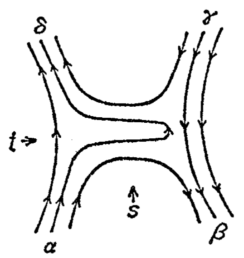

Fig. 2. Duality diagram illustrating the extra type of factorization in t-channel.

the second (the right vertex) written as

$$\bar{D}^{ijk} D_{ijk'} E_{kl}^{k'l} \tag{3·6}$$

is a new coupling of the 36-plet trajectories to decuplet baryons. The presence of such a coupling would imply that in the t-channel, not only the nonets but also the 36-plet meson trajectories may contribute and they may or may not be degenerate. It will be clear, however, that only the nonet part of the 36-plet trajectories are allowed to couple to decuplet baryons through (3·6) and thus there is still no mesons exchanged in t-channel, belonging to the $SU(3)$ representations higher than octet and singlet. Note that the coupling defined by (3·6) is distinct from the usual nonet coupling (2·4).*) Possible presence of the coupling of this type further suggests the following vertex for the interaction of the 36-plet with the ordinary mesons,

$$E_{ik}^{ij} M_j^{\ l} M_l^{\ k} \tag{3·7}$$

which is illustrated in Fig. 3, where again only the nonet part of 36-plet mesons comes into play. In the case when such a coupling scheme is indeed allowed, we can modify our solution of the FESR obtained before simply by adding terms corresponding to new factorizations.

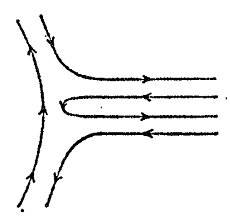

Fig. 3. Possible coupling of exotic mesons to two ordinary mesons.

§ 4. Extended Veneziano-type amplitude with Regge cuts

From our considerations made so far it will be clear that when exotic resonances are really absent in nature, the concept of duality between non-exotic resonances in s- and t-channels would no longer be applicable to baryon-antibaryon scattering. In such a case one is forced to give up the assumption (3) and introduces a modified duality concept involving cuts in addition to simple resonance poles, a possibility noted before by Pinsky.[11] In terms of the duality diagram given in Fig. 1 the simplest interpretation for the state represented by $(qq\bar{q}\bar{q})$ would be to identify it as the two-meson exchange process $(q\bar{q}+q\bar{q})$. The amplitude would then be such that *it has poles as well as cut singularities coming from baryon-antibaryon pair states in t-channel and only cut singularities in*

*) The residues for these trajectories, i.e. the product of coupling constants, we get from (2·6) and (3·6), are identical to those evaluated from (2·4). Consequently if the trajectories of the nonets and the 36-plets are degenerate, there is no distinction between the exchange of the nonet and that of the nonet part of the 36-plet.

s- (u-) channel coming from two-meson (baryon) states. We call this property the "pole-cut duality". It is to be distinguished from the property of higher order corrections recently formulated by Kikkawa, Sakita and Virasoro.[15),*)]

Our next task is then to go one step further and construct an explicit formula which exhibits such "pole-cut duality". For this purpose we extensively rely upon the duality diagram based on the quark model. The duality diagram (the quark-line diagram) was useful for analyzing the FESR, but it was also used for constructing generalized Veneziano formulae such as N-point functions and the amplitudes with loops. The rules to construct the generalized Veneziano formula in terms of the quark duality diagram have been given by Koba and Nielsen[16] in meson systems, according to which meson lines are represented by $q\bar{q}$ pairings. The rules, however, are not general enough for the cases in which baryons represented by three quark lines are involved. The reason is that in baryon-antibaryon scattering there are two "spectator" quarks, which, however, could form a $q\bar{q}$ meson and therefore may not really be regarded as "spectators". In this case not all of the restrictions ("the angular momentum conditions") among dual variables are derivable from the usual rule. In the case of meson-baryon scattering, baryons can effectively be considered as (qq) system and as such there arises no serious difficulty, since there is only one "spectator" quark involved.

Nevertheless, still it is possible to find a suitable rule for treating such a $q\bar{q}$ pairing and construct the amplitude with the desired properties. For simplicity we will show such an example in the case of scalar quarks and baryons, leaving all the detailed intuitive procedures for constructing the amplitude in the Appendix. Here we shall give a rough sketch.

In Fig. 1, there are many ways to pick out possible $q\bar{q}$ pairs and qqq sets, each representing a line in an extended duality diagram. We choose among them one of the simplest sets of lines.**) There are seven such lines. Cross lines are not independent and there must be certain relations, which are derived in the Appendix. Choosing x, y, u_1 and u_2 as independent variables, we write the following expression for the scattering amplitude:

$$T=\int d^4p\,dx\,dy\,du_1du_2[u_1(1-u_1)u_2(1-u_2)]^{-1}x^{-1-\alpha_0[-((P_1-P_2/2)+p)^2]}$$
$$\times y^{-1-\alpha_0[-((P_1-P_2/2)-p)^2]}z^{-\alpha_2(t)}{u_1}^{-\alpha_1[-((P_3+P_4/2)+p)^2]}$$
$$\times {v_1}^{-\alpha_1[-((P_3+P_4/2)-p)^2]}{u_2}^{-\alpha_1[-((P_1+P_2/2)+p)^2]}{v_2}^{-\alpha_2[-((P_1+P_2/2)-p)^2]}, \qquad (4\cdot1)$$

where α_0 and α_2 are the meson trajectories and α_1 is the (scalar) baryon trajectory with the universal slope α'. The dependent variables v_1, v_2 and z are given,

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*) We take the viewpoint that the amplitude with "pole-cut duality" corresponds to the "Born" amplitude in baryon-antibaryon scattering. Starting with such an amplitude one will be able to calculate higher order loop corrections.

**) See the Figures (in particular Fig. A5) in the Appendix.

in terms of x, y u_1 and u_2, by

$$v_i=(1-u_i)/[1-u_i(x+y-xy)], \qquad (i=1,2) \tag{4·2}$$

and

$$z=1-xyu_1u_2v_1v_2\,. \tag{4·3}$$

After integration over momentum p, we arrive at the final formula

$$T=\frac{i\pi^2}{(\alpha')^2}\int dxdydu_1du_2$$
$$\times\frac{(xy)^{-1-a_0}z^{-a_2}(u_1u_2v_1v_2)^{-a_1}}{u_1u_2(1-u_1)(1-u_2)\ln^2(xyu_1u_2v_1v_2)}\exp[-sf-ug-m^2h] \tag{4·4}$$

with

$$f=\frac{\alpha'}{\ln(xyu_1u_2v_1v_2)}(\ln x\ln y-\ln u_1u_2\ln v_1v_2), \tag{4·5}$$

$$g=\alpha'\left(\ln z+\frac{\ln(u_1/v_1)\ln(u_2/v_2)}{\ln xyu_1u_2v_1v_2}\right) \tag{4·6}$$

and

$$h=\frac{\alpha'}{\ln(xyu_1u_2v_1v_2)}(\ln xy\ln u_1u_2v_1v_2+4\ln u_1u_2\ln v_1v_2). \tag{4·7}$$

In Eq. (4·4), a_i stands for the intercept of the trajectory $\alpha_i(t)$,

$$a_i=\alpha_i(0), \tag{4·8}$$

and m is the mass of the baryon.

It is shown in the Appendix that the expression (4·4) has the following nice features:

1) The integrand of Eq. (4·1) has poles at

$$\alpha_i(p^2)=0, 1, 2, \cdots, \qquad (i=0,1)$$

and the amplitude (4·4) has poles at

$$\alpha_2(t)=0, 1, 2, \cdots$$

with correct residues.

2) The asymptotic behaviors are given by

$$T\sim s^{\alpha_2(t)}+s^{2\alpha_1(t/4)-1}/\ln s\,, \qquad \text{for fixed } t;$$
$$T\sim t^{2\alpha_0(s/4)-1}/\ln t\,, \qquad \text{for fixed } s;$$
$$T\sim s^{2\alpha_1(u/4)-1}/\ln s\,, \qquad \text{for fixed } u.$$

3) The FESR is automatically satisfied.

Thus the amplitude (4·4) is an explicit solution of the FESR with the "pole-cut duality". It has poles as well as cuts in t-channel and only cuts in s- and u-channels. It may be worthwhile to note that in the foregoing example, the series of poles in t-channel are embedded in the continuum coming from cuts in the same channel. Such situation may not be so serious at this stage, since we have not yet taken into account the unitarity corrections to our amplitudes.

Another point worth noticing is that our formula (4·4) has singularities in all channels (s-, t- and u-channels), thus, containing those coming from non-planar diagrams, though one usually attempts to construct the amplitude in "Born approximation", by using the Veneziano-type formula which has singularities coming from only planar diagrams. In this sense our solution, although similar in spirit, is different from Pinsky's example.[11]

In a more realistic case, one has to take into account signature factors as well as exchange degeneracies. This may be possible by adding appropriate terms*) similar to those given in (4·4). One will then find that in the baryon-baryon channel (u-channel in our case) total cross sections are flat as a consequence of factorization of the exchange degenerate meson and baryon trajectories.

§ 5. Summary and discussion

We have discussed the concept of duality as applied to baryon-antibaryon scattering from two different points of view. In each case, it was necessary to modify some of the assumptions underlying the usual analyses of the FESR.

When mesons with exotic quantum numbers are allowed to exist in nature, it is then possible to apply the duality between ordinary and exotic meson trajectories, and we find the simplest possible solution of the FESR for baryon-antibaryon scattering.

If one requires that ordinary meson trajectories obey the equal spacing rule with the nonet coupling residues all along the trajectories, it can then be shown that as a result of factorization these exotic meson trajectories, expressed as $qq\bar{q}\bar{q}$ states are classified according to the multiplets of 36, 18, $\overline{18}$ and 9 dimensional reducible representations of $SU(3)$. Trajectories belonging to each multiplet are split according to the rules consistent with the extended nonet-like mass formulae for mesons which lie on these trajectories. Similar results are also suggested from the PCAC consistency conditions applied to the scattering of 0^- mesons with the mesons of exotic quantum numbers.

These exotic trajectories will probably be degenerate with daughters of leading non-exotic trajectories, but it is rather difficult to predict their intercepts in a convincing manner.

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*) See the first footnote on p. 771. Internal symmetry factors can be introduced in a manner similar to that discussed in § 2.

Experimental observations of meson and baryon resonances with exotic quantum numbers will of course be very crucial for such attempts to work. In this connection, the selection rules for hadron couplings proposed by Freund, Rosner and Waltz[17] are to be noticed. It was suggested from the duality diagram for hadron vertices. It says that

(A) Each of the three hadrons meeting at a vertex exchanges at least one quark line with each of the remaining two hadrons.

(B) No quark line is connected with an antiquark line from the same hadron.

Under such selection rules, exotic mesons, represented in the quark model by $qq\bar{q}\bar{q}$ states, will only couple to baryons but not to ordinary mesons. As a result, there is no change about the conclusions in meson-meson scattering.

One may modify the above selection rules and drop the assumption (B).*) Then we are allowed to introduce the couplings of the type (3·7) mentioned in § 3 (b), under which the nonet part of $qq\bar{q}\bar{q}$ states can decay into two ordinary mesons. Since only the nonet part of $qq\bar{q}\bar{q}$ couples to ordinary mesons, the results from the analyses of the FESR remain unaltered as before.

One reason why we stick to $q\bar{q}$ excitation model for hadrons without the assumption (B) is that we want to take into account the level structures of hadrons in the duality diagram. It has been pointed out by several authors[8),16)] that the duality diagram based on the quark model is a good representation for constructing the Veneziano-type amplitudes. However, formulated as it is, the diagram is not sufficient to predict such rich level structures as suggested from the factorization of N-point Veneziano formulae.[18] When we introduce $q\bar{q}$ excitation model for hadrons without the assumption (B), it is then possible to interpret level structures of hadrons within the duality diagram based on the quark model, as recently suggested by one of us (H.Y.).[19]

In the case in which there are no resonances in exotic channels, we are forced to conclude that the ordinary concept of duality between (narrow) resonances is no longer applicable to baryon-antibaryon scattering. Based on the duality diagram for baryon-antibaryon scattering, we have then introduced the "pole-cut duality" and constructed an explicit formula with such modified duality. In doing so, it was necessary to generalize the rules for constructing the Feynman-like diagrams with loops.[15]

In particular, it is necessary to find the constraint given in (4·3). We believe that our constraint (4·3) is right, since it gives correct residues for the poles at $\alpha_2(t)=n$ $(n=0, 1, 2, \cdots)$ and correct asymptotic behavior, but we have not yet studied general rules for such constraints.

Finally, we note that our amplitude simultaneously represents the second

*) It is tempting to explain the observed split peaks of A_2 mesons as an interference between the ordinary $A_2(2^{++})$, corresponding to $q\bar{q}$ representation with $l=1$ and a new A_2, corresponding to the nonet part of $qq\bar{q}\bar{q}$ representation. Two nonets are distinguishable by studying the decay branching ratios, since, as noted before, the decay couplings into two mesons are different.

and the fourth order Feynman diagrams and therefore the corresponding coupling constants are no longer arbitrary. A question remains open as to whether or not such determination of the coupling constants is consistent with general bootstrap conditions for hadron scattering amplitudes.

It is to be hoped that our example is the first step toward a more general treatment of baryons in the duality theory.

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Appendix

Derivation of the "pole-cut dual" amplitude

We treat our problem on the basis of the quark duality diagram and take into account explicitly the qqq structure for baryons. In the first place, we interpret the rules for constructing the usual dual amplitudes stated in the duality-diagram language, in terms of quark-line language.

As an example, consider a 5-point scalar meson amplitude. The tree diagram and its quark-line representation are shown in Fig. (A·1). Treating all the quarks on an equal footing, we rewrite Fig. (A·1b) into Fig. (A·2), where the solid lines indicate all possible $q\bar{q}$ pairings. The duality diagram is then nothing but the diagram formed by the solid lines in Fig. (A·2). As another example, let us consider meson-baryon scattering, whose quark duality diagram is given in Fig. (A·3). In this case there are two kinds of lines represented by $q\bar{q}$ and qqq sets, each corresponding to meson and baryon exchange processes. The diagram formed by the solid lines is again the duality diagram.

Now let us see what would be a natural consequence when the above approach is applied to baryon-antibaryon scattering, as illustrated in Fig. (A·4). Here also there are only two kinds of dual lines corresponding to $q\bar{q}$ and qqq sets, as we are assuming no exotic resonances. The dual lines in Fig. (A·5) are denoted by hooks in order to avoid confusion. As the variables associated with the dual lines in Fig. (A·5) we take

$$\begin{aligned} x &: 15 \\ y &: 26 \\ z &: 34 \\ u_1 &: 146 \end{aligned} \tag{A·1}$$

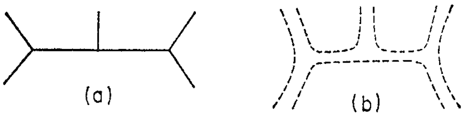
$$u_2 : 136$$
$$v_1 : 245$$
$$v_2 : 235\,,$$

where the quark lines are numbered according to Figs. (A·4) and (A·5). An example of "dual diagrams" and the corresponding Feynman diagrams are shown in Fig. (A·6).

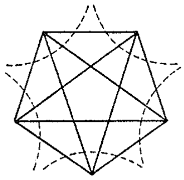
There are many possible lines other than those shown in (A·1). (Fig. (A·5).) Interchanging 1↔2 in (A·1), for instance, we get another set of variables

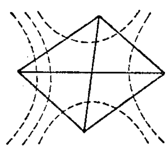
$$\begin{aligned} x' &: 25 \\ y' &: 16 \\ z &: 34 \\ u_1' &: 246 \\ u_2' &: 236 \\ v_1' &: 145 \\ v_2' &: 135\,. \end{aligned} \qquad (\mathrm{A}\cdot 2)$$

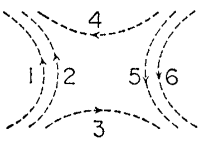
Other examples are



(A·1)

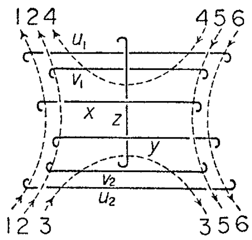

(A·2)


(A·3)

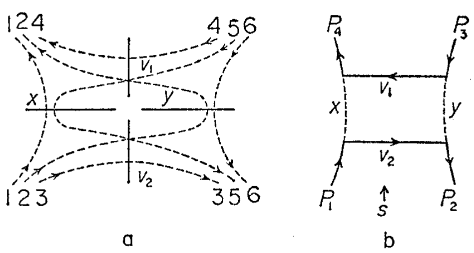


(A·4)

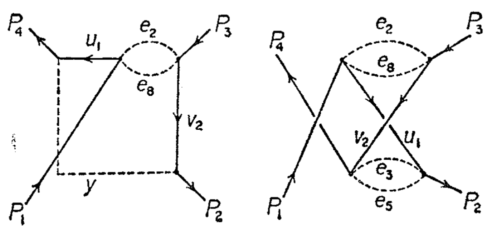
Figs. (A·1)～(A·4).



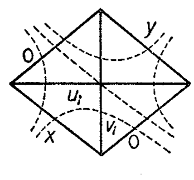
(A·5)

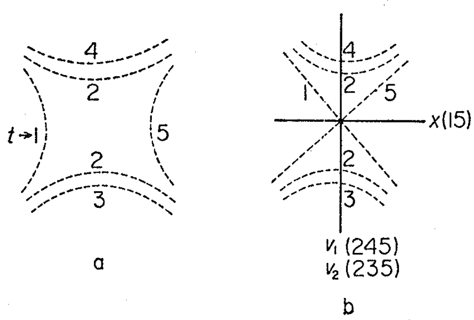


(A·6)


(A·7)

Figs. (A·5)～(A·7).


(A·8)



(A·9)

Figs. (A·8)～(A·9).

$$\begin{aligned} e_1 &: 14 \\ e_2 &: 24 \\ e_3 &: 45 \\ e_4 &: 46 \\ e_5 &: 13 \\ e_6 &: 23 \\ e_7 &: 35 \\ e_8 &: 36\,, \\ &\text{etc.} \end{aligned} \tag{A·3}$$

It is then not difficult to show that the set of variables chosen in (A·3) corresponds to Feynman diagrams with more than two loops. We show in Fig. (A·7) two such examples. In the following, we do not take into account such sets of variables simply because our main concern is to find the simplest possible example exhibiting the "pole-cut duality". In a similar manner, the set (A·2), which is equivalent to the set (A·1), is also put outside of our consideration. Thus we are going to construct the duality amplitude by the set (A·1) alone.

The constraints among dual variables are now to be investigated. We note from Fig. (A·6) that the amplitude in this case is connected through s-channel unitary with the off-mass-shell meson-baryon scattering. As a result, one may pick out the relevant meson-baryon scattering with the variables shown in Fig. (A·8), and applies the rule of Kikkawa, Sakita and Virasoro.[15] One will then find

$$v_i = \frac{1-u_i}{1-u_i(x+y-xy)}\,. \qquad (i=1,2) \tag{A·4a}$$

The restrictions (A·4a) are still insufficient for our purpose, as the meson state corresponding to the line z is not specified in terms of other variables. It has the property such that if one of the lines x, y, u_1, u_2, v_1 and v_2 is on the mass shell, then $z \neq 0$. We *assume* therefore that

$$z = 1 - xyu_1u_2v_1v_2\,. \tag{A·4b}$$

Indeed this choice will be found to be a correct one when we examine analytic structures of the amplitude.

We take the four variables x, y, u_1 and u_2 as independent integration variables. Since u_i and v_i must enter symmetrically into the expression for the amplitude in question, we use the following relation [derived from (A·4a)] to write down the explicit formula;

$$\frac{du_i}{u_i(1-u_i)} = -\frac{dv_i}{v_i(1-v_i)}\,. \qquad (i=1,2) \tag{A·5}$$

The expression for the amplitude is then obtained:

$$\begin{aligned}T= \int d^4p\,dx\,dy\,\frac{du_1}{u_1(1-u_1)}\,\frac{du_2}{u_2(1-u_2)}\,x^{-1-\alpha_0[-((P_1-P_2/2)+p)^2]} \\ \times y^{-1-\alpha_0[-((P_1-P_2/2)-p)^2]} z^{-\alpha_2(t)} u_1{}^{-\alpha_1[-((P_3+P_4/2)+p)^2]} \\ \times v_1{}^{-\alpha_1[-((P_3+P_4/2)-p)^2]} u_2{}^{-\alpha_1[-((P_1+P_2/2)+p)^2]} \\ \times v_2{}^{-\alpha_1[-((P_1+P_2/2)-p)^2]},\end{aligned} \tag{A·6}$$

where α_0 and α_2 are the meson trajectories and α_1 the (scalar) baryon trajectory with $\alpha_i(0)<0$ $(i=0,1,2)$, and $s=-(P_1-P_2)^2$, $t=-(P_1-P_4)^2$ and $u=-(P_1+P_3)^2$. The argument of the trajectory α_i is determined from the following procedure: Assign a momentum to each quark and simply sum the momenta of the constituent quarks; for example, $P_1=p_1+p_2+p_3$, etc., where p_i is the momentum attached to the i-th quark. The exponents of the dual variables are so chosen that the poles of the corresponding particles occur at $\alpha_i(p^2)=0, 1, 2, \cdots$. It is now straightforward to perform the integration in (A·6) over the Euclidean loop momentum p, and we arrive at the expression (4·4) given in § 4.

Next the high-energy asymptotic behavior is investigated, making use of the technique of the Mellin transform.[20)]

$s\to\infty$, t *fixed*

$$\begin{aligned}T(l,t) &\equiv \int_0^\infty ds\,T(s+i\varepsilon,t)\,s^{-l-1} \\ &= \frac{i\pi^2}{(\alpha')^2}\int dx\,dy\,du_1\,du_2\,\frac{(xy)^{-1-\alpha_0}z^{-\alpha_2}(u_1u_2v_1v_2)^{-\alpha_1}}{u_1u_2(1-u_1)(1-u_2)\ln^2(xyu_1u_2v_1v_2)} \\ &\quad\times\Gamma(-l)f^l\exp[-tg-m^2h].\end{aligned} \tag{A·7}$$

The right-most singularities of $T(l,t)$ are produced in the following two cases:

(i) $z\sim0$; x, y, u_i, $v_i\sim1$,

(ii) $u_i\sim0$; x, y, z, $v_i\sim1$.

Contribution to the integral (A·7) coming from the region (i) is estimated by changing the variables,

$$\begin{aligned}x&=1-\rho\lambda\,, \\ y&=1-\rho(1-\lambda), \\ u_i&=1-\rho\eta_i\,, \\ &(0\leq\rho\leq\rho_0\ll1)\end{aligned} \tag{A·8}$$

giving the expression

$$T(l,t)\simeq\frac{i\pi^2}{(\alpha')^2}\Gamma(-l)\frac{\rho_0{}^{l-\alpha_2(t)}}{l-\alpha_2(t)}\int d\lambda d\eta_1 d\eta_2 F(\lambda,\eta_1,\eta_2) \tag{A·9}$$

with

$$F(\lambda,\eta_1,\eta_2)=\frac{[1+\eta_1+\eta_2+\lambda(1-\lambda)(1/\eta_1+1/\eta_2)]^{-\alpha_2(t)-2}}{\eta_1\eta_2}$$
$$\times\left\{\alpha'\lambda(1-\lambda)\frac{-1+(\eta_1+\eta_2)(1/\eta_1+1/\eta_2)}{1+\eta_1+\eta_2+\lambda(1-\lambda)(1/\eta_1+1/\eta_2)}\right\}^l. \tag{A·10}$$

Therefore case (i) represents a Regge pole located at $l=\alpha_2(t)$, which gives the asymptotic behavior $\sim s^{\alpha_2(t)}$. On the other hand, case (ii) gives after change of the variables the expression

$$T(l,t)\simeq\frac{-i\pi^2}{2(\alpha')^2}\int d\lambda d\xi d\eta\Gamma(-l)[\lambda(1-\lambda)]^{-\alpha_1(t/4)-1}(-\alpha'\xi\eta)^l$$
$$\times\int_{-\infty}^{2\alpha_1(t/4)-1}d\beta\frac{e^{-c(l-\beta)}}{l-\beta}, \tag{A·11}$$

where $c\equiv-\ln\rho_0$. (A·11) shows a Mandelstam branch point at $l=2\alpha_1(t/4)-1$, giving the asymptotic behavior $\sim s^{2\alpha_1(t/4)-1}/\ln s$.

The fixed pole at $l=-1$ also appears in (A·11) when one integrates over ξ or η (the Gribov-Pomeranchuk pole). It moves, however, into the second sheet when it passes the elastic threshold to be consistent with unitarity.[21]

Here one may ask why such a Mandelstam cut behavior appears in the amplitude, since when one looks at Fig. (A·6a), it seems to indicate a vanishing third double spectral function in t-channel.[22]

In order to show that the third spectral function is non-vanishing, we reproduce the left-half of the amplitude of Fig. (A·6a) in Fig. (A·9a). Twisting then the diagram of Fig. (A·9a) into that given in Fig. (A·9b), we find that singularities corresponding to the third spectral representation indeed exist.

$t\to\infty$, s fixed

The right-most singularity of the Mellin transformed function in this case occurs at

$$x\sim0,\ y\sim0,\ z\sim1,\ u_i/v_i\sim1\,. \tag{A·12}$$

Repeating procedures similar to those given above we find a Mandelstam branch point at $l=2\alpha_0(s/4)-1$ with a similar nature of discontinuity. The asymptotic behavior is, therefore, $\sim t^{2\alpha_0(s/4)-1}/\ln t$.

$s\to\infty$, u fixed

The right-most singularities of the Mellin transformed function occur at

(i) $u_1\sim0,\ v_2\sim0;\ u_2,\ v_1,\ x,\ y,\ z\sim1$

and

$$\text{(ii)} \quad u_2 \sim 0,\ v_1 \sim 0;\ u_1,\ v_2,\ x,\ y,\ z \sim 1.$$

In both cases we find the same Mandelstam branch point at $l = 2\alpha_1(u/4) - 1$ and the asymptotic behavior is, therefore, given by $s^{2\alpha_1(u/4)-1}/\ln s$.

Summarizing the results, we have found that the amplitude (A·6) has the following properties:

1) poles at α_0, α_1, $\alpha_2 = 0, 1, 2, \cdots$ with correct residues.

2) pole and/or cut asymptotic behavior.

3) The FESR is automatically satisfied by construction, when appropriate averaging procedures are assumed to be made.

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